Differential and Integral Equations, Volume 6, Number 5, September 1993, pp. 1009–1024.

EXPONENTIAL STABILITY OF A DIFFUSION EQUATION WITH ABSORPTION

WOLFGANG ARENDT AND CHARLES J.K. BATTY†
Equipe de Mathématiques - URA CNRS 741, Université de Franche-Comté
25030 Besançon Cedex, France

(Submitted by: J.A. Goldstein)

0. Introduction. Consider a diffusion equation of the form

$$\begin{cases} u_t(t,x) = \Delta u(t,x) - V(x)u(t,x) & (t > 0, x \in \mathbb{R}^N) \\ u(0,x) = f(x), \end{cases} \tag{0.1}$$

where the absorption term $V \in L^1_{\rm loc}(\mathbb{R}^N)$ is positive. In a previous article [2] we studied asymptotic stability of the solutions of (0.1). Here we investigate how big the absorption has to be in order that the equation (0.1) is exponentially stable in L^p-norm; i.e., given $1 \le p < \infty$, there exist $M \ge 0$, $\varepsilon > 0$ such that

$$\|u(t,\cdot)\|_{L^p} \le Me^{-\varepsilon t}\|f\|_{L^p} \qquad (t \ge 0)$$

for all $f \in L^p(\mathbb{R}^N)$. This property does not depend on p. It does depend on the size of V in a very specific way which we will describe in the following. We distinguish two cases.

In Section 1 potentials in $L^1 + L^\infty$ are considered. Let $\mathcal{G}$ denote the set of all open subsets G of $\mathbb{R}^N$ which contain arbitrarily large balls. Then the equation (0.1) is exponentially stable if and only if

$$\int_G V(x)dx = \infty \quad \text{for all } G \in \mathcal{G} \tag{0.2}$$

Arbitrary positive potentials in $L^1_{\rm loc}(\mathbb{R}^N)$ are considered in Section 3. It is no longer possible to describe exponential stability by the behavior of V on large balls (unless $N = 1$). We replace $\mathcal{G}$ by the class $\mathcal{O}$ of all open sets Ω in $\mathbb{R}^N$ for which the heat equation in $L^2(\Omega)$ with Dirichlet boundary conditions

$$\begin{cases} u_t(t,x) = \Delta u(t,x) & (t > 0,\ x \in \Omega); \\ u(t,\cdot) \in H^1_0(\Omega) & (t > 0); \\ u(0,x) = f(x) & (x \in \Omega), \end{cases} \tag{0.3}$$

Received for publication November 1992.
†Permanent Address: St. John's College, Oxford 0X1 3JP, England.
AMS Subject Classifications: 35B35, 47D06, 35J10.

is not exponentially stable. Formulated in a different way, an open set $\Omega \subset \mathbb{R}^N$ is in $\mathcal{O}$ if and only if Poincaré's inequality fails in $H_0^1(\Omega)$ (see also Section 2). One always has $\mathcal{G} \subset \mathcal{O}$, but $\mathcal{G} \neq \mathcal{O}$ if $N \geq 2$. In Section 3 we show that

$$\int_\Omega V dx = \infty \quad \text{for all } \Omega \in \mathcal{O} \tag{0.4}$$

is a necessary condition for exponential stability of (0.1). We give an example of an absorption term $V \in L^1_{\text{loc}}$ such that (0.2) is satisfied but $\int_\Omega V\,dx = 0$ for one set Ω in $\mathcal{O}$. Thus condition (0.2) is no longer sufficient for exponential stability if $V \notin L^1 + L^\infty$.

All proofs given here are of analytic nature. Parallel to this work the second author studies the problem by probabilistic methods [3]. In particular, it is proved in [3] that condition (0.4) is also sufficient for exponential stability.

The above formulation of the problem is the most intuitive. As is well-known (see Kato [8] and Voigt [13]) the problem is governed by a semigroup $S = (S(t))_{t \geq 0}$ on $L^p(\mathbb{R}^N)$ with generator $\Delta - V$. This semigroup is frequently called a Schrödinger semigroup even though it governs the heat equation with absorption (0.1), whereas the Schrödinger equation is governed by $(S(it))_{t \in \mathbb{R}}$ ($p = 2$). We refer to Simon's survey article [12] for further information about Schrödinger semigroups and their relation to quantum mechanics. The fact that (0.1) is exponentially stable can be reformulated by saying that the type of S is negative (see Section 1).

1. Potentials in $L^1 + L^\infty$. Let $1 \leq p \leq \infty$. By $T_p = (T_p(t))_{t \geq 0}$ we denote the Gaussian semigroup on $L^p(\mathbb{R}^N)$ given by

$$(T_p(t)f)(x) = (4\pi t)^{-N/2} \int f(y) \exp(-(x-y)^2/4t)\,dy.$$

Note that T_p is a C_0-semigroup for $1 \leq p < \infty$ and T_∞ is weak*-continuous as the adjoint semigroup of T_1. By Δ_p we denote the generator of T_p, i.e., $D(\Delta_p) = \{f \in L^p : \Delta f \in L^p\}$; $\Delta_p f = \Delta f$ (in the sense of distributions).

Let $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$. We define $\Delta_p - V$ for $1 \leq p < \infty$ as follows: let $D(A_{\min}) = \mathcal{D}(\mathbb{R}^N)$ (the test functions on $\mathbb{R}^N$) and $A_{\min} f = \Delta f - Vf$. Then $A_{\min}$ is closable in $L^p(\mathbb{R}^N)$ and we set $\Delta_p - V := \overline{A_{\min}}$ in $L^p(\mathbb{R}^N)$. Then $\Delta_p - V$ generates a holomorphic semigroup $S_p = (S_p(t))_{t \geq 0}$ on $L^p(\mathbb{R}^N)$ which we denote occasionally by $S_p(t) = e^{t(\Delta_p - V)}$. Then

$$0 \leq S_p(t) \leq T_p(t) \qquad (t \geq 0)$$

in the sense of positive operators; i.e., $0 \leq f \in L^p$ implies $0 \leq S_p(t)f \leq T_p(t)f$. The semigroups S_p interpolate; i.e.,

$$S_p(t)f = S_q(t)f \qquad (f \in L^p \cap L^q,\ t \geq 0) \tag{1.1}$$

for $1 \leq p, q \leq \infty$, if one sets $S_\infty(t) = S_1(t)'$. The adjoint operator of $\Delta_1 - V$ is denoted by $\Delta_\infty - V$. We refer to T. Kato [8] and J. Voigt [14] for all this.

If $\mathcal{U} = (\mathcal{U}(t))_{t \geq 0}$ is a C_0-semigroup with generator A we denote by

$$\omega(A) = \inf\{w \in \mathbb{R} : \sup_{t \geq 0} e^{-wt} \|\mathcal{U}(t)\| < \infty\}$$

the *growth bound* (or *type*) of $\mathcal{U}$ and by

$$s(A) = \sup\{Re\lambda : \lambda \in \sigma(A)\}$$

the *spectral bound* of A. One always has $s(A) \leq \omega(A)$. The semigroup $\mathcal{U}$ or its generator A is called *exponentially stable* if $\omega(A) < 0$. This is the case if and only if $\|\mathcal{U}(t)\| < 1$ for some $t > 0$.

Definition 1.1. Let $G \subset \mathbb{R}^N$. We say that G *contains arbitrarily large balls* if for any $r > 0$ there exists $x \in \mathbb{R}^N$ such that the ball $B(x,r) := \{y \in \mathbb{R}^N : |x-y| < r\}$ is included in G. By $\mathcal{G}$ we denote the set of all open subsets of $\mathbb{R}^N$ which contain arbitrarily large balls.

Theorem 1.2. *Let $0 \leq V \in L^1 + L^\infty$. The following are equivalent:*

(i) $s(\Delta_p - V) < 0$ *for some* $p \in [1,\infty)$;

(ii) $\|S_p(t)\| < 1$ *for all* $t > 0$, $p \in [1,\infty)$;

(iii) $\int_G V dx = \infty$ *for all* $G \in \mathcal{G}$.

Remark 1.3. It is known that $s(\Delta_p - V) = \omega(\Delta_p - V)$ for all $p \in [1,\infty)$; see Simon [11], Hempel-Voigt [7] and Voigt [13]. We will not use this fact for the proof of Theorem 1.2.

Proposition 1.4. *Let $0 \leq V \in L^1_{\rm loc}(\mathbb{R}^N)$. The following are equivalent:*

(i) *There exist $r > 0$, $c > 0$ such that $\int_{B(x,r)} V(y)\,dy \geq c$ for all $x \in \mathbb{R}^N$.*

(ii) *There exists $c > 0$ such that $\int_G V(y)\,dy \geq c$ for all $G \in \mathcal{G}$.*

(iii) $\liminf_{n\to\infty} \int_{B(x_n,r_n)} V(x)\,dx > 0$ *whenever* $x_n \in \mathbb{R}^N$, $r_n > 0$ *such that* $\lim_{n\to\infty} r_n = \infty$.

(iv) $\int_G V(x)\,dx = \infty$ *for all* $G \in \mathcal{G}$.

Proof: (i) $\Rightarrow$ (iv). Assume (i) and let $G \in \mathcal{G}$. For $n \in \mathbb{N}$ there exist $x_1, \cdots, x_n \in \mathbb{R}^N$ such that $B(x_i,r) \cap B(x_j,r) = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{n} B(x_i,r) \subset G$. Hence

$$\int_G V\,dx \geq \sum_{i=1}^{n} \int_{B(x_i,r)} V\,dx \geq n \cdot c.$$

Consequently, $\int_G V dx = \infty$.

(iv) $\Rightarrow$ (ii) trivial.

(ii) $\Rightarrow$ (iii). If (iii) is false, then there exist $x_n \in \mathbb{R}^N$ and $r_n \to \infty$ such that $\int_{B(x_n,r_n)} V\,dx \leq 2^{-n}$. Let $G_m = \bigcup_{n>m} B(x_n,r_n) \in \mathcal{G}$. Then $\int_{G_m} V\,dx \leq 2^{-m}$. So (ii) does not hold.

(iii) $\Rightarrow$ (i) this is clear. ☐

Spectral bound and growth bound can be easily described for operators associated with forms.

Let H be a Hilbert space and a a positive, symmetric closed form with dense domain $D(a)$. Let A be the operator associated with a, i.e.,

$$D(A) = \{u \in D(a) : \text{there exists } v \in H \text{ such that } a(u,\varphi) = (v \mid \varphi)_H \text{ for all } \varphi \in D(a)\},$$
$$Au = v.$$

Then A is self-adjoint and form positive, so $-A$ generates a C_0-semigroup $\mathcal{U}$ of self-adjoint operators. Moreover, $D(A)$ is dense in the Hilbert space $(D(a), \|\cdot\|_a)$, where

$$\|u\|_a^2 = \Big((u \mid u)_H + a(u,u)\Big)^{\frac{1}{2}}.$$

It can easily be seen from the spectral theorem that

$$\begin{aligned} s(-A) = \omega(-A) &= \inf\{(Au \mid u)_H : u \in D(A), \|u\|_H = 1\} \\ &= \inf\{a(u,u) : u \in D(a), \|u\|_H = 1\} \\ &= \inf\{a(u,u) : u \in D, \|u\|_H = 1\} \end{aligned}$$

for any form-core D of a.

The operator $\Delta_2 - V$ ($V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$) is associated with the closure of the form b on $L^2(\mathbb{R}^N)$ given by

$$b(u,v) = \int_{\mathbb{R}^N} \nabla u \nabla v + \int_{\mathbb{R}^N} Vuv, \quad D(b) = C_c^\infty(\mathbb{R}^N) =: \mathcal{D}(\mathbb{R}^N),$$

Hence

$$-s(\Delta_2 - V) = \inf\Big\{\int (\nabla u)^2 + \int Vu^2 : u \bullet \mathcal{D}(\mathbb{R}^N),\ \|u\|_{L^2} = 1\Big\}. \tag{1.2}$$

Lemma 1.5. *Let $x_n \in \mathbb{R}^N$, $r_n > 0$, and $\lim_{n\to\infty} r_n = \infty$. Then there exists a sequence $(v_n) \subset C_c^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} v_n \subset B(x_n, r_n)$, $\|v_n\|_{L^2} = 1$, $\lim_{n\to\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^N)^N} = 0$ and $\lim_{n\to\infty} \|v_n\|_\infty = 0$.*

Proof: Let $u \in \mathcal{D}(B(0,1))$ such that $\int u^2\,dx = 1$. Let $w_n(x) = n^{-N/2}u(n^{-1}x)$. Then $w_n \in \mathcal{D}(B(0,n))$, $\|w_n\|_{L^2} = 1$, $\|w_n\|_\infty \to 0$ and

$$\int (\nabla w_n)^2\,dx \to 0 \quad (n \to \infty).$$

By a translation one obtains $v_n \in \mathcal{D}(B(x_n, r_n))$ with the desired properties. □

Proof of Theorem 1.2: (i) $\Rightarrow$ (iii). Let $p \in [1,\infty)$ such that $s(\Delta_p - V) < 0$. Let $\lambda > s(\Delta_p - V)$, $\frac{1}{p} + \frac{1}{q} = 1$. Since $\Delta_2 - V$ is selfadjoint, it follows that $\lambda \in \varrho(\Delta_q - V)$ and $R(\lambda, \Delta_q - V) = R(\lambda, \Delta_p - V)' \geq 0$. Furthermore, from the interpolation property (1.1), it follows that $R(\lambda, \Delta_q - V)$ and $R(\lambda, \Delta_p - V)$ coincide on $L^p \cap L^q$. With the help of the Riesz-Thorin theorem one concludes that $\lambda \in \varrho(\Delta_2 - V)$ and

$R(\lambda, \Delta_2 - V) \geq 0$. Since $\lambda > s(\Delta_p - V)$ was arbitrary we obtain that $s(\Delta_2 - V) < 0$ (see [9, C-III Theorem 1.1]).

It follows from (1.2) that

$$C := \inf\left\{ \int (\nabla u)^2 + \int V u^2 : u \in \mathcal{D}(\mathbb{R}^N), \int u^2 = 1 \right\} > 0.$$

Let $G \in \mathcal{G}$. Choosing $v_n \in \mathcal{D}(\mathbb{R}^N)$ as in Lemma 1.5, we obtain

$$0 < C \leq \int_G (\nabla v_n)^2 + \int_G V v_n^2 \leq \int_G (\nabla v_n)^2 + \int_G V \cdot \|v_n\|_\infty^2.$$

Since

$$\lim_{n\to\infty} \int_G (\nabla v_n)^2 = 0$$

and

$$\lim_{n\to\infty} \|v_n\|_\infty^2 = 0$$

it follows that

$$\int_G V = \infty.$$

(iii) $\Rightarrow$ (ii). Let $0 \leq V \in L^1 + L^\infty$ such that

$$\int_G V = \infty \quad \text{for all } G \in \mathcal{G}.$$

Since

$$0 \leq e^{t(\Delta_p - V_1)} \leq e^{t(\Delta_p - V_2)}$$

if $0 \leq V_2 \leq V_1$, we can assume that $V \in L^\infty(\mathbb{R}^N)$. By the variation of constants formula we have

$$S_\infty(t)\mathbf{1} = T_\infty(t)\mathbf{1} - \int_0^t T_\infty(t-s) V S_\infty(s)\mathbf{1}\, ds = \mathbf{1} - \int_0^t T_\infty(t-s) V S_\infty(s)\mathbf{1}\, ds.$$

Iterating this once yields

$$S_\infty(t)\mathbf{1} = \mathbf{1} - \int_0^t T_\infty(s) V\, ds + \int_0^t T_\infty(t-s) V \int_0^s T_\infty(s-r) V S_\infty(r)\mathbf{1}\, dr\, ds. \quad (1.3)$$

Let $t > 0$. We can assume that $\|V\|_\infty < \frac{1}{t}$. (In fact, otherwise we replace V by λV where $\lambda \in (0,1)$. Since $e^{t(\Delta_1 - V)} \leq e^{t(\Delta_1 - \lambda V)}$ $(t \geq 0)$, $\|e^{t(\Delta_1 - \lambda V)}\| < 1$ implies $\|e^{t(\Delta_1 - V)}\| < 1$).

Since $S_\infty(s)\mathbf{1} \leq T_\infty(s)\mathbf{1} = \mathbf{1}$ and $V \leq \|V\|_\infty \mathbf{1}$, we have

$$\begin{aligned}
&\int_0^t T_\infty(t-s) V \int_0^s T_\infty(s-r) V S_\infty(r)\mathbf{1}\, dr\, ds \\
&\leq \|V\|_\infty \int_0^t T_\infty(t-s) V \int_0^s T_\infty(s-r)\mathbf{1}\, dr\, ds \\
&= \|V\|_\infty \int_0^t s T_\infty(t-s) V\, ds = \|V\|_\infty \int_0^t (t-s) T_\infty(s) V\, ds \\
&\leq \|V\|_\infty t \int_0^t T_\infty(s) V\, ds.
\end{aligned}$$

Using (1.3) we obtain

$$S_\infty(t)1 \le 1 - \int_0^t T_\infty(s)V\,ds + t\|V\|_\infty \int_0^t T_\infty(s)V\,ds.$$

Thus

$$S_\infty(t)1 \le 1 - (1 - t\|V\|_\infty) \int_0^t T_\infty(s)V\,ds. \tag{1.4}$$

Now

$$(T_\infty(s)V)(x) = \int_{\mathbb{R}^N} p_s(x-y)V(y)\,dy,$$

where $p_s(z) = (4\pi s)^{-N/2}\exp(-z^2/4s)$. By hypothesis, there exist $c > 0$, $r > 0$ such that

$$\int_{B(x,r)} V(y)\,dy \ge c \quad \text{for all } x \in \mathbb{R}^N$$

(see Proposition 1.4). Let $\varepsilon = \inf\limits_{\substack{|z|\le r \\ 0<s\le t}} p_s(z) > 0$. Then

$$(T_\infty(s)V)(x) \ge \varepsilon \cdot c \quad (x \in \mathbb{R}^N,\ 0 < s \le t).$$

Hence

$$\int_0^t (T_\infty(s)V)(x)\,ds \ge t\varepsilon \cdot c \quad \text{for all } x \in \mathbb{R}^N.$$

It follows from (1.4) that $(S_\infty(t)1)(x) \le 1 - (1 - t\|V\|_\infty)t\varepsilon c$ $(x \in \mathbb{R}^N)$ and so $\|S_1(t)\| = \|S_\infty(t)\| = \|S_\infty(t)1\|_{L^\infty} < 1$. Moreover, from the Riesz-Thorin theorem, it follows that $\|S_p(t)\| < 1$ for all $p \in [1,\infty]$. So (ii) is proved. The implication (ii) $\Rightarrow$ (i) is obvious. □

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^N$ be open. The following are equivalent:*

(i) *Ω almost contains large balls (i.e., there exist $x_n \in \mathbb{R}^N$, $r_n \to \infty$ such that $mes(B(x_n,r_n)\backslash\Omega) \longrightarrow 0$ $(n \to \infty)$).*

(ii) *There exist $x_n \in \mathbb{R}^N$, $r_n > 0$ such that $r_n \to \infty$ $(n \to \infty)$ and $\sup\limits_{n\in\mathbb{N}} \text{mes}\,(B(x_n,r_n)\backslash\Omega) < \infty$.*

(iii) *$s(\Delta - k1_{\Omega^c}) = 0$ for all k ● $\mathbb{N}$.*

(iv) *$s(\Delta - V) = 0$ for all $0 \le V \in L^1 + L^\infty$ satisfying $\int_\Omega V dx < \infty$.*

Here $s(\Delta - V) = s(\Delta_p - V)$, which is independent of $p \in [1,\infty]$ (see Remark 1.3).

Proof: (i) $\Rightarrow$ (ii) this is trivial.

(ii) $\Rightarrow$ (iv). By hypothesis there exist balls $B(x_n,r_n)$ such that $\lim\limits_{n\to\infty} r_n = \infty$, $\sup\limits_{n\in\mathbb{N}} \text{mes}\,(B(x_n,r_n\backslash\Omega) =: M < \infty$. Using Lemma 1.5 we find $v_n \in \mathcal{D}(\mathbb{R}^N)$ satisfying $\text{supp}\, v_n \subset B(x_n,r_n)$, $\|v_n\|_{L^2} = 1$, $\lim\limits_{n\to\infty} \|\nabla v_n\|_{(L^2)^N} = 0$, and $\lim\limits_{n\to\infty} \|v_n\|_\infty = 0$.

Let $0 \le V_1 \in L^1$, $0 \le V_\infty \in L^\infty$ such that $V = V_1 + V_\infty$ and suppose that $\int_\Omega V\,dx < \infty$. Then

$$\limsup_{n\to\infty}\Big(\int(\nabla v_n)^2 + \int V v_n^2\Big) = \limsup_{n\to\infty}\int V v_n^2$$
$$\le \limsup_{n\to\infty}\Big(\int_\Omega V\Big)\|v_n\|_\infty^2 + \limsup_{n\to\infty}\Big(\int_{\Omega^c} V_1\Big)\|v_n\|_\infty^2 + \limsup_{n\to\infty}\int_{\Omega^c}(V_\infty v_n^2)$$
$$\le \|V_\infty\|_\infty \limsup_{n\to\infty}\|v_n\|_\infty^2 \text{ mes } (B(x_n, r_n)\backslash\Omega) = 0.$$

Hence $s(\Delta - V) = 0$ by (1.2).

(iv) $\Rightarrow$ (iii) this is trivial.

(iii) $\Rightarrow$ (i). If (i) does not hold, then for any sequence of balls $B(x_n, r_n)$ with $\lim_{n\to\infty} r_n = \infty$ one has $\liminf_{n\to\infty}$ mes $(B(x_n, r_n)\setminus\Omega) > 0$. Hence

$$\liminf_{n\to\infty} k\int_{B(x_n, r_n)} 1_{\Omega^c}\,dx > 0 \quad (k > 0).$$

By Proposition 1.4 and Theorem 1.2 this implies $s(\Delta - k1_{\Omega^c}) < 0$ for all $k > 0$. $\square$

Definition 1.7. Let $c > 0$. A measurable set $E \subset \mathbb{R}^N$ satisfies inequality $M(c)$ if

$$\int_E u^2 \le c\|u\|_{L^2(\mathbb{R}^N)}\|\nabla u\|_{L^2(\mathbb{R}^N)^N} \qquad (\mathrm{M}(c))$$

for all $u \in \mathcal{D}(\mathbb{R}^N)$. By $\mathcal{M}$ we denote the set of all measurable subsets E of $\mathbb{R}^N$ for which there exists $c > 0$ such that $M(c)$ holds.

A set of the form

$$\{x = (x_1, \cdots, x_N) \in \mathbb{R}^N : a < x_j < b\},$$

where $-\infty < a < b < \infty$ and $j \in \{1, \cdots, N\}$ or a rotation and translation of such a set is called a *strip* of width $b - a$.

Proposition 1.8. *Let S be a strip of width $c > 0$, then S satisfies* M(c).

Proof: (a)[1] Let $u \in \mathcal{D}(\mathbb{R})$. Then

$$u(x)^2 = \int_{-\infty}^x (u(y)^2)'dy = 2\int_{-\infty}^x uu'\,dy;$$

and

$$u(x)^2 = -\int_x^\infty (u^2)'dy = -2\int_x^\infty uu'dy.$$

Hence

$$u(x)^2 = \int_{-\infty}^x uu'dy - \int_x^\infty uu'dy \le \int_{-\infty}^{+\infty}|uu'|dy \le \|u\|_{L^2(\mathbb{R})}\|u'\|_{L^2(\mathbb{R})}.$$

[1] This proof is due to Ph. Bénilan and replaces a more complicated one of the authors.

Consequently,

$$\int_a^b u(x)^2 dx \le (b-a)\|u\|_{L^2(\mathbb{R})}\|u'\|_{L^2(\mathbb{R})},$$

which finishes the proof for $N = 1$.

(b) Let $N > 1$ and assume (without loss of generality) that $E = \{x : a < x_1 < b\}$. Then by (a)

$$\int_a^b u(x_1, x')^2\, dx_1 \le (b-a)\Big(\int_{\mathbb{R}} u(x_1,x')^2\, dx_1\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} \frac{\partial u}{\partial x_1}(x_1,x')^2 dx_1\Big)^{\frac{1}{2}}$$

for all $x' \in \mathbb{R}^{N-1}$. Integration with respect to x' yields with the help of the Cauchy-Schwarz inequality

$$\int_E u^2 dx \le (b-a)\cdot \|u\|_{L^2(\mathbb{R}^N)}\cdot \|\frac{\partial u}{\partial x_1}\|_{L^2(\mathbb{R}^N)} \le (b-a)\|u\|_{L^2(\mathbb{R}^N)}\cdot \|\nabla u\|_{L^2(\mathbb{R}^N)^N}.$$

Corollary 1.9. *Let S_j be a strip of width c_j ($j \in \mathbb{N}$). If $\sum_{j=1}^{\infty} c_j < \infty$, then*

$$\Omega = \bigcup_{j\in\mathbb{N}} S_j \in \mathcal{M}.$$

Theorem 1.2 shows that the size of V for large x is responsible for exponential stability. This is made more precise in the next result which shows that the part of V on a strip does not matter.

Theorem 1.10. *Let $0 \le V \in L^1 + L^\infty$ such that $s(\Delta_2 - V) < 0$. If $E \in \mathcal{M}$, then $s(\Delta_2 - V\cdot 1_{E^c}) < 0$, where $E^c = \mathbb{R}^N\backslash E$.*

Proof: By hypothesis

$$0 < \lambda = \inf\Big\{\int(\nabla u)^2 + \int Vu^2 : \int u^2 = 1,\ u \in \mathcal{D}(\mathbb{R}^N)\Big\}.$$

As a consequence of Theorem 1.2 $s(\Delta_2 - V) < 0$ if and only if $s(\Delta_2 - V - V_1) < 0$, where $0 \le V_1 \in L^1(\mathbb{R}^N)$. So we can assume that $V \in L^\infty(\mathbb{R}^N)$. Let $u \in \mathcal{D}(\mathbb{R}^N)$ such that $\int u^2 = 1$. If

$$\int_E Vu^2 \le \frac{\lambda}{2},$$

then

$$\int(\nabla u)^2 + \int V1_{E^c}u^2 \ge \frac{\lambda}{2}.$$

If not, then using inequality M(c) we have

$$\frac{\lambda}{2} \le \int_E Vu^2 \le \|V\|_\infty \cdot \int_E u^2 \le \|V\|_\infty\cdot c\cdot \Big(\int_{\mathbb{R}^N}(\nabla u)^2\Big)^{\frac{1}{2}}.$$

Hence

$$\int(\nabla u)^2 + \int V1_{E^c}u^2 \ge \min\Big\{\frac{\lambda}{2}, \Big(\frac{\lambda}{2\|V\|_\infty\cdot c}\Big)^2\Big\}.$$

2. Poincaré sets. Let $\Omega \subset \mathbb{R}^N$ be an open set. By Δ_Ω we denote the Dirichlet-Laplacian on $L^2(\Omega)$; i.e., Δ_Ω is defined by $D(\Delta_\Omega) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$ (where Δu is understood in the sense of distributions), $\Delta_\Omega u = \Delta u$ $(u \in D(\Delta_\Omega))$. The operator Δ_Ω is associated with the form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ given by

$$a(u,v) = \int_\Omega \nabla u \nabla v \, dx.$$

Hence

$$\begin{aligned}\lambda_\Omega : &= -\omega(\Delta_\Omega) = -s(\Delta_\Omega) \\ &= \inf\left\{ \int_\Omega (\nabla u)^2 dx : u \in H_0^1(\Omega), \int_\Omega u^2 = 1\right\} \\ &= \inf\left\{ \int_\Omega (\nabla u)^2 dx : u \in \mathcal{D}(\Omega), \int_\Omega u^2 = 1\right\} \quad \text{(cf. Section 1).}\end{aligned}$$

Note that

$$\lambda_\Omega \int_\Omega u^2 dx \le \int_\Omega (\nabla u)^2 dx \qquad (u \in H_0^1(\Omega)) \tag{2.1}$$

(Poincaré's inequality). If Ω is bounded, then λ_Ω is the smallest eigenvalue of $-\Delta_\Omega$.

Definition 2.1. An open subset Ω of $\mathbb{R}^N$ is a *Poincaré set* if $s(\Delta_\Omega) < 0$. By $\mathcal{P}$ we denote the set of all open Poincaré sets in $\mathbb{R}^N$.

It is immediately clear from the definitions that an open set Ω is in $\mathcal{P}$ whenever it is in $\mathcal{M}$.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be open. Consider the following assertions:*

(i) *Ω contains arbitrarily large balls.*

(ii) $\Omega \notin \mathcal{P}$.

(iii) *Ω almost contains large balls.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof: (i) $\Rightarrow$ (ii) follows from Lemma 1.5.

(ii) $\Rightarrow$ (iii) Suppose that Ω does not almost contain large balls. Then by Theorem 1.6 $s(\Delta - k1_{\Omega^c}) < 0$ for some $k \in \mathbb{N}$. Thus

$$\begin{aligned}0 < -s(\Delta - k1_{\Omega^c}) &= \inf\left\{ \int_{\mathbb{R}^N} (\nabla u)^2 + k\int_{\Omega^c} u^2 : u \in \mathcal{D}(\mathbb{R}^N), \int_{\mathbb{R}^N} u^2 dx = 1\right\} \\ &\le \inf\left\{ \int (\nabla u)^2 : u \in \mathcal{D}(\Omega), \int u^2 = 1\right\} \\ &= -s(\Delta_\Omega). \qquad \square\end{aligned}$$

The next proposition shows that (i) $\Longleftrightarrow$ (ii) if $N = 1$. We will see that (ii) $\not\Rightarrow$ (i) if $N \ge 2$. For any $N \ge 1$, (iii) $\not\Rightarrow$ (ii). In fact, $\Omega = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : x_1 \notin \mathbb{Z}\} \in \mathcal{P}$, but Ω almost contains large balls.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}$ be open. Then Ω is a Poincaré set if and only if*

$$\ell := \sup\{b - a : (a, b) \subset \Omega\} < \infty.$$

In that case $\lambda_\Omega = \frac{\pi^2}{\ell^2}$.

Proof: If $\Omega = (0, \ell)$, then it is easy to see that $\lambda_\Omega = \frac{\pi^2}{\ell^2}$ (with eigenfunction $\sin \frac{\pi}{\ell}x$) using translation $\lambda_{(a,b)} = \frac{\pi^2}{(b-a)^2}$. So assume that $b - a \leq \ell$ whenever $(a, b) \subset \Omega$. Let $u \in \mathcal{D}(\Omega)$. Then $u = \sum_{j=1}^n u_j$, where $u_j \in \mathcal{D}(a_j, b_j)$, $(a_j, b_j) \subset \Omega$, $(a_j, b_j) \cap (a_i, b_i) = \phi$ $(i \neq j)$; $i, j = 1, \dots n$. Hence

$$\frac{\pi^2}{\ell^2} \int u^2 = \frac{\pi^2}{\ell^2} \sum_{j=1}^n \int_{a_j}^{b_j} u_j^2 \leq \sum_{j=1}^n \frac{\pi^2}{(b_j - a_j)^2} \int_{a_j}^{b_j} u_j^2 \leq \sum_{j=1}^n \int_{a_j}^{b_j} (u_j')^2 dx = \int_\Omega (u')^2 dx.$$

Thus $\lambda_\Omega \geq \frac{\pi^2}{\ell^2}$. □

In order to show that (ii) $\not\Rightarrow$ (i) in Proposition 3.2 if $N \geq 2$ we need the following lemma.

Lemma 2.4. *Let $N \geq 2$, $x \in \mathbb{R}^N$. Let $\varepsilon > 0$. Then there exists $\eta \in \mathcal{D}(\mathbb{R}^N)$ such that $\|\eta\|_{H^1} \leq \varepsilon$ and $\eta(y) = 1$ for $y \in B(x, r)$ and for some $r > 0$ and $\eta(y) = 0$ for $|y - x| \geq \frac{1}{2}$.*

Proof: We can assume $x = 0$.

a) $N \geq 3$. Choose $\psi \in \mathcal{D}(\mathbb{R}^N)$ such that $\psi(y) = 1$ for $|y| \leq 1$ and $\psi(y) = 0$ for $|y| \geq 2$. Then for $n \in \mathbb{N}$, $\psi_n(y) = \psi(ny)$ defines a function satisfying $\psi_n(y) = 1$ for $|y| \leq \frac{1}{n}$, $\psi_n(y) = 0$ for $|y| \geq \frac{2}{n}$,

$$\int \psi_n^2 dy = \frac{1}{n^N} \int \psi^2 dy, \quad \int \left(\frac{\partial \psi_n}{\partial x_j}\right)^2 = n^{2-N} \int \left(\frac{\partial \psi}{\partial x_j}\right)^2.$$

So $\lim_{n \to 0} \|\psi_n\|_{H^1} = 0$ and the proof is finished in that case.

b) $N = 2$. Let $\psi \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ such that $\psi(x) \geq 0$ $(x \in \mathbb{R}^2)$, $\psi(x) = 0$ for $|x| \geq \frac{1}{3}$ and $\psi(x) = \left(\log \frac{1}{|x|}\right)^{1/4}$ for $0 < |x| < \frac{1}{4}$. Then $\psi \in H^1(\mathbb{R}^2)$ (cf. [4, IX Remark 17, p. 170]).

Let $\psi_n(x) = \inf\{\psi(x), n\}$. Then $\psi_n \in H^1(\mathbb{R}^N)$; in fact, $\frac{\partial \psi_n}{\partial x_j} = 1_{[\psi(x) < n]} \frac{\partial \psi}{\partial x_j}$ and so $\|\psi_n\|_{H^1} \leq \|\psi\|_{H^1}$ (cf. [5, IV §7 Prop. 6]).

There exist $r_n \in (0, \frac{1}{6})$ such that $\psi_n(x) = n$ for $|x| \leq 2r_n$. Let $\rho_n \in \mathcal{D}(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \rho_n(y) dy = 1$ and $\rho_n(y) = 0$ if $|y| \geq r_n$. Then $\eta_n = \frac{1}{n} \rho_n * \psi_n \in \mathcal{D}(\mathbb{R}^2)$ and $\eta_n(x) = 1$ for $|x| \leq r_n$. Moreover, $\|\eta_n\|_{H^1(\mathbb{R}^2)} \leq \frac{1}{n} \|\psi_n\|_{H^1} \leq \frac{1}{n} \|\psi\|_{H^1} \to 0$ $(n \to \infty)$. □

Example 2.5. (the Swiss cheese): Let $N \geq 2$. For $z \in \mathbb{Z}^N$ let $r_z > 0$ such that $\lim_{|z| \to \infty} r_z = 0$. Define $\Omega = \mathbb{R}^N \setminus \bigcup_{z \in \mathbb{Z}^N} \overline{B}(z, r_z)$. Then $\Omega \notin \mathcal{P}$.

Remark. However, Ω does not contain large balls. So (ii) $\not\Rightarrow$ (i) in Proposition 2.2 if $N \geq 2$.

Proof: Let $\varepsilon > 0$. We have to show that there exists $\psi \in \mathcal{D}(\Omega)$, $\psi \neq 0$, such that $\|\nabla\psi\|_{(L^2)^N} \leq \varepsilon\|\psi\|_{L^2}$.

There exists $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $\|\varphi\|_{L^2} = 1$, $\|\nabla\varphi\|_{(L^2)^N} \leq \varepsilon$, and the same remains true if φ is replaced by φ_x given by $\varphi_x(y) = \varphi(x-y)$ $(x, y \in \mathbb{R}^N)$. So it suffices to show that

$$\inf\{\|\varphi_x - \psi\|_{H^1} : \psi \in \mathcal{D}(\Omega),\ x \in \mathbb{R}^N\} = 0. \tag{2.2}$$

There exists $c \geq 0$ such that $\|\eta\varphi_x\|_{H^1} \leq c\|\eta\|_{H^1}$ for all $\eta \in \mathcal{D}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$.

Let $k \in \mathbb{N}$ such that $\operatorname{supp}\varphi \subset B(0,k)$. There exists $\ell \in \mathbb{N}$ such that for any $x \in \mathbb{R}^N$ there are at most ℓ numbers $z \in \mathbb{Z}^N$ such that $B(x,k) \cap B(z, r_z) \neq \emptyset$.

Let $\delta > 0$ be arbitrary. We observe that $\lim_{|x|\to\infty} \max\{r_z : B(x,k) \cap B(z,r_z) \neq \emptyset\} = 0$. So by Lemma 2.4, fixing x with $|x|$ sufficiently large, there exists $\eta \in \mathcal{D}(\mathbb{R}^N)$ such that $\|\eta\|_{H^1} \leq \delta$ and $\eta \equiv 1$ on $\overline{B}(z, r_z)$ whenever $B(z,r_z) \cap B(x,k) \neq \emptyset$. Since $\operatorname{supp}\varphi_x \subset B(x,k)$, it follows that $\psi = \varphi_x - \eta\varphi_x \in \mathcal{D}(\Omega)$ and $\|\psi - \varphi_x\|_{H^1} = \|\eta\varphi_x\|_{H^1} \leq c\|\eta\|_{H^1} \leq c\delta$. □

Remark 2.6. Lemma 2.4 implies that for $N \geq 2$, $H_0^1(\Omega) = H_0^1(\Omega\backslash\{a\})$ for any open set $\Omega \subset \mathbb{R}^N$ and $a \in \Omega$ (cf. [4, IX.4 Remark 18, p. 171]). In fact, let $\varphi \in \mathcal{D}(\Omega)$. For $\varepsilon > 0$ there exists $\eta \in \mathcal{D}(\mathbb{R}^N)$ such that $\eta \equiv 1$ in a neighborhood of a and $\|\eta\|_{H^1(\Omega)} \leq \varepsilon$. So $\varphi(1-\eta) \in \mathcal{D}(\Omega\backslash\{a\})$ and $\|\varphi - \varphi(1-\eta)\|_{H^1} = \|\varphi\eta\|_{H^1} \leq \text{const}\,\|\eta\|_{H^1} \leq \text{const}\,\varepsilon$, where the constant does not depend on η. So the completion of $\mathcal{D}(\Omega\backslash\{a\})$ and $\mathcal{D}(\Omega)$ with respect to H^1 are the same.

3. Potentials in $L^1_{\text{loc}}(\mathbb{R}^N)$.

In this section we investigate exponential stability of heatflow with arbitrary positive absorption in $L^1_{\text{loc}}(\mathbb{R}^N)$. The following theorem is the main result. A necessary condition is established which is stronger then condition (iii) of Theorem 1.2.

Theorem 3.1. *Let $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$. If $s(\Delta - V) < 0$, then*

$$\int_\Omega V\,dx = \infty$$

for every open set $\Omega \subset \mathbb{R}^N$ which is not a Poincaré set.

Remark. Here $s(\Delta - V) = s(\Delta_p - V) = \omega(\Delta_p - V)$ which is independent of $p \in [1,\infty)$.

Theorem 3.1 in conjunction with Example 2.5 shows that the characterization given in Section 1 (Theorem 1.2) is no longer valid if $V \notin L^1 + L^\infty$. In fact, suppose that $N \geq 2$ and let $\Omega \subset \mathbb{R}^N$ be the Swiss cheese of Example 2.5. Then $\Omega \notin \mathcal{P}$. Let $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $V_{|\Omega} = 0$. It follows from Theorem 3.1 that $s(\Delta - V) = 0$. However, choosing V such that $V_{|\Omega} = 0$ but $\int_H V\,dx = 1$ for every "hole" $H \subset \mathbb{R}^N\backslash\Omega$ we have $\int_G V\,dx = \infty$ for all G in $\mathcal{G}$.

For the proof of Theorem 3.1 we need some preparation. If $\Omega_1, \Omega_2 \in \mathcal{P}$, then, in general, $\Omega_1 \cup \Omega_2 \notin \mathcal{P}$ (for example, $\mathbb{R} = \Omega_1 \cup \Omega_2 \notin \mathcal{P}$ for $\Omega_1 = \mathbb{R}\backslash 2\mathbb{Z} \in \mathcal{P}$ and $\Omega_2 = \mathbb{R}\backslash 2\mathbb{Z}+1 \in \mathcal{P}$). However, letting $B^\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, B) < \delta\}$ for $B \subset \mathbb{R}^N$, $\delta > 0$, the following holds.

Proposition 3.2. *Let $\Omega \in \mathcal{P}$ and let $B \subset \mathbb{R}^N$ be open such that $B^\delta \in \mathcal{M}$ for some $\delta > 0$. Then $\Omega \cup B \in \mathcal{P}$.*

Proof: Assume that $\Omega \cup B \notin \mathcal{P}$. Then there exist $u_n \in \mathcal{D}(\Omega \cup B)$ such that

$$\int u_n^2 = 1, \quad \lim_{n\to\infty} \int (\nabla u_n)^2\, dx = 0.$$

Let $\delta > 0$ such that $B^\delta \in \mathcal{M}$ and choose $\varphi \in C^\infty(\mathbb{R}^N)$ such that $\varphi = 1$ on $\mathbb{R}^N \backslash B^\delta$ and $\varphi = 0$ on B^ε for some $\varepsilon > 0$ and $\sup_{x\in\mathbb{R}^N} |\nabla\varphi(x)|^2 < \infty$ (one may take $\varphi = 1 - 1_{B^{\delta/2}} * \rho$, where $\rho \in \mathcal{D}(B(0, \delta/2))$ such that $\int \rho = 1$). Then, since $B^\delta \in \mathcal{M}$,

$$\int_{B^\delta} u_n^2 \le \text{const}\, \|\nabla u_n\|_{L^2(\mathbb{R}^N)^N} \to 0 \quad (n \to \infty).$$

Hence

$$\lim_{n\to\infty} \int_{\Omega\backslash B^\delta} u_n^2 = 1.$$

Note that $\varphi u_n \in \mathcal{D}(\Omega)$. Since $\Omega \in \mathcal{P}$ we obtain,

$$\begin{aligned}\int_{\Omega\backslash B^\delta} u_n^2 &= \int_{\Omega\backslash B^\delta} (\varphi u_n)^2 \le \int_\Omega (\varphi u_n)^2 \le \text{const} \int_\Omega (\nabla(\varphi u_n))^2 \\ &\le 2\,\text{const} \int_\Omega [(\nabla\varphi)^2 u_n^2 + \varphi^2 (\nabla u_n)^2] \\ &\le 2\,\text{const}\, \|(\nabla\varphi)^2\|_\infty \int_{B^\delta} u_n^2 + 2\,\text{const}\, \|\varphi^2\|_\infty \int_\Omega (\nabla u_n)^2 \longrightarrow 0 \quad (n \to \infty).\end{aligned}$$

This is a contradiction since

$$\lim_{n\to\infty} \int_{\Omega\backslash B^\delta} u_n^2 = 1. \qquad \square$$

Corollary 3.3. *Let $\Omega \in \mathcal{P}$ and let $\widehat{\Omega} \subset \mathbb{R}^N$ be open. If $\widehat{\Omega}$ is contained in a strip, then $\Omega \cup \widehat{\Omega} \in \mathcal{P}$.*

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^N$ be open such that $\Omega \notin \mathcal{P}$. Then there exist bounded open sets $\Omega_n \subset \Omega$ such that Ω_n is of class C^∞, $dist(\Omega_n, \Omega_j) \ge 1$ for $j = 1, \dots, n-1$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} s(\Delta_{\Omega_n}) = 0$.*

Consequently, there exist $u_n \in H_0^1(\Omega_n)$ such that $\|u_n\|_{L^2(\Omega_n)} = 1$, $\Delta u_n = -\lambda_n u_n$ with $\lambda_n = -s(\Delta_{\Omega_n})$. Moreover, $u_n \in L^\infty(\Omega_n)$ and $\sup_{n\in\mathbb{N}} \|u_n\|_\infty < \infty$.

Proof: Let $\varepsilon_n \downarrow 0$. There exists $v \in \mathcal{D}(\Omega)$ such that

$$\int v^2 = 1, \quad \int (\nabla v)^2 < \varepsilon_1.$$

Let $\Omega_1 = \{x \in \Omega : v(x) \ne 0\}$. Then Ω_1 is a bounded open set of class C^∞ such that $\lambda_1 < \varepsilon_1$.

Assume that $n \in \mathbb{N}$ and $\Omega_1, \dots, \Omega_n$ are constructed such that Ω_n is bounded open of class C^∞, $\operatorname{dist}(\Omega_i, \Omega_j) \geq 1$ for $i, j \in \{1, \dots, n\}$, $i \neq j$ and $\lambda_j < \varepsilon_j$, $j = 1, \dots n$. Let

$$K = \Big\{x \in \mathbb{R}^N : \operatorname{dist}\big(x, \bigcup_{j=1}^{n} \Omega_j\big) \leq 1\Big\}.$$

Then K is compact. So it follows from Corollary 3.3 that $\widehat{\Omega} = \Omega \setminus K \notin \mathcal{P}$. Consequently, there exists $v \in \mathcal{D}(\widehat{\Omega})$ such that

$$\int v^2 = 1, \quad \int (\nabla v)^2 < \varepsilon_{n+1}.$$

Let $\Omega_{n+1} = \{x : v(x) \neq 0\}$. Then Ω_{n+1} is a bounded open set of class C^∞ and $\operatorname{dist}(\Omega_{n+1}, \Omega_j) \geq 1$ for $j = 1 \dots n$. We have proved the first assertion.

Since Δ_{Ω_n} has compact resolvent, there exist $u_n \in D(\Delta_{\Omega_n})$ such that

$$\Delta u_n = -\lambda_n u_n, \quad \|u_n\|_{L^2} = 1.$$

It remains to show that $\sup_{n \in \mathbb{N}} \|u_n\|_\infty < \infty$. Let

$$O = \bigcup_{j=1}^{n} \Omega_j$$

and $k > \frac{N}{4}$. Then $u_n \in D(\Delta_O^k)$ and $\|\Delta_O^m u_n\| = \lambda_n^m$ $(m = 0, \dots, k)$. Hence $(u_n)_{n \in \mathbb{N}}$ is bounded in $D(\Delta_O^k)$ for the graph norm. But $D(\Delta_O^k) \hookrightarrow H^{2k}(O) \hookrightarrow L^\infty(O)$, see [4, Théorème IX.25 and Corollary IX.15]. □

Proof of Theorem 3.1: Assume that there exists $O \subset \mathbb{R}^N$ open such that $O \notin \mathcal{P}$ and $\int_O V < \infty$. By Lemma 3.4 there exist open sets $\Omega_n \subset O$ such that $\Omega_n \cap \Omega_m = \emptyset$ for $n \neq m$ and there exist $u_n \in H_0^1(\Omega_n) \cap L^\infty$ such that

$$\int u_n^2 = 1, \quad \int (\nabla u_n)^2 \to 0 \quad (n \to \infty)$$

and $M := \sup_{u \in \mathbb{N}} \|u_n\|_\infty < \infty$. Consequently,

$$\int (\nabla u_n)^2 + \int V u_n^2 \leq \int (\nabla u_n)^2 + M^2 \int_{\Omega_n} V \to 0 \quad (n \to \infty)$$

since

$$\sum_{n=1}^{\infty} \int_{\Omega_n} V \leq \int_O V < \infty.$$

Hence $s(\Delta_2 - V) = 0$ by (1.2). □

If $N = 1$, then Theorem 1.2 also holds for general potentials.

Theorem 3.5. *Let $N = 1$ and $0 \le V \in L^1_{\text{loc}}(\mathbb{R})$. The following are equivalent*

(i) $s(\Delta - V) < 0$.

(ii) *There exist $k \in \mathbb{N}$, $\alpha > 0$ such that $\int_{x-k}^{x+k} V\,dy \ge \alpha$ for all $x \in \mathbb{R}$.*

(iii) *$\int_G V\,dy = \infty$ for all $G \in \mathcal{G}$.*

Proof: (i) $\Rightarrow$ (iii) follows from Theorem 3.1.

(iii) $\Rightarrow$ (ii). If (ii) does not hold, there exist $x_n \in \mathbb{R}$ such that

$$\int_{x_n-n}^{x_n+n} V dy \le 2^{-n} \quad (n \in \mathbb{N}).$$

Then

$$G = \bigcup_{n\in\mathbb{N}} (x_n - n, x_n + n) \in \mathcal{G}$$

but

$$\int_G V dy \le 1.$$

(ii) $\Rightarrow$ (i). There exist $k \in \mathbb{N}$ and $\alpha > 0$ such that

$$\int_{nk}^{(n+1)k} V(y)dy \ge \alpha \quad \text{for all } n \in \mathbb{Z}.$$

Let $u \in \mathcal{D}(\mathbb{R})$ and $n \in \mathbb{Z}$. Choose $x_0 \in [nk, (n+1)k]$ such that

$$|u(x_0)| = \inf\{|u(x)| : nk \le x \le (n+1)k\}.$$

Then

$$\begin{aligned} u(x)^2 &= \Big(u(x_0) + \int_{x_0}^{x} u'(y)\,dy\Big)^2 \le \Big(|u(x_0)| + |x - x_0|^{\frac12}\Big(\int_{x_0}^{x} u'(y)^2\,dy\Big)^{\frac12}\Big)^2 \\ &\le 2u(x_0)^2 + 2|x - x_0| \int_{x_0}^{x} u'(y)^2 dy \\ &\le 2u(x_0)^2 + 2k \int_{nk}^{(n+1)k} u'(y)^2 dy \quad (nk \le x \le (n+1)k). \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{kn}^{k(n+1)} u(x)^2 dx &\le 2ku(x_0)^2 + 2k^2 \int_{nk}^{(n+1)k} u'(y)^2 dy \\ &\le \frac{2k}{\alpha} u(x_0)^2 \int_{kn}^{k(n+1)} V(y)\,dy + 2k^2 \int_{nk}^{(n+1)k} u'(y)^2 dy \\ &\le \frac{2k}{\alpha} \int_{kn}^{k(n+1)} u(y)^2 V(y)\,dy + 2k^2 \int_{nk}^{(n+1)k} u'(y)^2 dy. \end{aligned}$$

Summing over n yields with $\beta = \max\{\frac{2k}{\alpha}, 2k^2\}$,

$$\int_{\mathbb{R}} u(y)^2 dy \le \beta\Big\{\int_{\mathbb{R}} u^2 V + \int_{\mathbb{R}} u'^2\Big\}.$$

Thus, $\inf\{\int u'^2 + \int u^2 V : \int u^2 = 1, u \in \mathcal{D}(\mathbb{R})\} \geq \frac{1}{\beta}$. □

Next we consider Theorem 1.10 for general potentials.

Example 3.6. Let $N = 1$. There exist $E \in \mathcal{M}$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R})$ such that $s(\Delta_2 - V) < 0$ but $s(\Delta - V1_{E^c}) = 0$ and so (in view of Theorem 1.10) $s(\Delta - V_k) = 0$ for all $k \in \mathbb{N}$, with $V_k = \inf\{V, k\}$. In fact, let $E = \bigcup_{n=1}^{\infty} [2n, 2n+r_n]$, where $0 < r_n < 1$. Let $V(x) = \frac{1}{r_n}$ if $x \in [2n, 2n + r_n]$ and $V = 0$ on $\mathbb{R}\backslash E$. Then, $\int_G V dx = \infty$ for all $G \in \mathcal{G}$ and so $s(\Delta_2 - V) < 0$. If $\sum_{n=1}^{\infty} r_n < \infty$, then $E \in \mathcal{M}$ (by Corollary 1.9).

The preceding example shows that alteration of V on a set in $\mathcal{M}$ may change the property of exponential stability. However, we have the following theorem.

Theorem 3.7. *Let $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $s(\Delta - V) < 0$. Let $B \subset \mathbb{R}^N$ be measurable such that $B^\delta \in \mathcal{M}$ for some $\delta > 0$. Then $s(\Delta - V1_{B^c}) < 0$.*

Proof: Since $s(\Delta_2 - V) < 0$, there exists $c > 0$ such that

$$\|u\|^2_{L^2} = \int u^2 \leq c\Big(\int (\nabla u)^2 + \int V u^2\Big) \quad \text{for all } u \in \mathcal{D}(\mathbb{R}^N).$$

Assume that $s(\Delta_2 - V1_{B^c}) = 0$. Then there exist $u_n \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\int u_n^2 = 1, \quad \int (\nabla u_n)^2 + \int_{B^c} V u_n^2 \to 0 \quad (n \to \infty).$$

Since $B^\delta \in \mathcal{M}$,

$$\int_{B^\delta} u_n^2 \leq \text{const}\Big(\int (\nabla u_n)^2\Big)^{\frac{1}{2}} \longrightarrow 0 \quad (n \to \infty). \tag{3.1}$$

Let $\varphi \in C^\infty(\mathbb{R}^N)$ such that $\varphi = 1$ on $\mathbb{R}^N \backslash B^\delta$ and $\varphi = 0$ on B and φ, $(\nabla\varphi)^2 \in L^\infty$ (cf. proof of Proposition 3.2). Then

$$\begin{aligned}\int_{\mathbb{R}^N\backslash B^\delta} u_n^2 &\leq \int_{\mathbb{R}^N} (\varphi u_n)^2 \leq c\Big\{\int (\nabla(\varphi u_n))^2 + \int V(\varphi u_n)^2\Big\} \\ &\leq c\Big\{2\int (\nabla\varphi)^2 u_n^2 + 2\int \varphi^2 (\nabla u_n)^2 + \int V\varphi^2 u_n^2\Big\} \\ &\leq c\Big\{2\int_{B^\delta\backslash B} u_n^2 \|(\nabla\varphi)^2\|_\infty + 2\int (\nabla u_n)^2 \cdot \|\varphi\|^2_\infty + \|\varphi^2\|_\infty \int_{B^c} V u_n^2\Big\} \to 0\end{aligned}$$

by (3.1). This together with (3.1) contradicts that $\int u_n^2 = 1$.

Remark. If B is included in a finite union of strips, then $B^\delta \in \mathcal{M}$ for all $\delta > 0$.

We conclude with a similar result for the special potential $V = \varepsilon\mathbf{1}$.

Proposition 3.8. *Let $\Omega \subset \mathbb{R}^N$ be open such that $\Omega^\delta \in \mathcal{P}$ for some $\delta > 0$. Then $s(\Delta_2 - \varepsilon 1_{\Omega^c}) < 0$ for all $\varepsilon > 0$.*

Proof: Assume that $s(\Delta_2 - \varepsilon 1_{\Omega^c}) = 0$. Then there exist $u_n \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\int u_n^2 = 1 \quad \text{and} \quad \int (\nabla u_n)^2 + \varepsilon \int_{\Omega^c} u_n^2 \to 0 \quad (n \to \infty).$$

Let $\varphi \in C^\infty(\mathbb{R}^N)$ such that $\varphi, (\nabla\varphi)^2 \in L^\infty$, $\varphi = 1$ on Ω and $\varphi = 0$ on $\mathbb{R}^N \backslash \Omega^\delta$. Then

$$\begin{aligned}\int_\Omega u_n^2 &\le \int (\varphi u_n)^2 \le \text{const}\Big(\int \nabla(\varphi u_n)^2\Big)\\ &\le 2\,\text{const}\Big(\int (\nabla\varphi)^2 u_n^2 + \int \varphi^2(\nabla u_n)^2\Big)\\ &\le 2\,\text{const}\Big(\int_{\Omega^\delta\backslash\Omega} u_n^2 \|(\nabla\varphi)^2\|_\infty + \int (\nabla u_n)^2 \|\varphi^2\|_\infty\Big) \to 0.\end{aligned}$$

This leads to a contradiction since

$$\int_{\Omega^c} u_n^2 \to 0 \quad (n \to \infty)$$

and

$$\int u_n^2 = 1.$$

Remark. The hypothesis on Ω in Proposition 3.8 is weaker than that in Theorem 3.7. In fact, for $\Omega \subset \mathbb{R}$ it is easy to see that $\Omega^\delta \in \mathcal{M}$ for some $\delta > 0$ if and only if Ω is bounded.

REFERENCES

[1] R. Adams, "Sobolev Spaces," Acad. Press, 1975.

[2] W. Arendt, C.J.K. Batty, and Ph. Bénilan, *Asymptotic stability of Schrödinger semigroups on $L^1(\mathbb{R}^N)$*, Math. Z., 209 (1992), 511–518.

[3] C.J.K. Batty, *Asymptotic stability of Schrödinger semigroups: Path integral methods*, Math. Ann., 292 (1992), 457–492.

[4] H. Brezis, "Analyse Fonctionnelle," Masson, 1983.

[5] R. Dautray, J.L. Lions, "Analyse Mathématique et Calcul Numérique," Masson, 1987.

[6] E.B. Davies, "Heat Kernels and Spectral Theory," Cambridge University Press, 1989.

[7] R. Hempel, J. Voigt, *On the L_p-spectrum of Schrödinger operators*, J. Math. Anal. Appl., 121 (1987), 138–159.

[8] T. Kato, *L_p-theory of Schrödinger operators with a singular potential*, in "Aspects of Positivity in Functional Analysis," R. Nagel, U. Schlotterbeck, M. Wolff (eds.), North Holland, Amsterdam, 1986.

[9] R. Nagel, (ed.), "One-Parameter Semigroups of Positive Operators," Springer LN 1184, Berlin, 1986.

[10] M. Reed, B. Simon, "Methods of Modern Mathematical Physics," I-IV, Academic Press, London, 1978.

[11] B. Simon, *Brownian motion, L^p properties of Schrödinger operators and localization of binding*, J. Functional Anal., 35 (1980), 215–229.

[12] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc., 7 (1982), 447–526.

[13] J. Voigt, *Interpolation for positive C_0-semigroups on L^p-spaces*, Math. Z., 188 (1985), 283–286.

[14] J. Voigt, *Absorption semigroups, their generators and Schrödinger semigroups*, J. Operator Theory, 20 (1988), 117–131.