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RESEARCH ARTICLE

Interpolation of Semigroups and Integrated Semigroups*

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0. Introduction

In 1971, S. G. Krein, G. I. Laptev and G. A. Cvetkova [K-L-C] proved that any linear (unbounded) operator A on a Banach space E such that the resolvent set contains a half-line (w,∞), generates a C_o-semigroup on a certain (maximal) subspace Z of E (see also [Ka], [M-O-O], [Ne3]). This is a very general result, expressing the popular belief that linear dynamic systems are good-natured in some sense.

The fact that no information on the size of Z is available in general, suggests a comparison of these operators according to the actual size of Z; with $Z=E$ as the best possible case and with $Z=\{0\}$ as the worst, but still possible one (see [Be]). On the good side of this scale, the cases $D(A^k)\subset Z$ $(k\in\mathbb{N})$ are of particular interest. In this paper we want to show that this situation is actually characteristic for generators of k-times integrated semigroups which were introduced in order to treat the abstract Cauchy problem $u'(t)=Au(t)$, $u(0)=x$ in cases where the resolvent of the operator A exists and has polynomial growth in a right half-plane. Such operators frequently occur if one studies differential operators in $L^p(\mathbb{R}^n)$, $(1\le p\le\infty)$, systems of linear partial differential equations or higher order Cauchy problems, to mention just a few instances (see [Ar1], [Ar2], [A-K], [dL], [K-H], [Ne1], [Ne2], [Ne3], [N-S], [Oh], [So], [T-M1], [T-M2], [Th]). It turned out that integrated semigroups share many properties with C_o-semigroups. The purpose of this paper is to show that every C_o-semigroup on a Banach space F induces integrated semigroups on continuously embedded subspaces and that, conversely, every integrated semigroup on F can be "sandwiched" by C_o-semigroups on extrapolation- and interpolation spaces.

In order to make this more precise we introduce some notation. An operator A on a Banach space E is the generator of k-times integrated semigroup (where $k\in\mathbb{N}_0$) if there exist $w\ge 0$ and $S(\cdot):[0,\infty)\to\mathcal{L}(E)$ strongly continuous such that (w,∞) is contained in the resolvent set of A, and

$$(\mu I-A)^{-1}x=\mu^k\int_0^\infty e^{-\mu t}S(t)x\,dt \qquad (x\in E,\mu>w).$$

The function $S(\cdot)$ is called *k-times integrated semigroup.* If there exists $M\ge 0$ such that $\|S(t)\|\le Me^{wt}$ for all $t\ge 0$, then $S(\cdot)$ is called exponentially bounded of type w. Thus, if $S(\cdot)$ is of type w, it is also of type w' for all $w'>w$.

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This notion differs slightly from the usual one in the case $k = 0$, but will be convenient for our purposes.

An operator A generates a 0-times integrated semigroup if and only if A generates a C_o-semigroup (see [Ar1]).

For Banach spaces E, F we write $E \hookrightarrow F$ if E is a subspace of F and the inclusion is continuous. We write $E \hookrightarrow_d F$ if $E \hookrightarrow F$ and E is dense in F.

Let B be an operator on F with domain $D(B)$. We denote by $\rho(B)$ the resolvent set and by $R(\mu, B) := (\mu I - B)^{-1}$ the resolvent of B in μ. If $E \hookrightarrow F$, then we denote by B_E the "part of B in E" defined by $D(B_E) := \{x \in D(B) \cap E : Bx \in E\}$, $B_E x := Bx$.

In particular, if $R \in \mathcal{L}(F)$ (the space of all bounded linear operators) such that $RE \subset E$, then R_E is the usual "restriction" of R to E. It follows from the closed graph theorem that $R_E \in \mathcal{L}(E)$.

If B is a closed operator on F, then $D(B^k)$ is a Banach space for the graph norm $|x|_{B^k} := |x| + |Bx| + ... + |B^k x|$, $(k \in \mathbb{N}_0)$. This Banach space is denoted by $[D(B)^k]$. Clearly, $[D(B^k)] \hookrightarrow F$. The part of B in $[D(B^k)]$ is denoted by B_k.

If B generates a C_o-semigroup $T(\cdot)$ on F, then B_k generates a C_o-semigroup $T_k(\cdot)$ on $[D(B^k)]$ and $T_k(t)$ coincides with the restriction of $T(t)$ to $[D(B^k)]$ for all $t \geq 0$ (see [Na]).

Now we can state the main results.

Theorem 0.1. (Interpolation Theorem) *Let B be the generator of a C_o-semigroup $T(\cdot)$ on a Banach space F.*

(a) *Assume that E is a Banach space such that $[D(B^k)] \hookrightarrow E \hookrightarrow F$ for some $k \in \mathbb{N}$. In the case $k \geq 2$ assume in addition that $R(\mu_o, B)E \subset E$ for some $\mu_o \in \rho(B)$. Then B_E generates an exponentially bounded, k-times integrated semigroup $S_E(\cdot)$ on E.*

(b) *Assume that $D(B) \neq F$. Then, given any $k \in \mathbb{N}$, there exists a Banach space E such that $[D(B^k)] \hookrightarrow E \hookrightarrow F$ and B_E generates an exponentially bounded, k-times integrated semigroup, but not a $(k-1)$-times integrated semigroup.*

The next result is a converse of Theorem 1. Given the generator A of a k-times integrated semigroup we construct a maximal inscribed space on which the part of A acts as a generator of a C_o-semigroup. An extrapolation space is obtained as well.

Theorem 0.2. (Extrapolation Theorem) *Let A be the generator of an exponentially bounded, k-times integrated semigoup of type $w > 0$ on a Banach space E. Let $\alpha > w$. Then there exists a generator B of C_o-semigroup of type α on a Banach space F such that*

(a) $[D(B^k)] \hookrightarrow E \hookrightarrow_d F$ *and* $A = B_E$;

(b) *the Banach space $[D(B^k)]$ is maximal unique in the following sense : If W is a Banach space such that $W \hookrightarrow E$ and A_W generates a C_o-semigroup of type α on W, then $W \hookrightarrow [D(B^k)]$.*

Note that the operator A is not necessarily densely defined. Theorem 0.2 says that A is "sandwiched" by the C_o-semigroup generators B and B_k. In combination with Theorem 0.1 one obtains the following characterization.

Corollary 0.3. *Let A be a densely defined operator on a Banach space E with non empty resolvent set. Then the following statements are equivalent.*

(a) *A generates an exponentially bounded, k-times integrated semigroup on E.*
(b) *There exists a Banach space G such that $[D(A^k)] \hookrightarrow G \hookrightarrow E$, and A_G generates a C_o-semigroup on G.*
(c) *There exists a generator B of a C_o-semigroup on a Banach space F such that $[D(B^k)] \hookrightarrow E \hookrightarrow F$, $R(\mu, B)E \subset E$ for some $\mu \in \rho(B)$ and $A = B_E$.*

These results show that the concepts of integrated semigroups and C_o-semigroups are the same up to the choice of the Banach space. However, in many instances it turns out that it is relatively easy to prove that an operator A generates an integrated semigroup on a "nice" Banach space E, whereas the construction of the inter- or extrapolation spaces on which C_o-semigroups are generated is quite tedious, if not impossible (see [A-K], [Ne1], [Ne2]).

The paper is organized as follows. Section 1 contains the basic properties of integrated semigroups which are needed later while in Section 2 and 3 the main results are proved.

1. Preliminaries

At first we define Laplace transforms of operator-valued functions. Let E be a Banach space and $S(\cdot) : [0,\infty) \to \mathcal{L}(E)$ be a strongly continuous function. For $\mu \in \mathbb{C}$ and $b \geq 0$ we define the operator $\int_0^b e^{-\mu t} S(t)dt \in \mathcal{L}(E)$ by

$$\left(\int_0^b e^{-\mu t} S(t)dt\right)x := \int_0^b e^{-\mu t} S(t)x dt \qquad (x \in E),$$

where $\int_0^b e^{-\mu t} S(t)x dt$ is the usual Riemann integral. Let $S_1(t) := \int_0^t S(s)ds$ $(t \geq 0)$. If $S(\cdot)$ is exponentially bounded, then the Laplace transform

$$\int_0^\infty e^{-\mu t} S(t)dt := \lim_{b\to\infty} \int_0^b e^{-\mu t} S(t)dt$$

of $S(\cdot)$ exists. The converse statement does not hold (see [Do], p. 38). We show that the Laplace transform of $S(\cdot)$ exists if and only if the once integrated function $S_1(\cdot)$ is exponentially bounded.

Proposition 1.1. a) *If $|S_1(t)| \leq Me^{wt}$ for some $M, w \geq 0$ and all $t \geq 0$, then*

$$\lim_{b\to\infty} \int_0^b e^{-\mu t} S(t)dt$$

exists in the operator norm for $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > w$ *and*

$$\int_0^\infty e^{-\mu t} S(t)dt = \mu \int_0^\infty e^{-\mu t} S_1(t)dt.$$

b) *Conversely, if $\sup_{b\geq 0} \left|\int_0^b e^{-\mu t} S(t)dt\right| < \infty$, then there exists $M \geq 0$ such that*

$|S_1(t)| \le Me^{\operatorname{Re}\mu t}$ *(if* $\operatorname{Re}\mu > 0$ *or* $\mu = 0$*) and* $|S_1(t)| \le M(1+t)$ *(if* $\operatorname{Re}\mu = 0$ *and* $\mu \ne 0$*) for all* $t \ge 0$.

Proof. a) From the assumptions it follows that

$$\left|\int_t^{t+h} e^{-\mu s} S_1(s)ds\right| \le M(e^{-(\operatorname{Re}\mu - w)t})/(\operatorname{Re}\mu - w) \to 0$$

for $t \to \infty$ uniformly in $h \ge 0$ for all $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > w$. Hence $\int_0^\infty e^{-\mu s} S_1(s)ds$ converges in the operator norm. Integrating by parts one obtains

$$\int_0^t e^{-\mu s} S(s)ds = e^{-\mu t} S_1(t) + \mu \int_0^t e^{-\mu t} S_1(s)ds \to \mu \int_0^\infty e^{-\mu s} S_1(s)ds$$

for $t \to \infty$ with respect to the operator norm.

b) By assumption, the operator family $B(t) := \int_0^t e^{-\mu t} S(s)ds$ is norm bounded. The statement follows from

$$S_1(t) = \int_0^t e^{\mu s} e^{-\mu s} S(s)ds = e^{\mu t} B(t) - \mu \int_0^t e^{\mu s} B(s)ds. \qquad \blacksquare$$

Next we define integrated semigroups and their generators.

Definition 1.2. Let $k \in \mathbb{N}_0$. An operator A on a Banach space E is called a *generator* of a k-times integrated semigroup if there exist $M, w \ge 0$ and a strongly continuous function $S(\cdot)\colon [0,\infty) \to \mathcal{L}(E)$ satisfying

$$\left|\int_0^t S(s)ds\right| \le Me^{wt} \qquad (t \ge 0)$$

such that $(w,\infty) \subset \rho(A)$ and $R(\mu, A) = \mu^k \int_0^\infty e^{-\mu t} S(t)dt$ for $\mu > w$. The function $S(\cdot)$ is called the *k-times integrated semigroup generated by A*. If there exists $C \ge 0$ and w' such that $|S(t)| \le Ce^{w't} (t \ge 0)$, then $S(\cdot)$ is called *exponentially bounded of type w'*. $\blacksquare$

We do not know whether in the situation of Definition 1.2 the function $S(\cdot)$ is automatically exponentially bounded. Thus the definition we give here might be more general than the one given in [Ar1] or [Ne1], where the function $S(\cdot)$ is always assumed to be exponentially bounded. However, by Proposition 1.1, if A generates a k-times integrated semigroup $S(\cdot)$, then A generates the exponentially bounded, $(k+1)$-times integrated semigroups $S_1(t) = \int_0^t S(s)ds$.

Moreover, the Cauchy problem with respect to A

$$\mathrm{CP}(A) \qquad u'(t) = Au(t), \quad u(0) = x, \quad u(\cdot) \in C^1([0,\infty), E) \cap C([0,\infty), D(A))$$

has at most one solution for all $x \in E$ (see [Ar1] or [Ne1]).

Let A be the generator of a k-times integrated semigroup $S(\cdot)$ on E.

Define $T(t) \in \mathcal{L}([D(A^k)], E)$, $(t \geq 0)$ by

(1.1) $T(t)x := S(t)A^k x + (t^{k-1}/(k-1)!)A^{k-1}x + \ldots + tAx + x \qquad (x \in D(A^k)).$

For $x \in D(A^{k+1})$ the function $u(\cdot) := T(\cdot)x$ is the unique solution of CP(A). In fact, by the proof of Prop. 3.3 in [Ar1] one has

$$(1.2) \quad \int_0^t S(s)y\,ds \in D(A) \text{ and } A\int_0^t S(s)y\,ds = S(t)y - (t^k/k!)y \quad (y \in E, t \geq 0),$$

$$(1.3) \qquad S(t)Ay = AS(t)y \qquad (y \in D(A)).$$

In particular, for $y \in D(A)$, $S(\cdot)y \in C^1([0,\infty), E) \cap C([0,\infty), D(A))$ and

$$(1.4) \qquad \frac{d}{dt}S(t)y = S(t)Ay + (t^{k-1}/(k-1)!)y \qquad (y \in D(A)).$$

Now it follows immediately that $T(\cdot)x = (d^k/dt^k)S(\cdot)x$ for $x \in D(A^k)$ and that $T(\cdot)x$ is a solution of CP(A) whenever $x \in D(A^{k+1})$. Moreover, if $u(\cdot)$ is a solution of CP(A) with $|u(t)| \leq \text{const}\cdot e^{\alpha t}$ for some $\alpha \geq 0$, then for $\alpha < \mu \in \rho(A)$

$$(1.5) \qquad R(\mu, A)u(0) = \int_0^\infty e^{-\mu t}u(t)\,dt \qquad \text{(see [Ne1, 4.6])}.$$

Next we discuss the "rescaling" of integrated semigroups (see also [dL]).

Proposition 1.3. *Let $k \in \mathbb{N}_0$ and $r \in \mathbb{R}$. If A generates an exponentially bounded, k-times integrated semigroup, then $A - rI$ generates an exponentially bounded, k-times integrated semigroup.*

Proof. Let $S(\cdot)$ be of type w. There exists a polynomial $p(\cdot)$ of degree $k-1$ such that $\sum_{j=1}^{k}\binom{k}{j}r^j\mu^{-j} = \int_0^\infty e^{-\mu t}p(t)\,dt$, $(\mu > 0)$. Define a strongly continuous and exponentially bounded function $S_r(\cdot)\colon [0,\infty) \to \mathcal{L}(E)$ by

$$S_r(t) := e^{-rt}S(t) + \int_0^t p(t-s)e^{-rs}S(s)\,ds \qquad (t \geq 0).$$

Let $\mu > \max(0, w-r)$. It follows from Fubini's Theorem that

$$\begin{aligned}\int_0^\infty e^{-\mu t}S_r(t)\,dt &= \int_0^\infty e^{-(\mu+r)t}S(t)\,dt + \int_0^\infty e^{(-\mu+r)t}S(t)\,dt \int_0^\infty e^{-\mu s}p(s)\,ds \\ &= \left(1 + \sum_{j=1}^{k}\binom{k}{j}r^j\mu^{-j}\right)(\mu+r)^{-k}R(\mu+r, A) \\ &= (1 + r/\mu)^k(\mu+r)^{-k}R(\mu+r, A) = \mu^{-k}R(\mu, A-rI). \qquad \blacksquare\end{aligned}$$

2. Interpolation of semigroups

We first prove Theorem 0.1 for $k = 1$; i.e., we consider interpolation spaces between the given Banach space F and the domain of the C_o-semigroup generator B. For later purposes it is convenient to consider not only generators of C_o-semigroups, but also generators of integrated semigroups. Thus, for $k = 1$, Theorem 0.1 is the special case of the following theorem for $m = 0$.

Theorem 2.1. *Let $m \in \mathbb{N}_0$ and let B generate an m-times integrated semigroup $T(\cdot)$ on a Banach space F. Assume E is a Banach space such that $[D(B)] \hookrightarrow E \hookrightarrow F$. Then B_E generates an $(m+1)$-times integrated semigroup $S_E(\cdot)$ on E. Moreover, if $T(\cdot)$ is exponentially bounded, then $S_E(\cdot)$ is exponentially bounded.*

We will use the following lemma, which is easy to prove.

Lemma 2.2. *Let A be an operator on E and let B be an operator on F such that $E \hookrightarrow F$. Assume that there exists $\mu \in \rho(B)$ such that $R(\mu, B)E \subset E$. Then $A = B_E$ if and only if $\mu \in \rho(A)$ and $R(\mu, A) = R(\mu, B)_E$.*

Proof of Theorem 2.1. Let $T(\cdot): [0, \infty) \to \mathcal{L}(F)$ be the m-times integrated semigroup generated by B. Define $S(t) := \int_0^t T(s)ds \in \mathcal{L}(F)$ $(t \geq 0)$. By (1.2), $S(t)F \subset D(B)$ and so $S(t)E \subset E$ for $t \geq 0$. Set $S_E(t) := S(t)_E \in \mathcal{L}(E)$. By (1.2), $BS(t) = T(t) - (t^m/m!)I$. It follows that $S(\cdot)x \in C([0,\infty), [D(B)])$ for all $x \in F$. Consequently, $S_E(\cdot): [0, \infty) \to \mathcal{L}(E)$ is strongly continuous.

We show that $\int_0^t S_E(s)ds$ is exponentially bounded (in $\mathcal{L}(E)$). By Proposition 1.1, there exist $M, w > 0$ such that $|S(t)|_{\mathcal{L}(F)} \leq Me^{wt}$ $(t \geq 0)$. Let $x \in E$. Then we have $\int_0^t S_E(s)xds = \int_0^t S(s)xds \in D(B)$ and $B\int_0^t S_E(s)ds = S(t)x - (t^{m+1}/(m+1)!)x$. Since $[D(B)] \hookrightarrow E \hookrightarrow F$ we obtain

$$\begin{aligned}
\left|\int_0^t S_E(s)xds\right|_E &\leq \text{const} \cdot \left|\int_0^t S(s)xds\right|_B \\
&= \text{const} \cdot \left(\left|\int_0^t S(s)xds\right|_F + \left|B\int_0^t S(s)xds\right|_F\right) \\
&= \text{const} \cdot \left(\left|\int_0^t S(s)xds\right|_F + \left|S(t)x - (t^{m+1}/(m+1)!)x\right|_F\right) \\
&\leq \text{const} \cdot \left(\frac{M}{w}e^{wt}\,|x|_F + Me^{wt}\,|x|_F + (t^{m+1}/(m+1)!\,|x|_F\right).
\end{aligned}$$

Consequently, $\sup_{t\geq 0}\left|e^{-wt}\int_0^t S_E(s)xds\right|_E < \infty$, $(x \in E)$. By the uniform boundedness principle, $\left|\int_0^t S_E(s)ds\right|_{\mathcal{L}(E)} \leq Ce^{wt}$, $(t \geq 0)$ for C suitable. Similarily, one shows that $|T(t)|_{\mathcal{L}(F)} \leq Me^{wt}$, $(t \geq 0)$ implies that $|S_E(t)|_{\mathcal{L}(E)} \leq Ce^{wt}$.

Since $R(\mu, B)E \subset E$ for $\mu \in \rho(B)$, one has $\rho(B) \subset \rho(B_E)$ and $R(\mu, B_E) = R(\mu, B)_E$ for all $\mu \in \rho(B)$. For $\mu > w$ define

$$R(\mu) := \mu^{m+1}\int_0^\infty e^{-\mu t}S_E(t)dt \in \mathcal{L}(E).$$

Integrating by parts one obtains

$$R(\mu)x = \mu^m\int_0^\infty e^{-\mu t}T(t)xdt = R(\mu, B)x = R(\mu, B_E)x \quad (x \in E, \mu > w).$$

By definition, this means that B_E generates the $(m+1)$-times integrated semigroup $S_E(\cdot)$ on E. ∎

Next we prove statement (a) of Theorem 0.1 for arbitrary k. We will frequently use the following fact. Let B be a closed operator and $k \in \mathbb{N}$. If $\mu \in \rho(B)$, then $x \to |(\mu I - B)^k x|$ defines an equivalent norm on $[D(B^k)]$.

Proof of Theorem 0.1. We show that $R(\mu, B)E \subset E$ for all $\mu \in \rho(B)$. Let $\mu, \mu_0 \in \rho(B)$. Iterating the resolvent equation
$R(\mu, B) = R(\mu_0, B) + (\mu_0 - \mu)R(\mu_0, B)R(\mu, B)$ yields

$$R(\mu, B) = \sum_{j=1}^{k-1} (\mu_0 - \mu)^{j-1} R(\mu_0, B)^j + (\mu_0 - \mu)^{k-1} R(\mu_0, B)^{k-1} R(\mu, B). \tag{2.1}$$

By assumption or by $R(\mu_0, B)F \subset D(B)$ (for $k = 1$), we have $R(\mu_0, B)E \subset E$ and $R(\mu_0, B)^{k-1} R(\mu, B)E \subset D(B^k) \subset E$. This proves the claim.

Let B generate a C_0-semigroup $T(\cdot)$ with $|T(t)| \leq Me^{wt}$ $(t \geq 0)$ for some $M, w > 0$. Considering $B - rI$ if necessary instead of B, we can assume that $0 \in \rho(B)$ (see Proposition 1.3). The exponentially bounded, k-times semigroup generated by B is given by $S(t) := \int_0^t ((t-s)^{k-1}/(k-1)!)T(s)ds$. It follows from (1.1) that

$$S(t) = B^{-k}T(t) - \sum_{i=0}^{k-1} (t^i/i!) B^{-k+i}. \tag{2.2}$$

Consequently, $S(t)E \subset E$, $S_E(t) := S(t)_E \in \mathcal{L}(E)$ and $S_E(\cdot): [0, \infty) \to \mathcal{L}(E)$ is strongly continuous. One obtains from (2.2) that

$$\begin{aligned}
|S_E(t)x|_E &\leq \sum_{i=0}^{k-1} (t^i/i!) \left|B^{-k+i}\right|_{\mathcal{L}(E)} |x|_E + \left|B^{-k}T(t)x\right|_E \\
&\leq \text{const} \cdot \left(e^{wt} |x|_E + \left|B^{-k}T(t)x\right|_{B^k}\right) \leq \text{const} \cdot e^{wt} |x|_E + \text{const} \cdot |T(t)x|_F \\
&\leq \text{const} \cdot e^{wt} |x|_E + \text{const} \cdot e^{wt} |x|_F \leq \text{const} \cdot e^{wt} |x|_E .
\end{aligned}$$

Hence, by the uniform boundedness principle, $S_E(\cdot)$ is exponentially bounded. Now one proceeds as in the proof of Theorem 2.1. ∎

We show by an example that in case $k > 1$ the hypothesis of E being invariant under the resolvent cannot be omitted in Theorem 0.1 (a).

Example 2.3. Let B generate a C_0-semigroup on a Banach space F. Assume that $D(B) \neq F$. Let $w \in F \backslash D(B)$ and $E := D(B^2) + \mathbb{R}.w$. Then E is a Banach space for the norm $|x + cw|_E := |x|_{B^2} + |cw|_F$. Clearly, $[D(B^2)] \hookrightarrow E \hookrightarrow F$. But B_E does not generate a k-times integrated semigroup for any $k \in \mathbb{N}$. In fact, assume that there exists $\mu \in \rho(B_E) \cap \rho(B)$. Then $R(\mu, B)E \subset E$. So there are $x \in D(B^2)$, $c \in \mathbb{R}$ such that $R(\mu, B)w = x + cw$. Hence $cw = R(\mu, B)w - x \in D(B)$. Thus $c = 0$. But then $R(\mu, B)w = x \in D(B^2)$. This implies $w \in D(B)$, which is a contradiction. ∎

Next we will prove statement (b) of Theorem 0.1. For that we need the following two lemmas.

Lemma 2.4. *Let $S(\cdot)$ be an exponentially bounded, $(k+1)$-times integrated semigroup on F with generator B. Then B generates a k-times integrated semigroup if and only if $S(\cdot)x \in C([0, \infty), [D(B)])$ for all $x \in F$.*

Proof. Assume that $S(\cdot)x \in C([0,\infty),[D(B)])$ for all $x \in F$. By (1.2), $T(\cdot)x := d/dtS(\cdot)x \in C([0,\infty),E)$ for all $x \in F$. Hence

$$R(\mu,B) = \mu^{k+1}\int_0^\infty e^{-\mu t}S(t)dt = \mu^k\int_0^\infty e^{-\mu t}T(t)dt$$

for μ large. By definition, B generates the k-times integrated semigroup $T(\cdot)$. The converse follows from (1.2). ∎

Lemma 2.5. *Let $S(\cdot)$ be an exponentially bounded, k-times integrated semigroup on F with generator B. Assume that $D(B) \neq F$ in the case $k = 0$ and that B does not generate a $(k-1)$-times integrated semigroup in the case $k > 0$. Then there exists a Banach space E such that $[D(B)] \hookrightarrow E \hookrightarrow F$ and such that B_E generates an exponentially bounded $(k+1)$-times integrated semigroup but not a k-times integrated semigroup on E.*

Proof. By Lemma 2.4, there is $w \in F$ such that $S(\cdot)w \notin C([0,\infty),[D(B)])$. By (1.3), $w \notin D(B)$ so that $E := [D(B)] + \mathbb{R}.w$ is a direct sum. Define $|x+cw|_E := |x|_B + |cw|_F$. Then $[D(B)] \hookrightarrow E \hookrightarrow F$. By Theorem 2.1, B_E generates a $(k+1)$-times integrated semigroup on E. Suppose that B_E generates a k-times integrated semigroup $S_E(\cdot)$ on E. Then

$$\mu^k\int_0^\infty e^{-\mu t}S_E(t)ydt = R(\mu,B_E)y = R(\mu,B)y = \mu^k\int_0^\infty e^{-\mu t}S(t)ydt$$

($y \in E$, μ large). So it follows from the uniqueness theorem for Laplace transforms that $S_E(\cdot)y = S(\cdot)y$ for all $y \in E$. Consequently $S(\cdot)y \in C([0,\infty),E)$ for all $y \in E$. In particular, there exists $S_1(\cdot)w \in C([0,\infty),[D(B)])$ and $c(\cdot) \in C([0,\infty))$ such that $S(\cdot)w = S_1(\cdot)w + c(\cdot)w$. Hence $h(\cdot) := \int_0^\cdot (S(s)w - c(s)w)ds \in C([0,\infty),[D(B)])$. By (1.2), $\int_0^t S(s)wds \in D(B)$ for $t \geq 0$. Hence $(\int_0^t c(s)ds)w \in D(B)$ for all $t \geq 0$. Since $w \notin D(B)$, we conclude $c(\cdot) = 0$. But then $S(\cdot)w = S_1(\cdot)w \in C[0,\infty),[D(B)])$. This contradicts Lemma 2.4. ∎

Proof of Theorem 0.1(b). Let B be the generator of a C_0-semigroup $T(\cdot)$ on a Banach space F with $D(B) \neq F$. By Lemma 2.5, there exists a Banach space E_1 such that $[D(B)] \hookrightarrow E_1 \hookrightarrow E_0 := F$ and such that the part B_1 of B in E_1 generates an exponentially bounded, 1-times integrated semigroup, but not a 0-times integrated semigroup. By Lemma 2.5, there exists a Banach space E_2 such that $[D(B_1)] \hookrightarrow E_2 \hookrightarrow E_1$ and such that the part B_2 of B_1 in E_2 generates an exponentially bounded, 2-times integrated semigroup, but not a 1-times integrated semigroup. Then B_2 is the part of B in E_2. Hence we found a Banach space E_2 such that $[D(B^2)] \hookrightarrow E_2 \hookrightarrow F$ and such that the part of B in E_2 generates an exponentially bounded, 2-times integrated semigroup, but not a 1-times integrated semigroup. Proceeding in this manner one obtains inductively Banach spaces E_k such that $[D(B^k)] \hookrightarrow E_k \hookrightarrow F$ and such that the part of B in E_k generates an exponentially bounded k-times integrated semigroup, but not a $(k-1)$-times integrated semigroup. ∎

3. Extrapolation of integrated semigroups

In this section we prove Theorem 0.2 and Corollary 0.3. Let $k \in \mathbb{N}$ and $S(\cdot)$ be a k-times integrated semigroup on E of type $w > 0$ with generator A. Let $\mu_0 > \alpha > w$ be fixed. Define F to be the completion of E with respect to the norm

$$|x|_F := \sup_{t\geq 0}\left|e^{-\alpha t}T(t)R(\mu_0,A)^k x\right|_E, \tag{3.1}$$

where $T(\cdot)$ is given by (1.1). Since $\alpha > 0$, it follows that $|x|_F \leq \text{const} \cdot |x|_E$ $(x \in E)$. Thus $E \hookrightarrow_d F$. Next we show that

$$|(\mu - \alpha)R(\mu, A)x|_F \leq |x|_F \quad (\mu > \alpha, x \in E). \tag{3.2}$$

Let $y \in D(A^k)$ and $t \geq 0$. Then $u(s) := T(t+s)R(\mu_0, A)y$, $(s \geq 0)$ is a solution of CP(A) for the initial value $x = T(t)R(\mu_0, A)y$.
It follows from (1.5) that $R(\mu_0, A)R(\mu, A)T(t)y = R(\mu_0, A)\int_0^\infty e^{-\mu s}T(t+s)y ds$ $(\mu > \alpha)$. Hence

$$R(\mu, A)T(t)y = \int_0^\infty e^{-\mu s}T(t+s)yds \qquad (\mu > \alpha) \tag{3.3}$$

for all $y \in D(A^k)$. Let $x \in E$ and $\mu > \alpha$. Using (3.3) one obtains

$$\begin{aligned}
&\left|e^{-\alpha t}T(t)R(\mu_0, A)^k R(\mu, A)x\right|_E = \left|e^{-\alpha t}\int_0^\infty e^{-\mu s}T(t+s)R(\mu_0, A)^k x ds\right|_E \\
&= \left|\int_0^\infty e^{-(\mu-\alpha)s}e^{-\alpha(t+s)}T(t+s)R(\mu_0, A)^k x ds\right|_E \\
&\leq \int_0^\infty e^{-(\mu-\alpha)s}ds\,|x|_F = |x|_F/(\mu - \alpha).
\end{aligned}$$

Since $t \geq 0$ is arbitrary this implies (3.2). It follows from (3.2) that $R(\mu, A)$ has a unique extension $R(\mu) \in \mathcal{L}(F)$ satisfying

$$|(\mu - \alpha)R(\mu)|_{\mathcal{L}(\mathcal{F})} \leq 1 \qquad (\mu > \alpha). \tag{3.4}$$

Then $\{R(\mu) : \mu > \alpha\}$ is a pseudo resolvent on F. We show that

$$\lim_{\mu\to\infty} |\mu R(\mu)y - y|_F = 0 \qquad (y \in F). \tag{3.5}$$

Because of (3.4) and density it sufficies to show (3.5) for $y \in E$. Let $x = R(\mu_0, A)^k y$. Then, by (3.3),

$$\begin{aligned}
|\mu R(\mu)y - y|_F &= \sup_{t\geq 0}\left|e^{-\alpha t}(\mu R(\mu, A)T(t)x - T(t)x)\right|_E \\
&= \sup_{t\geq 0}\left|e^{-\alpha t}\int_0^\infty \mu e^{-\mu s}(T(t+s)x - T(t)x)ds\right|_E .
\end{aligned}$$

Let $\varepsilon > 0$. Since $S(\cdot)$ is of type $w > 0$ there exists $M \geq 0$ such that $|T(t+s)x - T(t)x| \leq Me^{w(t+s)}$ for all $s, t \geq 0$. Since $\alpha > w$, there exists $q > 0$ such that

$$\begin{aligned}
&\sup_{t\geq q}\left|e^{-\alpha t}\int_0^\infty \mu e^{-\mu s}(T(t+s)x - T(t)x)ds\right|_E \\
&< \text{const} \cdot \mu \cdot e^{(w-\alpha)q}/(\mu - w) < \varepsilon/2
\end{aligned} \tag{3.6}$$

for all $\mu > \alpha$. Since $T(\cdot)$ is uniformly continuous on compact intervals, there exists $\delta > 0$ such that $|T(t+s)x - T(t)x|_F \leq \varepsilon/2$ for $t \in [0, q], s \in [0, \delta]$. Hence

$$\sup_{0\leq t\leq q}\left|e^{-\alpha t}\int_0^\delta \mu e^{-\mu s}(T(t+s)x - T(t)x)ds\right|_E < \varepsilon/2 \qquad (\mu > \alpha). \tag{3.7}$$

Since $S(\cdot)$ is of type w,

$$\sup_{0\le t\le q}\left|e^{-\alpha t}\int_\delta^\infty \mu e^{-\mu s}(T(t+s)x-T(t)x)ds\right|_E \le \text{const}\cdot\mu\cdot e^{(w-\mu)\delta}/(\mu-w)\to 0$$

for $\mu\to\infty$. This together with (3.6), (3.7) shows that $\overline{\lim\limits_{\mu\to\infty}}|\mu R(\mu)y-y|_F\le\varepsilon$. Since $\varepsilon>0$ is arbitrary, (3.5) is proved.

It follows from (3.5) that $\mathrm{Ker}R(\mu)=\{0\}$, ($\mu>0$; observe that $\mathrm{Ker}R(\mu)$ is independent of $\mu>\alpha$ because of the resolvent equation). Hence there exists an operator B on F such that $(\alpha,\infty)\subset\rho(A)$ and $R(\mu)=R(\mu,B)$ for $\mu>\alpha$ (see the proof of Theorem 1.9.3 in [Pa]). Because of (3.5) the domain of B is dense in F. It follows from (3.4) and the Hille-Yosida Theorem that B generates a C_0-semigroups of type α on F.

Since by construction $R(\mu,B)E\subset E$ and $R(\mu,B)_E=R(\mu,A)$ $(\mu>\alpha)$, it follows by Lemma 2.2 that $A=B_E$. Taking $t=0$ in (3.1) one obtains

$$\left|R(\mu_0,A)^k y\right|_E\le|y|_F \qquad (y\in E). \tag{3.8}$$

Let $x\in D(B^k)$ and $y\in F$ with $x=R(\mu_0,B)^k y$. Then there exist $y_n\in E$ such that $y_n\to y$ in F. By (3.8), $R(\mu_0,A)^k y_n$ is a Cauchy seqence in E. Hence $x=F-\lim R(\mu_0,B)^k y_n=E-\lim R(\mu_0,A)^k y_n\in E$. This implies $D(B^k)\subset E$ and $\left|R(\mu_0,B)^k y\right|_E\le|y|_F$ for all $y\in F$. Consequently $|x|_E\le\left|(\mu_0-B)^k x\right|_F$ for all $x\in D(B^k)$. This shows that $[D(B^k]\hookrightarrow E$.

We have proved part a) of Theorem 0.2. Before giving the proof of part b) we observe that $G:=[D(B^k)]$ is a Banach space with the norm $|x|_G:=\left|(\mu_0-B)^k x\right|_F$ (which is equivalent to the graph norm). Since $A=B_E$, it follows $A_G=B_k$ (the part of B in $[D(B^k)]$). Moreover, $R(\mu_0,B)^k$ is an isometric isomorphism from F onto G which coincides with $R(\mu_0,A)^k$ on E. Since E is dense in F, it follows that $D(A^k)=R(\mu_0,B)^k E$ is dense in G. For $x\in D(A^k)$ the norm is given by

$$|x|_G:=\sup_{t\ge0}\left|e^{-\alpha t}T(t)x\right|_E, \tag{3.9}$$

where $T(\cdot)$ is given by (1.1). Now we prove the maximality assertion (b). Assume that $W\hookrightarrow E$ such that A_W generates a C_0-semigroup $T_W(\cdot)$ of type α on W. Then $D(A_W^k)\subset D(A^k)$ and for $x\in D(A_W^k)$ one has (by (3.5) for $t=0$)

$$\int_0^\infty e^{-\mu s}T_W(s)x\,ds=R(\mu,A_W)x=R(\mu,A)x=\int_0^\infty e^{-\mu s}T(s)x\,ds \qquad (\mu>\alpha).$$

So the uniqueness of the Laplace transform implies that $T(\cdot)x=T_W(\cdot)x$. Consequently, $|x|_G:=\sup\limits_{t\ge0}\left|e^{-\alpha t}T(t)x\right|_E\le\text{const}\cdot\sup\limits_{t\ge0}\left|e^{-\alpha t}T_W(t)x\right|_W\le\text{const}\cdot|x|_W$ for all $x\in D(A_W^k)$ since $T_W(\cdot)$ is of type α. This implies that

$$W=W\text{-closure of } D(A_W^k) \quad\hookrightarrow\quad G\text{-closure of } D(A^k)=G. \qquad ■$$

Remark 3.1. a) In the situation of Theorem 0.2 one has $\rho(A)=\rho(B)$. In fact, by the construction itself it follows that $R(\mu,B)E\subset E$ for $\mu>\alpha$. Hence $\rho(B)\subset\rho(A)$ Theorem 0.1. Conversely, assume that $\mu\in\rho(A)$. Then it follows from (3.1) that $R(\mu,A)$ has a continuous extension $R(\mu)$ to F. It is easy to see that $R(\mu)=R(\mu,B)$.

b) One might define the norm $|\cdot|_G$ on $D(A^k)$ directly by formula (3.9), and then define the space G as the completion of $(D(A^k), |\cdot|_G)$. Doing so, one has to prove that G can be identified with a subspace of E. It is this point which was missed in [Ke] and [Ne1]. The proofs given there can be "repaired" if one replaces the operators $T(t)$ by their closures $(\mu_0 - A)^k T(t) R(\mu_0, A)^k$ with domain $\{x \in E : T(t)R(\mu_0, A)^k x \in D(A^k)\}$. However, these proofs a far more technical than the one given above.

Proof of Corollary 0.3. The implications (a) $\to$ (c) follow from Theorem 0.2. Choosing $G := [D(B^k)]$ in (c) one sees that (c) $\to$ (b). If (b) holds, then, for every initial value $x \in D(A^{k+1}) \subset D(A_G)$, there exists a unique solution

$$u(\cdot, x) \in C^1([0,\infty), E) \subset C^1([0,\infty), E)$$

of CP(A) with $u(t,x) \in D(A_G) \subset D(A)$ and

$$|u(t,x)| \leq \text{const} \cdot |u(t,x)|_G \leq \text{const} \cdot e^{\alpha t} |x|_G \leq \text{const} \cdot e^{\alpha t} |x|_{A^k}.$$

With this, statement (a) follows from Theorem 4.2 in [Ne1]. ∎

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