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VECTOR-VALUED TAUBERIAN THEOREMS AND ASYMPTOTIC BEHAVIOR OF LINEAR VOLTERRA EQUATIONS*

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Abstract. The asymptotic behavior of the solutions of linear Volterra equations in a Banach space X of the form

$$(*) \qquad u(t) = f(t) + \int_0^t a(t-\tau)Au(\tau)d(\tau), \quad t \geq 0$$

is studied, in particular that of the resolvent $S(t)$ for $(*)$; here $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and A is a closed linear operator in X with dense domain. A complete characterization of the existence of $\lim_{t\to 0} S(t)x = Px$ for all $x \in X$ in the sense of Abel is obtained, and the nature of the ergodic limit P is studied. By means of vector-valued Tauberian theorems for the Laplace transform, a general result on convergence of $S(t)$ in the strong sense is derived. Several examples are given which illustrate this result, and also an application to the theory of linear viscoelasticity is presented.

Key words. Volterra equations, resolvents, asymptotic behavior, Laplace transform, ergodic limit, Abel-limit, Cesaro-limit, Tauberian theorems, C_0-semigroups, cosine families, viscoelasticity

AMS(MOS) subject classifications. primary 45N05, 45K05, 44A10; secondary 47D05, 47G05, 76A10

1. Introduction. Let X be a Banach space, $a \in L^1_{\text{loc}}(\mathbb{R}_+)$, A a closed linear operator in X with dense domain $D(A)$, and consider the abstract linear Volterra equation in X

$$(1.1) \qquad u(t) = f(t) + \int_0^t a(t-\tau)Au(\tau)d\tau, \qquad t \geq 0,$$

where $f : \mathbb{R}_+ \to X$ is continuous, $\mathbb{R}_+ = [0, \infty)$. X_A denotes the Banach space $D(A)$ equipped with the graph norm $|.|_A$ of A. A function $u \in C(\mathbb{R}_+; X_A)$ satisfying (1.1) on $\mathbb{R}_+$ is called a *strong solution* of (1.1), while $u \in C(\mathbb{R}_+; X)$ is a *mild solution* of (1.1) if $a * u \in C(\mathbb{R}_+; X_A)$ holds and

$$(1.2) \qquad u(t) = f(t) + A\int_0^t a(t-\tau)u(\tau)d\tau, \qquad t \geq 0,$$

is satisfied on $\mathbb{R}_+$. A family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ of bounded linear operators in X is called a *resolvent* for (1.1) if $S(t)$ commutes with A and satisfies the *resolvent equation*

$$(1.3) \qquad S(t)x = x + \int_0^t a(t-\tau)AS(\tau)xd\tau, \qquad t \geq 0, \quad x \in D(A).$$

Once a resolvent $S(t)$ for (1.1) is known to exist, it is unique, and the solution of (1.1) is represented by the *variation of parameters formula*

$$(1.4) \qquad u(t) = \frac{d}{dt}\int_0^t S(t-\tau)f(\tau)d\tau, \qquad t \geq 0,$$

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whenever u is a mild solution of (1.1), then $S * f \in C^1(\mathbb{R}_+; X)$ and u is represented by (1.4).

By now the question of existence of a resolvent for (1.1) has been settled for many different classes of pairs (a, A); for a general exposition of the theory, we refer to Prüss [33]. Here we always assume the existence of a resolvent $S(t)$ for (1.1) which is in addition of subexponential growth, i.e., which satisfies

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log |S(t)| \leq 0. \tag{1.5}$$

It is the purpose of this paper to study the asymptotic behavior of the solutions of (1.1), in particular that of the resolvent $S(t)$ itself. More precisely, the existence of the limits $\lim_{t\to\infty} u(t) = u(\infty)$ and $\lim_{t\to\infty} S(t) = P$ in various senses are investigated, and the nature of the limits $u(\infty)$ and P are discussed.

Our approach is based on the theory of vector-valued Laplace transforms. A well-known Abelian theorem shows that if $\lim_{t\to\infty} S(t)x = Px$ for all $x \in X$, then

$$H(\lambda) = \hat{S}(\lambda) = \int_0^\infty S(t)e^{-\lambda t}dt, \qquad \mathrm{Re}\ \lambda > 0, \tag{1.6}$$

satisfies

$$A - \lim_{t\to\infty} S(t)x := \lim_{\lambda\to 0+} \lambda H(\lambda)x = Px \quad \text{for all } x \in X. \tag{1.7}$$

Therefore it is natural to study first the existence of the Abelian limit P of $S(t)$ as well as its properties. This will be done in §4, where we also apply some elementary vector-valued Tauberian theorems to deduce the convergence of $S(t)$ in the ordinary sense from existence of the ergodic limit $P \in \mathcal{B}(X)$; for that, several strong assumptions on $S(t)$ are needed. Once the Abelian limit $P \in \mathcal{B}(X)$ of $S(t)$ is known to exist, it follows easily that $A - \lim_{t\to\infty} u(t) = u(\infty)$ also exists whenever $f(t)$ admits an Abelian limit $f(\infty)$ and then $u(\infty) = Pf(\infty)$ holds.

The main result of this paper, the General Convergence Theorem stated and proved in §5, gives sufficient conditions for the strong convergence of $S(t)$ to its ergodic limit P as $t \to \infty$. For the special case $a(t) \equiv 1$ and A the generator of a bounded C_0-semigroup $T(t)$ in X, we have $S(t) = T(t)$ and the result reduces to a stability theorem for C_0-semigroups obtained recently by Arendt and Batty [2] and independently by Lyubich and Phong [28]; cf. also §7. The proof of the General Convergence Theorem relies on the complex Tauberian theory for the vector-valued Laplace transform. In fact, it is very much inspired by the proof of Arendt and Batty [2] for the semigroup case. However, due to the more complicated structure of the Laplace transform $H(\lambda)$ of the resolvent $S(t)$ for (1.1), i.e.,

$$H(\lambda) = \frac{1}{\lambda}(I - \hat{a}(\lambda)A)^{-1}, \qquad \mathrm{Re}\ \lambda > 0, \tag{1.8}$$

the Tauberian arguments involved are more delicate and differ from those employed in the proof of Arendt and Batty [2].

Since Abelian and Tauberian theorems for the vector-valued Laplace transform are at the heart of our approach, and since there is no coherent presentation of this material available in the literature, we have included two sections on this matter. Section 2 contains the basic Abelian theorem, as well as the vector-valued extension of the classical real Tauberian theorems due to Hardy–Littlewood, Wiener, Pitt, and

Karamata; cf. Doetsch [15] and Widder [42] for their classical statements. Complex Tauberian theorems are presented in §3. Here a condition on the Laplace transform $\hat{f}$ of f is given which implies convergence of $f(t)$ $(t \to \infty)$. A first result of this type had been given in 1938 by Ingham [20], but recently a simple new technique of proof due to Newman [30] has led to considerable extensions; see Korevaar [25], Allan, O'Farrell, and Ransford [1], Arendt and Batty [2], Ransford [34], and Batty [3].

Section 6 is devoted to an elaboration of several examples and special cases of the theory developed in §§3–5. In particular, several classes of kernels are presented, for which the assumptions of the General Convergence Theorem reduce to boundedness of $S(t)$ (which is necessary for the existence of the strong limit of $S(t)$ anyway) and to a spectral condition that cannot be relaxed (and to some extent is also necessary). In §7, we apply our results to the theory of linear viscoelasticity. Here we show that if A generates a uniformly bounded cosine family and $a(t)$ is of the form

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(\tau)d\tau, \qquad t \geq 0 \tag{1.9}$$

with $a_0, a_\infty \geq 0$, $a_1(t) \geq 0$ nonincreasing, $\log a_1(t)$ convex, and $\lim_{t\to\infty} a_1(t) = 0$, then $S(t)$ converges strongly as $t \to \infty$ if in addition $N(A)^\perp \cap N(A') = \{0\}$ and $a(t) \not\equiv a_\infty t$ hold. This result shows that any viscoelastic fluid in a smooth domain $\Omega \subset \mathbb{R}^n$ with compact boundary is asymptotically stable in the strong sense, whether Ω is bounded or not. It has been shown in Prüss [32] that viscoelastic fluids are uniformly asymptotically stable if and only if $A = P\Delta$ is invertible. This is always true for bounded domains Ω, but it is in general not the case for unbounded domains; cf. §7 for these concepts and further discussion.

2. Abelian and real Tauberian theorems. Throughout this section, $(X, |\ \ |)$ is a Banach space and $f \in L^1_{\text{loc}}([0,\infty), X)$ is such that

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt := \lim_{b\to\infty} \int_0^b e^{-\lambda t} f(t)dt$$

exists for Re $\lambda > 0$ (this is equivalent to $\sup_{t\geq 0} e^{-\lambda t}|\int_0^t f(s)ds| < \infty$ for all $\lambda > 0$).

DEFINITION 2.1. Let $f_\infty \in X$. The function f converges to f_∞ in the sense of Cesaro $(t \to \infty)$ if $C - \lim_{t\to\infty} f(t) := \lim_{t\to\infty}(1/t)\int_0^t f(s)ds = f_\infty$, and f converges to f_∞ in the sense of Abel $(t \to \infty)$ if $A - \lim_{t\to\infty} f(t) := \lim_{\lambda\to 0+} \lambda\hat{f}(\lambda) = f_\infty$.

The following *Abelian theorem* is easy to prove (see [19, Thm. 18.2.1]). It will be convenient to introduce $F(t) = \int_0^t f(s)ds$ as an auxiliary function.

THEOREM 2.2. *Let f_∞, $F_\infty \in X$.*

(a) *If $\lim_{t\to\infty} f(t) = f_\infty$, then $C - \lim_{t\to\infty} f(t) = f_\infty$.*

(b) *If $C - \lim_{t\to\infty} f(t) = f_\infty$, then $A - \lim_{t\to\infty} f(t) = f_\infty$.*

(c) *If $\lim_{t\to\infty} F(t) = F_\infty$, then $\lim_{\lambda\to 0+} \hat{f}(\lambda) = F_\infty$.*

Note that (c) is a special case of (b) since $\hat{f}(\lambda) = (F')^\wedge(\lambda) = \lambda\hat{F}(\lambda)$ $(\lambda > 0)$.

A result if called a *Tauberian theorem* if a condition on f is given under which the converse implications of (a), (b), or (c) are valid. Such theorems are presented in sections A, B, C, D, and E respectively. Most of these results are well known at least in the numerical case. We include proofs here for the sake of completeness.

Our main objective is to find conditions under which Abelian convergence implies convergence (see §D).

A. Conditions under which $C-\lim_{t\to\infty} f(t) = f_\infty$ implies $\lim_{t\to\infty} f(t) = f_\infty$. A vector-valued function f is called *feebly oscillating* (when $t \to \infty$) if

$$\lim_{\substack{t,s\to\infty \\ t/s\to 1}} |f(t) - f(s)| = 0$$

(cf. [19, Def. 18.3.1], [43, Def. 8.4]).

Example 2.3. Assume that $t|f(t)| \le M$ for $t \ge \tau$, where $\tau \ge 0$. Then F is feebly oscillating. In fact, $|F(t) - F(s)| \le \int_s^t r|f(r)|(dr/r) \le M \log(t/s)$ for $t \ge s \ge \tau$.

THEOREM 2.4. *Assume that f is feebly oscillating and let $f_\infty \in X$. If $C-\lim_{t\to\infty} f(t) = f_\infty$, then $\lim_{t\to\infty} f(t) = f_\infty$.*

Proof. We can suppose that $f_\infty = 0$. Let $\epsilon > 0$. There exist $\delta > 0$, $t_0 > 0$ such that $|f(s) - f(t)| < \epsilon$ whenever $s, t > t_0$, $s \in [t - \delta t, t + \delta t]$. Hence $|f(t) - (1/2\delta t)\int_{t(1-\delta)}^{t(1+\delta)} f(s)ds| = |(1/2\delta t)\int_{t(1-\delta)}^{t(1+\delta)} (f(t) - f(s))ds| \le \epsilon$ if $t \ge t_0$. Since

$$\lim_{t\to\infty} \frac{1}{2\delta t}\int_{t(1-\delta)}^{t(1+\delta)} f(s)ds = 0,$$

we conclude $\lim_{t\to\infty} f(t) = 0$. □

B. Conditions under which $A-\lim_{t\to\infty} f(t) = f_\infty$ implies $C-\lim_{t\to\infty} f(t) = f_\infty$. The following result is a particular case of [19, Thms. 18.3.3, 18.3.2].

THEOREM 2.5. *Let $f_\infty \in X$. Assume that $f \in L^\infty([\tau,\infty); X)$ for some $\tau \ge 0$. If $A - \lim_{t\to\infty} f(t) = f_\infty$ then $C - \lim_{t\to\infty} f(t) = f_\infty$.*

Proof. 1. We first assume that $\tau = 0$. For $\beta > 0$ let $e_\beta(t) = \beta e^{-\beta t}(t > 0)$. Then span $\{e_\beta : \beta > 0\}$ is dense in $L^1[0,\infty)$ (in fact, if $g \in L^\infty[0,\infty)$ such that $0 = \langle e_\beta, g\rangle = \beta\hat{g}(\beta)$ for all $\beta > 0$, then $g = 0$ almost everywhere by uniqueness theorem for Laplace transforms). By hypothesis $\lim_{\alpha\to\infty}\int_0^\infty e^{-s} f(\alpha s)ds = \lim_{\lambda\to 0+}\int_0^\infty e^{-s} f(s/\lambda)ds = \lim_{\lambda\to 0+}\lambda\int_0^\infty e^{-\lambda s} f(s)ds = f_\infty$. Hence

$$\lim_{\alpha\to\infty}\langle e_\beta, f(\alpha\cdot)\rangle = \lim_{\alpha\to\infty}\beta\int_0^\infty e^{-s\beta} f(\alpha s)ds = \lim_{\alpha\to\infty}\int_0^\infty e^{-s} f(\frac{\alpha}{\beta}s)ds = f_\infty = \langle e_\beta, f_\infty\rangle$$

for all $\beta > 0$. It follows that $\lim_{\alpha\to\infty}\langle h, f(\alpha\cdot)\rangle = f_\infty\int_0^\infty h(t)dt$ for all $h \in L^1[0,\infty)$. Letting $h = X_{[0,1]}$ we obtain $\lim_{\alpha\to\infty} 1/\alpha\int_0^\alpha f(s)ds = \lim_{\alpha\to\infty}\int_0^1 f(\alpha s)ds = \lim_{\alpha\to\infty}\langle h, f(\alpha\cdot)\rangle = f_\infty$.

2. If $\tau > 0$ the result follows by applying 1. to $g(t) = f(t + \tau)$. □

Another result of this type involves an order condition. We assume in the following theorem that X is an ordered Banach space with normal cone X_+ (i.e., X_+ is a closed convex cone such that $X_+ \cap (-X_+) = \{0\}$ and $X'_+ - X'_+ = X'$ where X'_+ denotes the dual cone; see [5] for details). For example, X may be a Banach lattice.

THEOREM 2.6. *Assume that $f(t) \ge 0$ (i.e., $f(t) \in X_+$) for $t \ge 0$. If $A - \lim_{t\to\infty} f(t) = f_\infty$, then $C - \lim_{t\to\infty} f(t) = f_\infty$.*

Karamata's proof of this result (see [43, Thm. 8.5.3]) goes through in the vector-valued case described above. A very short and elegant proof in the scalar case is given by König [24].

C. Conditions under which $\lim_{\lambda\to 0+}\hat{f}(\lambda) = F_\infty$ implies $\lim_{t\to\infty} F(t) = F_\infty$. The following theorem is due to Hardy and Littlewood in the numerical case; see [43, Thm. 8.4.3].

THEOREM 2.7. *Let $F_\infty \in X$. Assume that for some $\tau \ge 0$*

$$M = \sup_{t\ge\tau} t|f(t)| < \infty. \tag{2.1}$$

If $\lim_{\lambda\to 0+} \hat{f}(\lambda) = F_\infty$, *then* $\lim_{t\to\infty} F(t) = F_\infty$.

Proof. 1. We first assume that $\tau = 0$. For $t > 0$ we have

$$\begin{aligned}
|F(t) - \hat{f}(\frac{1}{t})| &= |\int_0^t f(s)[1 - e^{-s/t}]ds - \int_t^\infty f(s)e^{-s/t}ds| \\
&\leq M(\int_0^t [1 - e^{-s/t}]/s^{-1}ds + \int_t^\infty s^{-1}e^{-s/t}ds) \\
&\leq M[\sup_{0<s<t} t[1 - e^{-s/t}]/s + \int_1^\infty e^{-r}r^{-1}dr] \\
&\leq M[\sup_{0<x\leq 1} (1 - e^{-x})/x + \int_1^\infty e^{-r}r^{-1}dr] < \infty.
\end{aligned}$$

Since $\lim_{t\to\infty} \hat{f}(1/t) = F_\infty$, it follows that F is bounded. But $A - \lim_{t\to\infty} F(t) = \lim_{\lambda\to 0+} \hat{f}(\lambda) = F_\infty$. So it follows from Theorem 2.5 that $C - \lim_{t\to\infty} F(t) = F_\infty$. The function F is slowly oscillating (see Example 2.3). Hence $\lim_{t\to\infty} F(t) = F_\infty$ by Theorem 2.4.

2. If $\tau > 0$ the result follows from 1. by considering $f(t+\tau)$ instead of $f(t)$. □

D. Conditions under which $A - \lim_{t\to\infty} f(t) = f_\infty$ **implies** $\lim_{t\to\infty} f(t) = f_\infty$. Since $\lambda\hat{F}(\lambda) = \hat{f}(\lambda)$, any Tauberian theorem of type D yields one of type C. Conversely, if $f \in C^1([\tau,\infty), X)$ we can apply a result of type C to the function $f'(t+\tau)$ and obtain a Tauberian theorem of type D.

Following an idea of Batty [3] we apply instead Tauberian theorems of type C to the function f_δ defined by

$$f_\delta(t) = (f(t+\delta) - f(t))/\delta \qquad (t \geq 0) \tag{2.2}$$

for some $\delta > 0$. The following implications hold.

LEMMA 2.8. *Let* $f_\infty \in X$, $\delta > 0$.

(i) $$\lim_{t\to\infty} f(t) = f_\infty$$

$$\Downarrow$$

(ii) $$\lim_{t\to\infty} \int_t^{t+\delta} f(s)ds = f_\infty\delta$$

$$\Updownarrow$$

(iii) $$\lim_{t\to\infty} \int_0^t f_\delta(s)ds = f_\infty - \frac{1}{\delta}\int_0^\delta f(s)ds$$

$$\Downarrow$$

(iv) $$\lim_{\lambda\to 0+} \hat{f}_\delta(\lambda) = f_\infty - \frac{1}{\delta}\int_0^\delta f(s)ds$$

$$\Updownarrow$$

(v) $$A - \lim_{t\to\infty} f(t) = f_\infty.$$

Proof. Since $|(1/\delta)\int_t^{t+\delta} f(s)ds - f_\infty| = |(1/\delta)\int_t^{t+\delta}(f(s) - f_\infty)ds| \leq \sup_{s\geq t} |f(s) - f_\infty|$, (i) implies (ii), and (ii) is equivalent to (iii) since $\int_0^t f_\delta(s)ds = (1/\delta)\int_t^{t+\delta} f(s)ds - (1/\delta)\int_0^\delta f(s)ds$.

By Theorem 2.2(c), (iii) implies (iv). Since

$$\hat{f}_\delta(\lambda) = \frac{1}{\delta\lambda}(e^{\lambda\delta} - 1)\lambda\hat{f}(\lambda) - \frac{e^{\lambda\delta}}{\delta}\int_0^\delta e^{-\lambda s} f(s)ds, \tag{2.3}$$

(iv) is equivalent to (v). □

Let $f_\infty \in X$. We say, f is *B-convergent* to f_∞, or simply write $B - \lim_{t\to\infty} f(t) = f_\infty$, if (ii) of Lemma 2.8 holds for all $\delta > 0$. A vector-valued function f is called *slowly oscillating* (when $t \to \infty$) if

$$\lim_{\substack{t,s\to\infty \\ t-s\to 0}} |f(t) - f(s)| = 0.$$

PROPOSITION 2.9. *Let $f_\infty \in X$.*

(a) *If $B - \lim_{t\to\infty} f(t) = f_\infty$, then $A - \lim_{t\to\infty} f(t) = f_\infty$.*

(b) *If $\lim_{t\to\infty} f(t) = f_\infty$, then $B - \lim_{t\to\infty} f(t) = f_\infty$.*

(c) *If f is slowly oscillating then $B - \lim_{t\to\infty} f(t) = f_\infty$ implies $\lim_{t\to\infty} f(t) = f_\infty$.*

Proof. (a) and (b) follow from Lemma 2.8. Assume that $B - \lim_{t\to\infty} f(t) = f_\infty$. Then

$$\begin{aligned}\overline{\lim_{t\to\infty}}|f(t) - f_\infty| &\le \overline{\lim_{t\to\infty}}|f(t) - \frac{1}{\delta}\int_t^{t+\delta} f(s)ds| \\ &= \overline{\lim_{t\to\infty}}|\frac{1}{\delta}\int_t^{t+\delta}(f(t) - f(s))ds| \le \overline{\lim_{t\to\infty}} \sup_{t\le s\le t+\delta} |f(t) - f(s)|.\end{aligned}$$

Hence if f is slowly oscillating, we obtain $\overline{\lim}_{t\to\infty}|f(t) - f_\infty| = 0$ by letting $\delta \downarrow 0$. □

Every feebly oscillating function is slowly oscillating (this is obvious from the definitions); moreover, f is slowly oscillating whenever there exists $\tau \ge 0$ such that $f = g + h$, where $g \in UC([\tau,\infty);X)$ (the space of all uniformly continuous functions on $[\tau,\infty)$ with values in X), and $h \in L^\infty([\tau,\infty);X)$ converges to zero as $t \to \infty$.

Remark. In order that $B - \lim_{t\to\infty} f(t) = f_\infty$ it suffices that (ii) holds for all $\delta \in (0, \delta_0)$ for some $\delta_0 > 0$. In fact, if (ii) holds for $\delta > 0$ and $\eta > 0$ it does so for $\delta + \eta$.

Now we are able to deduce from Theorem 2.7 the following Tauberian theorem of type D.

THEOREM 2.10. *Let $f_\infty \in X$. Assume that for some $\delta > 0$*

$$\overline{\lim_{t\to\infty}} \sup_{t\le s\le t+\delta} t|f(t) - f(s)| < \infty. \tag{2.4}$$

If $A - \lim_{t\to\infty} f(t) = f_\infty$ then $\lim_{t\to\infty} f(t) = f_\infty$.

Proof. Assumption (2.4) implies that f_δ satisfies (2.1) for $\delta > 0$ small enough and also that f is slowly oscillating. Since $A - \lim_{t\to\infty} f(t) = f_\infty$, it follows that (iv) of Lemma 2.8 is satisfied. We conclude (iii) from Theorem 2.7 so that $B - \lim_{t\to\infty} f(t) = f_\infty$. Hence $\lim_{t\to\infty} f(t) = f_\infty$ by Proposition 2.9. □

Note that (2.4) is satisfied whenever $f \in C^1([\tau,\infty),X)$ and $|tf'(t)| \le M$ for $t \ge \tau$; in fact, $|f(t) - f(s)| = |\int_t^s f'(r)dr| \le M\log(s/t) \le M(s-t)/t$ for $t < s$.

Applying Theorem 2.10 to F we obtain an improvement of Theorem 2.7.

COROLLARY 2.11. *Let $F_\infty \in X$. Assume that for some $\delta > \infty$*

$$\overline{\lim_{t\to\infty}} \int_t^{t+\delta} r|f(r)|dr < \infty. \tag{2.5}$$

If $\lim_{\lambda\to 0+} \hat{f}(\lambda) = F_\infty$, *then* $\lim_{t\to\infty} F(t) = F_\infty$.

Proof. We have

$$\begin{aligned}\overline{\lim_{t\to\infty}} \sup_{t\le s\le t+\delta} t|F(t) - F(s)| &\le \overline{\lim_{t\to\infty}}\, t \int_t^{t+\delta} |f(s)|ds \\ &\le \overline{\lim_{t\to\infty}} \int_t^{t+\delta} s|f(s)|ds < \infty.\end{aligned}$$

Hence F satisfies (2.4) and the conclusion follows from Theorem 2.10. □

E. Power series. Let $p(z) = \sum_{n=0}^\infty a_n z^n$ be a power series, where $a_n \in X$, which converges for $|z| < 1$. Defining $f \in L^1_{\mathrm{loc}}([0,\infty); X)$ by

$$f(t) = a_n \quad \text{if } t \in [n, n+1) \tag{2.6}$$

the preceding results yield Tauberian theorems for p. In fact,

$$\hat{f}(\lambda) = \left(\frac{1-e^{-\lambda}}{\lambda}\right) \sum_{n=0}^\infty a_n e^{-\lambda n} \qquad (\operatorname{Re} \lambda > 0). \tag{2.7}$$

From Theorem 2.7 Hardy's theorem can be obtained.

THEOREM 2.12. *Assume that* $\sup\{n|a_n| : n \in \mathbb{N}_0\} < \infty$, *and let* $b_\infty \in X$. *If* $\lim_{z\uparrow 1} p(z) = b_\infty$, *then* $\sum_{n=0}^\infty a_n = b_\infty$.

The special case when $\lim_{n\to\infty} na_n = 0$ had been proven by Tauber [38] (in the scalar case) and was the starting point of Tauberian theory.

In the case of power series, theorems of type C and D are equivalent. In fact, let $b_n = \sum_{k=0}^n a_k$, or equivalently, $a_0 = b_0$, $a_n = b_n - b_{n-1}$ $(n = 1, 2\cdots)$. Then $q(z) = \sum_{n=0}^\infty b_n z^n$ has the same radius of convergence as $p(z)$. The formula for the Cauchy product yields

$$\frac{1}{z-1} \sum_{k=0}^\infty a_k z^k = \sum_{k=0}^\infty z^k \cdot \sum_{k=0}^\infty a_k z^k = \sum_{k=0}^\infty b_k z^k \qquad (|z| < 1);$$

that is,

$$\sum_{k=0}^\infty a_k z^k = (1-z) \sum_{k=0}^\infty b_k z^k \qquad (|z| < 1).$$

Thus $A - \lim_{n\to\infty} b_n := \lim_{z\uparrow 1}(1-z) \sum_{k=0}^\infty b_k z^k = \lim_{z\uparrow 1} \sum_{k=0}^\infty a_k z^k$ whenever one of the limits exists. So we obtain the following.

COROLLARY 2.13. *Let $b_n \in X$ be such that* $\sup\{n|b_n - b_{n-1}| : n \in \mathbb{N}\} < \infty$. *If* $A - \lim_{n\to\infty} b_n = b_\infty$, *then* $\lim_{n\to\infty} b_n = b_\infty$.

3. Complex Tauberian theorems. We assume throughout this section that $f \in L^1_{\mathrm{loc}}([0,\infty); X)$ is such that

$$\hat{f}(\lambda) = \lim_{b\to\infty} \int_0^b e^{-\lambda t} f(t)dt =: \int_0^\infty e^{-\lambda t} f(t)dt$$

exists for Re $\lambda > 0$. In this section we consider conditions on $\hat{f}$ (rather than on f) in order to establish Tauberian theorems. The following theorem (of type C, see §2) is a variant of [2, Thm. 4.1].

THEOREM 3.1. *Let $f_\infty \in X$. Assume that $f \in L^\infty([\tau,\infty);X)$ for some $\tau \geq 0$ and that $(\hat{f}(\lambda) - F_\infty)/\lambda$ has a continuous extension to $\bar{\mathbb{C}}_+ \backslash iE$, where $E \subset \mathbb{R}$ is a closed null set and $0 \notin E$. If for all $R > 0$*

$$M(R) := \sup_{\substack{\eta\in E\\ |\eta|\leq R}} \sup_{t\geq 0} \left| \int_0^t \exp(-i\eta s) f(s)ds \right| < \infty, \tag{3.1}$$

then $\lim_{t\to\infty} F(t) = F_\infty$.

Here and in the sequel we let $F(t) = \int_0^t f(s)ds$, $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \text{ Re } \lambda > 0\}$, and $\bar{\mathbb{C}}_+$ the closure of $\mathbb{C}_+$. Note that the hypothesis implies that $\hat{f}(\lambda)$ has a continuous extension to $\bar{\mathbb{C}}_+ \backslash iE$ and that $\hat{f}(0) = F_\infty$. In particular, $A - \lim_{t\to\infty} F(t) = F_\infty$.

The proof of [2, Thm. 4.1] works for Theorem 3.1 as well if the basic estimate Lemma 5.2 which will be proved in §5 is used instead of [2, Lemma 3.1].

For $E = \emptyset$ Theorem 3.1 is a version of a theorem due to Ingham [20]. A very short and elegant proof based on an ingenious contour argument due to Newman [30] is given by Korevaar [25]. In [2] the technique of Newman and Korevaar has been extended in order to treat singularities in $i\,\mathbb{R}$. Whereas in [2] it is assumed that $\hat{f}$ has a holomorphic extension to $\bar{\mathbb{C}}_+ \backslash iE$, our slightly more general version is more natural in view of the applications to Volterra equations we have in mind (see §5).

We give several comments on Theorem 3.1, starting with the case when $E = \emptyset$.

Remark 3.2. *Quantitative estimates.* Korevaar's argument actually yields the following more precise result. Assume that $f \in L^\infty([\tau,\infty);X)$ for some $\tau > 0$ and that $F_\infty \in X$ such that $(\hat{f}(\lambda) - F_\infty)/\lambda$ has a continuous extension to $\mathbb{C}_+ \cup i[-R,R]$ where $R > 0$. Then

$$\overline{\lim_{t\to\infty}} |F(t) - F_\infty| \leq \frac{2}{R} \overline{\lim_{t\to\infty}} |f(t)|. \tag{3.2}$$

Proof. In fact, Korevaar shows (a special case of Lemma 5.2 below)

$$\overline{\lim_{t\to\infty}} |F(t) - F_\infty| \leq \frac{2}{R} \sup_{t\geq 0} |f(t)|. \tag{3.3}$$

Applying this to $g(t) = f(t+s)$ with $s \geq \tau$, we have $\hat{g}(\lambda) = e^{\lambda s}[\hat{f}(\lambda) - \int_0^s e^{-\lambda r} f(r)dr]$ (Re $\lambda > 0$) so that $(\hat{g}(\lambda) - G_\infty)/\lambda$ has a continuous extension to $\mathbb{C}_+ \cup i[-R,R]$ with $G_\infty = F_\infty - \int_0^s f(r)dr$. Hence, by (3.3)

$$\overline{\lim_{t\to\infty}} |F(t) - F_\infty| = \overline{\lim_{t\to\infty}} \left| \int_0^{t+s} f(r)dr - F_\infty \right| = \overline{\lim_{t\to\infty}} |G(t) - G_\infty| \leq \sup_{t\geq s} \frac{2}{R} |f(t)|.$$

Letting $s \to \infty$ yields (3.2). □

Quantitative estimates in the case $E \neq \emptyset$ are given in Batty [3].

Remark 3.3. *Convergence of the Laplace integral at regular points.* Assume that $\lim_{t\to\infty} |f(t)| = 0$. If $\hat{f}$ has a holomorphic extension to $\mathbb{C}_+ \cup U$, where U is a neighborhood of $i\eta \in i\mathbb{R}$, then

$$\hat{f}(i\eta) = \int_0^\infty e^{-i\eta s} f(s)ds = \lim_{t\to\infty} \int_0^t e^{-i\eta s} f(s)ds. \tag{3.4}$$

To see this, it suffices to replace $f(t)$ in (3.2) by $e^{-i\eta t}f(t)$.

Remark 3.4. *Riesz's theorem on power series* [40, Thm. 7.3]. Let $a_n \in X$ be such that $\lim_{n\to\infty} |a_n| = 0$ and let $p(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < 1$). If p has a holomorphic extension to $D \cup U$ ($D = \{z \in \mathbb{C} : |z| < 1\}$) where U is an open neighborhood of $z_0 \in \Gamma := \{z \in \mathbb{C} : |z| = 1\}$, then $p(z_0) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n z_0^n$. This is obtained by applying (3.4) to the function f defined by (2.6).

Next we establish a complex Tauberian theorem of type D. The following is a variant of [3, Cor. 2.6].

THEOREM 3.5. *Assume that f is slowly oscillating, let $f_\infty \in X$, and suppose that $\hat{f}(\lambda) - (f_\infty/\lambda)$ has a continuous extension to $\bar{\mathbb{C}}_+ \backslash iE$, where $E \subset \mathbb{R}$ is a closed null set such that $0 \notin E$. If for all $R > 0$,*

$$M(R) := \sup_{\eta \in E\cap[-R,R]} \sup_{t\geq 0} \left| \int_0^t \exp(-i\eta s) f(s)ds \right| < \infty, \tag{3.5}$$

then $\lim_{t\to\infty} f(t) = f_\infty$.

Remark. The assumption implies that $A - \lim_{t\to\infty} f(t) = f_\infty$.

Proof. The function f_δ defined by (2.2) is eventually bounded for $\delta > 0$ sufficiently small. Let $c := f_\infty - (1/\delta)\int_0^\delta f(s)ds$. Then by (2.3),

$$\begin{aligned}
(\hat{f}_\delta(\lambda) - c)/\lambda &= \frac{1}{\delta\lambda}(e^{\lambda\delta} - 1)\hat{f}(\lambda) - \frac{e^{\lambda\delta}}{\lambda\delta}\int_0^\delta e^{\lambda s} f(s)ds - \frac{c}{\lambda} \\
&= \frac{1}{\delta\lambda}(e^{\lambda\delta} - 1)(\hat{f}(\lambda) - \frac{f_\infty}{\lambda}) + \left[\frac{1}{\delta\lambda}(e^{\lambda\delta} - 1) - 1\right] f_\infty/\lambda \\
&\qquad - \frac{e^{\lambda\delta}}{\delta}\int_0^\delta \frac{e^{-\lambda s} - 1}{\lambda} f(s)ds - \frac{e^{\lambda\delta} - 1}{\lambda\delta}\int_0^\delta f(s)ds.
\end{aligned}$$

Since the functions $(1/\delta\lambda)(e^{\lambda\delta} - 1)$, $[(1/\delta\lambda)(e^{\lambda\delta} - 1) - 1]/\lambda$, and $(e^{-\lambda s} - 1)/\lambda$ are entire, it follows that $(\hat{f}_\delta(\lambda) - c)/\lambda$ has a continuous extension to $\bar{\mathbb{C}}_+ \backslash iE$. Moreover,

$$\begin{aligned}
\left| \int_0^t \exp(-i\eta s) f_\delta(s)ds \right| &= \left| \delta^{-1} \int_0^t \exp(-i\eta s)(f(s+\delta) - f(s)ds \right| \\
&= \delta^{-1} \left| \int_\delta^{\delta+t} \exp i\eta(\delta - s) f(s)ds - \int_0^t \exp(-i\eta s) f(s)ds \right| \\
&\leq 3\delta^{-1} M(R) \quad \text{for all } \eta \in E \cap [-R, R].
\end{aligned}$$

It follows from Theorem 3.1 that $\lim_{t\to\infty} \int_0^t f_\delta(s)ds = c$. Hence $B - \lim_{t\to\infty} f(t) = f_\infty$ by Lemma 2.8. It follows from Proposition 2.9(c) that $\lim_{t\to\infty} f(t) = f_\infty$. ◻

Remark 3.6. (a) If in Theorem 3.5, instead of f slowly oscillating, we merely assume that $f_\delta \in L^\infty([\tau,\infty); X)$ for all $\delta > 0$, then we obtain $B - \lim_{t\to\infty} f(t) = f_\infty$

(b) However, if f is not slowly oscillating, then f does not converge in general, even if f is bounded. An example is the function $f(t) = T(t)y$ from [2, proof of Ex. 2.5]. The function f is bounded and $\hat{f}$ has a holomorphic extension to $\bar{\mathbb{C}}_+$. However, $f(t)$ does not converge for $t \to \infty$.

Next we consider the case where $0 \in E$. For simplicity we assume $f_\infty = 0$. We let $\text{Lip}([\tau,\infty), X) = \{f : [\tau,\infty) \to X : f \text{ is Lipschitz continuous }\}$.

THEOREM 3.7. *Assume that $f \in \mathrm{Lip}([\tau,\infty);X)$, for some $\tau \geq 0$. Suppose that $\hat{f}(\lambda)$ has a continuous extension to $\bar{\mathbb{C}}_+\backslash iE$, where E is a closed null set, $0 \in E$, and that for each $R \geq 0$*

$$M(R) := \sup_{\eta\in E\cap[-R,R]} \sup_{t\geq 0} \left| \int_0^t \exp(-i\eta s) f(s) ds \right| < \infty. \tag{3.6}$$

Then $\lim_{t\to\infty} f(t) = 0$.

Remark. Since $0 \in E$, condition (3.6) implies that $C - \lim_{t\to\infty} f(t) = 0$.

Proof. We first show that $f \in L^\infty([\tau,\infty);X)$. There exists $L \geq 0$ such that $|f(t) - f(s)| \leq L|t-s|$ for all $s,t \geq \tau$. Let $\varphi \in X'$, $|\varphi| \leq 1$. Then, by the Taylor expansion for $F(t) = \int_0^t f(r)dr$ in $s \geq \tau$, we have

$$\langle F(s+1), \varphi\rangle = \langle F(s),\varphi\rangle + \langle f(s),\varphi\rangle + \int_s^{s+1} (s+1-r)\frac{d}{dr}\langle f(r),\varphi\rangle dr.$$

Hence

$$\begin{aligned}|\langle f(s),\varphi\rangle| &\leq |\langle F(s+1),\varphi\rangle| + |\langle F(s),\varphi\rangle| + \int_s^{s+1} (s+1-r)|\frac{d}{dr}\langle f(r),\varphi\rangle| dr \\ &\leq 2M(0) + L\int_s^{s+1}(s+1-r)dr \leq 2M(0) + \frac{L}{2}.\end{aligned}$$

Fix $\mu \in \mathbb{R}\backslash E$ and define $g(t) = e^{i\mu t}f(t)$. Then $g \in \mathrm{Lip}([\tau,\infty);X)$ and $\hat{g}(\lambda) = \hat{f}(\lambda - i\mu)(\mathrm{Re}\ \lambda > 0)$. Hence $\hat{g}(\lambda)$ has a continuous extension to $\bar{\mathbb{C}}_+\backslash iE'$ where $E' = E + \mu$. Moreover, for $\eta' = \eta + \mu \in E' \cap [-R,R]$ we have $|\int_0^t \exp(-i\eta' s)g(s)ds| = |\int_0^t \exp(-i\eta s)f(s)ds| \leq M(R+|\mu|)$ for all $t \geq 0$. Since $0 \notin E'$, the assertion follows from Theorem 3.5. □

Applying Theorem 3.5 to power series we obtain a variant of a result due to Allan, O'Farrell, and Ransford [1]. We let $\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$.

THEOREM 3.8. *Let $b_n \in X$ be such that $\sup\{|b_n| : n \in \mathbb{N}_0\} < \infty$ and set $p(z) = \sum_{n=0}^\infty b_n z^n$ for $|z| < 1$. Assume that p has a continuous extension to $\bar{D}\backslash F$, where $F \subset \Gamma$ is a closed null set.*

If $\sup_{z\in F}\sup_{N\in\mathbb{N}} |\sum_{n=0}^N b_n z^n| < \infty$, then $\lim_{n\to\infty} b_n = 0$.

Remark. The hypothesis of the theorem directly implies $A - \lim_{n\to\infty} b_n = 0$ if $1 \notin F$ and $C - \lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/n \sum_{k=0}^{n-1} b_k = 0$ if $1 \in F$.

Proof. Replacing b_n by $b_n w^{-n}$ for some $w \in \Gamma\backslash F$ if necessary, we may assume that $1 \notin F$. Let $f(t) = b_n$ for $t \in [n,n+1)$. Then $\hat{f}(\lambda) = [(1-e^{-\lambda})/\lambda]\sum_{n=0}^\infty b_n e^{-\lambda n}$ has a continuous extension to $\mathbb{C}_+\backslash iE$ where $E = \{\eta \in \mathbb{R} : e^{-i\eta} \in F\}$. Moreover, for $t \in [n,n+1)$ we have $\int_0^t \exp(-i\eta s)f(s)ds = \sum_{m=0}^n b_m \exp(-i\eta m)(1-\exp(-i\eta))/i\eta + b_n\exp(-i\eta n)(1-\exp(-i\eta(t-n)))/i\eta$, so that (3.5) is satisfied (since $0 \notin E$). It follows from Theorem 3.5 and Remark 3.6 that $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\int_n^{n+1} f(s)ds = B - \lim_{t\to\infty} f(t) = 0$. □

It is implied by Riesz's theorem (Remark 3.4) that in the situation of Theorem 3.8 we have $p(z) = \sum_{n=0}^\infty b_n z^n$ for all regular $z \in \Gamma$ (and this is precisely what is shown in [1], assuming that p has a holomorphic extension to $\bar{D}\backslash F$).

An immediate consequence of Theorem 3.8 is the Katznelson–Tzafriri theorem (which actually was the motivation of the work by Allan, O'Farrell, and Ransford [1]).

THEOREM 3.9. (*Katznelson–Tzafrifi* [22].) *Let* $T \in \mathcal{L}(X)$ *such that* $\sup_{n\geq 0} |T^n| < \infty$ *and* $\sigma(T) \cap \Gamma \subset \{1\}$. *Then* $\lim_{n\to\infty} |(T-I)T^n| = 0$.

Proof. Let $p(z) = \sum_{n=0}^{\infty}(T-I)T^n z^n = (T-I)(I-zT)^{-1}, |z| < 1$. Since $\sup_{n\geq 0} |\sum_{n=0}^{N}(I-T)T^n| = \sup_{N\geq 0} |I - T^{N+1}| < \infty$, the hypotheses of Theorem 3.8 are satisfied for $b_n = (I-T)T^n$ and $F = \{1\}$. □

We are going to prove a continuous version of the Katznelson–Tzafriri theorem. Formally, it is expected that $(T^n)_{n\geq 0}$ has to be replaced by $(T(t))_{t\geq 0}$ and $T - I = (T(1) - I)/1$ by A, the generator of $(T(t))_{t\geq 0}$. We make this more precise. A C_0-semigroup $(T(t))_{t\geq 0}$ on X is called *eventually differentiable* if there exists $\tau > 0$ such that $T(\tau)X \subset D(A)$. Note that then $T(t)X \subset D(A)$ and $AT(t) \in \mathcal{L}(X)$ for all $t \geq \tau$.

THEOREM 3.10. *Let* $(T(t))_{t\geq 0}$ *be a bounded, eventually differentiable semigroup with generator* A. *The following are equivalent.*

(i) $\lim_{t\to\infty} |AT(t)| = 0$;

(ii) $\sigma(A) \cap i\mathbb{R} \subset \{0\}$.

Proof. Let $M = \sup_{t\geq 0} |T(t)|$. Assume that (ii) holds and that $\tau > 0$ such that $T(\tau)X \subset D(A)$. Then $T(t)X \subset D(A^2)$ for all $t \geq 2\tau$. Let $f : [0,\infty) \to \mathcal{L}(X)$ be given by $F(t) = AT(t + 2\tau)$. Then

$$\begin{aligned} |f(t) - f(s)| &= \left| \int_t^s \frac{d}{dr} f(r)dr \right| = \left| \int_t^s A^2T(r+2\tau)dr \right| \\ &= \left| \int_t^s T(r)A^2T(2\tau)dr \right| \leq M|A^2T(2\tau)||s-t|, \qquad s,t \geq 0, \end{aligned}$$

so that f is Lipschitz continuous. Moreover, $\hat{f}(\lambda) = R(\lambda, A)AT(2\tau)$ has a continuous extension to $\bar{\mathbb{C}}_+\backslash\{0\}$. Since $|\int_0^t f(s)ds| = |T(t+2\tau) - T(t)| \leq 2M \quad (t \geq 0)$, it follows from Theorem 3.7 that $\lim_{t\to\infty} |AT(t)| = \lim_{t\to\infty} |f(t)| = 0$.

Conversely, assume that (i) holds. (a) We show that $\lambda e^{t\lambda} \in \sigma(AT(t))$ for all $t \geq \tau$ whenever $\lambda \in \sigma(A) \cap i\mathbb{R}$. In fact, let $\lambda \in \sigma(A) \cap i\mathbb{R}$; then $\lambda \in \sigma_p(A) \cup \sigma_c(A)$ since λ is a boundary point of $\rho(A)$. Hence, there exist $x_n \in D(A)$, $|x_n| = 1$ such that $\lim_{t\to\infty} |(\lambda - A)x_n| = 0$. Consequently,

$$\begin{aligned} (\lambda e^{t\lambda} - AT(t))x_n &= \lambda(e^{\lambda t} - T(t))x_n + T(t)(\lambda - A)x_n \\ &= \lambda e^{\lambda t} \int_0^t e^{-\lambda s}T(s)(\lambda - A)x_n ds + T(t)(\lambda - A)x_n \to 0, \end{aligned}$$

as $n \to \infty$. Thus $\lambda e^{\lambda t} \in \sigma(AT(t))$ for $t \geq \tau$.

(b) Let $\eta \in \mathbb{R}$ be such that $i\eta \in \sigma(A)$. Then by (a), $i\eta e^{i\eta t} \in \sigma(AT(t))$ for $t \geq \tau$. Consequently, $|\eta| = |i\eta e^{i\eta t}| \leq |AT(t)| \to 0$ as $t \to \infty$, i.e., $\eta = 0$. □

As another application of Theorem 3.7 we obtain the following result which in some sense is complementary to Theorem 3.10.

THEOREM 3.11. *Let* $U(t)$ *be* C_0*-group with generator* A *and suppose that* $\sup_{t\geq 0} |U(t)x| < \infty$ *for all* $x \in D_\infty := \cap_{n\geq 0} D(A^n)$. *If* $\sigma(A) \cap i\mathbb{R} \subset \{0\}$, *then* $U(t) = I$ *for all* $t \in \mathbb{R}$.

Proof. Let $x \in D_\infty$ and $f(t) = AU(t)x = U(t)Ax$. Then f is Lipschitz continuous since $|f(t) - f(s)| = |\int_s^t (d/dr)U(r)Axdr| = |\int_s^t U(r)A^2xdr| \leq |t-s| \sup_{r\geq 0} |U(r)A^2x|$, and $|\int_0^t f(s)dr| = |U(t)x - x|$ is bounded for $t \geq 0$. For Re $\lambda > 0$ we have $\hat{f}(\lambda) \in D(A)$ and $(\lambda - A)\hat{f}(\lambda) = Ax$. Hence $\hat{f}(\lambda) = (\lambda - A)^{-1}Ax$ whenever $\lambda \in \varrho(A)$, Re $\lambda > 0$. This shows that $\hat{f}(\lambda)$ has a continuous extension to $\bar{\mathbb{C}}_+\backslash\{0\}$. It follows from Theorem 3.7 that $\lim_{t\to\infty} U(t)Ax = \lim_{t\to\infty} f(t) = 0$.

So far we have shown that $\lim_{t\to\infty} U(t)Ax = 0$ for all $x \in D_\infty$. We will deduce from this that $Ax = 0$ for all $x \in D_\infty$ and hence $A = 0$ since D_∞ is a core. In fact, D_∞ is a Fréchet space for the topology defined by the norms $p_n(x) = |x|+|Ax|+\cdots+|A^n x|$, $n \in \mathbb{N} \cup \{0\}$. We show that there exists $k \in \mathbb{N}$ such that

$$|U(t)x| \leq kp_k(x) \tag{3.7}$$

for all $x \in D_\infty$, $t \in \mathbb{R}$. If this is false, there exist $x_m \in D_\infty$, $t_m \in \mathbb{R}$ such that $p_m(x_m) = 1$ and $|U(t_m)x_m| \geq m$, $m \in \mathbb{N}$. Let $Y_k = \{x \in D_\infty : |U(t)x| \leq kp_k(x)$ for all $t \in \mathbb{R}\}$. Then Y_k is closed in D_∞ and $\cup_{k\geq 0} Y_k = D_\infty$. So by Baire's theorem there exists $k \in \mathbb{N}$ such that Y_k has a nonempty interior; i.e., we find $a \in D_\infty$, $\epsilon > 0$, $\ell \geq k$ such that $p_\ell(a - x) \leq \epsilon$ implies $|U(t)x| \leq kp_k(x)$ for all $t \in \mathbb{R}$. Consequently,

$$\begin{aligned} m \cdot \epsilon/p_\ell(x_m) - |U(t_m)a| &\leq \epsilon/p_\ell(x_m)|U(t_m)x_m| - |U(t_m)a| \leq |U(t_m)(a - \epsilon/p_\ell(x_m)x_m| \\ &\leq kp_k(a - (\epsilon/p_\ell(x_m))x_m) \leq k(p_k(a) + \epsilon p_k(x_m)/p_\ell(x_m)) \\ &\leq k(p_k(a) + \epsilon) \quad \text{since } \ell \geq k, \end{aligned}$$

hence,

$$\begin{aligned} \epsilon m &\leq p_\ell(x_m)[k(p_k(a) + \epsilon) + |U(t_m)a|] \leq p_m(x_m)[k(p_k(a) + \epsilon) + |U(t_m)a|] \\ &= [k(p_k(a) + \epsilon) + |U(t_m)a|] \quad \text{for all } m \geq \ell. \end{aligned}$$

But $(U(t_m)a)_{m\geq 0}$ is bounded in X, a contradiction. So (3.7) is proved.

Let $x \in D_\infty$. Then by (3.7)

$$\begin{aligned} |Ax| &= |U(-t)U(t)Ax| \leq kp_k(U(t)Ax) \\ &= k\{|U(t)Ax| + |U(t)A^2x| + \cdots + |U(t)A^{k+1}x|\} \to 0 \quad \text{as } t \to \infty. \end{aligned}$$

Hence $Ax = 0$ for all $x \in D_\infty$. □

4. Real ergodic theorems for Volterra equations. Throughout the remainder of the paper, we make the assumptions of the Introduction. In particular, $(S(t))_{t\geq 0}$ denotes the resolvent governing (1.1). Recall that we assume

$$\sup_{t\geq 0} |e^{-\lambda t} S(t)| < \infty$$

for all $\lambda > 0$. By $\hat{S}(\lambda) = \int_0^\infty e^{-\lambda t} S(t)dt$, Re $\lambda > 0$, we denote the Laplace transform of $S(t)$. In addition, we assume that the (complex-valued) kernel $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is Laplace transformable, i.e., there exists $\alpha \geq 0$ such that $\int_0^\infty e^{-\alpha t}|a(t)|dt < \infty$. We let $\hat{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)dt$ (Re $\lambda \geq \alpha$). If (1.3) holds, then the closedness of A implies

$$\int_0^t a(t-s)S(s)x\,ds \in D(A) \quad \text{and} \quad S(t)x = x + A\int_0^t a(t-s)S(s)x\,ds \tag{4.1}$$

for all $x \in X$. Moreover, due to the assumptions above we have the following proposition.

PROPOSITION 4.1. (a) *$\hat{a}(\lambda)$ has a meromorphic extension to $\mathbb{C}_+$.*

(b) *$\hat{a}(\lambda) \neq 0$ on $\mathbb{C}_+$ if A is unbounded.*

(c) *$\lambda\hat{S}(\lambda) = (I - \hat{a}(\lambda)A)^{-1}$ for all $\lambda \in \mathbb{C}_+$ such that λ is not a pole of $\hat{a}$.*

We refer to [33] for the proof of (4.1) and Proposition 4.1.

COROLLARY 4.2. *If $\hat{a}$ has a pole in $\mathbb{C}_+$, then $R(A)$ is closed and $X = N(A) \oplus R(A)$.*

Remark. Here $R(A) := \{Ax : x \in D(A)\}$ denotes the *range* and $N(A) := \{x \in D(A) : Ax = 0\}$ the *kernel* of A.

Proof. Assume that $\lambda_0 \in \mathbb{C}_+$ is a pole of $\hat{a}$ of order n; then $1/\hat{a}(\lambda)$ maps a neighborhood of λ_0 onto a neighborhood of zero. It follows from Proposition 4.1(c) that there exists $\epsilon > 0$ such that $V := \{z \in \mathbb{C} : 0 < |z| < \epsilon\} \subset \varrho(A)$. Moreover, $|((1/\hat{a}(\lambda)) - A)^{-1}| = |\hat{a}(\lambda)\lambda\hat{S}(\lambda)| \leq$ const $|\hat{a}(\lambda)|$ near λ_0. Hence $|(z-A)^{-1}| \leq \text{const}/|z|$ $(z \in V)$. Thus zero is at most a pole of order 1 of $(z-A)^{-1}$. Now the claim follows from [41, Chap. VIII.8]. □

In order to study the asymptotic behavior of the resolvent, we use the following terminology.

DEFINITION 4.3. The resolvent S is called (a) *uniformly* (*strongly, weakly*) *Abel-ergodic* if $\lim_{\lambda\to 0+} \lambda\hat{S}(\lambda) = P$ exists in the uniform (respectively, strong, weak) operator topology;

(b) *uniformly* (*strongly, weakly*) *Cesaro-ergodic* if $\lim_{t\to\infty} 1/t \int_0^t S(s)ds = P$ exists uniformly (respectively, strongly, weakly);

(c) *uniformly* (*strongly, weakly*) *ergodic* if $\lim_{t\to\infty} S(t) = P$ exists uniformly (respectively, strongly, weakly).

Notation. We shall use the abbreviation (i,J)-ergodic where i runs through the symbols u, s, w with obvious meaning, and J runs through A, C, E. Then the following implication scheme holds.

$$\begin{array}{ccccc}
(u,A) & \Leftarrow & (u,C) & \Leftarrow & (u,E) \\
\Downarrow & & \Downarrow & & \Downarrow \\
(s,A) & \Leftarrow & (s,C) & \Leftarrow & (s,E) \\
\Downarrow & & \Downarrow & & \Downarrow \\
(w,A) & \Leftarrow & (w,C) & \Leftarrow & (w,E)
\end{array}$$

Our goal is to characterize (i, J) ergodicity of $S(t)$ in terms of the operator A and the kernel a; or, at least, to find sufficient conditions. We need the following.

PROPOSITION 4.4. *Let B be a densely defined linear operator on X, $\mu_n \in \mathbb{C}$ such that $\lim_{n\to\infty} |\mu_n| = \infty$, $1/\mu_n \in \varrho(B)$ and $\sup_{n\geq 0} |(I - \mu_n B)^{-1}| < \infty$. Then*

(a) $N(B) \cap \overline{R(B)} = \{0\}$.

(b) *The following are equivalent.*

(i) $\lim_{n\to\infty}(I - \mu_n B)^{-1} = P$ *exists strongly;*

(ii) $\lim_{n\to\infty}(I - \mu_n B)^{-1} = P$ *exists weakly;*

(iii) $N(B) \oplus \overline{R(B)} = X$;

(iv) $N(B)^\perp \cap N(B') = \{0\}$.

If this is the case, then P is the projection onto $N(B)$ along $\overline{R(B)}$.

(c) *If X is reflexive, the equivalent conditions of* (b) *are automatically satisfied.*

(d) *Assume that the equivalent conditions of* (b) *hold. Then the following are equivalent.*

(i) $R(B)$ *is closed;*

(ii) $\lim_{n\to\infty}(I - \mu_n B)^{-1} = P$ *in* $\mathcal{L}(X)$;

(iii) $\lim_{n\to\infty}(I - \mu_n B)^{-2} = P$ *in* $\mathcal{L}(X)$.

This result is well known; we refer to [41, Chap. VIII.4] and [19, Chap. XVIII]. We add the analogous properties of $(I - \mu_n B)^{-1}$ at zero.

PROPOSITION 4.5. *Let B be an operator on X, $0 \neq \mu_n \in \mathbb{C}$ such that $1/\mu_n \in \varrho(B)$, $\lim_{n\to\infty} \mu_n = 0$ and $\sup_{n\geq 0} |(I - \mu_n B)^{-1}| < \infty$.*

(a) *The following are equivalent.*

(i) *$D(B)$ is dense in X;*

(ii) *$\lim_{n\to\infty}(I - \mu_n B)^{-1} = I$ strongly;*

(iii) *$\lim_{n\to\infty}(I - \mu_n B)^{-1} = I$ weakly.*

(b) *The following are equivalent.*

(i) *$D(B) = X$;*

(ii) *$\lim_{n\to\infty}(I - \mu_n B)^{-1} = I$ in $\mathcal{L}(X)$;*

(iii) *$\lim_{n\to\infty}(I - \mu_n B)^{-2} = I$ in $\mathcal{L}(X)$.*

For the proof we refer to [19, Chap. XVIII].

Strong and weak Abel ergodicity of the resolvent $S(T)$ of (1.1) are characterized as follows.

THEOREM 4.6. *The following are equivalent.*

(i) *$S(t)$ is strongly Abel ergodic.*

(ii) *$S(t)$ is weakly Abel ergodic.*

(iii) (a) *$|\lambda\hat{S}(\lambda)|$ is bounded on $(0,1]$;*

(b) *$\lim_{\lambda\to 0+} \hat{a}(\lambda) =: \hat{a}(0)$ exists in $\mathbb{C} \cup \{\infty\}$;*

(c) *$N(A)^{\perp} \cap N(A') = \{0\}$ if $\hat{a}(0) = \infty$.*

Moreover, if these equivalent conditions are satisfied, then $\lim_{\lambda\to 0} \lambda\hat{S}(\lambda) = (I - \hat{a}(0)A)^{-1}$ in $\mathcal{L}(X)$ if $0 \neq \hat{a}(0) \in \mathbb{C}$, $\lim_{\lambda\to 0+} \lambda\hat{S}(\lambda) = I$ strongly if $\hat{a}(0) = 0$, and $\lim_{\lambda\to 0} \lambda\hat{S}(\lambda) = P$ strongly if $\hat{a}(0) = \infty$, where P denotes the projection onto $N(A)$ along $\overline{R(A)}$. If X is reflexive, then (c) in (iii) can be omitted.

Proof. (ii) $\Rightarrow$ (iii). Assume that $w - \lim_{\lambda\to 0+} \lambda\hat{S}(\lambda)x = w - \lim_{\lambda\to 0+}(I - \hat{a}(\lambda)A)^{-1}x = Px$ for all $x \in X$. Then

$$\sup_{\lambda\in(0,1]} |(I - \hat{a}(\lambda)A)^{-1}| < \infty. \tag{4.2}$$

Choose a sequence $\lambda_n \to 0$ such that $\mu_n = \hat{a}(\lambda_n) \to \mu_\infty \in \mathbb{C} \cup \{\infty\}$. We distinguish three cases.

Case 1. $0 < |\mu_\infty| < \infty$.

Then, by (4.2), $\mu_\infty^{-1} \in \varrho(A)$ and $P = (I - \mu_\infty A)^{-1} = \lim_{n\to\infty}(I - \mu_n A)^{-1}$ in $\mathcal{L}(X)$.

Case 2. $\mu_\infty = 0$.

Then $\lim_{n\to\infty}(I - \mu_n A)^{-1} = I$ strongly by Proposition 4.5.

Case 3. $\mu_\infty = \infty$.

It follows from Proposition 4.4 and 4.5, that $(I - \mu_n A)^{-1} \to P$ strongly, where P is the projection onto $N(A)$ along $\overline{R(A)}$.

Now suppose that there exists another sequence $\lambda_n' \to 0$ such that $\hat{a}(\lambda_n') \to \mu_\infty' \neq \mu_\infty$, $\mu_\infty' \in \mathbb{C} \cup \{\infty\}$. Since $A \neq 0$, the limit operators P and P' are different. But this is impossible since $P = \lim_{\lambda\to 0+} \lambda\hat{S}(\lambda) = P'$. This shows that $\hat{a}(0) := \lim_{\lambda\to 0+} \hat{a}(\lambda)$ exists in $\mathbb{C} \cup \{\infty\}$. We have proved (iii). It follows from Propositions 4.4 and 4.5 that (iii) implies (i). $\square$

From the preceding proof, we also obtain the following characterization of uniform Abel ergodicity.

THEOREM 4.7. *$S(t)$ is uniformly Abel ergodic if and only if the following four conditions hold.*

(a) *$|\lambda\hat{S}(\lambda)|$ is bounded on $(0,1]$;*

(b) *$\lim_{\lambda\to 0+} \hat{a}(\lambda) =: \hat{a}(0)$ exists in $\mathbb{C} \cup \{\infty\}$;*

(c) *if $\hat{a}(0) = \infty$ then $R(A)$ is closed and $X = N(A) \oplus R(A)$;*

(d) $\hat{a}(0) \neq 0$ *if A is unbounded.*

COROLLARY 4.8. *Suppose that $(\lambda_0 - A)^{-1}$ is compact for some $\lambda_0 \in \varrho(A)$. We assume that $\hat{a}(0) \neq 0$ if A is unbounded. If $S(t)$ is (w, A)-ergodic, then $S(t)$ is (u, A)-ergodic.*

Proof. This follows from Theorem 4.7 and Theorem 4.6 since $R(A)$ is closed because of the compactness of $(\lambda_0 - A)^{-1}$. □

It is instructive to classify Abel ergodicity by the limits of $\hat{a}(\lambda)$ as $\lambda \to 0+$. Assume that $\hat{a}(0) = \lim_{\lambda \to 0+} \hat{a}(\lambda) \in \mathbb{C} \cup \{\infty\}$ exists.

Case 1. $\hat{a}(0) = 0$. Then

(a) $S(t)$ is (u, A)-ergodic iff A is bounded; and

(b) $S(t)$ is (s, A)-ergodic iff $(I - \hat{a}(\lambda)A)^{-1}$ is bounded for $\lambda \to 0+$.

The ergodic limit then is $P = I$.

Case 2. $\hat{a}(0) \neq 0, \infty$. Then $S(t)$ is (u, A)-ergodic iff it is (s, A)-ergodic iff $(I - \hat{a}(\lambda)A)^{-1}$ is bounded for $\lambda \to 0+$ iff $\hat{a}(0)^{-1} \in \rho(A)$.

The ergodic limit then is $P = (I - \hat{a}(0)A)^{-1}$.

Case 3. $\hat{a}(0) = \infty$. Then

(a) $S(t)$ is (u, A)-ergodic iff $\overline{\lim}_{\lambda \to 0+} |(I - \hat{a}(\lambda)A)^{-1}| < \infty$, $N(A)^{\perp} \cap N(A') = \{0\}$ and $R(A)$ is closed;

(b) $S(t)$ is (s, A)-ergodic iff $\overline{\lim}_{\lambda \to 0+} |(I - \hat{a}(\lambda)A)^{-1}| < \infty$, and $N(A)^{\perp} \cap N(A') = \{0\}$.

The ergodic limit P is then the projection onto $N(A)$ along $\overline{R(A)}$.

In particular, we obtain the following necessary conditions.

COROLLARY 4.9. *If $A - \lim_{t \to \infty} S(t) = 0$ strongly, then $\lim_{\lambda \to 0+} \hat{a}(\lambda) = \infty$ and $0 \notin \sigma_p(A) \cup \sigma_p(A')$.*

Proof. For the second assertion observe that $Ax = 0$ implies $S(t)x = x$ $(t \geq 0)$ and so $x = 0$. This shows $N(A) = 0$. Hence $N(A') = N(A)^{\perp} \cap N(A') = \{0\}$. Thus $0 \notin \sigma_p(A) \cup \sigma_p(A')$. □

Next, we consider Cesaro ergodicity.

THEOREM 4.10. (a) *If $S(t)$ is bounded and (w, A)-ergodic, then $S(t)$ is (s, C)-ergodic.*

(b) *Suppose that X is an ordered Banach space with normal and generating cone. If $S(t) \geq 0$ $(t \geq 0)$ and $S(t)$ is (w, A)-ergodic, then $S(t)$ is (s, C)-ergodic.*

Proof. This follows from Theorem 2.5 and 2.6. □

Finally, we consider ergodicity of $S(t)$. We say that $S(t)$ is a *bounded analytic resolvent* if there exists a bounded, analytic extension of S to a sector $\Sigma(\theta) = \{z : |\arg z| < \theta\}$ for some $\theta \in (0, \pi/2)$.

Remark 4.11. Equation (1.1) is governed by a bounded analytic resolvent if and only if the following conditions are satisfied for some $\theta \in (0, \pi/2)$.

(a) $\hat{a}$ admits a meromorphic extension to $\Sigma(\theta + \pi/2)$.

(b) $\hat{a}(\lambda) \neq 0$ if A is unbounded and $1/\hat{a}(\lambda) \in \varrho(A)$ for all $\lambda \in \Sigma(\theta + \pi/2)$ with $\hat{a}(\lambda) \neq 0$.

(c) $|(I - \hat{a}(\lambda)A)^{-1}|$ is bounded on $\Sigma(\theta + \pi/2)$.

We refer to [33] for a proof.

In the semigroup case $(a(t) \equiv 1)$ the notion of bounded analytic resolvent coincides with that of a bounded analytic semigroup.

PROPOSITION 4.12. *Assume that S is a bounded analytic resolvent. Then there exists $M \geq 0$ such that*

$$t|S'(t)| \leq M \quad \textit{for all } t > 0$$

(see [33, Cor. 2.1] for a proof).

THEOREM 4.13. *Assume that $S(t)$ is a bounded analytic resolvent. If $S(t)$ is weakly Abel ergodic, then $S(t)$ is strongly ergodic. Moreover, $S(t)$ is even uniformly ergodic, if in addition $(\lambda_0 - A)^{-1}$ is compact for some $\lambda_0 \in \varrho(A)$.*

Proof. It follows from Theorem 4.6 that $S(t)$ is (s, A)-ergodic, and by Corollary 4.8 that S is (u, A)-ergodic if $(\lambda_0 - A)^{-1}$ is compact for some $\lambda_0 \in \varrho(A)$. Let $f(t) = S(t)$ $(t \geq 0)$. Then $f : \underline{(0, \infty)} \to \mathcal{L}(X)$ is analytic and bounded, hence $f \in L^\infty([0, \infty); \mathcal{L}(X))$. Moreover, $\overline{\lim}_{t\to\infty} t|f'(t)| < \infty$ (by Proposition 4.12). So the claim follows from Theorem 2.10. □

Example 4.14. Consider the kernel $a(t) = t^{\alpha-1}/\Gamma(\alpha)$ where $\alpha \in (0, 2]$ and assume that (1.1) is well posed. For $\alpha = 1$ this means that A generates a C_0-semigroup, for $\alpha = 2$, that A generates a cosine function. We assume again that $\sup_{t\geq 0} |e^{-\lambda t} S(t)| < \infty$ for all $\lambda > 0$. Since $\hat{a}(\lambda) = \lambda^{-\alpha}$, it follows that $\Sigma(\alpha\frac{\pi}{2}) \subset \varrho(A)$. Moreover, $\lim_{\lambda\to 0+} \hat{a}(\lambda) = \infty$ and $\lambda\hat{S}(\lambda) = (I - \lambda^{-\alpha}A)^{-1} = \lambda^\alpha(\lambda^\alpha - A)^{-1}$. Thus Abel ergodicity is the same for all $\alpha \in (0, 2]$:

(a) $S(t)$ is (s, A)-ergodic iff $\sup_{\mu\in(0,1]} |\mu(\mu - A)^{-1}| < \infty$ and $N(A') \cap N(A)^\perp = \{0\}$.

(b) $S(t)$ is (u, A)-ergodic iff (a) holds and $R(A)$ is closed.

In order to characterize strong ergodicity we assume $\alpha < 2$ and $\Sigma(\theta) \subset \varrho(A)$, $|\mu(\mu - A)^{-1}| \leq M$ on $\Sigma(\theta)$ for some $\theta \in (\alpha, \pi/2, \pi)$. Then (1.1) is governed by a bounded analytic resolvent (Remark 4.11). If $N(A') \cap N(A)^\perp = \{0\}$, it follows from Theorem 4.13 that $\lim_{t\to\infty} S(t) = P$ strongly, where P is the projection onto $N(A)$ along $\overline{R(A)}$.

Finally, we consider Volterra equations on $L^\infty = L^\infty(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) denotes a positive measure space; this Banach space plays an exceptional role.

THEOREM 4.15. *If $X = L^\infty$, then the well-posedness of* (1.1) *implies that A is bounded.*

Remark. Conversely, if A is bounded, then (1.1) is well posed for every kernel.

Theorem 4.15 is due to Lotz [26] in the case $a(t) = 1$, where A is the generator of a C_0-semigroup (see also [29, A-II.3]); for the special case of contraction semigroups it was obtained independently by Coulhon [11]; and for positive semigroups it is due to Kishimoto and Robinson [23].

The reasons for the phenomenon expressed in Theorem 4.15 are two properties of $X = L^\infty$, namely,

(DP) $$x_n \to 0 \quad \text{in } (X, \sigma(X, X')) \quad \text{and} \quad x'_n \to 0 \quad \text{in } (X', \sigma(X', X'')) \quad \text{imply } \langle x_n, x'_n\rangle \to 0$$

and

(G) $$x'_n \to 0 \quad \text{in } (X', \sigma(X', X)) \quad \text{implies } x'_n \to 0 \quad \text{in } (X', \sigma(X', X''))$$

(see [36, Chaps. II.9.7 and II.10.4]). The first property is called the *Dunford–Pettis property*; a space satisfying the second is called a *Grothendieck space*. For further details on the background in geometry of Banach spaces, we refer to [26] (see also [27], [12]).

The key of the proof of Theorem 4.5 is the following result due to Lotz [26, Thm. 2].

LEMMA 4.16. *Let X satisfy* (G) *and* (DP). *Suppose $T_n \in \mathcal{L}(X)$ is such that $\lim_{n\to\infty} T_n = 0$ strongly and $\lim_{n\to\infty} T'_n = 0$ strongly. Then $\lim_{n\to\infty} |T_n^2| = 0$.*

Using this lemma we obtain the following general result which contains Theorem 4.15 as a special case.

THEOREM 4.17. *Assume that X satisfies* (G) *and* (DP). *Let B be an operator on X and (μ_n) be a sequence in $\mathbb{C}\backslash\{0\}$ such that $1/\mu_n \in \varrho(B)$, $\sup_{n\geq 0} |(I-\mu_n B)^{-1}| < \infty$ and $\lim_{n\to\infty} |\mu_n| = 0$. If $\overline{D(B)} = X$, then B is bounded.*

Proof. Let $J_n = (1-\mu_n B)^{-1}$. Then $\lim_{n\to\infty} J_n = I$ strongly by Proposition 4.5. Hence, $\sigma(X', X) - \lim_{n\to\infty} J_n' x' = x'$ and so by (G), $\sigma(X', X'') - \lim_{n\to\infty} J_n' x' = x'$ for all $x' \in X'$. It follows from Proposition 4.5 that $\lim_{n\to\infty} J_n' = I$ strongly. Now we deduce from Proposition 4.16 that $\lim_{n\to\infty} |(J_n - I)^2| = 0$ which implies $D(B) = X$ by Proposition 4.5. □

Next we consider ergodicity of (1.1) in L^∞.

THEOREM 4.18. *If $X = L^\infty$ and $S(t)$ is weakly Abel ergodic, then $S(t)$ is uniformly Abel ergodic.*

Remark. Since by our general assumption (1.1) is well posed, A is bounded in the situation of Theorem 4.18 (by Theorem 4.15).

We first show the following.

THEOREM 4.19. *Assume that X satisfies* (G) *and* (DP). *Let B be an operator on X such that $1/\mu_n \in \varrho(B)$ for a sequence $(\mu_n) \subset \mathbb{C}$ such that $\lim_{n\to\infty} |\mu_n| = \infty$. If $\lim_{n\to\infty}(I-\mu_n B)^{-1} = P$ weakly, then $\lim_{n\to\infty}(I-\mu_n B)^{-1} = P$ in $\mathcal{L}(X)$.*

Proof. By Proposition 4.4 we have $X = N(B) \oplus \overline{R(B)}$. We can assume $N(B) = 0$ and $P = 0$. Moreover, since $J_n := (I-\mu_n B)^{-1} \to 0$ strongly, it follows that $J_n' x' \to 0$ for $\sigma(X', X)$ and so by (G) for $\sigma(X', X'')$ for all $x' \in X'$. It follows from Proposition 4.4 that $\lim_{n\to\infty} J_n' = I$ strongly. Thus $\lim_{n\to\infty} |J_n^2| = 0$ by Lemma 4.16. This implies $R(B) = X$ by Proposition 4.4(d). □

Proof of Theorem 4.18. Assume that $S(t)$ is (w, A)-ergodic on L^∞. If $\hat{a}(0) \in \mathbb{C}$, then $S(t)$ is (u, A)-ergodic by Theorem 4.7 (note that A is bounded). If $\hat{a}(0) = \infty$, then S is (u, A)-ergodic by Theorem 4.19. □

Remark. Lotz [26] investigates ergodic properties of discrete semigroups $(T^n)_{n\geq 0}$ where T is a bounded linear operator on L^∞.

5. A general convergence theorem for Volterra equations. This section contains the main theorem which is based on the complex methods introduced in §3.

We assume throughout that a is a kernel as described in §4, A is a linear closed densely defined operator and that the Volterra equation (1.1) is well posed and governed by the resolvent $S(t)$, which is bounded.

Then we know in particular that $\hat{a}$ has a meromorphic extension to $\mathbb{C}_+$. For later purposes (§§6 and 7) we set

$$\varrho(a) := \{i\mu : \mu \in \mathbb{R}, \hat{a} \text{ has a continuous extension to } \mathbb{C}_+ \cup i[\mu-\epsilon, \mu+\epsilon] \text{ with values in } \mathbb{C} \cup \{\infty\} \text{ for some } \epsilon > 0\} \tag{5.1}$$

and still denote by $\hat{a}$ the continuous extension of $\hat{a}$ to $\mathbb{C}_+ \cup \varrho(a)$.

In this section, though, we assume throughout that

$$\varrho(a) = i\mathbb{R}. \tag{5.2}$$

Moreover, we assume that $S(t)$ is strongly Abel ergodic, and set

$$\lim_{\lambda\to 0+} \lambda \hat{S}(\lambda) = Q \tag{5.3}$$

Remark. Since by assumption $S(t)$ is bounded, this is automatically satisfied if X is reflexive (see Thm. 4.4).

From Theorem 4.6, we know the following. If $\hat{a}(0) \in \mathbb{C}$, then $Q = (I - \hat{a}(0)A)^{-1}$; if $\hat{a}(0) = \infty$, then $X = N(A) \oplus \overline{R(A)}$ and $Q = P$, the projection onto $N(A)$ along $\overline{R(A)}$. The following "resolvent set $\varrho(a, A)$ of (a, A)" plays an important role.

$$\varrho(a, A) := \Big\{ i\eta \in i\mathbb{R} : \text{ there exists } \epsilon > 0 \text{ such that } \tfrac{1}{\lambda}[(1 - \hat{a}(\lambda)A)^{-1} - Q] \\ \text{has a strongly continuous extension to } \mathbb{C}_+ \cup i[\eta - \epsilon, \eta + \epsilon] \Big\}. \tag{5.4}$$

Now we are able to formulate the General Convergence Theorem. It is valid for arbitrary kernels (satisfying (5.2)). In the forthcoming sections it will be shown that, for many interesting classes of kernels, hypotheses (H2) and (H3) are automatically satisfied so that (H1) remains to be verified in order to conclude that $S(t)$ is strongly ergodic. Note that in the reflexive case (H1) reduces to a condition on the spectral behavior of (a, A) on $i\mathbb{R}$: the singular set iE has to be countable and $1/\hat{a}(i\eta) \notin \sigma_p(A')$ whenever $\eta \in E$ such that $\hat{a}(i\eta) \neq 0, \infty$ (by $\sigma_p(A')$ we denote the point spectrum of the adjoint A' of A).

THEOREM 5.1. *Assume* (5.2), (5.3), *and suppose the following three hypotheses are satisfied.*

(H1) *The singular set $iE := i\mathbb{R}\backslash\varrho(a, A)$ is countable and $\mu \in E\backslash\{0\}$, $\hat{a}(i\mu) \neq 0, \infty$ implies $R(I - \hat{a}(i\mu)A) = X$; $\mu \in E\backslash\{0\}$, $\hat{a}(i\mu) = \infty$ implies $X = N(A) \oplus \overline{R(A)}$.*

(H2) *For all $\mu \in E$ there exists $C(\mu) \geq 1$ such that $|\int_0^t e^{-i\mu s}(a * S(s) - \hat{a}(i\mu)S(s))Ax\,ds| \leq C(\mu)|x|_A$ for all $x \in D(A)$ if $\hat{a}(i\mu) \in \mathbb{C}$, and $|\int_0^t e^{-i\mu s}S(s)Ax\,ds| \leq C(\mu)|x|_A$ for all $x \in D(A)$ if $\hat{a}(i\mu) = \infty$.*

(H3) *There exist $\tau \geq 0$, $M \geq 0$ such that $|S'(t)x| \leq M|x|_A$ ($x \in D(A)$, $t \geq \tau$), and $|S(t)| \leq M$ ($t \geq 0$).*

Then $\lim_{t\to\infty} S(t)x = Qx$ for all $x \in X$, where $Q = (I - \hat{a}(0)A)^{-1}$ if $\hat{a}(0) \in \mathbb{C}$, and Q is the projection onto $N(A)$ along $\overline{R(A)}$ if $\hat{a}(0) = \infty$.

We start with the following estimate which is a variant of [2, Lemma 3.1].

LEMMA 5.2. *Let $f : [0, \infty) \to X$ be measurable, $|f(t)| \leq M_0$ ($t \geq 0$). Let $R > 0$. Assume that $\hat{f}(\lambda)/\lambda$ (which is defined for* Re $\lambda > 0$) *has a continuous extension to $\mathbb{C}_+ \cup i([-R, R]\backslash \bigcup_{j=1}^n (\xi_j - \epsilon_j, \xi_j + \epsilon_j))$ where $\xi_j \in \mathbb{R}$, $\epsilon_j > 0$ such that the intervals $(\xi_j - \epsilon_j, \xi_j + \epsilon_j)$ ($j = 1 \cdots n$) are pairwise disjoint and $0 \notin \bigcup_{j=1}^n [\xi_j - \epsilon_j, \xi_j + \epsilon_j] \subset (-R, R)$. Furthermore, suppose that for $j = 1, \cdots, n$ there exist $\eta_j \in (\xi_j - \epsilon_j, \xi_j + \epsilon_j)$ such that*

$$M_j = \sup_{t \geq 0} \Big| \int_0^t \exp(-i\eta_j s) f(s) ds \Big| < \infty \qquad (j = 1 \cdots n).$$

Then,

$$\overline{\lim_{t\to\infty}} \Big| \int_0^t f(s) ds \Big| \leq \frac{2M_0}{R} \prod_{j=1}^n a_j + 12 \sum_{j=1}^n M_j \delta_j \prod_{\substack{k=1 \\ k \neq j}}^n b_{jk}, \tag{5.5}$$

where

$$\begin{aligned} a_j &= (1+\epsilon_j^2(R-|\xi_j|)^{-2})\xi_j^2(\xi_j^2-\epsilon_j^2)^{-1}; \\ b_{jk} &= (1+\epsilon_k^2(|\xi_j-\xi_k|-\epsilon_j)^{-2})\xi_k^2(\xi_k^2-\epsilon_k^2)^{-1} \qquad (k\neq j); \\ \delta_j &= \epsilon_j\xi_j^2(|\xi_j|-\epsilon_j)^{-1}(\xi_j^2-\epsilon_j^2)^{-1}. \end{aligned} \tag{5.6}$$

Proof. We modify the proof of [2, Lemma 3.1] in the following way, keeping the notation used there (cf. also [25, 2.2]). The paths γ_j are replaced by straight lines on the imaginary axis ($j=0,\cdots,n$). Applying (a slight extension of) Cauchy's theorem to $g(\lambda)=\hat f(\lambda)$, we have $0=-(1/2\pi i)\int_\gamma h(z)(g(z)/z)e^{tz}dz$. Moreover, g_t being entire implies

$$\int_0^t f(s)ds = g_t(0) = \frac{1}{2\pi i}\int_{|z|=R} h(z)g_t(z)e^{tz}\frac{dz}{z}$$

and

$$0=\sum_{j=1}^n \frac{1}{2\pi i}\int_{|z-i\eta_j|=\epsilon_j} h(z)g_t(z)e^{tz}\frac{dz}{z}.$$

Summing up, we obtain

$$\begin{aligned} \int_0^t f(s)ds &= \frac{1}{2\pi i}\int_{\substack{|z|=R\\ \mathrm{Re}\, z>0}} h(z)(g_t(z)-g(z))e^{tz}\frac{dz}{z} \\ &\quad+\sum_{j=1}^n\frac{1}{2\pi i}\int_{\substack{|z-i\eta_j|=\epsilon_j\\ \mathrm{Re}\, z>0}} h(z)(g_t(z)-g(z))e^{tz}\frac{dz}{z} \\ &\quad-\sum_{j=1}^n\frac{1}{2\pi i}\int_{\gamma_j} h(z)g(z)e^{tz}\frac{dz}{z}+\frac{1}{2\pi i}\int_{\substack{|z|=R\\ \mathrm{Re}\, z<0}} h(z)g_t(z)e^{tz}\frac{dz}{z} \\ &\quad+\sum_{j=1}^n\frac{1}{2\pi i}\int_{\substack{\mathrm{Re}\, z<0\\ |z-i\eta_j|=\epsilon_j}} h(z)g_t(z)e^{tz}\frac{dz}{z} \end{aligned}$$

Now the third term converges to zero ($t\to\infty$) by the Riemann–Lebesgue lemma; the other estimates are given in [2, Lemma 3.1]. □

We put Lemma 5.2 in a different form (corresponding to Tauberian theorems of type D) keeping the definition (5.6) throughout this section.

LEMMA 5.3. *Let $\varphi\in L^1_{\mathrm{loc}}([0,\infty),X)\cap C^1([\tau,\infty),X)$ where $\tau\geq 0$. Assume that $\hat\varphi(\lambda)$ has a continuous extension to $K:=\mathbb{C}_+\cup i\,([-R,R]\setminus\bigcup_{j=1}^n(\eta_j-\epsilon_j,\eta_j+\epsilon_j))$ where $\eta_j\in\mathbb{R}$, $\epsilon_j>0$ such that the intervals $(\eta_j-\epsilon_j,\eta_j+\epsilon_j)$ are pairwise disjoint ($j=1\cdots n$) and $0\notin\bigcup_{j=1}^n[\eta_j-\epsilon_j,\eta_j+\epsilon_j]\subset(-R,R)$. Suppose that*

$$N_0:=\sup_{t\geq\tau}|\varphi'(t)|+|\varphi(\tau)|<\infty$$

and

$$N_j:=\sup_{t\geq\tau}\Big|\int_\tau^t e^{-i\eta_j s}\varphi'(s)ds\Big|+|\varphi(\tau)|<\infty, \qquad \textit{for } j=1,\cdots,n.$$

Then,

$$\overline{\lim_{t\to\infty}}|\varphi(t)| \le \frac{2N_0}{R}\prod_{j=1}^{n} a_j + 12\sum_{j=1}^{n} N_j\delta_j \prod_{\substack{k=1\\k\ne j}}^{n} b_{jk}.$$

Proof. (a) We assume that $\tau = 0$. Let $f(t) = \varphi'(t) + \varphi(0)\exp(-t)$. Then $\hat{f}(\lambda)/\lambda = \hat{\varphi}(\lambda) - \varphi(0)/(1+\lambda)$ has a continuous extension to K. Moreover, $|f(t)| \le |\varphi'(t)| + |\varphi(0)| \le N_0$ $(t \ge 0)$ and $|\int_0^t \exp(-i\eta_j s)f(s)ds| = |\int_0^t \exp(-i\eta_j s)\varphi'(s)ds + \varphi(0)\int_0^t \exp(-i\eta_j s)\exp(-s)ds| \le N_j$ $(t \ge 0)$, $j = 1\cdots n$. Since $\int_0^t f(s)ds = \varphi(t) - \varphi(0)\exp(-t)$ one has $\overline{\lim}_{t\to\infty}|\varphi(t)| = \overline{\lim}_{t\to\infty}|\int_0^t f(s)ds|$. So the claim follows from Lemma 5.2.

(b) If $\tau \ge 0$ is arbitrary we apply (a) to $\psi(t) = \varphi(t+\tau)$. □

Proof of Theorem 5.1. Since $S(t)x = x$ on $N(A)$ we can assume that $P = Q = 0$ in the case when $\hat{a}(0) = \infty$. Choose $\nu_0 \in \varrho(A)$ and let $L = (\nu_0 - A)^{-1}$. Let $0 \le \mu_0 \in \mathbb{R}\backslash E$ be fixed. Let $R > \mu_0$ such that $\pm R \notin E$. We set $E_0 = E \cap [\mu_0 - R, \mu_0 + R]$. For every ordinal α, we define inductively subsets E_α of E in the following way. Suppose that E_β has been defined for all $\beta < \alpha$. We let E_α be the set of all cluster points of $E_{\alpha-1}$, if α has a predecessor $\alpha - 1$, and $E_\alpha = \bigcap_{\beta<\alpha} E_\beta$ if not.

For $\mu \in E$, $\mu \ne 0$ we define

$$B(\mu) = \begin{cases} \dfrac{1-\hat{a}(i\mu)A}{C(\mu)} i\mu & \text{if } |\hat{a}(i\mu)| \le 1 \\[2ex] \left(\dfrac{1}{\hat{a}(i\mu)} - A\right) i\mu/C(\mu) & \text{if } |\hat{a}(i\mu)| > 1, \end{cases}$$

where $C(\mu)$ is the constant from hypothesis (H2), and

$$B(0) = \begin{cases} A & \text{if } \hat{a}(0) = \infty \\ 1 - \hat{a}(0)A & \text{if } \hat{a}(0) \in \mathbb{C}. \end{cases}$$

We shall prove the following.

Inductive statement. If $\mu_j \in E_\alpha$, $\epsilon_j > 0$ $(j = 1, \cdots, n)$ such that $(\mu_j - \epsilon_j, \mu_j + \epsilon_j)$ are pairwise disjoint,

$$E_\alpha \subset \bigcup_{j=1}^{n} (\mu_j - \epsilon_j, \mu_j + \epsilon_j) \quad \text{and } \mu_0 \notin \bigcup_{j=1}^{n} [\mu_j - \epsilon_j, \mu_j + \epsilon_j] \subset (\mu_0 - R, \mu_0 + R),$$

then

$$\overline{\lim_{t\to\infty}}|(S(t) - Q)LUx| \le \frac{2N_0|Ux|}{R}\prod_{j=1}^{n} a_j + 12\sum_{j=1}^{n} C_0|U_jx|\delta_j \prod_{\substack{k=1\\k\ne j}}^{n} b_{jk} \tag{5.7}$$

for all $x \in D(A^n)$, where

$$U = \prod_{j=1}^{n} B_j; \quad U_j = \prod_{\substack{k=1\\k\ne j}}^{n} B_k, \quad B_j = B(\mu_j);$$

the constants a_j, δ_j, b_{jk} are given by (5.6), and C_0, N_0 are constants which will be defined below and do not depend on μ_j, ϵ_j.

It is part of the inductive statement that

$$\varlimsup_{t\to\infty} |(S(t)-Q)Lx| \leq \frac{2N_0}{R}|x| \tag{5.8}$$

for all $x \in X$ if $E_\alpha = \emptyset$ (which is (5.7) with the convention that the empty product is 1 and the empty sum zero).

Once the inductive statement has been established, the theorem is proved as follows. Since E_α is compact and countable, E_α is either empty or contains isolated points, so that $E_\alpha = \emptyset$ or $E_{\alpha+1} \neq E_\alpha$. Thus it follows that for some α (at most ω_1), $E_\alpha = \emptyset$. Hence, (5.8) holds. We can choose $0 < R \notin E \cup -E$ arbitrarily large. Thus $\lim_{t\to\infty} |(S(t)-Q)Lx| = 0$ for all $x \in X$. Since $R(L) = D(A)$ is dense in X and $S(t)$ is bounded the claim follows.

It remains to prove the inductive statement.

(1) $\alpha = 0$. Let $\mu_j \in E_0$, $\epsilon_j > 0$ such that

$$E_0 \subset \bigcup_{j=1}^{n} (\mu_j - \epsilon_j, \mu_j + \epsilon_j) \text{ and } \mu_0 \notin \bigcup_{j=1}^{n} [\mu_j - \epsilon_j, \mu_j + \epsilon_j] \subset (\mu_0 - R, \mu_0 + R),$$

according to the statement. Let $y \in X$ and set $\varphi(t) = e^{-i\mu_0 t}(S(t)-Q)Ly$ $(t \geq 0)$. We verify that φ satisfies the hypotheses of Lemma 5.3 (after specification of y).

For $\operatorname{Re}\lambda > 0$ we have

$$\hat{\varphi}(\lambda) = \hat{S}(\lambda + i\mu_0)Ly - \frac{1}{i\mu_0 + \lambda}QLy = \frac{1}{\lambda + i\mu_0}((1 - \hat{a}(\lambda + i\mu_0)A)^{-1} - Q)Ly.$$

Set $\eta_j = \mu_j - \mu_0$ $(j = 1, \cdots, n)$. Then $0 \notin \cup_{j=1}^{n} [\eta_j - \epsilon_j, \eta_j + \epsilon_j] \subset (-R, R)$ and $\hat{\varphi}(\lambda)$ has a continuous extension to $\mathbb{C}_+ \cup i([-R, R] \setminus \bigcup_{j=1}^{n} (\eta_j - \epsilon_j, \eta_j + \epsilon_j))$.

Setting $C_L = |L| + |AL|$ we have $|Ly|_A \leq C_L|y|$. We have

$$\varphi'(t) = -i\mu_0 \exp(-i\mu_0 t)(S(t)-Q)Ly + \exp(-i\mu_0 t)S'(t)Ly.$$

Using (H3), we obtain

$$|\varphi'(t)| + |\varphi(\tau)| \leq N_0|y| \qquad (t \geq \tau) \tag{5.9}$$

with $N_0 = \mu_0(M + |Q|)|L| + MC_L + (M + |Q|)|L|$.

Now let $y = Ux = B_jU_jx$ where $x \in D(A^n)$, and observe that $|\eta_j| = |\mu_j - \mu_0| \leq R$, hence $|\mu_j| \leq R + \mu_0$. Moreover, since $C(\mu) \geq 1$, it follows from the definition of $B(\mu)$ that

$$|B(\mu)L| \leq |\mu|C_L \qquad (0 \neq \mu \in E); \tag{5.10}$$

in particular,

$$|B_jL| \leq (R + \mu_0)C_L \quad \text{if } \mu_j \neq 0. \tag{5.11}$$

Due to (1.1), hypothesis (H2) implies

$$\left| \int_0^t e^{-i\mu s} S(s)(1 - \hat{a}(i\mu)A)x\, ds \right| \leq C(\mu)|x|_A \qquad (x \in D(A))$$

if $\mu \in E\backslash\{0\}$ and $\hat{a}(i\mu) \in \mathbb{C}$. Consequently, it follows from (H2) that

$$\left|\int_0^t e^{-i\mu s} S(s)B(\mu)y\,ds\right| \le |\mu||y|_A \quad (y \in D(A), \quad \mu \in E\backslash\{0\}). \tag{5.12}$$

We estimate $\int_\tau^t e^{-i\mu_j s}\varphi'(s)ds$.

Case 1. $\mu_j \neq 0$, i.e., $\eta_j = \mu_j - \mu_0 \neq -\mu_0$.

$$\begin{aligned}
&\int_\tau^t \exp(-i\eta_j s)\varphi'(s)ds\\
&\quad = -i\mu_0 \int_\tau^t \exp(-i\mu_j s)(S(s)-Q)Ly\,ds + \int_\tau^t \exp(-i\mu_j s)S'(s)Ly\,ds\\
&\quad = -i\mu_0 \int_\tau^t \exp(-i\mu_j s)(S(s)-Q)Ly\,ds + \exp(-i\mu_j t)S(t)Ly\\
&\qquad\qquad - \exp(-i\mu_j \tau)S(\tau)Ly + i\mu_j \int_\tau^t \exp(-i\mu_j s)S(s)Ly\,ds\\
&\quad = i(\mu_j - \mu_0)\int_0^t \exp(-i\mu_j s)S(s)Ly\,ds + \frac{\mu_0}{\mu_j}(\exp(-i\mu_j\tau) - \exp(-i\mu_j t))QLy\\
&\qquad\qquad + \exp(-i\mu_0 t)S(t)Ly - \exp(-i\mu_j\tau)S(\tau)Ly\\
&\qquad\qquad -i(\mu_j - \mu_0)\int_0^\tau \exp(-i\mu_j s)S(s)Ly\,ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|\int_\tau^t \exp(-i\eta_j s)\varphi'(s)ds\right|\\
&\quad \le R\left|\int_0^t \exp(-i\mu_j s)S(s)B_j LU_j x\,ds\right|\\
&\qquad\qquad +2\frac{\mu_0}{|\mu_j|}|Q||B_j LU_j x| + 2M|B_j LU_j x| + R\tau M|B_j LU_j x|\\
&\quad \le R|\mu_j|.|LU_j x|_A + 2\mu_0|Q|C_L|U_j x| + 2M(R+\mu_0)C_L|U_j x|\\
&\qquad\qquad +R\tau M(R+\mu_0)C_L|U_j x|,
\end{aligned}$$

by (5.12), (5.10), and (5.11). Setting

$$C_1 := R(R+\mu_0)C_L + 2\mu_0|Q|C_L + 2M(R+\mu_0)C_L + R\tau M(R+\mu_0)C_L,$$

we obtain

$$\left|\int_\tau^t \exp(-i\eta_j s)\varphi'(s)ds\right| \le C_1|U_j x|. \tag{5.13}$$

Case 2. $\mu_j = 0$; that is, $\eta_j = -\mu_0$. Then,

$$\begin{aligned}
\int_\tau^t e^{-i\eta_j s}\varphi'(s)ds &= -i\mu_0\int_\tau^t (S(s)-Q)Ly\,ds + \int_\tau^t S'(s)Ly\,ds\\
&= -i\mu_0\int_0^t (S(s)-Q)y\,ds + S(t)Ly - S(\tau)Ly + i\mu_0\int_0^\tau (S(s)-Q)Ly\,ds.
\end{aligned}$$

We must distinguish two cases. (a) If $\hat{a}(0) = \infty$, then $Q = 0$ and $B_j = A$, $y = AU_jx$. Then,

$$\left|\int_\tau^t \exp(-i\eta_j s)\varphi'(s)ds\right| \le \mu_0 \left|\int_0^t S(s)ALU_jx\, ds\right| + 2M|ALU_jx| + \mu_0\tau M|ALU_jx|$$
$$\le \mu_0 C(0)|LU_jx|_A + 2MC_L|U_jx| + \mu_0\tau MC_L|U_jx|$$

by (H2). Hence, $|\int_\tau^t \exp(-i\eta_j s)\varphi'(s)ds| \le C_2|U_jx|$ if we set $C_2 := (\mu_0 C(0)C_L + 2MC_L + \mu_0\tau MC_L)$ if $0 \in E$ and $\hat{a}(0) = \infty$.

(b) If $\hat{a}(0) \in \mathbb{C}$, then $Q = (I - \hat{a}(0)A)^{-1}$, $B_j = Q^{-1}$, $y = Q^{-1}U_jx$ and so $(S(s) - Q)Ly = (S(s)Q^{-1} - I)LU_jx = (S(s)(I - \hat{a}(0)A) - I)LU_jx = S(s)LU_jx - LU_jx - \hat{a}(0)S(s)ALU_jx = A(a * S)(s)LU_jx - \hat{a}(0)S(s)ALU_jx$ by (4.1). Thus,

$$\left|\int_\tau^t \exp(-i\eta_j s)\varphi'(s)ds\right| \le \mu_0 \left|\int_0^t ((a*S)(s) - \hat{a}(0)S(s))ALU_jx\, ds\right| + 2M|Q^{-1}LU_jx|$$
$$+\mu_0\tau(M + |Q|)|Q^{-1}LU_jx|$$

$$\le \mu_0 C(0)|LU_jx|_A + 2M|(I - \hat{a}(0)A)L||U_jx| + \mu_0\tau(M + |Q|)|(I - \hat{a}(0)A)L||U_jx|$$

by (H2). Hence, $|\int_\tau^t \exp(-i\eta_j s)\varphi'(s)ds| \le C_2|U_jx|$ $(t \ge \tau)$ if we set

$$C_2 = \mu_0 C(0)C_L + 2M|(I - \hat{a}(0)A)L| + \mu_0\tau(M + |Q|) + \mu_0\tau(M + |Q|)|(I - \hat{a}(0)A)L|$$

in the case $0 \in E$, $\hat{a}(0) \in \mathbb{C}$. So far, we have proved that

$$\left|\int_\tau^t \varphi'(s)\exp(-i\eta_j s)ds\right| \le C_3|U_jx|, \quad j = 1, \cdots, n, \quad t \ge \tau$$

if we put $C_2 := 0$ in the case where $0 \notin E$ and $C_3 = \max\{C_1, C_2\}$ (see (5.13)). Finally, we let

$$C_4 = \begin{cases} MC_L & \text{if } 0 \in E, \hat{a}(0) = \infty \\ (M + |Q|)(I - \hat{a}(0)A)L| & \text{if } 0 \in E, \hat{a}(0) \in \mathbb{C} \\ 0 & \text{if } 0 \notin E \end{cases}$$
$$C_5 = (M + |Q|)(R + \mu_0)C_L$$
$$C_6 = \max\{C_4, C_5\}.$$

Then $|\varphi(\tau)| \le C_6|U_jx|$ $(j = 1 \cdots n)$. In fact, if $\mu_j \ne 0$, then

$$|\varphi(\tau)| \le (M + |Q|)|B_jLU_jx| \le C_5|U_jx|;$$

if $\mu_j = 0$ and $\hat{a}(0) = \infty$, then $Q = 0$, $B_j = A$, $y = AU_jx$, and so $|\varphi(\tau)| \le M|LAU_jx| \le MC_L|U_jx| = C_4|U_jx|$; if $\mu_j = 0$ and $\hat{a}(0) \in \mathbb{C}$, then $Q = (I - \hat{a}(0)A)^{-1}$, $y = Q^{-1}U_jx$ and so

$$|\varphi(\tau)| \le (M + |Q|)|(I - \hat{a}(0)A)L||U_jx| = C_4|U_jx|.$$

Letting $C_0 := \max\{C_3, C_6\}$, we finally have

$$\left|\int_\tau^t \varphi'(s)\exp(-i\eta_j s)ds\right| + |\varphi(\tau)| \le C_0|U_jx|$$

$(j = 1, \cdots, n)$. In view of (5.9), now the claim (5.7) follows from Lemma 5.3. This proves the inductive statement for $\alpha = 0$.

(2) Let α be an ordinal greater than zero and assume that the inductive statement holds for all ordinals $\beta < \alpha$. We show the statement to hold for α. Let $(\mu_j - \epsilon_j, \mu_j + \epsilon_j)$ $(j = 1, \cdots, n)$ be disjoint intervals such that $\mu_0 \notin \cup_{j=1}^{n} [\mu_j - \epsilon_j, \mu_j + \epsilon_j] \subset (\mu_0 - R, \mu_0 + R)$ and $E_\alpha \subset \Omega := \cup_{j=1}^{n} (\mu_j - \epsilon_j, \mu_j + \epsilon_j)$.

Case 1. $\alpha - 1$ does not exist. Then $E_\alpha = \cap_{\beta<\alpha} E_\beta$. Since Ω is open and E_0 compact, it follows that $E_\beta \subset \Omega$ for some $\beta < \alpha$. So (5.7) follows trivially from the inductive hypothesis.

Case 2. $\alpha - 1$ exists. Since E_α is the set of all accumulation points of $E_{\alpha-1}$, $E_{\alpha-1} \backslash E_\alpha$ is finite, say $E_{\alpha-1} \backslash E_\alpha = \{\mu_{n+1}, \cdots, \mu_{n+p}\}$. Let $\epsilon_j > 0$, $j = n+1, \cdots, n+p$ be small enough so that $\mu_0 \notin \cup_{j=1}^{n+p} [\mu_j - \epsilon_j, \mu_j + \epsilon_j] \subset (\mu_0 - R, \mu_0 + R)$. Since $E_{\alpha-1} \subset \cup_{j=1}^{n+p} (\mu_j - \epsilon_j, \mu_j + \epsilon_j)$, we conclude from the inductive hypothesis for $\alpha - 1$ that

$$\overline{\lim_{t\to\infty}} |S(t) - Q)LVy| \le \frac{2N_0|Vy|}{R} \prod_{j=1}^{n+p} a_j + 12 \sum_{j=1}^{n+p} C_0 |V_j y| \delta_j \prod_{\substack{k=1 \\ k\neq j}}^{n+p} b_{jk}$$

for all $y \in D(A^{n+p})$, where

$$V = \prod_{j=1}^{n+p} B_j, \quad V_j = \prod_{\substack{k=1 \\ k\neq j}}^{n+p} B_k, \quad B_j = B(\mu_j) \quad (j = 1 \cdots n+p).$$

Letting $\epsilon_j \downarrow 0$ for $j = n+1, \cdots, n+p$, we obtain

$$\overline{\lim_{t\to\infty}} |(S(t) - Q)LVy| \le 2(N_0/R)|Vy| \prod_{j=1}^{n} a_j + 12 \sum_{j=1}^{n} C_0 |V_j y| \delta_j \prod_{\substack{k=1 \\ k\neq j}}^{n} b_{jk}. \tag{5.14}$$

Letting $W = \prod_{j=n+1}^{n+p} B_j$, $U = \prod_{j=1}^{n} B_j$, $U_j = \prod_{\substack{k=1 \\ k\neq j}}^{n} B_k$, we can rewrite (5.14) as

$$\begin{aligned} \overline{\lim_{t\to\infty}} |(S(t) - Q)LUWy| &\le 2(N_0/R)|UWy| \prod_{j=1}^{n} a_j \\ &\quad + 12 \sum_{j=1}^{n} C_0 |U_j W y| \delta_j \prod_{\substack{k=1 \\ k\neq j}}^{n} b_{jk} \qquad (y \in D(A^{n+p})). \end{aligned} \tag{5.15}$$

Now the operators $B_j (j = n+1, \cdots, n+p)$ commute and have dense range by (H1). This implies that $WD(A^{n+p})$ is dense in $(D(A^n); |\ |_{A^n})$, where $|x|_{A^n} := |x| + |Ax| + \cdots + |A^n x|$ for $x \in D(A^n)$. Thus, given $x \in D(A^n)$ we find $y_m \in D(A^{n+p})$ such that $\lim_{m\to\infty} |Wy_m - x|_{A^n} = 0$, hence $\lim_{m\to\infty} UWy_m = Ux$ in $(D(A), |\ |_A)$. Setting $y = y_m$ in (5.15), we obtain (5.7) by letting $m \to \infty$. This completes the proof of Theorem 5.1. □

6. Some examples and illustrations. In this section we want to discuss several examples of kernels $a(t)$ and operators A to which the General Convergence Theorem applies and also to present conditions on the kernel $a(t)$ such that assumptions (H2) and (H3) of Theorem 6.1 are satisfied.

We begin with the semigroup case $a(t) \equiv 1$, $t \geq 0$. Then the resolvent $S(t)$ satisfying

$$S(t) = I + A\int_0^t a(\tau)S(t-\tau)d\tau, \qquad t \geq 0, \tag{6.1}$$

is the semigroup generated by A, i.e., $S(t) = e^{At}$. Therefore the relation $S'(t)x = S(t)Ax$ shows that (H3) is trivially satisfied whenever the semigroup is bounded. To verify (H2), observe that $\hat{a}(\lambda) = 1/\lambda$; hence, $\hat{a}(0) = \infty$ and $\hat{a}(i\mu) \in \mathbb{C}$ otherwise. For $\mu = 0$ we obtain

$$\int_0^t S(\tau)Ax\,d\tau = \int_0^t S'(\tau)x\,d\tau = S(t)x - x, \qquad t > 0,$$

and so (H2) is valid for $\mu = 0$. If $\mu \neq 0$ we get, via an integration by parts,

$$\begin{aligned}\int_0^t e^{-i\mu\tau}((a*S)(\tau) - \hat{a}(i\mu)S(\tau))Ax\,d\tau &= \int_0^t e^{-i\mu\tau}\left(\int_0^\tau S(s)ds - \frac{1}{i\mu}S(\tau)\right)Ax\,d\tau \\ &= \frac{1}{i\mu}e^{-i\mu t}(x - S(t)x);\end{aligned}$$

hence, (H2) is valid for all $\mu \in \mathbb{R}$. Since $E = \sigma(A) \cap i\mathbb{R}$, (H1) becomes $(\sigma_p(A) \cup \sigma_p(A')) \cap i\mathbb{R} \subset \{0\}$ and $N(A)^\perp \cap N(A') = \{0\}$. Thus, the General Convergence Theorem reduces for the case $a(t) \equiv 1$ to the following version of the stability theorem of Arendt and Batty [2], and Lyubich and Phong [28].

COROLLARY 6.1. *Suppose A generates a bounded C_0-semigroup in X, let $\sigma(A) \cap i\mathbb{R}$ be countable, $\sigma_p(A') \cap i\mathbb{R} \subset \{0\}$, and assume $N(A)^\perp \cap N(A') = \{0\}$. Then $\lim_{t\to\infty} S(t)x = Px$ for each $x \in X$, where P denotes the projection onto $N(A)$ along $\overline{R(A)}$.*

Next we show that condition (H2) for $\mu \neq 0$ is satisfied for a large class of kernels, provided the resolvent $S(t)$ is known to be bounded. We denote by $BV(\mathbb{R}_+)$ the space of all functions $a : \mathbb{R} \to \mathbb{R}$ of bounded variation, which are left-continuous and such that $a(t) = 0$ for $t \leq 0$.

PROPOSITION 6.2. *Suppose that the resolvent $S(t)$ of* (1.1) *is bounded, let $a(t)$ be of the form*

$$a(t) = \sum_{k=0}^n a_k(t), \qquad t > 0, \tag{6.2}$$

where

$$a_0, ta_0 \in L^1(\mathbb{R}_+) \quad \text{and} \quad \text{for } k = 1, \cdots, n,$$

$$a_k \in W^{k-1,1}_{\text{loc}}(\mathbb{R}_+), \quad a_k^{(k-1)} \in BV(\mathbb{R}_+), \quad \int_0^\infty t|da_k^{(k-1)}(t)| < \infty. \tag{6.3}$$

Then $\varrho(a) \supset i\mathbb{R}\backslash\{0\}$, $\hat{a}(i\mu) \in \mathbb{C}$ for all $\mu \in \mathbb{R}\backslash\{0\}$, and for each $\mu \in \mathbb{R}\backslash\{0\}$ there is constant $c(\mu)$, such that

$$\left|\int_0^t e^{-i\mu s}[(a*S)(s) - \hat{a}(i\mu)S(s)]Ax\,dx\right| \leq c(\mu)|x|_A, \qquad x \in D(A). \tag{6.4}$$

If $n = 0$ the assertions also hold for $\mu = 0$.

Proof. Adding suitable constants to the functions of bounded variation $b_k(t) = a_k^{(k-1)}(t)$, we may assume $a_k^{(i)}(0) = 0$ for all $0 \leq i \leq k-2 \leq n-2$. Let $b_0(t) = \int_0^t a_0(\tau)d\tau$. The familiar formula

$$\widehat{db}_k(\lambda) = (da_k^{(k-1)})^\wedge(\lambda) = \lambda^k \hat{a}_k(\lambda), \quad \operatorname{Re}\lambda > 0, \quad k = 1, \cdots, n,$$

by (6.2) yields the representation

$$\hat{a}(\lambda) = \sum_{k=0}^{n} \widehat{db}_k(\lambda)\lambda^{-k}, \qquad \operatorname{Re}\lambda > 0. \tag{6.5}$$

Since $b_k \in BV(\mathbb{R}_+)$, $k = 0, \cdots, n$, (6.5) shows that $\hat{a}(\lambda)$ admits a continuous extension at least to $\overline{\mathbb{C}}_+\backslash\{0\}$; hence, we obtain $\varrho(a) \supset i\mathbb{R}\backslash\{0\}$ and $\hat{a}(\lambda) \in \mathbb{C}$ for all $\lambda \in \overline{\mathbb{C}}_+\backslash\{0\}$. Integrating by parts k times leads to

$$\int_0^t e^{-i\mu\tau}(a_k * S)(\tau)d\tau = (i\mu)^{-k}\int_0^t e^{-i\mu\tau}(db_k * S)(\tau)d\tau - e^{-i\mu t}\sum_{j=0}^{k-1}(a_k^{(j)} * S)(t)(i\mu)^{-j-1};$$

hence, summation over k gives

$$\begin{aligned}
&\int_0^t e^{-i\mu\tau}[(a * S)(\tau) - \hat{a}(i\mu)S(\tau)]Ax\,d\tau \\
&= \sum_{k=0}^{n}(i\mu)^{-k}\int_0^t e^{-i\mu\tau}[(db_k * S)(\tau) - \widehat{db}_k(i\mu)S(\tau))]Ax\,d\tau \\
&\qquad - e^{-i\mu t}\sum_{k=1}^{n}\sum_{j=0}^{k-1}(a_k^{(j)} * S)(t)(i\mu)^{-j-1}Ax \\
&= \sum_{k=0}^{n}(i\mu)^{-k}(T_k(t) - e^{-i\mu t}R_k(t))Ax,
\end{aligned} \tag{6.6}$$

where

$$T_k(t) = \int_0^t e^{-i\mu\tau}[(db_k * S)(\tau) - \widehat{db}_k(i\mu)S(\tau)]d\tau \tag{6.7}$$

and

$$R_k(t) = \sum_{j=k}^{n}(a_j^{(k-1)} * S)(t), \qquad R_0(t) = 0. \tag{6.8}$$

To estimate $T_k(t)$, we write

$$\begin{aligned}
T_k(t) &= \int_0^t e^{-i\mu\tau}\int_0^\tau db_k(\tau - s)S(s)d\tau - \int_0^t e^{-i\mu\tau}\widehat{db}_k(i\mu)S(\tau)d\tau \\
&= \int_0^t S(s)e^{-i\mu s}\left(\int_s^t db_k(\tau - s)e^{-i\mu(\tau-s)} - \widehat{db}_k(i\mu)\right)ds, \\
&= -\int_0^t S(s)e^{-i\mu s}\left(\int_{t-s}^\infty db_k(\tau)e^{-i\mu\tau}\right)ds.
\end{aligned}$$

Hence,

$$|T_k(t)| \le M \int_0^t \int_{t-s}^\infty |db_k(\tau)| ds = M\left(\int_t^\infty \int_0^t ds |db_k(\tau)| + \int_0^t \int_{t-\tau}^t ds |db_k(\tau)| \right)$$
$$= M\left(t \int_t^\infty |db_k(\tau)| + \int_0^t \tau |db_k(\tau)| \right) \le M \int_0^\infty \tau |db_k(\tau)| = M_k < \infty,$$

where $M = \sup_{\tau \ge 0} |S(\tau)|$. To derive a bound on the $R_k(t)$, we expand $(a_k * S)(t)$ into a Taylor series up to order k,

$$(a_k * S)(t+h) - (a_k * S)(t) = \sum_{j=0}^{k-1} (a_k^{(j)} * S)(t) \frac{h^j}{j!} + \int_t^{t+h} (db_k * S)(\tau) \frac{(t+h-\tau)^{k-1}}{(k-1)!} d\tau.$$

Summing over k, we obtain with (6.1)

$$S(t+h)x - S(t)x = \sum_{k=1}^n \sum_{j=1}^k (a_k^{(j-1)} * SAx)(t) \frac{h^{j-1}}{(j-1)!}$$
$$+ \sum_{k=0}^n \int_t^{t+h} (db_k * SAx)(\tau) \frac{(t+h-\tau)^{k-1}}{(k-1)!} d\tau.$$

Since $S(t)$ is bounded and $b_k \in BV(\mathbb{R}_+)$ the polynomials

$$P_t(h)x = \sum_{j=1}^n \frac{h^{j-1}}{(j-1)!} \left(\sum_{k=j}^n a_k^{(j-1)} * SAx \right) = \sum_{j=1}^n R_{j-1}(t)Ax \frac{h^{j-1}}{(j-1)!}$$

are bounded, uniformly for $0 \le h \le 1$, $t \ge 0$; but this implies the existence of a constant $C > 0$ such that

$$|R_k(t)Ax| \le C|x|_A, \quad x \in D(A), \quad k = 1, \cdots, n. \tag{6.9}$$

The proof is now complete. □

A special case of Proposition 6.2 will be used in §7, namely, the following.

COROLLARY 6.3. *Suppose that the resolvent $S(t)$ for (1.1) is bounded; let $a(t)$ be of the form*

$$a(t) = b_0 + b_\infty t + \int_0^t b_1(s) ds, \qquad t > 0, \tag{6.10}$$

where $b_0, b_\infty \ge 0$ are constants and $b_1 \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is nonnegative, nonincreasing, and convex. Then $\varrho(a) \supset i\mathbb{R}\backslash\{0\}$, $\hat{a}(i\mu) \in \mathbb{C}$ for all $\mu \in \mathbb{R}\backslash\{0\}$ and (6.4) *holds for each $\mu \in \mathbb{R}\backslash\{0\}$.*

Proof. We may assume $\lim_{t\to\infty} b_1(t) = 0$, changing b_∞ otherwise. Let $t_0 > 0$ and

$$c_1(t) = \begin{cases} b_1(t) - b_1(t_0) & \text{for } t \le t_0, \\ 0 & \text{for } t \ge t_0, \end{cases}$$
$$c_3(t) = \begin{cases} 0 & \text{for } t \le t_0, \\ b_1(t_0) - b_1(t) & \text{for } t > t_0, \end{cases}$$

and define $a_0(t) = 0$,

$$a_1(t) = b_0 + \int_0^t c_1(\tau)d\tau, \qquad t > 0,$$

$$a_2(t) = (b_\infty + b_1(t_0))t, \qquad t > 0,$$

$$a_3(t) = -\int_0^t c_3(\tau)d\tau, \qquad t > 0.$$

Obviously, $a(t) = a_1(t) + a_2(t) + a_3(t)$, $a_1 \in BV(\mathbb{R}_+)$ and $da_1 = b_0\delta + c_1(t)dt$ has all moments since its support is compact; $a_2 \in W^{1,1}_{\text{loc}}(\mathbb{R}_+)$, $\dot{a}_2 = b_\infty + b_1(t_0) \in BV(\mathbb{R}_+)$ and $d\dot{a}_2 = (b_\infty + b_1(t_0))\delta$ also has all moments. a_3 belongs to $W^{2,1}_{\text{loc}}(\mathbb{R}_+)$ since b_1 is nonincreasing and convex, and $\ddot{a}_3(t) = -\dot{c}_3(t)$ for $t > t_0$, $\ddot{a}_3(t) = 0$, for $t < t_0$; moreover, by convexity, $-\dot{c}_3(t)$ is nonincreasing for $t > t_0$ and nonnegative, hence $\ddot{a}_3 \in BV(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ and in particular $d\ddot{a}_3$ admits a finite first moment, since $\ddot{a}_3(t)$ is nondecreasing, as integration by parts shows. □

The argument at the end of the proof of Proposition 6.2 also yields (H3), i.e., boundness of $S'(t)x$, whenever $S(t)$ is bounded and $a(t)$ is of the form (6.2), (6.3) with $a_0 = 0$. More precisely, we have the following.

PROPOSITION 6.4. *Suppose the resolvent $S(t)$ for (1.1) is bounded; let $a(t)$ be of the form*

$$a(t) = \sum_{k=1}^{n} a_k(t), \qquad t > 0, \tag{6.11}$$

where

$$a_k \in W^{k-1,1}_{\text{loc}}(\mathbb{R}_+), \quad a_k^{(k-1)} \in BV(\mathbb{R}_+), \quad k = 1, \cdots, n. \tag{6.12}$$

Then there is a constant $C > 0$, such that

$$|S'(t)x| \leq C|x|_A \quad \textit{for all } x \in D(A), \quad t > 0. \tag{6.13}$$

Proof. Equation (6.1) yields for $x \in D(A)$

$$S'(t)x = \sum_{k=2}^{n} (\dot{a}_k * SAx)(t) + (da_1 * SAx)(t) = R_2(t)Ax + (da_1 * SAx)(t), \qquad t > 0,$$

where $R_2(t)$ is given by (6.8). Since $a_1 \in BV(\mathbb{R}_+)$, estimate (6.9) yields the assertion. Observe that for the proof of (6.9) no moment condition was used. □

For the applications in §7 we shall need the following special case of Proposition 6.4.

COROLLARY 6.5. *Suppose the resolvent $S(t)$ for (6.1) is bounded; let $a(t)$ be of the form*

$$a(t) = b_0 + b_\infty t + \int_0^t b_1(\tau)d\tau, \qquad t > 0, \tag{6.14}$$

where $b_0, b_\infty \geq 0$ and $b_1 \in L^1_{\text{loc}}(\mathbb{R}_+)$ is nonnegative and nonincreasing. Then there is a constant $C > 0$ such that

$$|S'(t)x| \leq C|x|_A \quad \textit{for all } x \in D(A), \quad t > 0. \tag{6.15}$$

Proof. We may assume $\lim_{t\to\infty} b_1(t) = 0$. Define

$$a_1(t) = b_0 + \int_0^t c_1(\tau)d\tau, \qquad a_2(t) = (b_\infty + b_1(t_0))t - \int_0^t c_3(\tau)d\tau,$$

where $c_1(t)$ and $c_3(t)$ are defined as in the proof of Corollary 6.3. Then $a_1 \in BV(\mathbb{R}_+)$ and $\dot a_2 = b_\infty + b_1(t_0) - c_3(t) \in BV(\mathbb{R}_+)$; hence Proposition 6.4 applies and yields (6.15). □

Observe that in Proposition 6.4 we have to assume that the non-BV part of $a_0(t)$ of $a(t)$ in decomposition (6.2) is absent. This clearly restricts its applicability; however, in case $a_0 \neq 0$, Estimate (6.13) cannot be expected. In general, $S(t)x$ need not be differentiable at all. In this case, we must use the structure of $a_0(t)$ and A directly to obtain a bound on $S'(t)$.

The verification of (H2) for $\mu = 0$ is more difficult. If $n = 0$ in Proposition 6.2 then $\varrho(a) = i\mathbb{R}$, $\hat a(i\mu) \in \mathbb{C}$ for all $\mu \in \mathbb{R}$ and (6.4) remains valid for $\mu = 0$ as the proof given there shows (in fact, no integration by parts is needed). On the other hand, if $n \geq 1$ then generically $\hat a(0) = \infty$ as (6.5) shows (only one of the $\widehat{db_k}(0) = b_k(\infty)$, $k = 1, \cdots, n$ must be nonzero for $\hat a(0) = \infty$); then we have to prove that

$$U(t)Ax = \int_0^t S(\tau)Ax\, d\tau, \qquad t > 0, \tag{6.16}$$

is bounded by the graph norm $|x|_A$ of x. Since by (6.1) we obtain the relations

$$\hat U(\lambda)Ax = \frac{1}{\lambda \hat a(\lambda)}((S - I)x)^\wedge(\lambda) = \frac{1}{\lambda^2 \hat a(\lambda)}(\dot S)^\wedge(\lambda)x$$

for the Laplace transform of $U(t)Ax$, we see that $U(t)Ax$ will be bounded if there is $k \in BV(\mathbb{R}_+)$, such that $\widehat{dk}(\lambda) = (\lambda\hat a(\lambda))^{-1}$, or if there is $\ell \in BV(\mathbb{R}_+)$ such that $\widehat{d\ell}(\lambda) = (\lambda^2 \hat a(\lambda))^{-1}$ and $S'(t)x$ is bounded. It should be clear that more information on the kernel $a(t)$ must be available in order to achieve this, rather than just an expansion of the form (6.2) and (6.3). In §7 it will be shown how this can be done. Let us summarize.

PROPOSITION 6.6. *Suppose that the resolvent $S(t)$ for* (1.1) *is bounded, and assume either of the following.* (a) *There is $k \in BV(\mathbb{R}_+)$ such that $(\lambda\hat a(\lambda))^{-1} = \widehat{dk}(\lambda)$, $\lambda > 0$, i.e.,*

$$(k * a)(t) \equiv t, \qquad t > 0.$$

(b) *There is $\ell \in BV(\mathbb{R}_+)$ such that $(\lambda^2\hat a(\lambda))^{-1} = \widehat{d\ell}(\lambda)$, $\lambda > 0$, i.e.,*

$$(\ell * a)(t) = t^2/2, \qquad t > 0,$$

and, in addition, suppose that (6.13) *holds.*

Then $0 \in \varrho(a)$, $\hat a(0) = \infty$, and there is a constant $C > 0$ such that

$$\left|\int_0^t S(\tau)Ax\, d\tau\right| \leq C|x|_A \quad \text{for all } x \in D(A), \quad t \geq 0. \tag{6.17}$$

Consider now the cosine case, i.e., $a(t) \equiv t$ and A generating a bounded strongly continuous cosine family $C(t)$. Then we have $S(t) = C(t)$, $t \geq 0$, $a(t)$ is of the form (6.2), (6.3) and also of the form (6.11), (6.12). Since $\lambda^2\hat{a}(\lambda) = 1 = \widehat{d\ell}(\lambda)$ with $\ell(\lambda) = 1$ for $t > 0$, Propositions 6.2, 6.4, and 6.6 imply that (H2) and (H3) of the General Convergence Theorem are satisfied. Since $\sigma(A) \subset (-\infty, 0]$ (H1) becomes $\sigma(A)$ countable and $\sigma_p(A') \subset \{0\}$, $N(A)^\perp \cap N(A') = \{0\}$. Thus we have the following.

COROLLARY 6.7. *Suppose A generates a bounded, strongly continuous cosine family $C(t)$ in X, assume $\sigma(A)$ is at most countable, $\sigma_p(A') \subset \{0\}$ and $N(A)^\perp \cap N(A') = \{0\}$. Then $\lim_{t\to\infty} C(t)x = Px$ for all $x \in X$, where P denotes the projection onto $N(A)$ along $R(A)$.*

We conclude this section with an example which is such that none of the results of this section can be applied, although (H1), (H2), and (H3) hold, and so the General Convergence Theorem can still be used.

Example 6.8. Let X be a Hilbert space, A a dissipative operator in X such that $\varrho(A) \supset i\mathbb{R}$, and let $a(t) = \cos(t)$, $t > 0$. We claim that the resolvent $S(t)$ of (1.1) satisfies

$$\lim_{t\to\infty} S(t)x = x \quad \text{for all } x \in X. \tag{6.18}$$

To prove this we will apply the General Convergence Theorem of §5. Observe first that $\hat{a}(\lambda) = \lambda(\lambda^2+1)^{-1}$; hence $a(t)$ is not of the form (6.2), (6.3) in view of the poles $\lambda = \pm i$ of $\hat{a}(\lambda)$. For the Laplace transform of $S(t)$, we obtain

$$\lambda\hat{S}(\lambda) = (\lambda + 1/\lambda)(\lambda + 1/\lambda - A)^{-1}, \quad \operatorname{Re}\lambda \geq 0, \quad \lambda \neq 0, \tag{6.19}$$

which exists on $\overline{\mathbb{C}}_+\backslash\{0\}$, since A is dissipative and $\varrho(A) \supset i\mathbb{R}$, and the function $\varphi(\lambda) = \lambda + 1/\lambda$ maps $\overline{\mathbb{C}}_+\backslash\{0\}$ onto $\bar{\mathbb{C}}_+$. Furthermore, (6.19) yields

$$\lim_{\lambda\to 0+} \lambda\hat{S}(\lambda)x = \lim_{r\to\infty} r(r-A)^{-1}x = x \quad \text{for all } x \in X. \tag{6.20}$$

The set of singularities E of (a, A) consists only of the point zero and so we only have to prove that $S(t)$, $S'(t)A^{-1}$, and $V(t) = 1 * (a * S)(t)$ are bounded (existence of $S(t)$ follows, e.g., from the paper of Grimmer and Prüss [18] since $a(0+) > 0$ and $a(t)$ is smooth). Let $x \in D(A)$ and put $u(t) = V(t)x$; then it is easy to see that $u(t)$ satisfies

$$u'' = Au' - u + x, \qquad u(0) = u'(0) = 0. \tag{6.21}$$

Take the inner product of (6.21) with $u'(t)$ and integrate to the result

$$|u'(t)|^2 + |u(t) - x|^2 \leq |x|^2 + 2\int_0^t (Au'(s), u'(s))ds \leq |x|^2,$$

since A is dissipative and $u(0) = u'(0) = 0$. But this means

$$|(a * S)(t)x|^2 + |V(t)x - x|^2 \leq |x|^2, \qquad t > 0, \tag{6.22}$$

i.e., $V(t)$ and $(a*S)(t)$ are both bounded. Similarly, $u(t) = S(t)x$, $x \in D(A^2)$ satisfies (6.21) with initial values $u(0) = x$ and $u'(0) = Ax$; therefore, the same argument yields

$$|S'(t)x|^2 + |S(t)x - x|^2 \leq |Ax|^2, \qquad t > 0, \tag{6.23}$$

i.e., $S'(t)A^{-1}$ is bounded by 1 and so $S(t) = S'(t)A^{-1} + V(t)$ is bounded as well.

7. Applications to viscoelasticity. Let $\Omega \subset \mathbb{R}^n$ be a domain with compact and smooth boundary $\partial\Omega$ that is occupied by a linear incompressible viscoelastic fluid. Assuming the fluid at rest for $t \geq 0$, its velocity field $u(t,x)$ is governed for $t > 0$ by the following problem

$$(7.1)\quad \begin{cases} u_t(t,x) = \int_0^t \Delta u(t-\tau,x)da(\tau) - \nabla p(t,x) + g(t,x) & \\ (\nabla \circ u)(t,x) = 0 & \text{for } x \in \Omega, \quad t > 0, \\ u(t,x) = 0 & \text{for } x \in \partial\Omega, \quad t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

Here $p(t,x)$ denotes the (also unknown) hydrostatic pressure; $g(t,x)$ a (given) external force field, $u_0(x)$ the (given) initial velocity field (induced by a δ-perturbation at time $t = 0$); $\Delta, \nabla, \nabla\circ$ designate the Laplacian, gradient, divergence with respect to the x-variables, respectively. The *stress relaxation modulus* $da(t)$ of a linear viscoelastic material is of the general form

$$(7.2)\qquad a(t) = a_0 + a_\infty t + \int_0^t a_1(s)ds, \qquad t \geq 0,$$

where $a_0, a_\infty \geq 0$ are constants, and $a_1(t) \geq 0$ is nonincreasing and of positive type, $\lim_{t\to\infty} a_1(t) = 0$; for a viscoelastic fluid we even have $a_\infty = 0$ and $a_1 \in L^1(\mathbb{R}_+)$.

For the derivation of (7.1), the properties of the kernel $da(t)$, and more on the physical background of viscoelasticity, we refer to the monographs of Christensen [9], Renardy, Hrusa, and Nohel [35], and Pipkin [31].

Equation (7.1) can be rewritten as an abstract Volterra equation in a Banach space X of the form (1.1), i.e.,

$$(7.3)\qquad u(t) = \int_0^t a(t-\tau)Au(\tau)d\tau + f(t), \qquad t \geq 0,$$

where A denotes a closed linear operator in X with dense domain $D(A)$ and $f \in C(\mathbb{R}_+, X)$. In fact, we may choose $X = L_0^2(\Omega;\mathbb{R}^n)$, the space of all divergence-free L^2-vector fields, $A = P\Delta$, the Stokes operator with $D(A) = W^{2,2}(\Omega;\mathbb{R}^n) \cap W_0^{1,2}(\Omega;\mathbb{R}^n) \cap X$ (P denotes the Helmholtz projection in $L^2(\Omega;\mathbb{R}^n)$) and $f : \mathbb{R}_+ \to X$ is defined by $f(t) = u_0 + \int_0^t g(s)ds$. It is well known that the Stokes operator is self-adjoint and negative semidefinite, and hence gives rise to a bounded cosine family in X.

For the Helmholtz projection and the properties of the Stokes operator mentioned above, as well as others, we refer to the paper by Giga and Sohr [17], and to the monograph of Temam [39].

Existence of the resolvent in the general case relevant for the theory of viscoelasticity was first obtained in a Hilbert space setting by Carr and Hannsgen [7].

PROPOSITION 7.1. *Let X be a Hilbert space, A self-adjoint and negative semidefinite and let $a(t)$ be of the form* (7.2) *with $a_0, a_\infty \geq 0$, $a_1(t) \geq 0$ nonincreasing and of positive type with $a_1 \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $\lim_{t\to\infty} a_1(t) = 0$. Then* (7.3) *admits a resolvent $S(t)$ such that $|S(t)| \leq 1$ on $\mathbb{R}_+$.*

Actually, Carr and Hannsgen assumed in addition that $a_1(t)$ is convex; however, for existence this is not needed. The proof of Proposition 7.1 relies on the spectral decomposition of self-adjoint operators in Hilbert spaces and estimates on the solutions

$s(t;\mu)$ of the scalar equations

$$s(t) + \mu \int_0^t a(t-\tau)s(\tau)d\tau = 1, \quad t \geq 0, \quad \mu \geq 0. \tag{7.4}$$

A different approach was introduced in Prüss [32].

PROPOSITION 7.2. *Let X be a Banach space, A the generator of a bounded cosine family $C(t)$ in X, $a(t)$ of the form* (7.2) *with $a_0, a_\infty \geq 0$, $a_1 \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, $a_1(t) \geq 0$ nonincreasing and $\log a_1(t)$ convex, $\lim_{t\to\infty} a_1(t) = 0$. Then* (7.3) *is governed by a bounded resolvent $S(t)$.*

The proof of this result is based on the complete monotonicity of the functions $h(\lambda,\tau) = \exp(-\tau/\hat{a}(\lambda)^{1/2})/(\lambda\hat{a}(\lambda)^{1/2})$ with respect to $\lambda > 0$, for each fixed $\tau \geq 0$, on the representation formula

$$\hat{S}(\lambda) = \int_0^\infty C(\tau)h(\lambda,\tau)d\tau, \qquad \lambda > 0, \tag{7.5}$$

and on the generation theorem for resolvents due to Da Prato and Iannelli [13] and Grimmer and Prüss [18].

Here we are interested in the asymptotic behavior of the resolvent $S(t)$. Before we quote some known results, let us introduce the following definition.

DEFINITION 7.3. Suppose (7.3) admits a resolvent $S(t)$.

(i) Equation (7.3) is called *uniformly asymptotically stable* if there is $\varphi \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$ such that $|S(t)| \leq \varphi(t)$ on $\mathbb{R}_+$.

(ii) Equation (7.3) is called *asymptotically stable* if $S(t)x \to 0$ as $t \to \infty$ for each $x \in X$.

Carr and Hannsgen [7] obtained the following result.

THEOREM 7.4. *Let the assumptions of Proposition* 7.1 *be satisfied, and assume in addition that $a_1 \in C^1(0,\infty)$ and that $-\dot{a}_1(t)$ is nonincreasing and convex. If A is invertible and $a(t) \not\equiv a_\infty t$, then* (7.3) *is uniformly asymptotically stable.*

Observe that A must necessarily be invertible if (7.3) is uniformly asymptotically stable. In fact, if $0 \in \sigma(A)$, then $|\mu(\mu - A)^{-1}| \geq 1$ for each $\mu \in \varrho(A)$; on the other hand, $S(\cdot)x \in L^1(\mathbb{R}_+, X)$ for each $x \in X$ implies that $\hat{S}(\lambda) = (1/\lambda)(I - \hat{a}(\lambda)A)^{-1}$ is uniformly bounded for $\operatorname{Re}\lambda \geq 0$, i.e., $M \geq |\hat{S}(\lambda)| \geq 1/|\lambda|$ which is impossible. Also $a(t) \not\equiv a_\infty t$ is necessary for uniform asymptotic stability, since otherwise $S(t) = C(\sqrt{a_\infty}t)$ where $C(t)$ denotes the cosine family generated by A; but cosine families are never integrable.

There is a similar result for the situation of Proposition 7.2; see Prüss [32].

THEOREM 7.5. *Let the assumptions of Proposition* 7.2 *be satisfied. Then* (7.3) *is uniformly asymptotically stable if and only if $a(t) \not\equiv a_\infty t$ and A is invertible.*

In the case $A = P\Delta$, the Stokes operator in $L^2_0(\Omega;\mathbb{R}^n)$, A is invertible if the domain $\Omega \subset \mathbb{R}^n$ is bounded; thus Theorems 7.4 and 7.5 show that viscoelastic fluids with (sufficiently) convex stress relaxation moduli are always uniformly asymptotically stable, unless they are purely elastic, i.e., $da(t) = a_\infty dt$.

However, for unbounded domains Ω, the operator $A = P\Delta$ will in general not be invertible and so (7.3) is not uniformly asymptotically stable. As a consequence of our General Convergence Theorem and the results in §7, in this situation (7.3) will still be asymptotically stable, as the next theorems show.

THEOREM 7.6. *Let the assumption of Proposition* 7.2 *be satisfied, and assume in addition $a(t) \not\equiv a_\infty t$ and $N(A)^\perp \cap N(A') = \{0\}$. Then $\lim_{t\to\infty} S(t)x = Px$ for each $x \in X$, where P denotes the projection onto $N(A)$ along $\overline{R(A)}$.*

Proof. (a) By Proposition 7.2 there is a resolvent $S(t)$ for (7.3) which is bounded on $\mathbb{R}_+$. Since log-convex functions are convex, we see that Corollaries 6.3 and 6.5 apply; hence, $\varrho(a) \supset i\mathbb{R}\backslash\{0\}$ and (H2) holds for $\mu \neq 0$, and also (H3) is satisfied.

(b) We next compute the set E of singularities of $\hat{S}(\lambda)$. Since A generates a bounded cosine family, we have $\sigma(A) \subset (-\infty, 0]$. Convex functions are of positive type; hence, $\hat{a}(\lambda) \notin (-\infty, 0]$ for Re $\lambda > 0$. Therefore, $E \subset \{0\}$ will follow if we show that Im $\hat{a}(i\mu) \neq 0$ for $\mu \in \mathbb{R}$, $\mu \neq 0$. Since $ta_1(t) \leq \int_0^t a_1(\tau)d\tau \to 0$ as $t \to 0$, via an integration by parts, we obtain with $\mu > 0$,

$$-\mu \mathrm{Im}\ \hat{a}(i\mu) = a_0 + \mathrm{Re}\ \hat{a}_1(i\mu) = a_0 + \mu^{-1} \int_0^\infty (-\dot{a}_1(t)) \sin(\mu t) dt \geq 0, \tag{7.6}$$

since $a_1(t)$ is nondecreasing and convex (hence also absolutely continuous on $(0, \infty)$). Equality in (7.6) can only hold in case $a_0 = 0$, and $-\dot{a}_1(t)$ is constant on each of the intervals $(2k\pi\mu^{-1}; 2(k+1)\pi\mu^{-1})$; but this cannot happen since $a_1(t)$ is log-convex by assumption and is nontrivial, for otherwise, $a(t) \equiv a_\infty t$. Thus, $E \subset \{0\}$ holds.

(c) We next show $0 \in \varrho(a)$ and (H2) for $\mu = 0$. This will be done with the help of the following result.

LEMMA 7.7. *Let $a(t)$ satisfy the assumptions of Proposition 7.2 and define $g(\lambda) = \hat{a}(\lambda)^{-1/2}$ for* Re $\lambda \geq 0$. *Then there are k, $\ell \in BV(\mathbb{R}_+)$ such that*

$$\widehat{dk}(\lambda) = \frac{g(\lambda)}{1+g(\lambda)}, \quad \widehat{d\ell}(\lambda) = \widehat{dk}(\lambda)g(\lambda)/\lambda, \quad \mathrm{Re}\ \lambda \geq 0. \tag{7.7}$$

The proof of Lemma 7.7 is based on Bernstein's theorem and the Wiener–Levy theorem; see Prüss [32, pp. 341–342].

Observe that Lemma 7.7 yields $0 \in \varrho(a)$ and $\hat{a}(0) = \infty$, since $\widehat{dk}(\lambda)$ is continuous on $\overline{\mathbb{C}}_+$, $\hat{a}(\lambda) = (1/\widehat{dk}(\lambda) - 1)^2$ and $\lim_{\lambda \to 0+} \hat{a}(\lambda) = \hat{a}(0) = \infty$.

Now let $U(t) = \int_0^t S(\tau)d\tau$; then for $x \in D(A)$, we have

$$(UAx)^\wedge(\lambda) = \lambda^{-1}\hat{S}(\lambda)Ax = \lambda^{-2}(I - \hat{a}(\lambda)A)^{-1}Ax = \lambda^{-2}g(\lambda)^2(g(\lambda)^2 - A)^{-1}Ax$$

and

$$(S - I)^\wedge(\lambda)x = \hat{a}(\lambda)A\lambda^{-1}(I - \hat{a}(\lambda)A)^{-1} = \lambda^{-1}A(g(\lambda)^2 - A)^{-1}x,$$

as well as

$$(S')^\wedge(\lambda)x = \lambda\hat{S}(\lambda)x - x = \hat{a}(\lambda)A(I - \hat{a}(\lambda)A)^{-1}x = A(g(\lambda)^2 - A)^{-1}x.$$

These relations and the identity

$$\lambda^{-2}g(\lambda)^2 = \lambda^{-1}\widehat{d\ell}(\lambda)(1 + \widehat{dk}(\lambda)) + \widehat{d\ell}(\lambda)^2, \qquad \mathrm{Re}\ \lambda \geq 0,$$

yield

$$\hat{U}(\lambda)Ax = \widehat{d\ell}(\lambda)^2(\hat{S}')^\wedge(\lambda)x + \widehat{d\ell}(\lambda)(1 + \widehat{dk}(\lambda))(S - I)^\wedge(\lambda)x;$$

hence,

$$U(t)Ax = (d\ell * d\ell * S')(t)x + (d\ell + d\ell * dk) * (S - I)(t)x.$$

Since the measures $d\ell$ and dk are bounded, we obtain from the boundedness of $S(t)$ and $S'(t)$ on $D(A)$ the desired bound on $U(t)Ax$, i.e., (H2) holds.

(d) Finally, the assumption $N(A)^{\perp} \cap N(A') = \{0\}$ implies $\lambda\hat{S}(\lambda) \to P$ strongly as $\lambda \to 0+$ by Theorem 4.6; hence, the General Convergence Theorem applies and the proof is complete. □

The proof of Theorem 7.6 shows that boundedness of $U(t)Ax$ is the difficult thing to prove. This turns out to be even more difficult in the situation of Proposition 7.1, where the assumptions on $a_1(t)$ are weaker so that, in general, Bernstein's theorem can no longer be employed. We want to discuss this case now in some detail. So suppose that X is a Hilbert space, A negative semidefinite, and let $a(t)$ be of the form (7.2) with $a_0, a_\infty \geq 0$, $a_1 \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ nonnegative, nonincreasing of positive type, and $\lim_{t\to\infty} a_1(t) = 0$; let us exclude the cosine case $a(t) \equiv a_\infty t$ which has already been discussed in §6.

Proposition 7.1 shows the existence and boundedness of the resolvent $S(t)$, Corollary 6.5 yields the boundedness of $S'(t)$ on $D(A)$. That is, (H3) holds, and since $\lim_{\lambda\to 0+} \hat{a}(\lambda) = \hat{a}(0) = \infty$, we obtain $\lim_{\lambda\to 0+} \lambda\hat{S}(\lambda)x = Px$ for all $x \in X$ where P denotes the orthogonal projection onto $N(A)$. By means of the decomposition $a_1(t) = a_2(t) + a_3(t)$, where

$$(7.8) \qquad a_2(t) = (a_1(t) - a_1(t_0))_+, \quad \text{and} \quad a_3(t) = \min(a_1(t), a_1(t_0))$$

for $t > 0$, $a_2(t) = a_3(t) = 0$ for $t \leq 0$ as before, we obtain

$$(7.9) \qquad \hat{a}(\lambda) = a_0/\lambda + a_\infty/\lambda^2 + \hat{a}_2(\lambda)/\lambda + \widehat{da_3}(\lambda)/\lambda^2, \qquad \mathrm{Re}\ \lambda \geq 0,$$

and therefore, $\varrho(a) \supset i\mathbb{R}\backslash\{0\}$, $\hat{a}(i\mu) \in \mathbb{C}$ for all $\mu \in \mathbb{R}$, $\mu \neq 0$.

Since $a_1(t)$ is nonincreasing, it follows that for $\mu \neq 0$

$$\mu^2 \mathrm{Re}\ \hat{a}(i\mu) = -a_\infty + \mu \mathrm{Im}\ \hat{a}_1(i\mu) \leq 0$$

and even strictly if $a_1(t)$ is also continuous on $(0, \infty)$ or in case $a_\infty > 0$. On the other hand, we have for $\mu \neq 0$

$$-\mu \mathrm{Im}\ \hat{a}(i\mu) = a_0 + \mathrm{Re}\ \hat{a}_1(i\mu) \geq 0,$$

since $a_1(t)$ is of positive type and even strictly if $a_0 > 0$. Therefore, we have

$$E_0 = E\backslash\{0\} = \{\mu \in \mathbb{R}\backslash\{0\} : \ \mathrm{Re}\ \hat{a}_1(i\mu) = -a_0, \hat{a}(i\mu)^{-1} \in \sigma(A) \text{ or } \hat{a}(i\mu) = 0\}$$

Thus, the spectral assumption (H1) reduces to

(7.10) $\quad E_0$ is at most countable, and $\mu \in E_0$ implies $\hat{a}(i\mu)^{-1} \notin \sigma_p(A)$.

Observe that $E_0 = \emptyset$ if $a_0 > 0$ or if $\mathrm{Re}\ \hat{a}_1(i\mu) \neq 0$ for all $\mu \neq 0$.

By Proposition 6.2 and Corollary 6.3, it is also not difficult to verify (H2) for $\mu \in E_0$; in fact, either $-\int_0^\infty t\,da_1(t) < \infty$ or a_1 convex will be sufficient; note, however, that the former is equivalent to $a_1 \in L^1(\mathbb{R}_+)$ since $a_1(t) \geq 0$ is nonincreasing.

We turn now to the question whether $0 \in \varrho(a)$ and whether $U(t)Ax = \int_0^t S(\tau)Ax\,d\tau$ is bounded on $D(A)$. The first two cases will be a consequence of Proposition 6.6.

Case 1. $a_\infty > 0$ (a "solid"). This one is easy. In fact, if $a_\infty > 0$, then $g_1(\lambda) = (\lambda^2\hat{a}(\lambda))^{-1}$ is bounded and completely monotonic for $\lambda > 0$, since $a_1(t)$ is nonincreasing. Therefore, by Bernstein's theorem there is a function $\ell \in BV(\mathbb{R}_+)$

such that $g_1(\lambda) = \widehat{d\ell}(\lambda)$ for $t > 0$. Proposition 6.6 then implies $0 \in \varrho(a)$, $\hat{a}(0) = \infty$ and boundedness of $U(t)Ax$ on $D(A)$.

Case 2. $a_\infty = 0$, $a_0 > 0$, $a_1 \in L^1(\mathbb{R}_+)$ ("viscous fluid"). Here we use

$$g_2(\lambda) = \frac{1}{\lambda \hat{a}(\lambda)} = \frac{1}{a_0 + \hat{a}_1(\lambda)} = a_0^{-1}\left(1 - \frac{\hat{a}_1(\lambda)}{a_0 + \hat{a}_1(\lambda)}\right), \qquad \operatorname{Re} \lambda \geq 0.$$

Since $a_1 \in L^1(\mathbb{R}_+)$ is of positive type, $\operatorname{Re} \hat{a}_1(\lambda) \geq 0$ for $\operatorname{Re} \lambda \geq 0$, and so $a_0 + \hat{a}_1(\lambda)$ does not vanish on $\overline{\mathbb{C}}_+$. By the Paley–Wiener theorem there is a function $r \in L^1(\mathbb{R}_+)$ such that

$$g_2(\lambda) = a_0^{-1}(1 - \hat{r}(\lambda)), \qquad \operatorname{Re} \lambda \geq 0,$$

and so assumption (a) of Proposition 6.6 is satisfied; therefore, $0 \in \varrho(a)$, $\hat{a}(0) = \infty$, and $U(t)Ax$ is bounded on $D(A)$.

Case 3. $a_\infty = a_0 = 0$, $a_1 \in L^1(\mathbb{R}_+)$ (a "rigid fluid"). We assume in addition that a_1 is absolutely continuous on $(0, \infty)$ in this case. As before, decompose $a_1(t) = a_2(t) + a_3(t)$, where a_2, a_3 are as in (7.8) and $t_0 > 0$ is small enough for $\alpha = a_3(0+) = a_1(t_0) > 0$. Since

$$S(t)x - x = (a_1 * UAx)(t)$$

and

$$S'(t)x - (a_2 * SAx)(t) = (da_3 * UAx)(t) = \alpha U(t)Ax + (\dot{a}_3 * UAx)(t),$$

we obtain

$$\hat{U}(\lambda)Ax = (\alpha + \hat{a}_1(\lambda) + (\dot{a}_3)^\wedge(\lambda))^{-1}(S(t)x - x + S'(t)x - (a_2 * SAx)(t))^\wedge(\lambda).$$

By boundedness of $S(t)x$, $S'(t)x$, and $S(t)Ax$ on $D(A)$, and since $a_2 \in L^1(\mathbb{R}_+)$, it is sufficient to show that

$$g_3(\lambda) = (\alpha + \hat{a}_1(\lambda) + (\dot{a}_3)^\wedge(\lambda))^{-1} = (\alpha + \hat{b}(\lambda))^{-1} = \alpha^{-1}\left(1 - \frac{\hat{b}(\lambda)}{\alpha + \hat{b}(\lambda)}\right)$$

is the Laplace transform of a bounded measure.

By assumption, $\operatorname{Re} \hat{a}_1(\lambda) \geq 0$ for $\operatorname{Re} \lambda > 0$ and $\hat{a}_1(0) = \int_0^\infty a_1(t)dt > 0$; on the other hand, $|(\dot{a}_3)^\wedge(\lambda)| \leq a_3(0+) = \alpha$ since $\dot{a}_3(t) \leq 0$ on $(0, \infty)$, and equality only holds for $\lambda = 0$. Therefore, $\hat{b}(\lambda) = \hat{a}_1(\lambda) + (\dot{a}_3)^\wedge(\lambda) \neq -\alpha$ for $\operatorname{Re} \lambda \geq 0$, and so by the Paley–Wiener theorem there is a function $r \in L^1(\mathbb{R}_+)$, such that

$$g_3(\lambda) = \alpha^{-1}(1 - \hat{r}(\lambda))) = \widehat{dk}(\lambda), \qquad \operatorname{Re} \lambda \geq 0,$$

where $k(t) = \alpha^{-1}(1 - \int_0^t r(\tau)d\tau)$ belongs to $BV(\mathbb{R}_+)$. Thus, $U(t)Ax$ is bounded on $D(A)$, $0 \in \varrho(a)$, and $\hat{a}(0) = \infty$ follow in this case easily from (7.9).

Case 4. $a_\infty = 0$, $a_1 \notin L^1(\mathbb{R}_+)$. If a_1 is not integrable then we cannot apply the Paley–Wiener theorem directly to show that the functions $g_j(\lambda)$ in Case 2 and Case 3 above are Laplace transforms of bounded measures. However, it is enough to know that for every $\alpha > 0$ there is $r_\alpha \in L^1(\mathbb{R}_+)$ such that

$$\hat{r}_\alpha(\lambda) = \frac{\hat{a}_1(\lambda)}{\alpha + \hat{a}_1(\lambda)}, \qquad \operatorname{Re} \lambda \geq 0,$$

holds. Obviously, this is enough in case $a_0 > 0$; put $\alpha = a_0$ to see this. If $a_0 = 0$, rewrite $g_3(\lambda)$ as

$$g_3(\lambda) = \alpha^{-1}\left(1 - \frac{\hat{r}_\alpha(\lambda) + (\dot{a}_3)^\wedge(\lambda)\hat{dk}_\alpha(\lambda)}{1 + (\dot{a}_3)^\wedge(\lambda)\hat{dk}_\alpha(\lambda)}\right)$$

where $\widehat{dk}_\alpha(\lambda) = (\alpha + \hat{a}_1(\lambda))^{-1} = \alpha^{-1}(1 - \hat{r}_\alpha(\lambda))$, and apply Paley–Wiener to this representation.

Shea and Wainger [37] have shown that if in addition $a_1(t)$ is convex such $r_\alpha \in L^1(\mathbb{R}_+)$ exist; see also Jordan, Staffans, and Wheeler [21]. It is clear that then we also have $0 \in \varrho(a)$.

We summarize this in the following theorem.

THEOREM 7.8. *Let the assumptions of Proposition* 7.1 *be satisfied. In addition we assume that one of the following conditions is satisfied:* (a) $a_\infty > 0$;

(b) $a_1 \in L^1(\mathbb{R}_+)$, *and either* a_1 *is absolutely continuous on* $(0, \infty)$ *or* $a_0 > 0$;

(c) $a_1(t)$ *is convex on* $(0, \infty)$.

Moreover, suppose that the spectral condition (7.10) *is satisfied. Then* $\lim_{t\to\infty} S(t)x = Px$ *for each* $x \in X$, *where* P *denotes the orthogonal projection onto* $N(A)$.

Finally, we want to mention that for a_1 convex, Re $\hat{a}_1(i\mu) = 0$ if $-\dot{a}_1(t)$ is constant on each of the intervals $(2k\pi\mu^{-1}, 2(k+1)\pi\mu^{-1})$; in particular, $E_0 = \emptyset$ if $-\dot{a}_1(t)$ is nonincreasing and continuous on $(0, \infty)$.

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