Asymptotic stability of Schrödinger semigroups on \( L^1(\mathbb{R}^N) \)

Wolfgang Arendt, Charles J.K. Batty*, and Philippe Bénilan
Equipe de Mathématiques U.A. CNRS 741, Université de Franche-Comté, F-25030 Besançon Cedex, France

Received February 19, 1991; in final form July 17, 1991

Introduction

By a Schrödinger semigroup one understands a semigroup \( S_p = (S_p(t))_{t \geq 0} \) generated by \( \Delta - V \) on \( L^p(\mathbb{R}^N) \), where \( \Delta \) denotes the Laplacian and \( V \) is a potential in \( L^1_{\text{loc}}(\mathbb{R}^N) \). These semigroups have been investigated by several authors, see the survey article [14] by Simon for further information.

The purpose of the present paper is to study convergence of \( S_p(t) \) as \( t \to \infty \) for positive potentials. In order to develop an intuitive idea of the problem, it is helpful to see \( (S_p(t)f)(x) \) (\( x \in \mathbb{R}^N, t > 0 \)) as the solution of the heat equation with absorbing potential in \( \mathbb{R}^N \).

If \( p > 1 \), then \( \lim_{t \to \infty} S_p(t) = 0 \) strongly even if \( V = 0 \). More interesting is the case \( p = 1 \). In fact, given a positive initial value \( f \in L^1(\mathbb{R}^N) \), \( \|S_1(t)f\|_{L^1} \) means the total amount of heat at time \( t \). If \( V = 0 \), this quantity is constant, i.e. \( S_1(t) \) is isometric on the positive cone. Our aim is to investigate, for which absorptions \( V \) the semigroup \( S_1 \) is (asymptotically) stable, i.e. \( \lim_{t \to \infty} S_1(t) = 0 \) strongly. It is quite easy to see that this is the case for every non-vanishing \( V \) if the Neumann Laplacian on a bounded region (of class \( C^1 \)) is considered instead of \( \Delta \) on \( \mathbb{R}^N \) (Sect. 2). On \( \mathbb{R}^N \) the asymptotic behavior depends on the dimension. Whereas for \( N = 1, 2 \), \( S_1 \) is always stable if \( V \neq 0 \) (Theorem 3.2), for \( N \geq 3 \), there always exist non-zero potentials \( V \) such that \( S_1 \) does not converge. This property depends on the growth of \( V \) at infinity. If \( S_1 \) is stable in \( L^1(\mathbb{R}^N), N \geq 3 \), then necessarily

\[
\int_{|y| \geq 1} \frac{V(y)}{|y|^{N-2}} dy = \infty.
\]

Moreover, if \( V \) is radial, the converse holds as well: condition (\( \ast \)) implies stability of \( S_1 \).

The proofs given here (Theorem 3.6 and 3.7) are analytical, throughout. In a separate paper, the second author obtains these and other results by probabilistic methods.

*Permanent address: St. John's College, Oxford OX1 3JP, UK
1 Stability of positive semigroups

In this preliminary section we give an abstract characterization of stability for positive (i.e. positivity preserving) semigroups. Let $T = (T(t))_{t \geq 0}$ be a $C_0$-semigroup on a Banach space with generator $A$. We denote by $\sigma(A)$ the spectrum of $A$, $P\sigma(A)$ its point spectrum and by

$$s(A) = \sup\{\Re \lambda : \lambda \in \sigma(A)\}$$

the spectral bound of $A$. The number

$$\omega(A) := \inf\{\omega \in \mathbb{R} : \sup_{t \geq 0} e^{-\omega t} \|T(t)\| < \infty\}$$

is called the growth bound or type of $T$ (or of $A$). One always has $s(A) \leq \omega(A)$ and growth bound and spectral bound coincide whenever $T$ is holomorphic or $T$ is a positive semigroup on $L^1$ or $L^2$ (see [12]). We denote by $N(A) = \{x \in D(A) : Ax = 0\}$ the kernel of $A$ and by $N(A')$ the kernel of the adjoint $A'$ of $A$.

**Definition 1.1** We say that $T$ (or $A$) is stable if $\lim_{t \to \infty} T(t)x = 0$ for all $x \in E$. If $\omega(A) < 0$, then $T$ (or $A$) is called exponentially stable.

In the following we assume that $T$ is a positive $C_0$-semigroup on a Banach lattice $E$.

**Lemma 1.2** Assume that $T$ is bounded. If $N(A') \neq \{0\}$, then $N(A')^+ := N(A') \cap E_+^* \neq \{0\}$.

**Proof.**

a) Observe that $\sup_{0 \leq t \leq 1} \|\lambda R(\lambda, A)'\| < \infty$ since $T$ is bounded.

b) Let $\varphi \in E'$. Then

$$\lim_{\lambda \to 0} \langle Ax, \lambda R(\lambda, A)'\varphi \rangle = \lim_{\lambda \to 0} \langle \lambda^2 R(\lambda, A)x - \lambda x, \varphi \rangle = 0$$

for all $x \in D(A)$.

Thus, if $\psi$ is a w*-limit point of $\lambda R(\lambda, A)'\varphi$ for $\lambda \downarrow 0$, then $\psi \in N(A')$.

c) Let $\varphi \in N(A')$. Then $\lambda R(\lambda, A)'\varphi^+$ and $\lambda R(\lambda, A)'\varphi^-$ possess w*-limit points for $\lambda \downarrow 0$ (by a), which are in $N(A')^+$ (by b). If every such limit point is 0, then it follows that

$$\varphi = \lim_{\lambda \to 0} \lambda^2 R(\lambda, A)'\varphi = \lim_{\lambda \to 0} \lambda R(\lambda, A)'\varphi^+ - \lim_{\lambda \to 0} \lambda R(\lambda, A)'\varphi^- = 0. \quad \square$$

**Theorem 1.3** Let $T$ be a positive bounded $C_0$-semigroup with generator $A$ such that $\sigma(A) \cap i\mathbb{R}$ is countable. Then $T$ is stable if and only if $N(A')^+ = \{0\}$.

**Proof.** It follows from the proof of [12, C-III Corollary 4.3] that $P\sigma(A') \cap i\mathbb{R}$ is cyclic; in particular, $0 \notin P\sigma(A')$ implies $P\sigma(A') \cap i\mathbb{R} = \emptyset$. By Lemma 1.1, $0 \notin P\sigma(A')$ if and only if $N(A')^+ = \{0\}$. Now the theorem follows from the stability theorem [1] or [11]. \(\square\)

**Corollary 1.4** Assume that $T$ is a bounded positive semigroup which is eventually norm continuous. Then $T$ is stable if and only if $N(A')^+ = \{0\}$. 
This follows since for such a semigroup automatically \( \sigma(A) \cap i \mathbb{R} \subset \{0\} \), [12, C-III Corollary 2.13].

**Remark 1.5** The particular case where \( T \) is a bounded holomorphic semigroup can be settled by a simpler argument: in that case \( \sup_{t > 0} \| tA T(t) \| < \infty \); hence \( T(t)x \to 0 \) (\( t \to \infty \)) for all \( x \in R(A) \) (the range of \( A \)). Thus \( T \) is stable if and only if \( R(A) \) is dense, which in turn is equivalent to \( N(A') \neq \{0\} \).

## 2 The Neumann-Laplacian with absorbing potential on bounded domain

Let \( \Omega \subset \mathbb{R}^N \) be a bounded connected open set of class \( C^1 \). Denote by \( A_2 \) the Neumann-Laplacian on \( L^2(\Omega) \), i.e. \( A_2 \) is given by

\[
D(A_2) = \{ u \in H^1(\Omega) : \exists v \in L^2(\Omega), \int_{\Omega} \nabla u \nabla \phi \}
= -\int \nabla \phi \text{ for all } \phi \in H^1(\Omega) \}, A_2 u = v.
\]

(See [4, Chap. X] or [13, XIII, 15, p 263].) This operator generates a positive semigroup \( T_2 = (T_2(t))_{t \geq 0} \) on \( L^2(\Omega) \). Moreover, there exist positive contraction semigroups \( T_p \) on \( L^p(\Omega) \) (\( 1 \leq p < \infty \)) such that \( T_p(t)f = T_q(t)f \) (\( f \in L^p(\Omega) \cap L^q(\Omega) \), \( t \geq 0 \)). Let \( A_p \) be the generator of \( T_p \).

**Theorem 2.1** Let \( 0 \leq V \in L^\infty(\Omega) \), \( V \neq 0 \). Then \( A_p - V \) is exponentially stable \((1 \leq p < \infty) \).

**Proof.** Denote by \( S_p \) the semigroup generated by \( A_p - V \). The operator \( A_2 - V \) has compact resolvent. Assume that \( s(A_2 - V) = 0 \). Then there exists \( u \in D(A_2) \), \( \|u\|_{L^1} = 1 \) such that \( A_2 u = Vu \).

\[
\int_{\Omega} (\nabla u)^2 + \int_{\Omega} Vu^2 = (-A_2 u + Vu |_u) = 0.
\]

It follows that \( \nabla u = 0 \). Hence \( u = \text{const} \). Since \( \int_{\Omega} Vu^2 = 0 \), it follows that \( V = 0 \), contradiction.

We have shown that \( s(A_2 - V) < 0 \). So there exist \( \omega < 0 \), \( M \geq 0 \) such that \( \| S_2(t) \| \leq Me^{\omega t} \) (\( t \geq 0 \)). The semigroup \( T_2 \) is ultracontractive (see [3] or [6, 2.4]); in particular, \( T_2(1) \) is a bounded operator from \( L^2(\Omega) \) to \( L^\infty(\Omega) \). Consequently, \( T_1(1) \) is bounded from \( L^1(\Omega) \) into \( L^2(\Omega) \), by self-adjointness. Since \( 0 \leq s_1(t) \leq T_1(t) \), also \( s_1(t) \) maps \( L^1(\Omega) \) into \( L^2(\Omega) \). Thus for \( f \in L^1(\Omega) \),

\[
\| s_1(t) f \|_{L^1} = \| S_2(t - 1) s_1(1) f \|_{L^1} \leq M e^{\omega(t - 1)} \| S_1(1) \|_{L^1} \| f \|_{L^1} \text{ (} t \geq 1 \).
\]

So \( s_1 \) is exponentially stable. It follows from the Riesz-Thorin theorem that \( s_p \) is exponentially stable for \( 1 \leq p \leq 2 \); \( 2 \leq p < \infty \) this follows by duality.

## 3 Stability of the Schrödinger semigroup

By \( T_p = (T_p(t))_{t \geq 0} \) we denote the Gaussian semigroup on \( L^p(\mathbb{R}^N) \) (\( 1 \leq p \leq \infty \)); i.e. \( T_p \) is given by

\[
(T_p(t)f)(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{(x-y)^2}{4t}} f(y) dy \quad (t > 0).
\]
This defines a $C_0$ semigroup on $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$ and a $w^*$-continuous semigroup on $L^\infty(\mathbb{R}^N)$ for $p = \infty$. The generator $A_p$ of $T_p$ is given by

$$D(A_p) = \{ f \in L^p(\mathbb{R}^N); \; A f \in L^p(\mathbb{R}^N), \; A_p f = Af \; \; (1 \leq p \leq \infty) \}.$$  

**Proposition 3.1** If $1 < p < \infty$, then $T_p$ is stable.

**Proof.** Since $(T_p(t)f)(\xi) = e^{-\xi^2 t} \hat{f}(\xi)$ $(f \in L^2(\mathbb{R}^N))$ this is clear for $p = 2$. For $1 < p < 2$ and $f \in L^1 \cap L^2$ one has

$$\| T_p(t)f \|_{L^p} \leq \| T_1(t)f \|_{L^p}^{1-p} \| T_2(t)f \|_{L^2}^{2-2/p} \to 0 \quad (t \to \infty).$$

Since $T_p$ is bounded and $L^1 \cap L^2$ is dense the result follows. For $2 < p < \infty$ one can argue similarly. \[\square\]

However, the Gaussian semigroup $T_1$ on $L^1(\mathbb{R}^N)$ is not stable, in fact, it preserves the norm on the positive cone, $\| T_1(t)f \|_{L^1} = \| f \|_{L^1}$ for $0 \leq f \in L^1$.

Next we introduce an absorption $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$. Then the operator $A_1 - V$ with domain $D(A_1 - V) = D(A_1) \cap D(V)$ generates a bounded holomorphic semigroup $S_1$ on $L^1(\mathbb{R}^N)$ which is dominated by the Gaussian semigroup (see [10, 15]):

$$0 \leq S_1(t) \leq T_1(t) \quad (t \geq 0). \tag{3.1}$$

**Theorem 3.2** Let $N \in \{1, 2\}$ and let $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$. If $V \neq 0$, then $A_1 - V$ is stable.

For the proof we use the following notion. A function $\varphi \in L^1_{\text{loc}}(\mathbb{R}^N)$ is called subharmonic if $A_0 \varphi \geq 0$ (cf. [5, Chap. II]). It is well-known that for $\varphi \in L^\infty(\mathbb{R}^N)$ $(N$ arbitrary) $A_0 \varphi = 0$ implies that $\varphi$ is constant (cf. [5, II §2, Corollaire 1]). This remains true for subharmonic $\varphi$ if $N = 1$ or 2.

**Proposition 3.3** Let $N \in \{1, 2\}$ and let $\varphi \in L^\infty(\mathbb{R}^N)$ be subharmonic; then $\varphi$ is constant.

**Proof.** This follows from [8, Theorem 2.14, p. 67, 68] or [7, problem 2.14] if $\varphi$ is smooth. Applying this result to $\rho_n * \varphi$ instead of $\varphi$, in the general case ($\rho_n$ being a mollifier), the result follows. \[\square\]

**Proof of Theorem 3.2** Let $0 \leq \varphi \in N((A_1 - V)')$. Then $0 \leq \varphi \in L^\infty(\mathbb{R}^N)$ and $\Delta \varphi = V \varphi$ in $\mathcal{D}'(\mathbb{R}^N)$. Hence $\varphi$ is subharmonic. It follows from Proposition 3.3 that $\varphi$ is constant, say $\varphi(x) = C (x \in \mathbb{R}^N)$. Then $0 = \Delta \varphi = CV$. Hence $C = 0$ or $V \equiv 0$. It follows from Remark 1.5 that $A_1 - V$ is stable if $V \neq 0$. \[\square\]

The situation is different if $N \geq 3$. Then there always exist non-zero potentials $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $A - V$ is not stable.

**Proposition 3.4** For $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ let

$$N_V := \{ g \in L^\infty(\mathbb{R}^N) ; 0 \leq g \leq 1; \; \Delta g = V g \}.$$
Denote by $S$ the semigroup generated by $A_1 - V$ on $L^1(\mathbb{R}^N)$. Then the following holds.

a) $N_V$ has a maximal element $g_V$.

b) If $V \in L^1(\mathbb{R}^N)$, then $g_V = 1 - \int_0^\infty S(t)V dt$, where the improper integral converges in the $w^*$-sense in $L^\infty$ (note that $S(t)V \in L^\infty$ for $t > 0$).

c) $A_1 - V$ is stable if and only if $g_V = 0$.

d) If $V, \bar{V} \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $0 \leq V \leq \bar{V}$, then $g_V \leq g_{\bar{V}}$.

e) Let $V_n \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $0 \leq V_n \leq V_{n+1} \leq V$ and $\lim_{n \to \infty} V_n(x) = V(x)$ a.e. Then $w^* - \lim_{n \to \infty} g_{V_n} = g_V$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Then $S(t)\varphi \in D(A_1 - V) = D(A_1) \cap D(V)$ for all $t \geq 0$. Consequently,

$$\frac{d}{dt} \langle \varphi, S(t)'1 \rangle = \langle (A_1 - V)S(t)\varphi, 1 \rangle = - \langle S(t)\varphi, V \rangle \quad (t \geq 0). \quad (3.2)$$

Thus $\langle \varphi, S(t)'1 \rangle$ is decreasing for all $0 \leq \varphi \in \mathcal{D}(\mathbb{R}^N)$. It follows that $S(t)'1$ is decreasing.

Let $g_V = \inf_{t > 0} S(t)'1 = w^* - \lim_{t \to \infty} S(t)'1$. Then

$$S(s)'1 \leq g_V = w^* - \lim_{t \to \infty} S(t)'1 = g_V.$$ 

Hence $g_V \in N((A_1 - V)')$ and $0 \leq g_V \leq 1$. Since $\mathcal{D}(\mathbb{R}^N)$ is a core of $A_1 - V$ (see [10]) it follows that

$$N_V = \{g \in N((A_1 - V)'): 0 \leq g \leq 1\}. \quad (3.3)$$

Consequently $g_V \in N_V$. We show that $g_V$ is maximal. Let $g \in N_V$. Since $0 \leq g \leq 1$, it follows that $g = S(t)'g \leq S(t)'1$ ($t \geq 0$); consequently, $g \leq g_V$. So a) is proved.

If $V \in L^1(\mathbb{R}^N)$, then $S(t)\varphi = S(t)'\varphi$ for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$, hence $\langle S(t)\varphi, V \rangle = \langle \varphi, S(t)V \rangle$ ($t \geq 0$). So it follows from (3.2) that

$$\langle \varphi, S(t)'1 \rangle - \langle \varphi, 1 \rangle = \int_0^t \frac{d}{ds} \langle \varphi, S(s)'1 \rangle ds = - \int_0^t \langle \varphi, S(s)V \rangle ds.$$ 

Since $\varphi \in \mathcal{D}(\mathbb{R}^N)$ is arbitrary one concludes that $S(t)'1 = 1 - \int_0^t S(s)V ds$. One obtains b) by letting $t \to \infty$.

Assertion c) follows from (3.3) and Remark 1.5 since $N_V = \{0\}$ if and only if $g_V = 0$. Assertion d) follows from the fact that $\bar{S}(t) \leq S(t)$ if $S(t)$ denotes the semigroup generated by $\bar{V} \in L^1_{\text{loc}}$, $\bar{V} \geq V$.

It remains to prove e). Let $g_n = g_{V_n}$, then by d) $0 \leq g_{n+1} \leq g_n \leq 1$. Let $g = \inf g_n = w^* - \lim g_n$. Then for $\varphi \in \mathcal{D}(\mathbb{R}^N)$,

$$\langle \varphi, \Delta g \rangle = \langle \Delta \varphi, g \rangle = \lim \langle \Delta \varphi, g_n \rangle = \lim \langle \varphi, \Delta g_n \rangle = \lim \langle \varphi, V g_n \rangle = \langle \varphi, Vg \rangle$$

by the dominated convergence theorem. Hence $g \in N_V$. So by a) $g \leq g_V$. On the other hand $g_V \leq S(t)'1 \leq S(t)'1$ ($t \geq 0$) for all $n \in \mathbb{N}$. Hence $g_V \leq g_n$ ($n \in \mathbb{N}$).

Consequently, $g_V \leq g$. We have shown that $g = g_V$. \qed
**Proposition 3.5** Let $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ and let $\lambda > 0$. Then $A_1 - V$ is stable if and only if $A_1 - \lambda V$ is stable.

**Proof.** Suppose that $\lambda > 1$. Denote by $S$ (resp. $U$) the semigroup generated by $A_1 - V$ (resp. $A_1 - \lambda V$). The $0 \leq U(t) \leq S(t)$; hence $U$ is stable whenever $S$ is stable. Conversely assume that $A_1 - \lambda V$ is stable. Let $V_n = V 1_{B(e, n)}$ and denote by $S_n$ (resp. $U_n$) the semigroup generated by $A_1 - V_n$ (resp. $A_1 - \lambda V_n$). Let $\psi_n = g_{\lambda V_n}$ and $\phi_n = g_{\lambda V_n}$. Then $\psi_n = w^* - \lim \psi_n = 0$ by Proposition 3.4. Moreover, by Proposition 3.4e) $g_{\lambda V_n} = w^* - \lim \phi_n$. By Proposition 3.4b) we have

$$\psi_n = 1 - \lambda \int_0^\infty U_n(s) V_n ds;$$

hence

$$1 = \psi_n + \lambda \int_0^\infty U_n(s) V_n ds \leq \psi_n + \lambda \int_0^\infty S_n(s) V_n ds = \psi_n + \lambda (1 - \phi_n).$$

Letting $n \to \infty$, we obtain $1 \leq \lambda (1 - \phi_n)$. Consequently $C g_{\lambda V_n} \leq 1$ where $C = \lambda (\lambda - 1)$. So $C g_{\lambda V_n} \in N_{\lambda V_n}$. It follows that $C g_{\lambda V_n} \leq g_{\lambda V_n}$. Hence $g_{\lambda V_n} = 0$ since $C > 1$; i.e. $A_1 - V$ is stable. If $\lambda < 1$, it suffices to apply the preceding result to $\lambda V$.

**Theorem 3.6** Let $N \geq 3$. If $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfies

$$\int_{|y| \geq 1} \frac{V(y)}{|y|^{N-2}} dy < \infty$$

then $A_1 - V$ is not stable.

**Proof.** a) We show that there exists a measurable set $B \subset \mathbb{R}^N$ of positive measure such that

$$\sup_{x \in B} \int \frac{V(y)}{|x-y|^{N-2}} dy < \infty.$$  

In fact, let $C = \int_{|x| \leq 2} V(x) dx$. One has

$$\int_{|x| \leq 1} \left\{ \int_{|x-y| \leq 2} \frac{V(x-y) dy}{|y|^{N-2}} \right\} dx \leq$$

$$\int_{|y| \leq 3} \left\{ \int_{|x-y| \leq 2} V(x-y) dx \right\} \frac{dy}{|y|^{N-2}} = \frac{9}{2} C \omega_N,$$

where $\omega_N = \frac{2\pi^{N/2}}{I(N/2)}$ is the surface of the unit sphere in $\mathbb{R}^N$.

By Fubini's theorem there exists a measurable set $B \subset B(0, 1) = \{x \in \mathbb{R}^N: |x| \leq 1\}$ of positive measure such that

$$\sup_{x \in B} \int_{|y| \leq 2} \frac{V(y)}{|x-y|^{N-2}} dy = \sup_{x \in B} \int_{|x-y| \leq 2} \frac{V(x-y)}{|y|^{N-2}} dy < \infty.$$  

But for $|x| \leq 1$, $|y| > 2$, one has $|x-y| \geq |y| - |x| \geq \frac{1}{2}|y|$. Thus

$$\int_{|y| > 2} \frac{V(y)}{|x-y|^{N-2}} dy \leq 2^{N-2} \int_{|y| > 2} \frac{V(y)}{|y|^{N-2}} dy < \infty.$$
by (3.4). This together with (3.6) proves the claim.

b) For $0 \leq g \in L^1(\mathbb{R}^N)$ we have by Fubini's theorem

$$\int_0^\infty (T_1(t)g)(x) \, dt = \int_0^\infty \int_{\mathbb{R}^N} (4\pi t)^{-N/2} \exp(-\frac{(x-y)^2}{4t}) \, dt \, g(y) \, dy$$

$$= (E_N \ast g)(x) \quad (x \in \mathbb{R}^N)$$

where $E_N(x) = \frac{1}{(N-2)\omega_N} |x|^{2-N}$. In view of Proposition 3.5, we may assume that

$$q := \sup_{x \in B} (E_N \ast V)(x) < 1.$$ 

For $n \in \mathbb{N}$, let $V_n = V\mathbf{1}_{B(o,n)}$, $S_n$ the semigroup generated by $A_1 - V_n$ and $g_n = g_n \ast v_n$. Then $g\ast w = w^* - \lim_{n \to \infty} g_n$ by Proposition 3.4e); and by b)

$$g(\ast)(x) = 1 - \int_0^\infty (S_n(t)\mathbf{1}_{V_n})(x) \, dt \geq 1 - \int_0^\infty (T_1(t)\mathbf{1}_{V_n})(x) \, dt$$

$$= 1 - (E_N \ast V_n)(x) \geq 1 - (E_N \ast V)(x) \geq 1 - q \quad (x \in B).$$

Hence $g\ast w = \inf g_n \geq 1 - q$ on $B$ and so $g\ast w \neq 0$. \qed

The converse of Theorem 3.6 is not true in general. A characterization of stability will be given by the second author [2] by means of probabilistic methods (i.e. Wiener measure and the Feynman-Kac formula). For radial $V$, however, we obtain the following characterization.

**Theorem 3.7** Let $N \geq 3$, and let $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ be radial. Then $A_1 - V$ is stable if and only if

$$\int_{|y| \geq 1} \frac{V(y)}{|y|^{N-2}} \, dy = \infty.$$ 

**Proof.** Suppose that $A_1 - V$ is not stable. Then there exists $0 \leq g \in L^\infty(\mathbb{R}^N)$, $g \neq 0$, such that $Ag = Vg$. We can suppose that $V \neq 0$, so $g$ is not constant. Since $V$ is radial, we can suppose that $g$ is radial; otherwise we replace $g(x)$ by $	ilde{g}(x) = \int g(|x| y) \, d\sigma(y)$ where $d\sigma$ is the surface-measure on $S(0, 1) = \{ z \in \mathbb{R}^N : |z| = 1 \}$.

Hence $1/r^{N-1} (r^{N-1} g'(r))' = Vg$ in $\mathcal{D}'(0, \infty)$. This implies that $g \in C^1(0, \infty)$ and $r^{N-1} g'(r)$ is non-decreasing. We show that $g$ is non-decreasing. If not, there exists $r_0 > 0$ such that $g'(r_0) < 0$. Then $r^{N-1} g'(r) \leq r_0^{N-1} g'(r_0)$ on $(0, r_0)$. Hence for $r \in (0, r_0)$,

$$g(r) = g(r_0) + \int_r^{r_0} (-g'(s))r^{-N-1} \, ds.$$

This is not possible since $g$ is bounded.
Since \( g \) is non-constant, there exists \( r_0 > 0 \) such that \( g(r_0) > 0 \) and \( g'(r_0) > 0 \). Then

\[
N^{-1} g'(r) = r_0^{N-1} g'(r_0) + \int_{r_0}^{r} (s^{N-1} g'(s))' \, ds
\]

\[
= r_0^{N-1} g'(r_0) + \int_{r_0}^{r} s^{N-1} V(s) g'(s) \, ds \geq g(r_0) \int_{r_0}^{r} s^{N-1} V(s) \, ds.
\]

Consequently,

\[
\frac{1}{N-2} \int_{|y| \geq r_0} \frac{V(y)}{|y|^{N-2}} \, dy = \frac{1}{N-2} \int_{r_0}^{\infty} s V(s) \, ds = \frac{1}{N-2} \int_{r_0}^{\infty} s^{-N+2} s^{N-1} V(s) \, ds
\]

\[
= \int_{r_0}^{\infty} \int_{r_0}^{\infty} \frac{1}{r^{N-1}} s^{N-1} V(s) \, ds = \int_{r_0}^{\infty} \int_{r_0}^{r} s^{N-1} V(s) \, ds \frac{1}{r^{N-1}} \, dr \leq \frac{1}{g(r_0)} \int_{r_0}^{\infty} g'(r) \, dr < \infty
\]

since \( g \) is bounded.  

References