Approximation of multipliers by regular operators

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Communicated by Prof. A.C. Zaanen at the meeting of October 29, 1990

INTRODUCTION

For a (real or complex) $L_p$ ($1 \leq p < \infty$) the space $\mathcal{L}(L_p)$ of all regular operators on $L_p$ is defined as the linear span of the positive operators. If $p = 1$, then $\mathcal{L}(L_1)$ coincides with the space $\mathcal{L}(L_1)$ of all continuous linear operators; cf. [18; chap. IV, Theorem 1.5], [1; sec. 15]. In this paper we show that $\mathcal{L}(L_p)$ is not dense in $\mathcal{L}(L_p)$ if $1 < p < \infty$ and $\dim L_p = \infty$. In particular we show that the Hilbert transformation $T$ on $L_p(G)$, for $G = \mathbb{Z}, \mathbb{R}, \mathbb{T}$, is strongly non-regular, i.e. $T$ does not belong to $\mathcal{L}(L_p(G))^{\omega}$. The fundamental idea is to prove that whenever there would exist a sequence $(T_n)$ in $\mathcal{L}(L_p(G))$ approximating $T$, it could already be chosen such that $T_n$ commutes with translations. This is achieved by showing that there exists a positive projection of the space of all operators onto the subspace of translation invariant operators.

We also show that an operator on a Hilbert space which can be approximated by regular operators with respect to all orderings induced by choosing some orthonormal basis is of the form $K + \lambda I$, with compact $K$ and $\lambda \in \mathbb{C}$.

1. A POSITIVE PROJECTION ONTO THE SPACE OF ALL TRANSLATION INVARIANT OPERATORS

In this section let $G$ be a locally compact group. For $a \in G$ and a function $f : G \to \mathbb{C}$ we denote by $af$ the left translate of $f$ by $a$, $af(x) := f(ax)$ ($x \in G$).

Let $\lambda$ be a left Haar measure on $G$ (i.e. $\int f d\lambda = \int_a f d\lambda$ for all $f \in C_c(G)$,
Let \( L := L(G, \lambda) \) (\( K \)-valued, where \( K = \mathbb{R} \) or \( K = \mathbb{C} \)), in this section.

Let \( \mathcal{D}_1(L_p) \) denote the set of left translation invariant operators,
\[
\mathcal{D}_1(L_p) := \{ T \in \mathcal{L}(L_p); T(a f) = a(T f) \text{ for all } a \in G, f \in L_p \}.
\]

A function \( f: G \rightarrow K \) is right uniformly continuous if \( \sup_x |f(x) - yf(x)| \rightarrow 0 \) if \( y \) tends to the unit in \( G \). The Banach space of all bounded, right uniformly continuous functions will be denoted by \( C_{b,r}(G) \).

A left invariant mean on \( C_{b,r}(G) \) is a functional \( M \in C_{b,r}(G)^* \) satisfying
\[
M(1) = 1, M \geq 0, M(a f) = M(f) \text{ for all } f \in C_{b,r}(G), a \in G \text{ (cf. [10; §1.1]).}
\]
In this section we shall assume \( G \) to be amenable, i.e., there exists a left invariant mean \( M \) on \( C_{b,r}(G) \). This is in fact equivalent to the existence of (left) invariant means on other function spaces, e.g. \( L_{\infty}(G, \lambda) \); cf. [10; §2.2], [16; Theorem 4.19].

1.1. **Theorem.** Let \( 1 \leq p < \infty \). There exists a positive, contractive projection \( \mathcal{Q} \) from \( \mathcal{L}(L_p) \) onto \( \mathcal{D}_1(L_p) \).

**Remark.** Let \( 1 < p < \infty, 1/p + 1/q = 1 \). Then \( \mathcal{Q} \) can be defined by \( \mathcal{Q}(T) = \tilde{T} \),
\[
(1.1) \quad \langle \tilde{T} f, g \rangle = M(a \rightarrow \langle T(a f), a g \rangle)
\]
\((f \in L_p, g \in L_q)\), where \( \langle \ldots \rangle \) denotes the natural duality bracket between \( L_p \) and \( L_q \). If \( p = 1 \), then \( \mathcal{Q} \) can be defined in such a way that (1.1) is true for all \( f \in L_1, g \in C_0(G) \).

**Proof.** (i) We first consider the case \( 1 < p < \infty \). Let \( T \in \mathcal{L}(L_p) \). For \( f \in L_p, g \in L_q \), the function \( a \rightarrow \langle T(a f), a g \rangle \) is bounded, \( |\langle T(a f), a g \rangle| \leq ||T|| ||f||_p ||g||_q \) \((a \in G)\), and right uniformly continuous; cf. [12; Theorem (20.4)]. Therefore, a bilinear mapping
\[
B: L_p \times L_q \rightarrow K
\]
is defined by
\[
(1.2) \quad B(f, g) := M(a \rightarrow \langle T(a f), a g \rangle),
\]
and the properties of \( M \) imply \( |B| \leq ||T|| \). This implies that there exists a unique \( \tilde{T} \in \mathcal{D}_1(L_p) \) such that \( B(f, g) = \langle \tilde{T} f, g \rangle \) for all \( f \in L_p, g \in L_q \), and we have the estimate \( ||\tilde{T}|| = ||B|| \leq ||T|| \).

Next we show \( \tilde{T} \in \mathcal{D}_1(L_p) \). Let \( b \in G, f \in L_p, g \in L_q \). Then
\[
\langle \tilde{T}(b f), g \rangle = M(a \rightarrow \langle T(b a f), a g \rangle)
\]
\[
= M(a \rightarrow \langle T(b a f), b^{-1} b g \rangle)
\]
\[
= M(a \rightarrow \langle T(a f), b^{-1} a g \rangle)
\]
\[
= \langle \tilde{T} f, b^{-1} g \rangle = \langle b(\tilde{T} f), g \rangle.
\]
This shows $\hat{T}(\phi f) = \phi (\hat{T}f)$, and thus $\hat{T} \in L_1(L_\phi)$.
If $T \in L_1(L_\phi)$ then it is clear from the definition that $\hat{T} = T$ holds.
If $T \geq 0$ then $B(f, g) \geq 0$ for $f \geq 0, g \geq 0$, and this implies $\hat{T} \geq 0$.

(ii) Let $p = 1$. Again let $T \in L_1$. Then (1.2) defines a continuous bilinear form $B$ on $L_1 \times C_0(G)$ such that $|B| \leq |T|$. So there exists a linear mapping $\hat{T} : L_1 \to C_0(G)'$, $|\hat{T}| = |B|$, such that

$$\langle g, \hat{T}f \rangle = B(f, g) \quad (f \in L_1, g \in C_0(G)).$$

For $\mu \in C_0(G)'$, $a \in G$, let $a\mu \in C_0(G)'$ be defined by

$$\langle \phi, a\mu \rangle = \langle \phi, \mu \rangle \quad (\phi \in C_0(G)).$$

Then one sees as in (i) that $\hat{T}_a f = \phi (\hat{T}f)$ for all $f \in L_1, a \in G$. This implies in particular that the mapping $G \ni a \mapsto \phi (\hat{T}f) \in C_0(G)'$ is continuous. Therefore, by [12; (19.27)], the measure $\hat{T}f$ is absolutely continuous with respect to $\lambda$. This means that in fact $\hat{T}$ maps $L_1$ into $L_1$. It is clear from the definition that $\hat{T}$ is positive if $T$ is positive.

2. RELATION TO THE PROJECTION ONTO THE CENTER

In this section let $G$ be a locally compact Abelian group, $\lambda$ a Haar measure on $G$, and $\hat{\lambda}$ the Haar measure on the character group $\hat{G}$, normalized by the requirement that the Fourier transformation $F : L_2(G, \lambda) \to L_2(\hat{G}, \hat{\lambda})$ be unitary. For brevity we shall use the notations $X := L_2(G, \lambda), Y := L_2(\hat{G}, \hat{\lambda})$.

The Fourier transformation induces a bijective linear mapping $\mathcal{S} : \mathcal{L}(X) \to \mathcal{L}(Y)$ defined by

$$\mathcal{S}(T) := FTF^{-1}.$$

Recall that for an order complete Banach lattice $E$ the space $\mathcal{L}'(E)$ of regular operators is an order complete Banach lattice; cf. [18; chap. IV, §1]. The center $\mathcal{Z}(E)$ of $\mathcal{L}(E)$ is the linear span of the order interval $[-I, I]$, where $I$ is the identity operator; it is a band in $\mathcal{L}'(E)$. We refer to [1; sections 8 and 15] for these statements; the elements of $\mathcal{Z}(E)$ are also called orthomorphisms or multiplication operators.

For $E = Y$, the center $\mathcal{Z}(Y)$ coincides with the multiplication operators by $L_\infty$-functions. This follows from [22; Theorem 7] and the localizability of $\hat{\lambda}$ (cf. [6; Theorem 9.4.8] together with [14; sec. 14, M1]).

Since $G$ is commutative the set of operators commuting with translations will be denoted by $\mathcal{L}_t(X) := \mathcal{L}_t(X)$. These operators are also called multipliers for $X$; cf. [5], [15], [7]. We recall that $\mathcal{S}$ maps $\mathcal{L}_t(X)$ onto $\mathcal{Z}(Y)$; cf. [15; Theorem 4.1.1].

Let $M$ be an invariant mean on $C_{b,m}(G) := C_{b,m}(G)$ (note that $G$ is amenable; cf. [10; Theorem 1.2.1]), and let $\mathcal{P} : \mathcal{L}(X) \to \mathcal{L}_t(X)$ be the projection associated with $M$ via Theorem 1.1. We define $\hat{\mathcal{P}} : \mathcal{L}(Y) \to \mathcal{Z}(Y)$ by

$$\hat{\mathcal{P}} := \mathcal{S} \mathcal{P} \mathcal{S}^{-1}.$$

Then $\hat{\mathcal{P}}$ is a contractive projection onto $\mathcal{Z}(Y)$.  

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2.1. **Theorem.** The restriction of $\mathcal{A}$ to $\mathcal{L}'(Y)$ is the band projection onto $\mathcal{J}(Y)$. In particular, $\mathcal{A}$ is positive.

**Proof.** (i) Let $T \in \mathcal{L}(Y)_+$. We show that $\mathcal{A}(T) \in \mathcal{L}'(Y)$ and $|\mathcal{A}(T)| \leq T$. In fact, let $f, g \in Y, f \geq 0$. It is easy to check that then

$$\int \mathcal{A}(T) f g \, d\lambda = M(a \mapsto \int T(a f) \bar{g} \, d\lambda),$$

where, in the last expression, $a \in G$ is interpreted as a character on $G$. Therefore

$$|\int \mathcal{A}(T) f g \, d\lambda| \leq \sup_a |\int T(a f) \bar{g} \, d\lambda|$$

$$\leq \|Tf\|_g \, d\lambda.$$

Since this is true for all $g \in Y$, we obtain $|\mathcal{A}(T) f| \leq Tf$. Now $\mathcal{A}(T) \in \mathcal{L}'(Y)$ follows from [18; chap. IV, Proposition 1.6], and $|\mathcal{A}(T)| \leq T$ is obtained from [18; chap. IV, Theorem 1.8].

(ii) If $0 \leq T \in \mathcal{J}(Y)_d$ then (i) implies $\mathcal{A}(T) \in \mathcal{J}(Y)_d$. Since also $\mathcal{A}(T) \in \mathcal{J}(Y)$ we obtain $\mathcal{A}(T) = 0$. This implies that $\mathcal{A}$, restricted to $\mathcal{L}'(Y)$, is the band projection onto $\mathcal{J}(Y)$. 

2.2. **Remark.** For an order complete Banach lattice $E$ it was shown in [21] that the band projection $\mathcal{P}: \mathcal{L}'(E) \to \mathcal{J}(E)$ is contractive with respect to the operator norm, and can therefore be extended as a contraction to all of $\mathcal{L}(E)$. For $E := Y$, Theorem 2.1 shows that such an extension is given by $\mathcal{A}$.

2.3. **Example.** If $G$ is compact, then the normalized Haar measure is the unique invariant mean on $C(G)$. In this case the projection $\mathcal{A}$ has the following form: Since $G$ is discrete each operator $T \in \mathcal{L}(Y)$ corresponds to a matrix $(t_{\beta y})_{\beta, y \in \hat{G}}$. Then $\mathcal{A}(T) = (\delta_{\beta y} t_{\beta y})_{\beta, y \in \hat{G}}$, with the Kronecker delta $(\delta_{\beta y})$. Indeed,

$$\int \mathcal{A}(T) \chi_{\beta |Y} \, d\lambda$$

$$= \int \mathcal{A}(F^{-1} TF) \beta \, d\lambda$$

$$= \int_{a \in G} \mathcal{A}(F^{-1} TF(a \beta) \bar{y} \, d\lambda \, d\lambda(a)$$

$$= \int_{a \in G} \beta(a) \bar{y}(a) \, d\lambda(a) \int_{\hat{G}} \mathcal{A}(T \chi_{\beta |Y}) \, d\lambda$$

$$= \delta_{\beta y} t_{\beta y}.$$

3. **Strong Non-Regularity of Multiplier Operators**

In the following $G$ is a locally compact Abelian group and $L_p = L_p(G, \lambda)$ ($1 \leq p < \infty$). By $M(G)$ we denote the space of all bounded Baire measures on $G$.

Let $T \in \mathcal{L}(L_p)$. Then $T(L_p \cap L_2) \subset L_p \cap L_2$, and there exists a unique $\mathcal{T} \in L_{1}(\hat{G}, \hat{\lambda})$ such that

$$FTf = \mathcal{T} \cdot Ff$$

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for all $f \in L_p \cap L_2$. Moreover, $\|T\| \leq \|T\|$ (see [15; Theorems 4.1.1 and 4.1.3]). For $\mu \in M(G)$ we define the convolution operator $T_\mu$ on $L_p$ by $T_\mu f = \mu * f$ (where $\mu * f(x) = \int f(y^{-1} x) \, d\mu(y)$). Then $\mu \mapsto T_\mu$ is a bijective linear mapping from $M(G)$ onto $L'(L_p) \cap \mathcal{L}_i(L_p)$ (see [2; Proposition 3.3]). In particular, for $T \in \mathcal{L}_i(L_p)$ one has

\[
(3.1) \quad T \in \mathcal{L}'(L_p) \text{ if and only if } T \in B(\hat{G}),
\]

where $B(\hat{G}) := \{ \hat{\mu}; \mu \in M(G) \}$ is the Fourier-Stieltjes algebra of $\hat{G}$. Here $\hat{\mu}$ is defined by $\hat{\mu}(\gamma) = \int (x, \gamma) \, d\mu(x)$ ($\gamma \in \hat{G}$). Then $B(\hat{G}) \subset C_{b,u}(\hat{G})$ (cf. [17; 1.3.3]), and we denote by $B(\hat{G})$ the closure of $B(\hat{G})$ in $C_p(\hat{G})$ with respect to the uniform norm.

3.1. THEOREM. Let $T \in \mathcal{L}_i(L_p)$. If $T \in \mathcal{L}'(L_p)$, then $\hat{T} \in B(\hat{G})$.

REMARK. $\mathcal{L}'(L_p)$ is the closure of $L'(L_p)$ in $L'(L_p)$ with respect to the operator norm. We call $T \in \mathcal{L}(L_p)$ strongly non-regular if $T \notin \mathcal{L}'(L_p)$.

PROOF. By Theorem 1.1 there exists a positive, contractive projection $\mathcal{P}$ from $\mathcal{L}(L_p)$ onto $\mathcal{L}_i(L_p)$. Let $S \in \mathcal{L}'(L_p)$. Since $\mathcal{P}$ is positive, it follows that $\mathcal{P}(S) \in \mathcal{L}'(L_p) \cap \mathcal{L}_i(L_p)$. Hence $\mathcal{P}(S) \in B(\hat{G})$ and $\|T - S\| \geq \|\mathcal{P}(T - S)\| = \|T - \mathcal{P}(S)\| \geq \|\hat{T} - \mathcal{P}(S)\|_\infty$. We have shown that dist$(T, \mathcal{L}'(L_p)) \geq$ dist$(\hat{T}, B(\hat{G}))$, where the expression on the left side is the distance in $L'(L_p)$ with respect to the operator norm, and on the right side in $L_\infty(\hat{G}, \lambda)$.

3.2. REMARKS. (a) If $p = 2$, then $\|T\| = \|\hat{T}\|_\infty$ for all $T \in \mathcal{L}_i(L_2(G))$. The proof of the theorem shows that

\[
(3.2) \quad \text{dist}(T, \mathcal{L}'(L_2)) = \text{dist}(\hat{T}, B(\hat{G}))
\]

for all $T \in \mathcal{L}_i(L_2)$.

(b) It follows from Theorem 3.1 that

\[
(3.3) \quad \hat{T} \in C_{b,u}(\hat{G}) \text{ for all } T \in \mathcal{L}_i(L_p) \cap \mathcal{L}'(L_p).
\]

Moreover, $C_0(\hat{G}) \subset B(\hat{G})$ by [15; 1.2.4]; in particular $B(\hat{G}) = C(\hat{G})$ if $\hat{G}$ is compact.

3.3. EXAMPLE (Hilbert transformation on $L_p(\mathbb{R})$). Let $1 < p < \infty$. The Hilbert transformation $T$ on $L_p(\mathbb{R})$ is the operator $T \in \mathcal{L}_i(L_p(\mathbb{R}))$ given by

\[
\hat{T}(x) = -i \, \text{sgn } x \quad (x \in \mathbb{R})
\]

(see [7; sec. 6.7]). Since $\hat{T}$ is not continuous, it follows by (3.3) that $T$ is strongly non-regular.

3.4. EXAMPLE (Hilbert transformation on $l_p(\mathbb{Z})$). Let $1 < p < \infty$. The Hilbert transformation is the operator $T \in \mathcal{L}_i(l_p(\mathbb{Z}))$ given by
\[ \hat{T}(e^{it}) = \begin{cases} 
\frac{i(t + \pi)}{\pi} & \text{for } -\pi \leq t < 0, 
\frac{i(t - \pi)}{\pi} & \text{for } 0 < t < \pi 
\end{cases} \]

(cf. [7; sec. 6.7]). Since \( \hat{T} \) is not continuous, \( T \) is strongly non-regular.

3.5. EXAMPLE (Schrödinger group on \( L_2(\mathbb{R}^n) \)). Let \( A \) be the negative Laplace operator in \( L_2(\mathbb{R}^n) \), given by \( D(A) = W_2^2(\mathbb{R}^n), Af = -\Delta f \). For \( t \in \mathbb{R} \) let \( T_t := e^{-itA}. \) Then \( \hat{T}_t(\xi) = e^{-it|\xi|^2} \) (where \( |\xi|^2 = \xi_1^2 + \cdots + \xi_n^2 \)). For \( t \neq 0 \) the function \( \hat{T}_t \) is uniformly continuous, and therefore (3.3) implies that \( T_t \) is strongly non-regular.

In order to give an example on \( L_p(\mathbb{T}) \) we need another criterion; we refer to [13; Theorem in sec. 7.11] for a related fact.

3.6. LEMMA. Let \( \mu \in M(G), \gamma \in \hat{G}. \) Then
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \hat{\mu}(\gamma^j) = \mu([\gamma = 1]), \]
where \( [\gamma = 1] = \{ x \in G ; \gamma(x) = 1 \}. \)

PROOF. Since for \( x \in G, \)
\[ \frac{1}{n} \sum_{j=1}^{n} \gamma(x)^j = \begin{cases} 
1 & \text{if } \gamma(x) = 1, 
\frac{1}{n} \frac{\gamma(x)^{n+1} - \gamma(x)}{\gamma(x) - 1} & \text{if } \gamma(x) \neq 1, 
\end{cases} \]
and \( |1/n \sum_{j=1}^{n} \gamma(x)^j| \leq 1 (n \in \mathbb{N}), \) it follows from the dominated convergence theorem that
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \hat{\mu}(\gamma^j) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \gamma(x)^j \mu(dx) = \mu([\gamma = 1]). \]

3.7. PROPOSITION. Let \( m \in B(\hat{G}), \gamma \in \hat{G}. \) Then \( \lim_{n \to \infty} 1/n \sum_{j=1}^{n} m(\gamma^j) \) and \( \lim_{n \to \infty} 1/n \sum_{j=1}^{n} m(\gamma^{-j}) \) exist and are equal.

PROOF. The property is true for \( m \in B(\hat{G}) \) by Lemma 3.6 and preserved by uniform limits.

3.8. COROLLARY. Let \( m \in B(\mathbb{Z}). \) Then \( \lim_{n \to \infty} 1/n \sum_{j=1}^{n} m(j) \) and \( \lim_{n \to \infty} 1/n \times \sum_{j=1}^{n} m(-j) \) exist and are equal. In particular, if \( m(\infty) := \lim_{n \to \infty} m(n) \) and \( m(-\infty) := \lim_{n \to -\infty} m(n) \) exist, then \( m(\infty) = m(-\infty). \)

3.9. EXAMPLE (Hilbert transformation on \( L_p(\mathbb{T}) \)). Let \( 1 < p < \infty. \) The Hilbert transformation \( T \) on \( L_p(\mathbb{T}) \) is the operator \( T \in L(\mathcal{L}(L_p(\mathbb{T}))) \) given by
\[ \hat{T}(k) = i \text{ sgn } k \quad (k \in \mathbb{Z}) \]

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(see [7; sec. 6.7]. Since \( \lim_{k \to -\infty} \hat{T}(k) \neq \lim_{k \to -\infty} \hat{T}(k) \), the operator \( T \) is strongly non-regular by Corollary 3.8.

3.10. REMARK. Let \( G = \mathbb{R}, \mathbb{T} \) or \( \mathbb{Z} \) and let \( T \) be the Hilbert transformation on \( L_2(G) \). It follows from (3.2) and the expression for \( \hat{T} \) that \( \text{dist}(T, \mathcal{L}''(L_2)) \geq 1 \); i.e. \( T \) is orthogonal to \( \mathcal{L}''(L_2) \) in the sense of Birkhoff [4]. This property had been proved by Synnatzschke [20] for several other singular integral transformations (e.g. the Fourier transformation on \( L_2(\mathbb{R}) \)).

3.11. REMARK. By a similar proof as that of Lemma 3.6 one can show that for every \( m \in \overline{B}(\mathbb{R}) \) the limits \( \lim_{t \to -\infty} \frac{1}{t} \int_0^t m(s) \, ds \) and \( \lim_{t \to -\infty} \frac{1}{t} \int_0^t m(-s) \, ds \) exist and are equal.

3.12. REMARK. For the case that \( G \) is not compact it was shown in [9] that for \( 1 < q < p < 2 \), \( \mathcal{L}(L_q) \) is not dense in \( \mathcal{L}(L_p) \). Since \( \mathcal{L}(L_q) \cap \mathcal{L}''(L_p) = \mathcal{L}(L_1) \) for all \( p \) it follows that for \( 1 < p < 2 \) the regular operators are not dense in \( \mathcal{L}(L_p) \) (and the same for \( 2 < p < \infty \), by duality).

4. EXISTENCE OF STRONGLY NON-REGULAR OPERATORS ON ARBITRARY \( L_p \)-SPACES

In Examples 3.3, 3.4 and 3.9 it was shown that there exists a strongly non-regular operator – the Hilbert transformation – on \( L_p(G) \), for \( G = \mathbb{Z}, \mathbb{R}, \mathbb{T} \) \( 1 < p < \infty \). In the first part of this section we show that this implies the existence of strongly non-regular operators on any infinite dimensional \( L_p \)-space.

4.1. PROPOSITION. Let \( 1 \leq p < \infty \), and let \( (\Omega, \mathcal{A}, \mu) \) be a measure space for which \( L_p(\Omega, \mathcal{A}, \mu) \) is infinite dimensional.

(a) There exist an isometric lattice homomorphism \( J : l_p \to L_p \) and a positive contraction \( K : L_p \to l_p \) such that \( K \circ J = \text{id}_{l_p} \).

(b) There exist a positive isometry \( \mathcal{J} : \mathcal{L}(l_p) \to \mathcal{L}(L_p) \) and a positive contraction \( \mathcal{K} : \mathcal{L}(L_p) \to \mathcal{L}(l_p) \) such that \( \mathcal{K} \circ \mathcal{J} = \text{id}_{\mathcal{L}(l_p)} \).

PROOF. (a) The assumption implies that there exists a disjoint sequence \( (\Omega_n)_{n \in \mathbb{N}} \) in \( \mathcal{A} \) such that \( 0 < \mu(\Omega_n) < \infty \) \( (n \in \mathbb{N}) \). The mappings \( J, K \) defined by

\[
J((x_n)) := \sum_{n} x_n \mu(\Omega_n)^{-1/p} \chi_{\Omega_n},
\]

\[
Kf := (\int \frac{1}{\mu(\Omega_n)} f \, d\mu(\Omega_n))_{n \in \mathbb{N}}
\]

(where \( 1/p + 1/q = 1 \)) have the asserted properties.

(b) With \( J, K \) from part (a) the mappings \( \mathcal{J}, \mathcal{K} \) defined by

\[
\mathcal{J}(T) := J \circ T \circ K \quad (T \in \mathcal{L}(l_p)),
\]

\[
\mathcal{K}(T) := K \circ T \circ J \quad (T \in \mathcal{L}(L_p))
\]

are as asserted. 

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4.2. **COROLLARY.** With $\mathcal{A}, \mathcal{K}$ as in Proposition 4.1 we have $\mathcal{K}(\mathcal{L}'(l_p)) = \mathcal{L}'(l_p)$. For all $T \in \mathcal{L}(l_p)$ we have
\[
\text{dist}(T, \mathcal{L}'(l_p)) = \text{dist}(\mathcal{A}(T), \mathcal{L}'(l_p)).
\]

The proof is an easy consequence of the properties of $\mathcal{A}$ and $\mathcal{K}$.

As a consequence of Proposition 4.1 and Corollary 4.2 we obtain from the existence of strongly non-regular operators on $l_p$ (see Example 3.4) the following theorem.

4.3. **THEOREM.** Let $1 \leq p < \infty$ and let $(\Omega, \mathcal{A}, \mu)$ be a measure space such that $X := L_p(\Omega, \mathcal{A}, \mu)$ is infinite dimensional. The following are equivalent.

(i) $p = 1$,
(ii) $\mathcal{L}(X) = \mathcal{L}'(X)$,
(iii) $\mathcal{L}'(X) = \mathcal{L}'(X)$.

In the second part of this section we show that a certain matrix represents a strongly non-regular operator $A$ on $l_p$; more precisely we show $\text{dist}(A, \mathcal{L}'(l_p)) = ||A||$. Similar matrices were used in a related context; cf. [18; chap. IV, §1, Examples], [3; Abschnitt 2].

4.4. **LEMMA.** Let $1 \leq p \leq \infty$, $A \in \mathbb{K}^{n \times n}$. Then $||A||_p \leq n^{1/p} ||A||_p$, where $||A||_p$ denotes the norm of $A$ as an operator on $(\mathbb{K}^n, ||.||_p)$.

**PROOF.** Let $A = (a_{jk})$, and choose $e_{jk}$ such that $|e_{jk}| = 1$, $|a_{jk}| = e_{jk} a_{jk}$. Let $x = (x_j) \in \mathbb{K}^n$, $||x||_p = 1$. Then
\[
||A||_p = (\sum_{j=1}^n |A_{jk} x_k|^p)^{1/p} \leq n^{1/p} (\max_j \sum_k e_{jk} |a_{jk} x_k|) = n^{1/p} \max_j (Ax^j)_j
\]
(\text{where } x^j = (e_{jk} |x_k|)_k, \text{ and } (Ax^j)_j \text{ denotes the } j\text{-th component of } Ax^j)\)
\[
\leq n^{1/p} \max_j ||Ax^j||_p \leq n^{1/p} ||A||_p.
\]

For $n \in \mathbb{N}_0$ we define recursively $2^n \times 2^n$-matrices $B_n$, by
\[
B_0 := (1),
B_n := \begin{pmatrix} B_{n-1} & B_{n-1} \\ B_{n-1} & -B_{n-1} \end{pmatrix} (n \geq 1).
\]

4.5. **LEMMA.** Let $2 \leq p \leq \infty$, $1/p + 1/q = 1$. Then $||B_n||_p = 2^{n/q}$, $||B_n||_q = 2^n$ $(n \in \mathbb{N}_0)$.

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PROOF. For the first equality we note first the obvious equality \( \|B_n\|_\infty = 2^n \). Next we remark that \( 2^{-n/2}B_n \) is an orthogonal matrix, and this implies \( \|B_n\|_2 = 2^{n/2} \). Now the Riesz-Thorin convexity theorem implies \( \|B_n\|_p \leq 2^{n/q} \). Testing with the vector \((1, 1, \ldots, 1)\) shows equality.

The second equality is easy to show. □

Now we fix \( 2 \leq p < \infty, 1/p + 1/q = 1 \). We define \( A_n := 2^{-n/q}B_n \) (\( n \in \mathbb{N}_0 \)). Then \( \|A_n\| = 1, \|A_n\| = 2^{n/p} \) (\( n \in \mathbb{N}_0 \)). Further we define the operator \( A \in \mathcal{B}(l_p) \) by representing \( I_p \) as the \( l_p \)-sum of \( 2^n \)-dimensional spaces \( E_n := (\ell^\infty, \| \cdot \|_p) \) in the obvious way, and letting \( A \) act as \( A_n \) on \( E_n \). In the matrix representation,

\[
A = \begin{bmatrix}
A_0 & 0 & & \\
& A_1 & & \\
& & A_2 & \\
& & & \ddots
\end{bmatrix}.
\]

4.6. THEOREM. One has \( \text{dist}_{\mathcal{B}(l_p)}(A, \mathcal{B}'(l_p)) = \|A\| = 1 \). Also, \( A \) acts as an operator in \( l_q \), and for this operator \( \text{dist}_{\mathcal{B}(l_q)}(A, \mathcal{B}'(l_q)) = \|A\| = 1 \).

PROOF. The equality \( \|A\| = 1 \) is immediate from \( \|A_n\| = 1 \) (\( n \in \mathbb{N}_0 \)).

Let \( S \in \mathcal{B}(l_p) \) be such that \( \|A - S\| < 1 \). We are going to show that this implies \( S \in \mathcal{B}'(l_p) \). Let \( \varepsilon := \|A - S\| (< 1) \). Define \( S_n \) as the \( 2^n \times 2^n \)-submatrix of \( S \) occupying the same place as \( A_n \) in \( A \). Then, using Lemma 4.4, we obtain

\[
\|B_n \times (A_n - S_n)\| \leq \|A_n - S_n\| \leq 2^{n/p} \|A_n - S_n\| \leq 2^{n/p} \varepsilon,
\]

where \( \times \) denotes the Schur product, i.e., entry by entry multiplication of matrices; note that \( B_n \times A_n = A_n \) by the definitions. Therefore

\[
\|S_n\| \geq \|B_n \times S_n\| \geq \|B_n \times A_n\| - \|B_n \times (A_n - S_n)\| \geq 2^{n/p} - 2^{n/p} \varepsilon = 2^{n/p} (1 - \varepsilon) \to \infty
\]

for \( n \to \infty \). This shows \( S \in \mathcal{B}'(l_p) \).

The statements concerning \( A \) as an operator in \( l_q \) follow by duality, since \( A \) is (formally) symmetric. □

4.7. REMARK (a permanence property for strong non-regularity). Let \( E \) be a Banach lattice and \( T \in \mathcal{L}(E) \) be strongly non-regular. Then \( (\lambda - T)^{-1} \) is strongly non-regular for all \( \lambda \in \mathcal{P}_{\infty}(T) \) (the unbounded component of the resolvent set of \( T \)). In fact, if \( R(\lambda) := (\lambda - T)^{-1} \in \mathcal{B}'(E) \) for one \( \lambda \in \mathcal{P}_{\infty}(T) \) then \( R(\lambda) \in \mathcal{B}'(E) \) for all \( \lambda \in \mathcal{P}_{\infty}(T) \), since \( \mathcal{B}'(E) \) is a closed subalgebra of \( \mathcal{B}(E) \). Consequently, \( T = \lim_{\lambda \to \infty} (\lambda^2 R(\lambda) - \lambda) \in \mathcal{B}'(E) \).

Moreover, if \( E = L_2 \), then \( R(\lambda, T) \notin \mathcal{B}'(L_2) \) for all \( \lambda \in \mathcal{P}(T) \) since \( \mathcal{B}'(L_2) \) is a full subalgebra of \( \mathcal{B}(L_2) \) (see [23; 24.6]).
Let $H$ be a separable infinite dimensional complex Hilbert space. Then, given any orthonormal basis on $H$, one may introduce a lattice ordering on $H$ by identifying $H$ with $l_2$, and one may ask which operators are regular and which operators are in $\mathcal{D}_f$ for all of these orderings.

The first question was answered independently by Sourour [19] and Sunder (cf. [11; Theorem 16.5]): For $T \in \mathcal{D}(H)$ the following are equivalent.

(i) $UTU^{-1} \in \mathcal{D}_f(l_2)$ for all unitary $U: H \to l_2$;

(ii) there exist a Hilbert-Schmidt operator $S$ on $H$ and $\lambda \in \mathbb{C}$ such that $T = S + \lambda I$.

Sourour [19] also observed that in (i) one may replace $l_2$ by $L_2(0, 1)$.

Concerning the second question, recall that every compact operator can be approximated in the operator norm by operators of finite rank. Hence if $T = K + \lambda I$ where $K$ is compact and $\lambda \in \mathbb{C}$, then $UTU^{-1} \in \mathcal{D}_f$ for all unitary operators $U: H \to l_2$ as well as for all unitary operators $U: H \to L_2(0, 1)$. We shall now prove that the converse is also true.

5.1. THEOREM. Let $T \in \mathcal{D}(H)$ be such that one of the following properties holds.

(a) $UTU^{-1} \in \mathcal{D}_f(l_2)$ for all unitary operators $U: H \to l_2$;

(b) $UTU^{-1} \in \mathcal{D}_f(L_2(0, 1))$ for all unitary operators $U: H \to L_2(0, 1)$.

Then there exist a compact operator $K$ and $\lambda \in \mathbb{C}$ such that $T = K + \lambda I$.

We use the following result which we extract from the proof of [19; Lemma 2].

5.2. LEMMA. Let $T \in \mathcal{D}(H)$ be a selfadjoint operator which is not of the form $K + \lambda I$ with compact $K$, $\lambda \in \mathbb{R}$. Then there exist a selfadjoint Hilbert–Schmidt operator $S$ and infinite dimensional closed subspaces $H_1, H_2, H_3$ of $H$ such that $H = H_1 \oplus H_2 \oplus H_3$, $H_1, H_2, H_3$ are invariant under $T + S$, $(T + S)_{H_1} = \alpha I_{H_1}$, and $(T + S)_{H_2} = \beta I_{H_2}$, with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$ (where $(T + S)_{H_j}$ denotes the part of $T + S$ in $H_j$, for $j = 1, 2$).

PROOF OF THEOREM 5.1. Since $UTU^{-1} \in \mathcal{D}_f$ for all unitary $U: H \to l_2$ ($U: H \to L_2(0, 1)$, respectively) the same is true for $T^*$, $(T + T^*)/2$ and $(T - T^*)/2i$. So we may assume that $T$ is selfadjoint.

We assume that $T$ is not of the form $K + \lambda I$, and obtain $S, H_1, H_2, H_3, \alpha, \beta$ from Lemma 5.2. It is sufficient to find a unitary $U: H \to l_2$ ($U: H \to L_2(0, 1)$) such that $U(T + S)U^{-1} \in \mathcal{D}_f$. Without restriction we may assume $S = 0$. (Note that any Hilbert-Schmidt operator on $l_2$ or $L_2(0, 1)$ is regular.)

Now if $H_0$ is a closed subspace of $H$ such that dim $H_0 = \dim H_0^\perp = \infty$ and $TH_0 \subset H_0$ then also the part $T_0 := T_{H_0}$ of $T$ in $H_0$ has the property that $U_0 T_0 U_0^{-1} \in \mathcal{D}_f$ for all unitary $U_0: H_0 \to l_2$ ($U_0: H_0 \to L_2(0, 1)$). (In fact, let $U_0: H_0 \to l_2$ ($U_0: H_0 \to L_2(0, 1)$) be unitary. Consider a unitary extension $U: H \to l_2 \oplus l_2$ ($U: H \to L_2(0, 1) \oplus L_2(0, 1)$). By hypothesis there exist $T_0 \in \mathcal{D}_f$ such
that \( \lim_{n \to \infty} T_n = U T U^{-1} \). Since the orthogonal projection \( P \) of \( L_2(0, 1) \oplus L_2(0, 1) \), respectively) onto the first component is positive, the operators \( P T_n | L_2 \) (\( P T_n | L_2(0, 1) \)) are regular and converge to \( U_0 T_0 U_0^{-1} \).

We apply the previous remark to \( T_0 := T_{H_1 \oplus H_2} = \alpha I_{H_1} \oplus \beta I_{H_2} \). Now we consider the two cases (a) and (b) separately.

Case (a). We identify \( H_1 \) unitarily with \( L_2(0, \pi) \) and \( H_2 \) with \( L_2(\pi, 2\pi) \). Then \( T_0 \) is given by \( T_0 f = m f (f \in L_2(0, 2\pi)) \) where

\[
m(x) = \begin{cases} 
\alpha & \text{if } x \leq \pi, \\
\beta & \text{if } x > \pi.
\end{cases}
\]

Since \( m \) is not continuous it follows from Theorem 3.1 that \( F T_0 F^{-1} \) is strongly non-regular, where \( F : L_2(0, 2\pi) \to L_2(\mathbb{Z}) \) is the Fourier transformation. This is a contradiction.

Case (b). We identify \( H_1 \) unitarily with \( L_2(-\infty) \) and \( H_2 \) with \( L_2(\mathbb{N} \cup \{0\}) \). Then \( T_0 x = (m_n x_n)_{n \in \mathbb{Z}} \), where

\[
m_n = \begin{cases} 
\alpha & \text{if } n < 0, \\
\beta & \text{if } n \geq 0.
\end{cases}
\]

Since \( \lim_{n \to -\infty} m_n \neq \lim_{n \to -\infty} m_n \), it follows from Corollary 3.8 that \( F T_0 F^{-1} \) is strongly non-regular, where \( F : L_2(\mathbb{Z}) \to L_2(\mathbb{Z}) \) is the Fourier transformation. This is a contradiction.

5.3. REMARK. It was pointed out to the authors that Theorem 5.1 can also be obtained as a consequence of [8; Theorem 1 and Corollary 3]. However, the proof given here is more direct and elementary in the present context.

REFERENCES