Math. Z. 209, 511–518 (1992)

Mathematische Zeitschrift

Asymptotic stability of Schrödinger semigroups on $L^1(\mathbb{R}^N)$

Wolfgang Arendt, Charles J.K. Batty*, and Philippe Bénilan

Equipe de Mathématiques U.A. CNRS 741, Université de Franche-Comté, F-25030 Besançon Cedex, France

Received February 19, 1991; in final form July 17, 1991

Introduction

By a Schrödinger semigroup one understands a semigroup $S_p = (S_p(t))_{t \geqq 0}$ generated by $\Delta - V$ on $L^p(\mathbb{R}^N)$, where Δ denotes the Laplacian and V is a potential in $L^1_{\text{loc}}(\mathbb{R}^N)_+$. These semigroups have been investigated by several authors, see the survey article [14] by Simon for further information.

The purpose of the present paper is to study convergence of $S_p(t)$ as $t \to \infty$ for positive potentials. In order to develop an intuitive idea of the problem, it is helpful to see $(S_p(t)f)(x)$ $(x \in \mathbb{R}^N, t > 0)$ as the solution of the heat equation with absorbing potential in $\mathbb{R}^N$.

If $p > 1$, then $\lim_{t \to \infty} S_p(t) = 0$ strongly even if $V = 0$. More interesting is the case $p = 1$. In fact, given a positive initial value $f \in L^1(\mathbb{R}^N)$, $\|S_1(t)f\|_{L^1}$ means the total amount of heat at time t. If $V = 0$, this quantity is constant; i.e. $S_1(t)$ is isometric on the positive cone. Our aim is to investigate, for which absorptions V the semigroup S_1 is (asymptotically) stable, i.e. $\lim_{t \to \infty} S_1(t) = 0$ strongly. It is quite easy to see that this is the case for every non-vanishing V if the Neumann Laplacian on a bounded region (of class C^1) is considered instead of Δ on $\mathbb{R}^N$ (Sect. 2). On $\mathbb{R}^N$ the asymptotic behavior depends on the dimension. Whereas for $N = 1, 2$, S_1 is always stable if $V \neq 0$ (Theorem 3.2), for $N \geqq 3$, there always exist non-zero potentials V such that S_1 does not converge. This property depends on the growth of V at infinity. If S_1 is stable in $L^1(\mathbb{R}^N)$, $N \geqq 3$, then necessarily

$$(*) \qquad \int_{|y| \geqq 1} \frac{V(y)}{|y|^{N-2}} \, dy = \infty .$$

Moreover, if V is radial, the converse holds as well: condition $(*)$ implies stability of S_1.

The proofs given here (Theorem 3.6 and 3.7) are analytical, throughout. In a separate paper, the second author obtains these and other results by probabilistic methods.

---

* *Permanent address*: St. John's College, Oxford OX1 3JP, UK

1 Stability of positive semigroups

In this preliminary section we give an abstract characterization of stability for positive (i.e. positivity preserving) semigroups. Let $T = (T(t))_{t \geqq 0}$ be a C_0-semigroup on a Banach space with generator A. We denote by $\sigma(A)$ the spectrum of A, $P\sigma(A)$ its point spectrum and by

$$s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$$

the spectral bound of A. The number

$$\omega(A) := \inf\{\omega \in \mathbb{R} : \sup_{t \geqq 0} e^{-\omega t} \| T(t) \| < \infty\}$$

is called the growth bound or type of T (or of A). One always has $s(A) \leqq \omega(A)$ and growth bound and spectral bound coincide whenever T is holomorphic or T is a positive semigroup on L^1 or L^2 (see [12]). We denote by $N(A) = \{x \in D(A): Ax = 0\}$ the kernel of A and by $N(A')$ the kernel of the adjoint A' of A.

Definition 1.1 We say that T (or A) is *stable* if $\lim_{t \to \infty} T(t)x = 0$ for all $x \in E$. If $\omega(A) < 0$, then T (or A) is called *exponentially stable.*

In the following we assume that T is a positive C_0-semigroup on a Banach lattice E.

Lemma 1.2 *Assume that T is bounded. If $N(A') \neq \{0\}$, then $N(A')_+ := N(A') \cap E'_+ \neq \{0\}$.*

Proof. a) Observe that $\sup_{0 \leqq \lambda \leqq 1} \| \lambda R(\lambda, A)' \| < \infty$ since T is bounded.

b) Let $\varphi \in E'$. Then

$$\lim_{\lambda \downarrow 0} \langle Ax, \lambda R(\lambda, A)'\varphi \rangle = \lim_{\lambda \downarrow 0} \langle \lambda^2 R(\lambda, A)x - \lambda x, \varphi \rangle = 0$$

$$\text{for all } x \in D(A)\,.$$

Thus, if ψ is a w^*-limit point of $\lambda R(\lambda, A)'\varphi$ for $\lambda \downarrow 0$, then $\psi \in N(A')$.

c) Let $\varphi \in N(A')$. Then $\lambda R(\lambda, A)'\varphi^+$ and $\lambda R(\lambda, A)'\varphi^-$ possess w^*-limit points for $\lambda \downarrow 0$ (by a), which are in $N(A')_+$ (by b). If every such limit point is 0, then it follows that

$$\varphi = \lim_{\lambda \downarrow 0} \lambda R(\lambda, A)'\varphi = \lim_{\lambda \downarrow 0} \lambda R(\lambda, A)'\varphi^+ - \lim_{\lambda \downarrow 0} \lambda R(\lambda, A)'\varphi^- = 0\,. \quad \square$$

Theorem 1.3 *Let T be a positive bounded C_0-semigroup with generator A such that $\sigma(A) \cap i\mathbb{R}$ is countable. Then T is stable if and only if $N(A')_+ = \{0\}$.*

Proof. It follows from the proof of [12, C-III Corollary 4.3] that $P\sigma(A') \cap i\mathbb{R}$ is cyclic; in particular, $0 \notin P\sigma(A')$ implies $P\sigma(A') \cap i\mathbb{R} = \emptyset$. By Lemma 1.1, $0 \notin P\sigma(A')$ if and only if $N(A')_+ = \{0\}$. Now the theorem follows from the stability theorem [1] or [11]. $\square$

Corollary 1.4 *Assume that T is a bounded positive semigroup which is eventually norm continuous. Then T is stable if and only if $N(A')_+ = \{0\}$.*

This follows since for such a semigroup automatically $\sigma(A) \cap i\mathbb{R} \subset \{0\}$, [12, C-III Corollary 2.13].

Remark 1.5 The particular case where T is a bounded holomorphic semigroup can be settled by a simpler argument: in that case $\sup_{t>0} \|tAT(t)\| < \infty$; hence $T(t)x \to 0$ $(t \to \infty)$ for all $x \in R(A)$ (the range of A). Thus T is stable if and only if $R(A)$ is dense, which in turn is equivalent to $N(A') \neq \{0\}$.

2 The Neumann-Laplacian with absorbing potential on bounded domain

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set of class C^1. Denote by A_2 the Neumann-Laplacian on $L^2(\Omega)$, i.e. A_2 is given by

$$D(A_2) = \{u \in H^1(\Omega)\colon \exists v \in L^2(\Omega), \int_\Omega \nabla u \nabla \varphi = -\int v\varphi \quad \text{for all } \varphi \in H^1(\Omega)\},\ A_2 u = v\,.$$

(See [4, Chap. X] or [13, XIII, 15, p 263].) This operator generates a positive semigroup $T_2 = (T_2(t))_{t \geq 0}$ on $L^2(\Omega)$. Moreover, there exist positive contraction semigroups T_p on $L^p(\Omega)$ $(1 \leq p < \infty)$ such that $T_p(t)f = T_q(t)f$ $(f \in L^p(\Omega) \cap L^q(\Omega)$, $t \geq 0)$. Let Δ_p be the generator of T_p.

Theorem 2.1 *Let* $0 \leq V \in L^\infty(\Omega)$, $V \neq 0$. *Then* $\Delta_p - V$ *is exponentially stable* $(1 \leq p < \infty)$.

Proof. Denote by S_p the semigroup generated by $\Delta_p - V$. The operator $\Delta_2 - V$ has compact resolvent. Assume that $s(\Delta_2 - V) = 0$. Then there exists $u \in D(\Delta_2)$, $\|u\|_{L^2} = 1$ such that $\Delta_2 u = Vu$. Hence

$$\int_\Omega (\nabla u)^2 + \int_\Omega Vu^2 = (-\Delta_2 u + Vu|u) = 0\,.$$

It follows that $\nabla u = 0$. Hence $u \equiv$ const. Since $\int_\Omega Vu^2 = 0$, it follows that $V = 0$, contradiction.

We have shown that $\omega(\Delta_2 - V) = s(\Delta_2 - V) < 0$. So there exist $\omega < 0$, $M \geq 0$ such that $\|S_2(t)\| \leq Me^{\omega t}$ $(t \geq 0)$. The semigroup T_2 is ultracontractive (see [3] or [6, 2.4]); in particular, $T_2(1)$ is a bounded operator from $L^2(\Omega)$ to $L^\infty(\Omega)$. Consequently, $T_1(1)$ is bounded from $L^1(\Omega)$ into $L^2(\Omega)$, by self-adjointness. Since $0 \leq S_1(t) \leq T_1(t)$, also $S_1(t)$ maps $L^1(\Omega)$ into $L^2(\Omega)$. Thus for $f \in L^1(\Omega)$, $\|S_1(t)f\|_{L^1} = \|S_2(t-1)S_1(1)f\|_{L^1} \leq Me^{\omega(t-1)} \|S_1(1)\|_{\mathscr{L}(L^1, L^2)} \|f\|_{L^1}$ $(t \geq 1)$. So S_1 is exponentially stable. It follows from the Riesz-Thorin theorem that S_p is exponentially stable for $1 \leq p \leq 2$; $2 \leq p < \infty$ this follows by duality. □

3 Stability of the Schrödinger semigroup

By $T_p = (T_p(t))_{t \geq 0}$ we denote the Gaussian semigroup on $L^p(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$; i.e. T_p is given by

$$(T_p(t)f)(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy \quad (t > 0)\,.$$

This defines a C_0 semigroup on $L^p(\mathbb{R}^N)$ for $1 \leqq p < \infty$ and a w^*-continuous semigroup on $L^\infty(\mathbb{R}^N)$ for $p = \infty$. The generator Δ_p of T_p is given by

$$D(\Delta_p) = \{f \in L^p(\mathbb{R}^N);\ \Delta f \in L^p(\mathbb{R}^N)\},\ \Delta_p f = \Delta f \quad (1 \leqq p \leqq \infty)\,.$$

Proposition 3.1 *If* $1 < p < \infty$, *then* T_p *is stable.*

Proof. Since $(T_2(t)f)^\wedge(\xi) = e^{-\xi^2 t}\hat{f}(\xi)$ $(f \in L^2(\mathbb{R}^N))$ this is clear for $p = 2$. For $1 < p < 2$ and $f \in L^1 \cap L^2$ one has

$$\|T_p(t)f\|_{L^p} \leqq \|T_1(t)f\|_{L^1}^{2/p-1} \|T_2(t)f\|_{L^2}^{2-2/p} \to 0 \quad (t \to \infty)\,.$$

Since T_p is bounded and $L^1 \cap L^2$ is dense the result follows. For $2 < p < \infty$ one can argue similarly. □

However, the Gaussian semigroup T_1 on $L^1(\mathbb{R}^N)$ is not stable, in fact, it preserves the norm on the positive cone, $\|T_1(t)f\|_{L^1} = \|f\|_{L^1}$ for $0 \leqq f \in L^1$.

Next we introduce an absorption $0 \leqq V \in L^1_{\text{loc}}(\mathbb{R}^N)$. Then the operator $\Delta_1 - V$ with domain $D(\Delta_1 - V) = D(\Delta_1) \cap D(V)$ generates a bounded bolomorphic semigroup S_1 on $L^1(\mathbb{R}^N)$ which is dominated by the Gaussian semigroup (see [10, 15]):

$$0 \leqq S_1(t) \leqq T_1(t) \quad (t \geqq 0)\,. \tag{3.1}$$

Theorem 3.2 *Let* $N \in \{1, 2\}$ *and let* $0 \leqq V \in L^1_{\text{loc}}(\mathbb{R}^N)$. *If* $V \neq 0$, *then* $\Delta_1 - V$ *is stable.*

For the proof we use the following notion. A function $\varphi \in L^1_{\text{loc}}(\mathbb{R}^N)$ is called *subharmonic* if $\Delta\varphi \geqq 0$ (cf. [5, Chap. II]). It is well-known that for $\varphi \in L^\infty(\mathbb{R}^N)$ (N arbitrary) $\Delta\varphi = 0$ implies that φ is constant (cf. [5, II §2, Corollaire 1]). This remains true for subharmonic φ if $N = 1$ or 2.

Proposition 3.3 *Let* $N \in \{1, 2\}$ *and let* $\varphi \in L^\infty(\mathbb{R}^N)$ *be subharmonic; then* φ *is constant.*

Proof. This follows from [8, Theorem 2.14, p. 67, 68] or [7, problem 2.14] if φ is smooth. Applying this result to $\rho_n * \varphi$ instead of φ, in the general case (ρ_n being a mollifier), the result follows. □

Proof of Theorem 3.2 Let $0 \leqq \varphi \in N((\Delta_1 - V)')$. Then $0 \leqq \varphi \in L^\infty(\mathbb{R}^N)$ and $\Delta\varphi = V\varphi$ in $\mathscr{D}'(\mathbb{R}^N)$. Hence φ is subharmonic. It follows from Proposition 3.3 that φ is constant, say $\varphi(x) = C$ $(x \in \mathbb{R}^N)$. Then $0 = \Delta\varphi = CV$. Hence $C = 0$ or $V \equiv 0$. It follows from Remark 1.5 that $\Delta_1 - V$ is stable if $V \neq 0$. □

The situation is different if $N \geqq 3$. Then there always exist non-zero potentials $0 \leqq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $\Delta - V$ is not stable.

Proposition 3.4 *For* $0 \leqq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ *let*

$$N_V := \{g \in L^\infty(\mathbb{R}^N) : 0 \leqq g \leqq 1;\ \Delta g = Vg\}\,.$$

Denote by S the semigroup generated by $\Delta_1 - V$ on $L^1(\mathbb{R}^N)$. Then the following holds.

a) N_V *has a maximal element* g_V.

b) *If* $V \in L^1(\mathbb{R}^N)$, *then* $g_V = 1 - \int_0^\infty S(t)V\,dt$, *where the improper integral converges in the w*-sense in* L^∞ *(note that* $S(t)V \in L^\infty$ *for* $t > 0$*).*

c) $\Delta_1 - V$ *is stable if and only if* $g_V = 0$.

d) *If* $V, \tilde{V} \in L^1_{\text{loc}}(\mathbb{R}^N)$ *such that* $0 \leqq V \leqq \tilde{V}$, *then* $g_{\tilde{V}} \leqq g_V$.

e) *Let* $V_n \in L^1_{\text{loc}}(\mathbb{R}^N)$ *such that* $0 \leqq V_n \leqq V_{n+1} \leqq V$ *and* $\lim_{n\to\infty} V_n(x) = V(x)$ *a.e. Then* $w^* - \lim_{n\to\infty} g_{V_n} = g_V$.

Proof. Let $\varphi \in \mathscr{D}(\mathbb{R}^N)$. Then $S(t)\varphi \in D(\Delta_1 - V) = D(\Delta_1) \cap D(V)$ for all $t \geqq 0$. Consequently,

$$\frac{d}{dt}\langle \varphi, S(t)'1\rangle = \langle (\Delta_1 - V)S(t)\varphi, 1\rangle = -\langle S(t)\varphi, V\rangle \quad (t \geqq 0)\,. \tag{3.2}$$

Thus $\langle \varphi, S(t)'1\rangle$ is decreasing for all $0 \leqq \varphi \in \mathscr{D}(\mathbb{R}^N)$. It follows that $S(t)'1$ is decreasing.

Let $g_V = \inf_{t>0} S(t)'1 = w^* - \lim_{t\to\infty} S(t)'1$. Then

$$S(s)'g_V = w^* - \lim_{t\to\infty} S(t+s)'1 = g_V.$$

Hence $g_V \in N((\Delta_1 - V)')$ and $0 \leqq g_V \leqq 1$. Since $\mathscr{D}(\mathbb{R}^N)$ is a core of $\Delta_1 - V$ (see [10]) it follows that

$$N_V = \{g \in N((\Delta_1 - V)') : 0 \leqq g \leqq 1\}\,. \tag{3.3}$$

Consequently $g_V \in N_V$. We show that g_V is maximal. Let $g \in N_V$. Since $0 \leqq g \leqq 1$, it follows that $g = S(t)'g \leqq S(t)'1$ $(t \geqq 0)$; consequently, $g \leqq g_V$. So a) is proved.

If $V \in L^1(\mathbb{R}^N)$, then $S(t)\varphi = S(t)'\varphi$ for all $\varphi \in \mathscr{D}(\mathbb{R}^N)$, hence $\langle S(t)\varphi, V\rangle = \langle \varphi, S(t)V\rangle$ $(t \geqq 0)$. So it follows from (3.2) that

$$\langle \varphi, S(t)'1\rangle - \langle \varphi, 1\rangle = \int_0^t \frac{d}{ds}\langle \varphi, S(s)'1\rangle\,ds = -\int_0^t \langle \varphi, S(s)V\rangle\,ds\,.$$

Since $\varphi \in \mathscr{D}(\mathbb{R}^N)$ is arbitrary one concludes that $S(t)'1 = 1 - \int_0^t S(s)V\,ds$. One obtains b) by letting $t \to \infty$.

Assertion c) follows from (3.3) and Remark 1.5 since $N_V = \{0\}$ if and only if $g_V = 0$. Assertion d) follows from the fact that $\tilde{S}(t) \leqq S(t)$ if $\tilde{S}(t)$ denotes the semigroup generated by $\tilde{V} \in L^1_{\text{loc}}$, $\tilde{V} \geqq V$.

It remains to prove e). Let $g_n = g_{V_n}$, then by d) $0 \leqq g_{n+1} \leqq g_n \leqq 1$. Let $g = \inf g_n = w^* - \lim g_n$. Then for $\varphi \in \mathscr{D}(\mathbb{R}^N)$,

$$\begin{aligned}\langle \varphi, \Delta g\rangle = \langle \Delta\varphi, g\rangle &= \lim \langle \Delta\varphi, g_n\rangle = \lim \langle \varphi, \Delta g_n\rangle \\ &= \lim \langle \varphi, V_n g_n\rangle = \langle \varphi, Vg\rangle\end{aligned}$$

by the dominated convergence theorem. Hence $g \in N_V$. So by a) $g \leqq g_V$. On the other hand $g_V \leqq S(t)'1 \leqq S_n(t)'1$ $(t \geqq 0)$ for all $n \in \mathbb{N}$. Hence $g_V \leqq g_n$ $(n \in \mathbb{N})$. Consequently, $g_V \leqq g$. We have shown that $g = g_V$. □

Proposition 3.5 *Let $0 \leqq V \in L^1_{loc}(\mathbb{R}^N)$ and let $\lambda > 0$. Then $\Delta_1 - V$ is stable if and only if $\Delta_1 - \lambda V$ is stable.*

Proof. Suppose that $\lambda > 1$. Denote by S (resp. U) the semigroup generated by $\Delta_1 - V$ (resp. $\Delta_1 - \lambda V$). The $0 \leqq U(t) \leqq S(t)$; hence U is stable whenever S is stable. Conversely assume that $\Delta_1 - \lambda V$ is stable. Let $V_n = V 1_{B(o,n)}$ and denote by S_n (resp. U_n) the semigroup generated by $\Delta_1 - V_n$ (resp. $\Delta_1 - \lambda V_n$). Let $\psi_n = g_{\lambda V n}$ and $\varphi_n = g_{Vn}$. Then $w^* - \lim \psi_n = 0$ by Proposition 3.4. Moreover, by Proposition 3.4e) $g_V = w^* - \lim \varphi_n$. By Proposition 3.4b) we have $\psi_n = 1 - \lambda \int_0^\infty U_n(s) V_n \, ds$; hence

$$1 = \psi_n + \lambda \int_0^\infty U_n(s) V_n \, ds \leqq \psi_n + \lambda \int_0^\infty S_n(s) V_n \, ds = \psi_n + \lambda(1 - \varphi_n) .$$

Letting $n \to \infty$ we obtain $1 \leqq \lambda(1 - g_V)$. Consequently $C g_V \leqq 1$ where $C = \lambda/(\lambda - 1)$. So $C g_V \in N_V$. It follows that $C g_V \leqq g_V$. Hence $g_V = 0$ since $C > 1$; i.e. $\Delta_1 - V$ is stable. If $\lambda < 1$, it suffices to apply the preceding result to λV. □

Theorem 3.6 *Let $N \geqq 3$. If $0 \leqq V \in L^1_{loc}(\mathbb{R}^N)$ satisfies*

$$\int_{|y| \geqq 1} \frac{V(y)}{|y|^{N-2}} \, dy < \infty \tag{3.4}$$

then $\Delta_1 - V$ is not stable.

Proof. a) We show that there exists a measurable set $B \subset \mathbb{R}^N$ of positive measure such that

$$\sup_{x \in B} \int_{\mathbb{R}^N} \frac{V(y)}{|x - y|^{N-2}} dy < \infty . \tag{3.5}$$

In fact, let $C = \int_{|x| \leqq 2} V(x) dx$. One has

$$\int_{|x| \leqq 1} \left\{ \int_{|x-y| \leqq 2} \frac{V(x-y)dy}{|y|^{N-2}} \right\} dx \leqq$$

$$\int_{|y| \leqq 3} \left\{ \int_{|x-y| \leqq 2} V(x-y)dx \right\} \frac{dy}{|y|^{N-2}} = \frac{9}{2} C \omega_N ,$$

where $\omega_N = \dfrac{2\pi^{N/2}}{\Gamma(N/2)}$ is the surface of the unit sphere in $\mathbb{R}^N$.

By Fubini's theorem there exists a measurable set $B \subset B(0, 1) := \{x \in \mathbb{R}^N : |x| \leqq 1\}$ of positive measure such that

$$\sup_{x \in B} \int_{|y| \leqq 2} \frac{V(y)}{|x - y|^{N-2}} dy = \sup_{x \in B} \int_{|x-y| \leqq 2} \frac{V(x-y)}{|y|^{N-2}} dy < \infty . \tag{3.6}$$

But for $|x| \leqq 1$, $|y| > 2$, one has $|x - y| \geqq |y| - |x| \geqq \frac{1}{2}|y|$. Thus

$$\int_{|y| > 2} \frac{V(y)}{|x - y|^{N-2}} dy \leqq 2^{N-2} \int_{|y| > 2} \frac{V(y)}{|y|^{N-2}} dy < \infty$$

by (3.4). This together with (3.6) proves the claim.

b) For $0 \leqq g \in L^1(\mathbb{R}^N)$ we have by Fubini's theorem

$$\int_0^\infty (T_1(t)g)(x)\,dt = \int_{\mathbb{R}^N} \int_0^\infty (4\pi t)^{-N/2} \exp(-(x-y)^2/4t)\,dt\,g(y)\,dy$$
$$= (E_N * g)(x) \quad (x \in \mathbb{R}^N)$$

where $E_N(x) = \dfrac{1}{(N-2)\omega_N} |x|^{2-N}$. In view of Proposition 3.5, we may assume that

$$q := \sup_{x \in B} (E_N * V)(x) < 1 .$$

For $n \in \mathbb{N}$, let $V_n = V1_{B(o,n)}$, S_n the semigroup generated by $\Delta_1 - V_n$ and $g_n = g_{V_n}$. Then $g_V = w^* - \lim_{n\to\infty} g_n$ by Proposition 3.4e); and by b)

$$g_n(x) = 1 - \int_0^\infty (S_n(t)V_n)(x)\,dt \geqq 1 - \int_0^\infty (T_1(t)V_n)(x)\,dt$$
$$= 1 - (E_N * V_n)(x) \geqq 1 - (E_N * V)(x) \geqq 1 - q \quad (x \in B) .$$

Hence $g_V = \inf g_n \geqq 1 - q$ on B and so $g_V \neq 0$. □

The converse of Theorem 3.6 is not true in general. A characterization of stability will be given by the second author [2] by means of probabilistic methods (i.e. Wiener measure and the Feynman-Kac formula). For radial V, however, we obtain the following characterization.

Theorem 3.7 *Let $N \geqq 3$, and let $0 \leqq V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be radial. Then $\Delta_1 - V$ is stable if and only if*

$$\int_{|y| \geqq 1} \frac{V(y)}{|y|^{N-2}}\,dy = \infty .$$

Proof. Suppose that $\Delta_1 - V$ is not stable. Then there exists $0 \leqq g \in L^\infty(\mathbb{R}^N)$, $g \neq 0$, such that $\Delta g = Vg$. We can suppose that $V \neq 0$, so g is not constant. Since V is radial, we can suppose that g is radial; otherwise we replace $g(x)$ by $\tilde{g}(x) = \int_{S(0,1)} g(|x|\ y)\ d\sigma(y)$ where $d\sigma$ is the surface-measure on $S(0,1) = \{z \in \mathbb{R}^N : |z| = 1\}$.

Hence $1/r^{N-1}\ (r^{N-1} g'(r))' = Vg$ in $\mathscr{D}'(0,\infty)$. This implies that $g \in C^1(0,\infty)$ and $r^{N-1}g'(r)$ is non-decreasing. We show that g is non-decreasing. If not, there exists $r_0 > 0$ such that $g'(r_0) < 0$. Then $r^{N-1}g'(r) \leqq r_0^{N-1}g'(r_0)$ on $(0, r_0)$. Hence for $r \in (0, r_0)$,

$$g(r) = g(r_0) + \int_r^{r_0} (-g'(s))s^{N-1} \frac{ds}{s^{N-1}} \geqq$$
$$g(r_0) + (-g'(r_0)r_0^{N-1}) \int_r^{r_0} \frac{ds}{s^{N-1}} \to \infty \quad (r \to 0) .$$

This is not possible since g is bounded.

Since g is non-constant, there exists $r_0 > 0$ such that $g(r_0) > 0$ and $g'(r_0) > 0$. Then

$$r^{N-1}g'(r) = r_0^{N-1}g'(r_0) + \int_{r_0}^{r} (s^{N-1}g'(s))'\,ds$$

$$= r_0^{N-1}g'(r_0) + \int_{r_0}^{r} s^{N-1}\,V(s)g(s)\,ds \geqq g(r_0)\int_{r_0}^{r} s^{N-1}\,V(s)\,ds.$$

Consequently,

$$\frac{1}{N-2}\int_{|y|\geqq r_0}\frac{V(y)}{|y|^{N-2}}\,dy = \frac{1}{N-2}\int_{r_0}^{\infty} s\,V(s)\,ds = \int_{r_0}^{\infty}\frac{1}{N-2}s^{-N+2}s^{N-1}\,V(s)\,ds$$

$$= \int_{r_0}^{\infty}\int_{s}^{\infty}\frac{1}{r^{N-1}}\,dr\,s^{N-1}\,V(s)\,ds = \int_{r_0}^{\infty}\int_{r_0}^{r} s^{N-1}\,V(s)\,ds\,\frac{1}{r^{N-1}}\,dr \leqq \frac{1}{g(r_0)}\int_{r_0}^{\infty} g'(r)\,dr < \infty$$

since g is bounded. □

References

1. Arendt, W., Batty, C.J.K.: Tauberian theorems and stability of one-parameter semigroups. Trans Am. Math. Soc. **306**, 837–852 (1988)
2. Batty, C.J.K.: Asymptotic stability of Schrödinger semigroups: path integral methods Math. Ann. (to appear)
3. Bénilan Ph.: Opérateurs accrétifs et semi-groupes dans les espaces L^p. In: Fujita, H. (ed.) Funct. Anal. and Num. Anal. Japan-France Seminar. Tokyo and Kyoto 1976
4. Brézis, H.: Analyse fonctionnelle. Paris: Masson 1963
5. Dautray, R., Lions, J.-L.: Analyse mathématique et calcul numérique. Paris: Masson 1987
6. Davies, E.B.: Heat Kernels and spectral theory. Cambridge: Cambridge University Press 1989
7. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Berlin Heidelberg New York: Springer 1983
8. Hayman, W.K., Kennedy, P.B.: Subharmonic Functions. London: Academic Press 1976
9. Hempel, R., Voigt, J.: On the L_p-spectrum of Schrödinger operators. J. Math. Anal. Appl. **121**, 138–159 (1987)
10. Kato, T.: L_p-Theory of Schrödinger operators with a singular potential. In: Nagel, R., Schlotterbeck, U., Wolff, H. (eds.): Aspects of Positivity in Functional Analysis. Amsterdam: North Holland 1986
11. Lyubich, Yu.I., Vũ Quôc Phóng: Asymptotic stability of linear differential equations on Banach spaces. Stud. Math. **88**, 37–42 (1988)
12. Nagel, R. (ed.): One-parameter Semigroups of Positive Operators. (Lect. Notes Math., vol. 1184) Berlin Heidelberg New York: Springer
13. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, Vol. IV. London: Academic Press 1978
14. Simon, B.: Schrödinger semigroups. Bull. Am. Math. Soc. **7**, 447–526 (1982)
15. Voigt, J.: Absorption semigroups, their generators and Schrödinger semigroups. J. Oper. Theory **20**, 117–131 (1988)