Acta Applicandae Mathematicae **27**: 27–31, 1992.

Domination of Uniformly Continuous Semigroups

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(Reveived: 27 April 1992)

Abstract. We prove that a bounded operator on a Banach lattice, satisfying a growth condition, is regular. Also, we prove that the generator of a C_0-semigroup on such a lattice for which such an operator exists is bounded.

Mathematics Subject Classifications (1991): 47D03, 46A40

Key words: domination of operators, uniformly continuous semigroups

Introduction

The aim of this note is to prove the following theorem.

THEOREM 0.1 *Let $T = (T(t))_{t\geq 0}$ be a C_0-semigroup on a real or complex Banach lattice E. Assume that $B \in \mathcal{L}(E)$ is a bounded operator such that*

$$|e^{tB}x| \leq T(t)|x| \qquad (t \geq 0, \quad x \in E). \tag{1}$$

Then B is a regular operator. Moreover, the generator A of T is bounded.

Here a bounded operator B is called *regular* if B is dominated by a positive operator $C \in \mathcal{L}(E)$ (i.e. $|Bx| \leq C|x|$ $(x \in E)$). If E is order complete, then every regular operator B possesses a *modulus*; i.e. there exists a smallest positive operator $|B|$ dominating B (see Schaefer [9]).

In analogy one may ask whether a given C_0-semigroup S which is dominated by a positive C_0-semigroup possesses a *modulus semigroup*, i.e. a smallest positive semigroup $S^{\#}$ dominating S. This question is treated by Derndinger [6], Becker–Greiner [4] and Charissiadis [5] (see also Nagel [7, C-II]). With the help of a result of Derndinger [6] (see also [7, C-II, Thm. 4.17]) from the above theorem one obtains a positive answer for uniformly continuous semigroups.

COROLLARY 0.2 *Assume that E is order complete. Let $B \in \mathcal{L}(E)$ and assume that there exists a C_0-semigroup T satisfying* (1).

Then

$$|e^{tB}x| \le e^{tB^{\#}}|x| \le T(t)|x| \qquad (t \ge 0, \quad x \in E), \tag{2}$$

where $B^{\#} = \mathrm{Re}\, B_0 + |B_1|$ with $B_0 \in Z(E)$, $B_1 \in Z(E)^{\perp}$ such that $B = B_0 + B_1$.

Here $Z(E)$ denotes the center of E (see Zaanen [11] or Aliprantis–Burkinshaw [1] where $Z(E)$ is denoted by Orth(E)). Note that $Z(E)$ is a band in $\mathcal{L}^r(E)$ so that the decomposition of B is unique.

The proof of the theorem is based on perturbation arguments (cf. [2]). For further relations between perturbation and domination we refer to [3] and Rhandi [8].

The Proofs

Let A be the generator of a C_0-semigroup $T = (T(t))_{t \ge 0}$ on a Banach lattice E. Denote by $\omega(A)$ the type of T. Then $(\omega(A), \infty) \subset \rho(A)$. If T is positive, then

$$(\lambda - A)^{-1} \ge (\mu - A)^{-1} \ge 0 \qquad \text{whenever } \omega(A) < \lambda < \mu. \tag{3}$$

Conversely, if $(\lambda - A)^{-1} \ge 0$ for all $\lambda \ge w$ and some $w > \omega(A)$, then T is positive. For $\alpha > 0$ the operator αA generates the C_0-semigroup T_α given by $T_\alpha(t) = T(\alpha t)$. Note that T_α is positive if and only if T is positive.

The following is an easy consequence of the Trotter–Kato theorem.

LEMMA 0.3 *Let $B_n \in \mathcal{L}(E)$ $(n \in \mathbf{N} \cup \{\infty\})$ such that $\lim_{n\to\infty} B_n = B_\infty$ in the operator norm. Denote by S_n the semigroup generated by $A + B_n$. Then $\lim_{n\to\infty} S_n(t) = S_\infty(t)$ strongly for all $t \ge 0$.*

We will use the following result due to Derndinger [6] (see also [7, C-II, Lemma 4.18]).

PROPOSITION 0.4 *Let A be the generator of a positive C_0-semigroup on E. If $A \ge 0$ (i.e. $Ax \ge 0$ for all $0 \le x \in D(A)$), then A is bounded.*

The following generalization of Proposition 0.4 is the key step in the proof of the theorem.

PROPOSITION 0.5 *Let A be the generator of a positive semigroup and let $B \in \mathcal{L}(E)$ be a real bounded operator. If $A - B \ge 0$, then A is bounded and therefore regular* (*cf.* [7, C-II, Thm. 1.11], [10]).

Proof. By Proposition 0.4 it suffices to show that the semigroup generated by $A - B$ is positive. For $\alpha \ge 0$ the operator $A_\alpha = A + \alpha(A - B) = (1+\alpha)A - \alpha B$ generates a C_0-semigroup S_α. Let $M = \{\alpha \ge 0\colon S_\alpha \text{ is positive }\}$. We claim that $M = [0, \infty)$.

a) Let $\alpha \in M$. Then there exists $\varepsilon > 0$ such that $[\alpha, \alpha+\varepsilon) \subset M$. In fact, let $\lambda > \omega(A_\alpha)$. Let $C = A - B$ and $\varepsilon = \|CR(\lambda, A_\alpha)\|^{-1}$. Then by (3), for $0 < \delta < \varepsilon$ and $\mu \geq \lambda$, $\|\delta CR(\mu, A_\alpha)\| < 1$ and so $(I - \delta CR(\mu, A_\alpha))^{-1} = \sum_{n=0}^{\infty}(\delta CR(\mu, A_\alpha))^n \geq 0$ (where $R(\mu, A_\alpha) = (\mu - A_\alpha)^{-1}$). Since $(\mu - A_{\alpha+\delta}) = (\mu - A_\alpha - \delta C) = (I - \delta C(\mu - A_\alpha)^{-1})(\mu - A_\alpha)$, it follows that $R(\mu, A_{\alpha+\delta}) = R(\mu, A_\alpha)(I - \delta CR(\mu, A_\alpha))^{-1} \geq 0$ for all $\mu \geq \lambda$, $\delta \in (0, \varepsilon)$. Consequently $\alpha + \delta \in M$ for $\delta \in (0, \varepsilon)$.

b) Assume that $\gamma := \sup\{\alpha \geq 0\colon [0, \alpha) \subset M\} < \infty$. Note that for $\alpha \geq 0$ the semigroup S_α is positive if and only if the semigroup generated by $\frac{1}{1+\alpha}A_\alpha = A - \frac{\alpha}{1+\alpha}B$ is positive. Letting $\alpha \uparrow \gamma$ it follows from Lemma 0.3 that S_γ is positive. Now a) leads to a contradiction. We have shown that $A_\alpha = (1+\alpha)A - \alpha B$ generates a positive semigroup for all $\alpha \geq 0$. Consequently, the semigroup generated by $A - \frac{\alpha}{1+\alpha}B = \frac{1}{1+\alpha}A_\alpha$ is positive as well. Letting $\alpha \to \infty$ it follows from Lemma 0.3 that $A - B$ generates a positive semigroup. Q.E.D.

Proof of the Theorem. Since $|e^{tB}x| \leq T(t)x$ $(x \in E_+)$ by assumption, it follows that

$$(\operatorname{Re} B)x = \lim_{t\downarrow 0} \frac{\operatorname{Re}(e^{tB}x) - x}{t} \leq \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = Ax$$

for all $0 \leq x \in D(A)_+$. Hence $A - \operatorname{Re} B \geq 0$. This finishes the proof in the real case. If E is complex we merely conclude that $\operatorname{Re} B$ is regular. In order to show that $\operatorname{Im} B$ is regular we first assume that E is order complete. Then by [10] the band projection from $\mathcal{L}^r(E)$ onto $Z(E)$ has a contractive extension $\mathcal{P}\colon \overline{\mathcal{L}^r(E)} \to Z(E)$ (where the closure is understood in $\mathcal{L}(E)$). Since $|M| \leq \|M\|I$ for all $M \in Z(E)$ one has

$$|\mathcal{P}C| \leq \|C\|I \qquad (C \in \overline{\mathcal{L}^r(E)}). \tag{4}$$

Denote by $\mathcal{J}$ the identity mapping on $\mathcal{L}(E)$. Since $e^{tB} \in \mathcal{L}^r(E)$ $(t > 0)$, it follows that $B \in \overline{\mathcal{L}^r(E)}$. We already know that A is bounded. Since $\mathcal{J} - \mathcal{P}$ is positive on $\mathcal{L}^r(E)$, it follows from (1) that

$$|(\mathcal{J} - \mathcal{P})e^{tB}| \leq (\mathcal{J} - \mathcal{P})|e^{tB}| \leq (\mathcal{J} - \mathcal{P})e^{tA} \qquad (t \geq 0).$$

Since $(\mathcal{J} - \mathcal{P})I = 0$, it follows for $x \in E_+$ that

$$\begin{aligned}|(\mathcal{J} - \mathcal{P})Bx| &= \lim_{t\to 0} t^{-1}|(\mathcal{J} - \mathcal{P})e^{tB}x| \\ &\leq \lim_{t\to 0}\, t^{-1}(\mathcal{J} - \mathcal{P})e^{tA}x \\ &= (\mathcal{J} - \mathcal{P})Ax \leq (A + \|A\|)x.\end{aligned}$$

Hence $B - \mathcal{P}B$ is regular, and so B is regular. Moreover, $(\mathcal{J} - \mathcal{P})(\operatorname{Im} B) = \operatorname{Im}((\mathcal{J} - \mathcal{P})B) \leq A + \|A\|I$. Since by (4) $\mathcal{P}(\operatorname{Im} B) \leq \|\operatorname{Im} B\|I \leq \|B\|I$, it follows that

$$\operatorname{Im} B \leq A + (\|A\| + \|B\|)I. \tag{5}$$

Now assume that E is arbitrary. Since A is bounded and E' is order complete, applying the preceeding to A' and B' one obtains

$$\operatorname{Im} B' \leq A' + (\|A'\| + \|B'\|)I.$$

Hence (5) holds for A and B as well. Consequently, $\operatorname{Im} B$ is regular. Q.E.D.

REMARK 0.6 By sligth modifications of the proof one obtains the following more general result. Let A be a densely defined operator such that for some $w \in \mathbf{R}$, $[w, \infty) \subset \rho(A)$ and

$$\sup\{\|\lambda R(\lambda, A)\| : \lambda \geq w\} < \infty.$$

Assume that $B \in \mathcal{L}(E)$ such that $[w, \infty) \subset \rho(B)$ and

$$R(\lambda, B)x \leq R(\lambda, A)x \qquad (\lambda \geq w, \quad x \in E).$$

Then B is regular and A is bounded.

We conclude with pointing out two open questions. In the first we formulate a generalization of the theorem presented above.

QUESTION 1 Let $B \in \mathcal{L}(E)$ and assume that e^{tB} is regular for all $t \geq 0$. Does it follow that B is regular?

The second concerns the possibility of generalizing Proposition 0.4 and 0.5.

QUESTION 2 Let A be the generator of a C_0-semigroup S. Assume that
a) $D(A)_+$ is dense in E_+; and
b) $Ax \geq 0$ for all $x \in D(A)_+$.
Does it follow that T is positive?

If A generates a positive C_0-semigroup, then a) is satisfied. In fact, if $x \in E_+$, then $x = \lim_{\lambda\to\infty} \lambda R(\lambda, A)x \in \overline{D(A)_+}$.

However, as the following example shows, there exists a generator B of a C_0-semigroup such that $D(B)_+ = \{0\}$. Thus condition b) is trivially satisfied and does not imply positivity without additional conditions.

EXAMPLE 0.7 Let $E = L^p(0,1)$ $(1 \leq p < \infty)$ and let A be the generator of the right shift semigroup, i.e. A is given by $D(A) = \{f\colon [0,1] \to \mathbf{R}$ absolutely continuous, $f' \in E$, $f(1) = 0\}$, $Af = -f'$. Let $\Omega \subset (0,1)$ be measurable such that $\operatorname{meas}(\Omega \cap (a,b)) > 0$ and $\operatorname{meas}(\Omega^c \cap (a,b)) > 0$ for all intervals $\emptyset \neq (a,b) \subset (0,1)$. Let $\mathcal{J}\colon E \to E$ be given by $\mathcal{J}f = 1_\Omega f - 1_{\Omega^c} f$. Then $\mathcal{J}$ is an isometric isomorphism and $\mathcal{J}^{-1} = \mathcal{J}$. Let B be the generator of the semigroup $T = (\mathcal{J}\mathcal{U}(t)\mathcal{J})_{t\geq 0}$. Then $D(B)_+ = \{0\}$. In fact, let $0 \leq f \in D(B)$. Then $\mathcal{J}f \in D(A)$. Thus $\mathcal{J}f$ is continuous and $\mathcal{J}f \geq 0$ on Ω and ≤ 0 on Ω^c. Consequently, $\mathcal{J}f = 0$.

Acknowledgement

The authors are grateful to H. Raubenheimer for several stimulating discussions on this subject.

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