Indag. Mathem., N.S., 2 (2), 159-170 June 17, 1991

Approximation of multipliers by regular operators

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Communicated by Prof. A.C. Zaanen at the meeting of October 29, 1990

INTRODUCTION

For a (real or complex) L_p $(1 \le p < \infty)$ the space $\mathscr{L}^r(L_p)$ of all regular operators on L_p is defined as the linear span of the positive operators. If $p=1$, then $\mathscr{L}^r(L_p)$ coincides with the space $\mathscr{L}(L_p)$ of all continuous linear operators; cf. [18; chap. IV, Theorem 1.5], [1; sec. 15]. In this paper we show that $\mathscr{L}^r(L_p)$ is not dense in $\mathscr{L}(L_p)$ if $1<p<\infty$ and $\dim L_p = \infty$. In particular we show that the Hilbert transformation T on $L_p(G)$, for $G=\mathbb{Z}, \mathbb{R}, \mathbb{T}$, is *strongly non-regular*, i.e. T does not belong to $\overline{\mathscr{L}^r(L_p(G))}^{\mathscr{L}}$. The fundamental idea is to prove that whenever there would exist a sequence (T_n) in $\mathscr{L}^r(L_p(G))$ approximating T, it could already be chosen such that T_n commutes with translations. This is achieved by showing that there exists a positive projection of the space of all operators onto the subspace of translation invariant operators.

We also show that an operator on a Hilbert space which can be approximated by regular operators with respect to all orderings induced by choosing some orthonormal basis is of the form $K+\lambda I$, with compact K and $\lambda \in \mathbb{C}$.

1. A POSITIVE PROJECTION ONTO THE SPACE OF ALL TRANSLATION INVARIANT OPERATORS

In this section let G be a locally compact group. For $a \in G$ and a function $f: G \to \mathbb{C}$ we denote by ${}_af$ the *left translate of* f *by* a, ${}_af(x) := f(ax)$ $(x \in G)$.

Let λ be a left Haar measure on G (i.e. $\int f d\lambda = \int {}_a f d\lambda$ for all $f \in C_c(G)$,

$a \in G$; cf. [12; §15]). For $1 \le p < \infty$ we shall write $L_p := L_p(G, \lambda)$ ($\mathbb{K}$-valued, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), in this section.

Let $\mathscr{L}_{li}(L_p)$ denote the set of left translation invariant operators,

$$\mathscr{L}_{li}(L_p) := \{T \in \mathscr{L}(L_p); T({}_a f) = {}_a(Tf) \text{ for all } a \in G, f \in L_p\}.$$

A function $f: G \to \mathbb{K}$ is right uniformly continuous if $\sup_x |f(x) - {}_y f(x)| \to 0$ if y tends to the unit in G. The Banach space of all bounded, right uniformly continuous functions will be denoted by $C_{b,ru}(G)$.

A *left invariant mean* on $C_{b,ru}(G)$ is a functional $M \in C_{b,ru}(G)'$ satisfying $M(1) = 1, M \ge 0$, $M({}_a f) = M(f)$ for all $f \in C_{b,ru}(G), a \in G$ (cf. [10; §1.1]). In this section we shall assume G to be *amenable*, i.e., there exists a left invariant mean M on $C_{b,ru}(G)$. This is in fact equivalent to the existence of (left) invariant means on other function spaces, e.g. $L_\infty(G, \lambda)$; cf. [10; §2.2], [16; Theorem 4.19].

1.1. THEOREM. *Let $1 \le p < \infty$. There exists a positive, contractive projection $\mathscr{Q}$ from $\mathscr{L}(L_p)$ onto $\mathscr{L}_{li}(L_p)$.*

REMARK. Let $1 < p < \infty, 1/p + 1/q = 1$. Then $\mathscr{Q}$ can be defined by $\mathscr{Q}(T) = \tilde{T}$,

$$\langle \tilde{T} f, g \rangle = M(a \mapsto \langle T({}_a f), {}_a g \rangle) \tag{1.1}$$

($f \in L_p, g \in L_q$), where $\langle .,. \rangle$ denotes the natural duality bracket between L_p and L_q. If $p = 1$, then $\mathscr{Q}$ can be defined in such a way that (1.1) is true for all $f \in L_1, g \in C_0(G)$.

PROOF. (i) We first consider the case $1 < p < \infty$. Let $T \in \mathscr{L}(L_p)$. For $f \in L_p, g \in L_q$, the function $a \mapsto \langle T({}_a f), {}_a g \rangle$ is bounded, $|\langle T({}_a f), {}_a g \rangle| \le \|T\| \|f\|_p \|g\|_q$ ($a \in G$), and right uniformly continuous; cf. [12; Theorem (20.4)]. Therefore, a bilinear mapping

$$B: L_p \times L_q \to \mathbb{K}$$

is defined by

$$B(f, g) := M(a \mapsto \langle T({}_a f), {}_a g \rangle), \tag{1.2}$$

and the properties of M imply $\|B\| \le \|T\|$. This implies that there exists a unique $\tilde{T} \in \mathscr{L}(L_p)$ such that $B(f, g) = \langle \tilde{T} f, g \rangle$ for all $f \in L_p, g \in L_q$, and we have the estimate $\|\tilde{T}\| = \|B\| \le \|T\|$.

Next we show $\tilde{T} \in \mathscr{L}_{li}(L_p)$. Let $b \in G, f \in L_p, g \in L_q$. Then

$$\begin{aligned} \langle \tilde{T}({}_b f), g \rangle &= M(a \mapsto \langle T({}_a({}_b f)), {}_a g \rangle) \\ &= M(a \mapsto \langle T({}_{ba} f), {}_{b^{-1}ba} g \rangle) \\ &= M(a \mapsto \langle T({}_a f), {}_{b^{-1}a} g \rangle) \\ &= \langle \tilde{T} f, {}_{b^{-1}} g \rangle = \langle {}_b(\tilde{T} f), g \rangle. \end{aligned}$$

This shows $\tilde{T}({}_bf) = {}_b(\tilde{T}f)$, and thus $\tilde{T} \in \mathscr{L}_{1i}(L_p)$.

If $T \in \mathscr{L}_{1i}(L_p)$ then it is clear from the definition that $\tilde{T} = T$ holds.

If $T \geq 0$ then $B(f, g) \geq 0$ for $f \geq 0, g \geq 0$, and this implies $\tilde{T} \geq 0$.

(ii) Let $p = 1$. Again let $T \in \mathscr{L}(L_1)$. Then (1.2) defines a continuous bilinear form B on $L_1 \times C_0(G)$ such that $\|B\| \leq \|T\|$. So there exists a linear mapping $\tilde{T}: L_1 \to C_0(G)'$, $\|\tilde{T}\| = \|B\|$, such that

$$\langle g, \tilde{T}f \rangle = B(f, g) \qquad (f \in L_1, g \in C_0(G)).$$

For $\mu \in C_0(G)', a \in G$, let ${}_a\mu \in C_0(G)'$ be defined by

$$\langle \varphi, {}_a\mu \rangle = \langle {}_{a^{-1}}\varphi, \mu \rangle \qquad (\varphi \in C_0(G)).$$

Then one sees as in (i) that $\tilde{T}_af = {}_a(\tilde{T}f)$ for all $f \in L_1, a \in G$. This implies in particular that the mapping $G \ni a \mapsto {}_a(\tilde{T}f) \in C_0(G)'$ is continuous. Therefore, by [12; (19.27)], the measure $\tilde{T}f$ is absolutely continuous with respect to λ. This means that in fact $\tilde{T}$ maps L_1 into L_1. It is clear from the definition that $\tilde{T}$ is positive if T is positive. ■

2. RELATION TO THE PROJECTION ONTO THE CENTER

In this section let G be a locally compact Abelian group, λ a Haar measure on G, and $\hat{\lambda}$ the Haar measure on the character group $\hat{G}$, normalized by the requirement that the Fourier transformation $F: L_2(G, \lambda) \to L_2(\hat{G}, \hat{\lambda})$ be unitary. For brevity we shall use the notations $X := L_2(G, \lambda)$, $Y := L_2(\hat{G}, \hat{\lambda})$.

The Fourier transformation induces a bijective linear mapping $\mathscr{I}: \mathscr{L}(X) \to \mathscr{L}(Y)$ defined by

$$\mathscr{I}(T) := FTF^{-1}.$$

Recall that for an order complete Banach lattice E the space $\mathscr{L}^r(E)$ of regular operators is an order complete Banach lattice; cf. [18; chap. IV, § 1]. The *center* $\mathscr{Z}(E)$ of $\mathscr{L}(E)$ is the linear span of the order interval $[-I, I]$, where I is the identity operator; it is a band in $\mathscr{L}^r(E)$. We refer to [1; sections 8 and 15] for these statements; the elements of $\mathscr{Z}(E)$ are also called orthomorphisms or multiplication operators.

For $E = Y$, the center $\mathscr{Z}(Y)$ coincides with the multiplication operators by L_∞-functions. This follows from [22; Theorem 7] and the localizability of $\hat{\lambda}$ (cf. [6; Theorem 9.4.8] together with [14; sec. 14, M]).

Since G is commutative the set of operators commuting with translations will be denoted by $\mathscr{L}_i(X) := \mathscr{L}_{1i}(X)$. These operators are also called *multipliers* for X; cf. [5], [15], [7]. We recall that $\mathscr{I}$ maps $\mathscr{L}_i(X)$ onto $\mathscr{Z}(Y)$; cf. [15; Theorem 4.1.1].

Let M be an invariant mean on $C_{b,u}(G) := C_{b,ru}(G)$ (note that G is amenable; cf. [10; Theorem 1.2.1]), and let $\mathscr{Q}: \mathscr{L}(X) \to \mathscr{L}_i(X)$ be the projection associated with M via Theorem 1.1. We define $\hat{\mathscr{Q}}: \mathscr{L}(Y) \to \mathscr{Z}(Y)$ by

$$\hat{\mathscr{Q}} := \mathscr{I}\mathscr{Q}\mathscr{I}^{-1}.$$

Then $\hat{\mathscr{Q}}$ is a contractive projection onto $\mathscr{Z}(Y)$.

2.1. THEOREM. *The restriction of $\hat{\mathscr{Q}}$ to $\mathscr{L}^r(Y)$ is the band projection onto $\mathscr{Z}(Y)$. In particular, $\hat{\mathscr{Q}}$ is positive.*

PROOF. (i) Let $T \in \mathscr{L}(Y)_+$. We show that $\hat{\mathscr{Q}}(T) \in \mathscr{L}^r(Y)$ and $|\hat{\mathscr{Q}}(T)| \leq T$. In fact, let $f, g \in Y, f \geq 0$. It is easy to check that then

$$\int \hat{\mathscr{Q}}(T) f \bar{g} \, d\hat{\lambda} = M(a \mapsto \int T(af) \overline{ag} \, d\hat{\lambda}),$$

where, in the last expression, $a \in G$ is interpreted as a character on $\hat{G}$. Therefore

$$|\int \hat{\mathscr{Q}}(T) f \bar{g} \, d\hat{\lambda}| \leq \sup_a |\int T(af) \overline{ag} \, d\hat{\lambda}|$$

$$\leq \int Tf |g| \, d\hat{\lambda}.$$

Since this is true for all $g \in Y$, we obtain $|\hat{\mathscr{Q}}(T) f| \leq Tf$. Now $\hat{\mathscr{Q}}(T) \in \mathscr{L}^r(Y)$ follows from [18; chap. IV, Proposition 1.6], and $|\hat{\mathscr{Q}}(T)| \leq T$ is obtained from [18; chap. IV, Theorem 1.8].

(ii) If $0 \leq T \in \mathscr{Z}(Y)^d$ then (i) implies $\hat{\mathscr{Q}}(T) \in \mathscr{Z}(Y)^d$. Since also $\hat{\mathscr{Q}}(T) \in \mathscr{Z}(Y)$ we obtain $\hat{\mathscr{Q}}(T) = 0$. This implies that $\hat{\mathscr{Q}}$, restricted to $\mathscr{L}^r(Y)$, is the band projection onto $\mathscr{Z}(Y)$. ■

2.2. REMARK. For an order complete Banach lattice E it was shown in [21] that the band projection $\mathscr{P}: \mathscr{L}^r(E) \to \mathscr{Z}(E)$ is contractive with respect to the operator norm, and can therefore be extended as a contraction to all of $\mathscr{L}(E)$. For $E := Y$, Theorem 2.1 shows that such an extension is given by $\hat{\mathscr{Q}}$.

2.3. EXAMPLE. If G is compact, then the normalized Haar measure is the unique invariant mean on $C(G)$. In this case the projection $\hat{\mathscr{Q}}$ has the following form: Since $\hat{G}$ is discrete each operator $T \in \mathscr{L}(Y)$ corresponds to a matrix $(t_{\beta\gamma})_{\beta, \gamma \in \hat{G}}$. Then $\hat{\mathscr{Q}}(T) = (\delta_{\beta\gamma} t_{\beta,\gamma})_{\beta, \gamma \in \hat{G}}$, with the Kronecker delta $(\delta_{\beta\gamma})$. Indeed,

$$\int_{\hat{G}} \hat{\mathscr{Q}}(T) \chi_{\{\beta\}} \chi_{\{\gamma\}} \, d\hat{\lambda}$$

$$= \int_G \mathscr{Q}(F^{-1} TF) \beta \bar{\gamma} \, d\lambda$$

$$= \int_{a \in G} \int_G F^{-1} TF({}_a\beta) \overline{{}_a\gamma} \, d\lambda \, d\lambda(a)$$

$$= \int_{a \in G} \beta(a) \overline{\gamma(a)} \, d\lambda(a) \int_{\hat{G}} T\chi_{\{\beta\}} \chi_{\{\gamma\}} \, d\hat{\lambda} =$$

$$= \delta_{\beta\gamma} t_{\beta\gamma}.$$

3. STRONG NON-REGULARITY OF MULTIPLIER OPERATORS

In the following G is a locally compact Abelian group and $L_p = L_p(G, \lambda)$ $(1 \leq p < \infty)$. By $M(G)$ we denote the space of all bounded Baire measures on G.

Let $T \in \mathscr{L}_i(L_p)$. Then $T(L_p \cap L_2) \subset L_p \cap L_2$, and there exists a unique $\hat{T} \in L_\infty(\hat{G}, \hat{\lambda})$ such that

$$FTf = \hat{T} \cdot Ff$$

for all $f \in L_p \cap L_2$. Moreover, $\|\hat{T}\|_\infty \leq \|T\|$ (see [15; Theorems 4.1.1 and 4.1.3]). For $\mu \in M(G)$ we define the convolution operator T_μ on L_p by $T_\mu f = \mu * f$ (where $\mu * f(x) = \int f(y^{-1}x)\, d\mu(y)$). Then $\mu \mapsto T_\mu$ is a bijective linear mapping from $M(G)$ onto $\mathscr{L}^r(L_p) \cap \mathscr{L}_i(L_p)$ (see [2; Proposition 3.3]). In particular, for $T \in \mathscr{L}_i(L_p)$ one has

(3.1) $\quad T \in \mathscr{L}^r(L_p)$ if and only if $\hat{T} \in B(\hat{G})$,

where $B(\hat{G}) := \{\hat{\mu}; \mu \in M(G)\}$ is the Fourier-Stieltjes algebra of $\hat{G}$. Here $\hat{\mu}$ is defined by $\hat{\mu}(\gamma) = \int \overline{(x, \gamma)}\, d\mu(x)$ $(\gamma \in \hat{G})$. Then $B(\hat{G}) \subset C_{b,u}(\hat{G})$ (cf. [17; 1.3.3]), and we denote by $\overline{B(\hat{G})}$ the closure of $B(\hat{G})$ in $C_b(\hat{G})$ with respect to the uniform norm.

3.1. THEOREM. *Let $T \in \mathscr{L}_i(L_p)$. If $T \in \overline{\mathscr{L}^r(L_p)}$, then $\hat{T} \in \overline{B(\hat{G})}$.*

REMARK. $\overline{\mathscr{L}^r(L_p)}$ is the closure of $\mathscr{L}^r(L_p)$ in $\mathscr{L}(L_p)$ with respect to the operator norm. We call $T \in \mathscr{L}(L_p)$ *strongly non-regular* if $T \notin \overline{\mathscr{L}^r(L_p)}$.

PROOF. By Theorem 1.1 there exists a positive, contractive projection $\mathscr{Q}$ from $\mathscr{L}(L_p)$ onto $\mathscr{L}_i(L_p)$. Let $S \in \mathscr{L}^r(L_p)$. Since $\mathscr{Q}$ is positive, it follows that $\mathscr{Q}(S) \in \mathscr{L}^r(L_p) \cap \mathscr{L}_i(L_p)$. Hence $\widehat{\mathscr{Q}(S)} \in B(\hat{G})$ and $\|T - S\| \geq \|\mathscr{Q}(T - S)\| = \|T - \mathscr{Q}(S)\| \geq \|\hat{T} - \widehat{\mathscr{Q}(S)}\|_\infty$. We have shown that $\operatorname{dist}(T, \mathscr{L}^r(L_p)) \geq \operatorname{dist}(\hat{T}, B(\hat{G}))$, where the expression on the left side is the distance in $\mathscr{L}(L_p)$ with respect to the operator norm, and on the right side in $L_\infty(\hat{G}, \lambda)$. ∎

3.2. REMARKS. (a) If $p = 2$, then $\|T\| = \|\hat{T}\|_\infty$ for all $T \in \mathscr{L}_i(L_2(G))$. The proof of the theorem shows that

(3.2) $\quad \operatorname{dist}(T, \mathscr{L}^r(L_2)) = \operatorname{dist}(\hat{T}, B(\hat{G}))$

for all $T \in \mathscr{L}_i(L_2)$.

(b) It follows from Theorem 3.1 that

(3.3) $\quad \hat{T} \in C_{b,u}(\hat{G})$ for all $T \in \mathscr{L}_i(L_p) \cap \overline{\mathscr{L}^r(L_p)}$.

Moreover, $C_0(\hat{G}) \subset \overline{B(\hat{G})}$ by [15; 1.2.4]; in particular $\overline{B(\hat{G})} = C(\hat{G})$) if $\hat{G}$ is compact.

3.3. EXAMPLE (Hilbert transformation on $L_p(\mathbb{R})$). Let $1 < p < \infty$. The Hilbert transformation T on $L_p(\mathbb{R})$ is the operator $T \in \mathscr{L}_i(L_p(\mathbb{R}))$ given by

$$\hat{T}(x) = -i \operatorname{sgn} x \quad (x \in \mathbb{R})$$

(see [7; sec. 6.7]). Since $\hat{T}$ is not continuous, it follows by (3.3) that T is strongly non-regular.

3.4. EXAMPLE (Hilbert transformation on $l_p(\mathbb{Z})$). Let $1 < p < \infty$. The Hilbert transformation is the operator $T \in \mathscr{L}_i(l_p(\mathbb{Z}))$ given by

$$\hat{T}(e^{it}) = \begin{cases} i(t+\pi)/\pi & \text{for } -\pi \le t \le 0, \\ i(t-\pi)/\pi & \text{for } 0 < t < \pi \end{cases}$$

(cf. [7; sec. 6.7]). Since $\hat{T}$ is not continuous, T is strongly non-regular.

3.5. EXAMPLE (Schrödinger group on $L_2(\mathbb{R}^n)$). Let A be the negative Laplace operator in $L_2(\mathbb{R}^n)$, given by $D(A) := W_2^2(\mathbb{R}^n)$, $Af := -\Delta f$. For $t \in \mathbb{R}$ let $T_t := e^{-itA}$. Then $\hat{T}_t(\xi) = e^{-it|\xi|^2}$ (where $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$). For $t \ne 0$ the function $\hat{T}_t$ is not uniformly continuous, and therefore (3.3) implies that T_t is strongly non-regular.

In order to give an example on $L_p(\mathbb{T})$ we need another criterion; we refer to [13; Theorem in sec. 7.11] for a related fact.

3.6. LEMMA. *Let $\mu \in M(G), \gamma \in \hat{G}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \hat{\mu}(\gamma^j) = \mu([\gamma = 1]),$$

where $[\gamma = 1] = \{x \in G; \gamma(x) = 1\}$.

PROOF. Since for $x \in G$,

$$\frac{1}{n} \sum_{j=1}^{n} \bar{\gamma}(x)^j = \begin{cases} 1 & \text{if } \gamma(x) = 1. \\ \dfrac{1}{n} \dfrac{\bar{\gamma}(x)^{n+1} - \bar{\gamma}(x)}{\bar{\gamma}(x) - 1} & \text{if } \gamma(x) \ne 1, \end{cases}$$

$$\to \chi_{[\gamma=1]}(x) \quad (n \to \infty)$$

and $|1/n \sum_{j=1}^n \bar{\gamma}(x)^j| \le 1$ $(n \in \mathbb{N})$, it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \hat{\mu}(\gamma^j) = \lim_{n\to\infty} \int \frac{1}{n} \sum_{j=1}^{n} \bar{\gamma}(x)^j \, d\mu(x) = \mu([\gamma = 1]). \qquad \blacksquare$$

3.7. PROPOSITION. *Let $m \in \overline{B(\hat{G})}, \gamma \in \hat{G}$. Then $\lim_{n\to\infty} 1/n \sum_{j=1}^n m(\gamma^j)$ and $\lim_{n\to\infty} 1/n \sum_{j=1}^n m(\gamma^{-j})$ exist and are equal.*

PROOF. The property is true for $m \in B(\hat{G})$ by Lemma 3.6 and preserved by uniform limits. ∎

3.8. COROLLARY. *Let $m \in \overline{B(\mathbb{Z})}$. Then $\lim_{n\to\infty} 1/n \sum_{j=1}^n m(j)$ and $\lim_{n\to\infty} 1/n \times \sum_{j=1}^n m(-j)$ exist and are equal. In particular, if $m(\infty) := \lim_{n\to\infty} m(n)$ and $m(-\infty) := \lim_{n\to-\infty} m(n)$ exist, then $m(\infty) = m(-\infty)$.*

3.9. EXAMPLE (Hilbert transformation on $L_p(\mathbb{T})$). Let $1 < p < \infty$. The Hilbert transformation T on $L_p(\mathbb{T})$ is the operator $T \in \mathcal{L}_i(L_p(\mathbb{T}))$ given by

$$\hat{T}(k) = i \operatorname{sgn} k \qquad (k \in \mathbb{Z})$$

(see [7; sec. 6.7]. Since $\lim_{k\to\infty}\hat{T}(k)\neq\lim_{k\to-\infty}\hat{T}(k)$, the operator T is strongly non-regular by Corollary 3.8.

3.10. REMARK. Let $G=\mathbb{R},\mathbb{T}$ or $\mathbb{Z}$ and let T be the Hilbert transformation on $L_2(G)$. It follows from (3.2) and the expression for $\hat{T}$ that $\operatorname{dist}(T,\mathscr{L}^r(L_2))\geq 1$; i.e. T is orthogonal to $\mathscr{L}^r(L_2)$ in the sense of Birkhoff [4]. This property had been proved by Synnatzschke [20] for several other singular integral transformations (e.g. the Fourier transformation on $L_2(\mathbb{R})$).

3.11. REMARK. By a similar proof as that of Lemma 3.6 one can show that for every $m\in\overline{B(\mathbb{R})}$ the limits $\lim_{t\to\infty}1/t\int_0^t m(s)\,ds$ and $\lim_{t\to\infty}1/t\int_0^t m(-s)\,ds$ exist and are equal.

3.12. REMARK. For the case that G is not compact it was shown in [9] that for $1\leq q<p\leq 2$, $\mathscr{L}_i(L_q)$ is not dense $\mathscr{L}_i(L_p)$. Since $\mathscr{L}_i\cap\mathscr{L}^r(L_p)=\mathscr{L}_i(L_1)$ for all p it follows that for $1<p\leq 2$ the regular operators are not dense in $\mathscr{L}(L_p)$ (and the same for $2\leq p<\infty$, by duality).

4. EXISTENCE OF STRONGLY NON-REGULAR OPERATORS ON ARBITRARY L_p-SPACES

In Examples 3.3, 3.4 and 3.9 it was shown that there exists a strongly non-regular operator – the Hilbert transformation – on $L_p(G)$, for $G=\mathbb{Z},\mathbb{R},\mathbb{T}$ $1<p<\infty$. In the first part of this section we show that this implies the existence of strongly non-regular operators on any infinite dimensional L_p-space.

4.1. PROPOSITION. *Let* $1\leq p<\infty$, and let $(\Omega,\mathscr{A},\mu)$ *be a measure space for which* $L_p(\Omega,\mathscr{A},\mu)$ *is infinite dimensional.*

(a) *There exist an isometric lattice homomorphism* $J:l_p\to L_p$ *and a positive contraction* $K:L_p\to l_p$ *such that* $K\circ J=id_{l_p}$.

(b) *There exist a positive isometry* $\mathscr{J}:\mathscr{L}(l_p)\to\mathscr{L}(L_p)$ *and a positive contraction* $\mathscr{K}:\mathscr{L}(L_p)\to\mathscr{L}(l_p)$ *such that* $\mathscr{K}\circ\mathscr{J}=id_{\mathscr{L}(l_p)}$.

PROOF. (a) The assumption implies that there exists a disjoint sequence $(\Omega_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that $0<\mu(\Omega_n)<\infty$ $(n\in\mathbb{N})$. The mappings J,K defined by

$$J((x_n)):=\sum_n x_n\mu(\Omega_n)^{-1/p}\chi_{\Omega_n},$$

$$Kf:=((\int_{\Omega_n} f\,d\mu)\mu(\Omega_n)^{-1/q})_{n\in\mathbb{N}}$$

(where $1/p+1/q=1$) have the asserted properties.

(b) With J,K from part (a) the mappings $\mathscr{J},\mathscr{K}$ defined by

$$\mathscr{J}(T):=J\circ T\circ K \qquad (T\in\mathscr{L}(l_p)),$$

$$\mathscr{K}(T):=K\circ T\circ J \qquad (T\in\mathscr{L}(L_p))$$

are as asserted. ∎

4.2. COROLLARY. *With $\mathcal{J},\mathcal{K}$ as in Proposition* 4.1 *we have* $\mathcal{K}(\mathcal{L}^r(L_p))=\mathcal{L}^r(l_p)$. *For all* $T\in\mathcal{L}(l_p)$ *we have*

$$\operatorname{dist}(T,\mathcal{L}^r(l_p))=\operatorname{dist}(\mathcal{J}(T),\mathcal{L}^r(L_p)).$$

The proof is an easy consequence of the properties of $\mathcal{J}$ and $\mathcal{K}$.

As a consequence of Proposition 4.1 and Corollary 4.2 we obtain from the existence of strongly non-regular operators on l_p (see Example 3.4) the following theorem.

4.3. THEOREM. *Let* $1\le p<\infty$ *and let* $(\Omega,\mathcal{A},\mu)$ *be a measure space such that* $X:=L_p(\Omega,\mathcal{A},\mu)$ *is infinite dimensional. The following are equivalent.*

(i) $p=1$,

(ii) $\mathcal{L}(X)=\mathcal{L}^r(X)$,

(iii) $\mathcal{L}(X)=\overline{\mathcal{L}^r(X)}$.

In the second part of this section we show that a certain matrix represents a strongly non-regular operator A on l_p; more precisely we show $\operatorname{dist}(A,\mathcal{L}^r(l_p))=\|A\|$. Similar matrices were used in a related context; cf. [18; chap. IV, §1, Examples], [3; Abschnitt 2].

4.4. LEMMA. *Let* $1\le p\le\infty$, $A\in\mathbb{K}^{n\times n}$. *Then* $\||A|\|_p\le n^{1/p}\|A\|_p$, *where* $\|A\|_p$ *denotes the norm of A as an operator on* $(\mathbb{K}^n,\|\cdot\|_p)$.

PROOF. Let $A=(a_{jk})$, and choose ε_{jk} such that $|\varepsilon_{jk}|=1$, $|a_{jk}|=\varepsilon_{jk}a_{jk}$. Let $x=(x_j)\in\mathbb{K}^n$, $\|x\|_p=1$. Then

$$\||A|x\|_p=\Big(\sum_{j=1}^{n}\Big|\sum_{k=1}^{n}\varepsilon_{jk}a_{jk}x_k\Big|^p\Big)^{1/p}$$

$$\le n^{1/p}\Big(\max_j\sum_k\varepsilon_{jk}a_{jk}|x_k|\Big)$$

$$=n^{1/p}\max_j\,(Ax^j)_j$$

(where $x^j=(\varepsilon_{jk}|x_k|)_k$, and $(Ax^j)_j$ denotes the j-th component of Ax^j)

$$\le n^{1/p}\max_j\|Ax^j\|_p$$

$$\le n^{1/p}\|A\|_p. \qquad \blacksquare$$

For $n\in\mathbb{N}_0$ we define recursively $2^n\times 2^n$-matrices B_n, by

$$B_0:=(1),$$

$$B_n:=\begin{pmatrix}B_{n-1} & B_{n-1}\\ B_{n-1} & -B_{n-1}\end{pmatrix}\qquad (n\ge 1).$$

4.5. LEMMA. *Let* $2\le p\le\infty$, $1/p+1/q=1$. *Then* $\|B_n\|_p=2^{n/q}$, $\||B_n|\|_p=2^n$ $(n\in\mathbb{N}_0)$.

PROOF. For the first equality we note first the obvious equality $\|B_n\|_\infty = 2^n$. Next we remark that $2^{-n/2}B_n$ is an orthogonal matrix, and this implies $\|B_n\|_2 = 2^{n/2}$. Now the Riesz-Thorin convexity theorem implies $\|B_n\|_p \le 2^{n/q}$. Testing with the vector $(1, 1, \dots, 1)$ shows equality.

The second equality is easy to show. ■

Now we fix $2 \le p < \infty, 1/p + 1/q = 1$. We define $A_n := 2^{-n/q}B_n$ ($n \in \mathbb{N}_0$). Then $\|A_n\| = 1, \| |A_n| \| = 2^{n/p}$ ($n \in \mathbb{N}_0$). Further we define the operator $A \in \mathscr{L}(l_p)$ by representing l_p as the l_p-sum of 2^n-dimensional spaces $E_n := (\mathbb{K}^{2^n}, \|\cdot\|_p)$ in the obvious way, and letting A act as A_n on E_n. In the matrix representation,

$$A = \begin{pmatrix} A_0 & & & 0 \\ & A_1 & & \\ & & A_2 & \\ 0 & & & \ddots \end{pmatrix}.$$

4.6. THEOREM. *One has* $\operatorname{dist}_{\mathscr{L}(l_p)}(A, \mathscr{L}^r(l_p)) = \|A\| = 1$. *Also, A acts as an operator in l_q, and for this operator* $\operatorname{dist}_{\mathscr{L}(l_q)}(A, \mathscr{L}^r(l_q)) = \|A\| = 1$.

PROOF. The equality $\|A\| = 1$ is immediate from $\|A_n\| = 1$ ($n \in \mathbb{N}_0$).

Let $S \in \mathscr{L}(l_p)$ be such that $\|A - S\| < 1$. We are going to show that this implies $S \notin \mathscr{L}^r(l_p)$. Let $\varepsilon := \|A - S\|$ (<1). Define S_n as the $2^n \times 2^n$-submatrix of S occupying the same place as A_n in A. Then, using Lemma 4.4, we obtain

$$\|B_n \times (A_n - S_n)\| \le \| |A_n - S_n| \| \le 2^{n/p}\|A_n - S_n\| \le 2^{n/p}\varepsilon,$$

where "$\times$" denotes the Schur product, i.e., entry by entry multiplication of matrices; note that $B_n \times A_n = |A_n|$ by the definitions. Therefore

$$\begin{aligned} \| |S_n| \| &\ge \|B_n \times S_n\| \\ &\ge \|B_n \times A_n\| - \|B_n \times (A_n - S_n)\| \\ &\ge 2^{n/p} - 2^{n/p}\varepsilon = 2^{n/p} \quad (1 - \varepsilon) \to \infty \end{aligned}$$

for $n \to \infty$. This shows $S \notin \mathscr{L}^r(l_p)$.

The statements concerning A as an operator in l_q follow by duality, since A is (formally) symmetric. ■

4.7. REMARK (a permanence property for strong non-regularity). Let E be a Banach lattice and $T \in \mathscr{L}(E)$ be strongly non-regular. Then $(\lambda - T)^{-1}$ is strongly non-regular for all $\lambda \in \varrho_\infty(T)$ (the unbounded component of the resolvent set of T). In fact, if $R(\lambda) := (\lambda - T)^{-1} \in \overline{\mathscr{L}^r(E)}$ for one $\lambda \in \varrho_\infty(T)$ then $R(\lambda) \in \overline{\mathscr{L}^r(E)}$ for all $\lambda \in \varrho_\infty(T)$, since $\overline{\mathscr{L}^r(E)}$ is a closed subalgebra of $\mathscr{L}(E)$. Consequently, $T = \lim_{\lambda \to \infty} (\lambda^2 R(\lambda) - \lambda) \in \overline{\mathscr{L}^r(E)}$.

Moreover, if $E = L_2$, then $R(\lambda, T) \notin \overline{\mathscr{L}^r(L_2)}$ for all $\lambda \in \varrho(T)$ since $\overline{\mathscr{L}^r(L_2)}$ is a full subalgebra of $\mathscr{L}(L_2)$ (see [23; 24.6]).

5. STRONGLY NON-REGULAR OPERATORS ON HILBERT SPACE

Let H be a separable infinite dimensional complex Hilbert space. Then, given any orthonormal basis on H, one may introduce a lattice ordering on H by identifying H with l_2, and one may ask which operators are regular and which operators are in $\overline{\mathscr{L}^r}$ for all of these orderings.

The first question was answered independently by Sourour [19] and Sunder (cf. [11; Theorem 16.5]): For $T \in \mathscr{L}(H)$ the following are equivalent.

(i) $UTU^{-1} \in \mathscr{L}^r(l_2)$ for all unitary $U: H \to l_2$;

(ii) there exist a Hilbert-Schmidt operator S on H and $\lambda \in \mathbb{C}$ such that $T = S + \lambda I$.

Sourour [19] also observed that in (i) one may replace l_2 by $L_2(0,1)$.

Concerning the second question, recall that every compact operator can be approximated in the operator norm by operators of finite rank. Hence if $T = K + \lambda I$ where K is compact and $\lambda \in \mathbb{C}$, then $UTU^{-1} \in \overline{\mathscr{L}^r}$ for all unitary operators $U: H \to l_2$ as well as for all unitary operators $U: H \to L_2(0,1)$. We shall now prove that the converse is also true.

5.1. THEOREM. *Let $T \in \mathscr{L}(H)$ be such that one of the following properties holds.*

(a) $UTU^{-1} \in \overline{\mathscr{L}^r(l_2)}$ *for all unitary operators* $U: H \to l_2$;

(b) $UTU^{-1} \in \overline{\mathscr{L}^r(L_2(0,1))}$ *for all unitary operators* $U: H \to L_2(0,1)$.

Then there exist a compact operator K and $\lambda \in \mathbb{C}$ such that $T = K + \lambda I$.

We use the following result which we extract from the proof of [19; Lemma 2].

5.2. LEMMA. *Let $T \in \mathscr{L}(H)$ be a selfadjoint operator which is not of the form $K + \lambda I$ with compact K, $\lambda \in \mathbb{R}$. Then there exist a selfadjoint Hilbert-Schmidt operator S and infinite dimensional closed subspaces H_1, H_2, H_3 of H such that $H = H_1 \oplus H_2 \oplus H_3$, H_1, H_2, H_3 are invariant under $T + S$, $(T+S)_{H_1} = \alpha I_{H_1}$ and $(T+S)_{H_2} = \beta I_{H_2}$, with $\alpha, \beta \in \mathbb{R}, \alpha \neq \beta$ (where $(T+S)_{H_j}$ denotes the part of $T+S$ in H_j, for $j = 1, 2$).*

PROOF OF THEOREM 5.1. Since $UTU^{-1} \in \overline{\mathscr{L}^r}$ for all unitary $U: H \to l_2$ ($U: H \to L_2(0,1)$, respectively) the same is true for T^*, $(T+T^*)/2$ and $(T-T^*)/2i$. So we may assume that T is selfadjoint.

We assume that T is not of the form $K + \lambda I$, and obtain $S, H_1, H_2, H_3, \alpha, \beta$ from Lemma 5.2. It is sufficient to find a unitary $U: H \to l_2$ ($U: H \to L_2(0,1)$) such that $U(T+S)U^{-1} \notin \overline{\mathscr{L}^r}$. Without restriction we may assume $S = 0$. (Note that any Hilbert-Schmidt operator on l_2 or $L_2(0,1)$ is regular.)

Now if H_0 is a closed subspace of H such that $\dim H_0 = \dim H_0^{\perp} = \infty$ and $TH_0 \subset H_0$ then also the part $T_0 := T_{H_0}$ of T in H_0 has the property that $U_0 T_0 U_0^{-1} \in \overline{\mathscr{L}^r}$ for all unitary $U_0: H_0 \to l_2$ ($U_0: H_0 \to L_2(0,1)$). (In fact, let $U_0: H_0 \to l_2$ ($U_0: H_0 \to L_2(0,1)$) be unitary. Consider a unitary extension $U: H \to l_2 \oplus l_2$ ($U: H \to L_2(0,1) \oplus L_2(0,1)$). By hypothesis there exist $T_n \in \mathscr{L}^r$ such that $\lim_{n\to\infty} T_n = UTU^{-1}$. Since the orthogonal projection P of $l_2 \oplus l_2$ ($L_2(0,1) \oplus L_2(0,1)$, respectively) onto the first component is positive, the operators $PT_n|l_2$ ($PT_n|L_2(0,1)$) are regular and converge to $U_0 T_0 U_0^{-1}$.)

We apply the previous remark to $T_0 := T_{H_1 \oplus H_2} = \alpha I_{H_1} \oplus \beta I_{H_2}$. Now we consider the two cases (a) and (b) seperately.

Case (a). We identify H_1 unitarily with $L_2(0,\pi)$ and H_2 with $L_2(\pi, 2\pi)$. Then T_0 is given by $T_0 f = mf$ ($f \in L_2(0, 2\pi)$) where

$$m(x) = \begin{cases} \alpha & \text{if } x \leq \pi, \\ \beta & \text{if } x > \pi. \end{cases}$$

Since m is not continuous it follows from Theorem 3.1 that FT_0F^{-1} is strongly non-regular, where $F: L_2(0,2\pi) \to l_2(\mathbb{Z})$ is the Fourier transformation. This is a contradiction.

Case (b). We identify H_1 unitarily with $l_2(-\mathbb{N})$ and H_2 with $l_2(\mathbb{N} \cup \{0\})$. Then $T_0 x = (m_n x_n)_{n \in \mathbb{Z}}$, where

$$m_n = \begin{cases} \alpha & \text{if } n < 0, \\ \beta & \text{if } n \geq 0. \end{cases}$$

Since $\lim_{n\to\infty} m_n \neq \lim_{n\to-\infty} m_n$, it follows from Corollary 3.8 that FT_0F^{-1} is strongly non-regular, where $F: l_2(\mathbb{Z}) \to L_2(\mathbb{T})$ is the Fourier transformation. This is a contradiction. ■

5.3. REMARK. It was pointed out to the authors that Theorem 5.1 can also be obtained as a consequence of [8; Theorem 1 and Corollary 3]. However, the proof given here is more direct and elementary in the present context.

REFERENCES

1. Aliprantis, C.D. and O. Burkinshaw – Positive operators. Academic Press, Orlando, 1985.
2. Arendt, W. – On the o-sprectrum of regular operators and the spectrum of measures. Math. Z. **178**, 271–287 (1981).
3. Arendt, W. and H.-U. Schwarz – Ideale regulärer Operatoren und Kompaktheit positiver Operatoren zwischen Banachverbänden. Math. Nachr. **131**, 7–18 (1987).
4. Birkhoff, G. – Orthogonality in linear metric spaces. Duke Math. J. **1**, 169–178 (1935).
5. Brainerd, B. and R.E. Edwards – Linear operators which commute with translations, part I: representation theorems. J. Australian Math. Soc. **6**, 289–327 (1966).
6. Cohn, D.L. – Measure theory. Birkhäuser, Boston, 1980.
7. Edwards, R.E., G.I. Gaudry – Littlewood-Paley and multiplier theory. Springer-Verlag, Berlin, 1977.
8. Fong, C.K., C.R. Miers and A.R. Sourour – Lie and Jordan ideals of operators on Hilbert space. Proc. Amer. Math. Soc. **84**, 516–520 (1982).
9. Gaudry, G.I. and I.R. Inglis – Approximation of multipliers. Proc. Amer. Math. Soc. **44**, 381–384 (1974).
10. Greenleaf, F.P. – Invariant means on topological groups. Van Nostrand-Reinhold, New York, 1969.
11. Halmos, P.R. and V.S. Sunder – Bounded integral operators on L^2 spaces. Springer-Verlag, Berlin, 1978.
12. Hewitt, E. and K.A. Ross – Abstract harmonic analysis I. Springer-Verlag, Berlin, 1963.

13. Katznelson, Y. – An introduction to harmonic analysis. 2nd ed. Dover Publications, New York, 1976.
14. Kelley, J.L. – I. Namioka – Linear topological spaces. Van Nostrand, Princeton, 1963.
15. Larsen, R. – An introduction to the theory of multipliers. Springer-Verlag, Berlin, 1971.
16. Pier, J.-P. – Amenable locally compact groups. John Wiley & Sons, New York, 1984.
17. Rudin, W. – Fourier analysis on groups. Interscience Publishers, New York, 1962.
18. Schaefer, H.H. – Banach lattices and positive operators. Springer-Verlag, New York, 1974.
19. Sourour, A.R. – Operators with absolutely bounded matrices. Math. Z. **162,** 183–187 (1978).
20. Synnatzschke, J. – Zu einer Eigenschaft gewisser Integraltransformationen. Beiträge zur Analysis **15,** 93-98 (1981).
21. Voigt, J. – The projection onto the center of operators in a Banach lattice. Math. Z. **199,** 115–117 (1988).
22. Zaanen, A.C. – Examples of orthomorphisms. J. Approximation Theory **13,** 192–204 (1975).
23. Zelasko, W. – Banach algebras. Elsevier, Amsterdam, 1973.