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INTEGRATED SOLUTIONS TO IMPLICIT DIFFERENTIAL EQUATIONS

Abstract. This paper establishes in two completely different ways that the abstract degenerate initial value problem $\frac{d}{dt}(Mu(t)) + Lu(t) = f(t), 0 \leq t \leq \tau, Mu(0) = Mu_0$, has always an integrated solution, provided that for all complex numbers z in the half-plane $Rez \geq a > 0$ the operator pencil $P(z) = zM + L$ has a bounded inverse from the Banach space X to $\mathcal{D}(L)$, endowed with the graph-norm, and its norm has a polynomial growth there. Some applications to partial differential equations are given. A Trotter-Kato type result is proved, too.

1. Introduction

The main purpose of this paper is to show that the initial value problem

$$\frac{d}{dt}(Mu(t)) + Lu(t) = f(t), \qquad 0 \leq t \leq \tau, \tag{1.1}$$

$$Mu(0) = Mu_0, \qquad u_0 \in \mathcal{D}(L), \tag{1.2}$$

has always a "solution", provided that it is understood as an integrated solution, even if (1.1), (1.2) may be not salvable in strict sense since, e.g., (1.1) and (1.2) are not compatible.

Here L and M are two closed linear operators from the complex Banach space Y to the (complex) Banach space X, f is a continuous function from $[0,\tau]$ into X and u_0 belongs to the domain $\mathcal{D}(L)$ of L.

For our aim, it is not restrictive to assume that $\mathcal{D}(L)$ is contained into $\mathcal{D}(M)$ and L has a bounded inverse. Henceforth we shall suppose that these properties hold (besides in Section 4 where we consider slightly more general hypotheses). In all of this paper we also shall require that the operator pencil $P(z) = zM + L$ has a bounded inverse $P(z)^{-1}$ for any

$$z \in \Sigma_a : Rez \geq a > 0$$

and there are $K > 0, m = 0, 1, \ldots$, such that

$$\|LP(z)^{-1}\|_{\mathcal{L}(X)} \leq K(1 + |z|)^m, \quad z \in \Sigma_a. \tag{P}$$

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There are many concrete interesting examples of operators satisfying (P) and we list some of them.

EXAMPLE 1.1. Let L, M be two densely defined operators acting in the complex Hilbert space $H = X = Y$, with inner product $<,>$ and induced norm $|| \, ||$, such that M is non negative and selfadjoint in H, L and its adjoint L^* fulfil

$$(1.3) \qquad Re < Lu, u > \geq c_0 ||u||^2, \qquad u \in \mathcal{D}(L),$$

$$(1.4) \qquad Re < L^* f, f > \geq c_0 ||f||^2, \qquad f \in \mathcal{D}(L^*),$$

where c_0 is a positive constant.

Assume further that for all $z \in \Sigma_a$, the adjoint operator of $zM + L$ coincides with $\bar{z}M + L^*$. This happens, as it is well known, if M is bounded, for example. The estimate

$$Re < P(z)u, u > \geq c_0 ||u||^2, u \in \mathcal{D}(L), z \in \Sigma_a,$$

implies that $zM + L$ has a closed range and is one-to-one.

The hypothesis (1.4) on the adjoint assures that $zM + L$ has a bounded inverse and

$$||P(z)^{-1}||_{\mathcal{L}(X)} \leq c_0^{-1}.$$

If M is bounded, we conclude that (P) holds with $m = 1$.

If M is unbounded, more must be assumed in general, but if L itself is selfadjoint with $\mathcal{D}(L^{1/2}) \subseteq \mathcal{D}(M)$, the estimate

$$(1.5) \qquad \begin{aligned} ||(zM + L)u|| \, ||u|| &\geq Re < (zM + L)u, u > \geq < Lu, u > \\ &= ||L^{1/2}u||^2, \qquad u \in \mathcal{D}(L), \end{aligned}$$

implies that

$$||f|| \, ||P(z)^{-1} f|| \geq ||L^{1/2} P(z)^{-1} f||^2, \qquad f \in H,$$

and therefore

$$c_0^{-1} ||f||^2 \geq ||L^{1/2} P(z)^{-1} f||^2,$$

that is, there are $C_1, C_2 > 0$ with

$$||MP(z)^{-1} f|| \leq C_1 ||L^{1/2} P(z)^{-1} f|| \leq C_2 ||f||, \qquad f \in H.$$

Hence, (P) is again satisfied with $m = 1$.

For an application to first order symmetric systems of partial differential equations, see Favini [9, pp.450-451].The second case (M unbounded) covers various equations (1.1) where M, L are defined by suitable elliptic differential operators.

EXAMPLE 1.2. Let us suppose that $z = 0$ is a polar singularity of $(z + ML^{-1})^{-1} = L(zL + M)^{-1}$, that is, there are $k \in \mathbb{N}, \varepsilon > 0, \delta > 0$, such that $zL + M$ has a bounded inverse ($\in \mathcal{L}(X,Y)$) for all $0 < |z| \leq \varepsilon$ and

$$\|L(zL + M)^{-1}\|_{\mathcal{L}(X)} \leq \delta|z|^{-k}, 0 < |z| \leq \varepsilon.$$

Hence assumption (P) holds with $m = k - 1$.

It is then well known that one has the representation $X = N(T^k) \oplus R(T^k)$, where $T = ML^{-1}$, and $N(T)$, $R(T)$ denote the kernel and the range of the operator T, respectively. On the basis of this property a detailed study of (1.1), (1.2) has been done in Favini [10]. For another "algebraic" type approach to the problem in this situation, invoking the Drazin inverse of a linear operator, we refer to the very recent paper by Nashed and Zhao [13].

EXAMPLE 1.3. Maxwell's equations

$$rot E = -\frac{\partial B}{\partial t}, \qquad rot H = \frac{\partial D}{\partial t} + J, \tag{1.6}$$

in R^3, where E (respectively, H) denotes the electric (resp., magnetic) field intensity, B (resp., D) denotes the electric (resp., magnetic) flux density and J is the current density, when the medium which fills the space is supposed to be linear, anisotropic and nonhomogeneous, (that is, $D = \varepsilon E, B = \mu H, J = \sigma E + J'$, for some 3×3 matrices $\varepsilon(x)$, $\mu(x)$ and $\sigma(x), x \in R^3$, and J' is a given forced current density), (1.6) reads

$$\frac{\partial}{\partial t}(c(x)w) + \sum_{i=1}^{3} a_i(x)\frac{\partial w}{\partial x_i} + b(x)w = g(t,x) \quad \text{in } [0,\tau] \times R^3, \tag{1.7}$$

with

$$w = (E,H), \quad c(x) = \begin{bmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{bmatrix}, \quad b(x) = \begin{bmatrix} \sigma(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad g(t,x) = -(J'(t,x),0).$$

In the paper [16], by Yagi, (1.7) is formulated in the abstract form (1.1) by taking $X = Y = (L^2(R^3))^6$. Under the assumptions that $\varepsilon(x), \mu(x)$ and $\sigma(x), x \in R^3$, are real matrices the components of which are bounded measurable functions in $R^3, \varepsilon(x)$ is symmetric and ≥ 0 for every $x \in R^3$, there exist $\delta > 0, \gamma \geq 0$ such that

$$(\{\gamma\varepsilon(x) + \sigma(x)\}\xi, \xi) \geq \delta \sum_{i=1}^{3} \xi_i^2, \qquad \xi = (\xi_1, \xi_2, \xi_3) \in R^3,$$

uniformly in $x \in R^3$, $\mu(x)$ is symmetric and $\geq \delta > 0$ uniformly in $x \in R^3$, if L, M are defined by

$$\mathcal{D}(L) = \{v \in X; \sum_{i=1}^{3} a_i \frac{\partial v}{\partial xi} \in X\}, \qquad Lv = \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} + b(x)v,$$

$M = K^2, K$ the multiplication operator by $\sqrt{c(x)}$, then by formula (5.12) in Yagi [16, p.404],

$$Re < Lv, v > \geq \delta||v||^2 + \lambda_0||Kv||^2, \qquad v \in \mathcal{D}(L), \tag{1.8}$$

with $\lambda_0 \leq -max\{\gamma, 1\}$. In turn, this reads

$$Re < Lv - \lambda_0 Mv, v > \geq \mathcal{D}||v||^2, \qquad v \in \mathcal{D}(L).$$

Fix $\lambda_0 \leq -max\{\gamma, 1\}$. The change of variable $u = e^{-\lambda_0 t}v$ transforms (1.1) to

$$\frac{d}{dt}(Mv(t)) + (L - \lambda_0 M)v(t) = e^{\lambda_0 t}f(t), \qquad 0 \leq t \leq \tau.$$

Arguing as in Example 1.1 and taking into account (1.8), one deduces that

$$Re < Lv - \lambda_0 Mv + \lambda Mv, v > \geq \delta||v||^2, \qquad v \in \mathcal{D}(L), \qquad Re\lambda \geq 0,$$

and $||(\lambda M + L - \lambda_0 M)v|| \geq \delta||v||$, $\forall v \in \mathcal{D}(L)$, $Re\lambda$ sufficiently large.
Always applying the argument in [16, p. 405], we see that if $(\lambda M + L - \lambda_0 M)^* w = 0$ $= (\overline{\lambda} M + L^* - \lambda_0 M)w = 0$, then $w \in \mathcal{D}(L)$ and therefore

$$Re < L^* w, w >= Re(\lambda_0 - \lambda)||Kw||^2 = Re < w, Lw > \geq \delta||w||^2 + \lambda_0||Kw||^2$$

implies

$$-Re\lambda||Kw||^2 \geq \delta||w||^2.$$

Since $Re\lambda \geq a > 0$, we conclude that $Kw = w = 0$.
Hence, we deduce that (P) is verified with $m = 1$.
For other different approaches to this type of equations we refer to Duvaut and Lions [8] and to Povoas [15].

EXAMPLE 1.4. If $X = Y$ and M has a bounded inverse, (P) says that

$$||(z + LM^{-1})^{-1}||_{\mathcal{L}(X)} \leq C(1 + |z|)^{m-1}, z \in \Sigma_a.$$

It in fact suffices to notice that $(z + LM^{-1})^{-1} = MP(z)^{-1} = z^{-1}\{I - LP(z)^{-1}\}$, for all $z \in \Sigma_a$. Therefore $-LM^{-1}$ generates an integrated semigroup too, see [2] and [14]. We also refer to the even more general theory of regularized semigroups developed in Hieber, Holderrieth and Neubrander [11, Theorem 3.4, p.372].
On the other hand, (see Chazarain [6, p.403] and [3, Section 4]), this means that $-LM^{-1}$ generates a distribution semigroup with exponential growth.

The main result motivating this paper is contained in Section 2. In Section 3 we establish the continuous dependence of the integrated solution to (1.1) on the operators L and M, generalizing Trotter-Kato theorem for C_0-semigroups. This theorem has to be compared with the one by Busenberg and Wu [5] on integrated semigroups and is obtained using a very recent convergence property established in Barbu and Favini [4]. In Section 4, on the basis of a complex Laplace representation theorem due to Arendt and Kellermann [3], we give a slightly different version of Theorem 2.1, obtaining a little bit more regularity. Moreover, in this latter approach it is not needed that L is invertible. A similar approach is also used by Abdelaziz and Neubrander [1], but their results are different from ours.

2. Integrated solutions to (1.1), (1.2)

To begin with, we formulate a previous existence and uniqueness theorem relative to (1.1), (1.2) obtained in Favini [9], generalizing the operational method by Da Prato and Grisvard [7] concerning parabolic equations.

PROPOSITION 2.1. *Let us assume (P). If* $f \in C^{(m+2)}([0,\tau];X), f^{(j)}(0) = 0$ *for* $j = 0, 1, \ldots, m+1$, *and* $Mu_0 \in R(T^{m+2})$, *with* $T = ML^{-1}$, *then (1.1), (1.2) has a unique strict solution.*

By a strict solution u of (1.1), (1.2) we mean a function $u \in C([0,\tau];\mathcal{D}(L))$ such that $Mu \in C^{(1)}([0,\tau];X)$ and (1.1), (1.2) hold.
Of course, if $k \in \mathbb{N} \cup \{0\}, C^{(k)}([0,\tau];X)$ denotes the set of all k-times continuously differentiable X-valued functions on $[0,\tau]$. We let $C^{(0)}([0,\tau];X) = C([0,\tau];X)$ and $C^{(\infty)}([0,\tau];X) = \bigcap_k C^{(k)}([0,\tau];X)$.

One sees that in general both regularity (for arbitrary f) and compatibility relations between $f^{(j)}(0)$ and u_0, for some j, are necessary to have a solution, even if (1.1) is required to hold for $t > 0$. Nevertheless it is a well known fact in applied mathematics, for example, in control theory, that also not compatible systems (1.1), (1.2) allow a corresponding "answer" u in some weak sense. Our aim is to clarify the meaning of this answer.

If $M = I$, the identity operator in $X = Y$, the notion of integrated solution to (1.1), (1.2) has been introduced guaranteeing that it exists for all $f \in C([0,\tau];X)$ and any $u_0 \in X$, see [2, 3, 12].

We shall present an analogous definition of k-integrated solution to (1.1), (1.2), $k = 0, 1, \ldots$, and we shall show that under (P) it always exists, whatever are f and $u_0 \in X$, see [2, 3, 12].

In order to motivate our notion of integrated solution, we observe that if u is a strict solution of (1.1), (1.2), the position

$$u_1(t) = \int_0^t u(s)ds$$

and

$$u_k(t) = \int_0^t u_{k-1}(s)ds, \quad k = 2, 3, \ldots$$

leads to the relation

$$Mu_k(t) = -\int_0^t Lu_k(s)ds + \frac{t^k}{k!}Mu_0 + f_{k+1}(t), \qquad k = 1, 2, \ldots, \tag{2.1}$$

where $f_{k+1}(t) = \int_0^t \frac{(t-s)^k}{k!} f(s)ds$.

On the other hand, if we introduce $Lu_k(t) = v_k(t), T = ML^{-1}$, (2.1) can also be written as

$$Tv_k(t) = -\int_0^t v_k(s)ds + \frac{t^k}{k!}w_0 + f_{k+1}(t), \qquad k = 1, 2, 3, \dots, \tag{2.2}$$

with $w_0 = Mu_0$.

We are in a position to introduce

DEFINITION 2.1. *A function $u \in C([0,\tau]; \mathcal{D}(L))$ such that (2.1) holds (with u instead of u_k), is a k-integrated solution of (1.1), (1.2).*

Equivalently, we could say that u is a k-integrated solution to (1.1), (1.2) if $v = Lu$ is a k-integrated solution to

$$\frac{d}{dt}(Tv(t)) = -v(t) + f(t), \qquad 0 \le t \le \tau, \tag{2.3}$$

$$Tv(0) = w_0. \tag{2.4}$$

REMARK 2.2. One could think of an alternative definition of integrated solution to (1.1), (1.2), in more strict analogy to the case of M=I, weaking the (apparently restrictive) assumption that the function u in Definition 2.1 belongs to $C([0,\tau]; \mathcal{D}(L))$. Precisely, we would require that $u \in C([0,\tau]; Y), u(t) \in \mathcal{D}(M)$ for all $t \in [0,\tau], Mu \in C([0,\tau]; X)$ and

$$Mu(t) = -L\int_0^t u(s)ds + \frac{t^k}{k!}w_0 + f_{k+1}(t). \tag{2.5}$$

If $S = L^{-1}M(\in \mathcal{L}(Y))$, (2.5) reads equivalently

$$Su(t) = -\int_0^t u(s)ds + \frac{t^k}{k!}Su_0 + L^{-1}f_{k+1}(t). \tag{2.6}$$

Notice that the integral $\int_0^t u(s)ds \in \mathcal{D}(L)$. Hence equation (2.6) is considered in the space $\mathcal{D}(L)$, endowed with the graph norm.

But as an operator from $\mathcal{D}(L)$ into itself, the operator S satisfies exactly assumption (P), since

$$||(zS+I)^{-1}f||_{\mathcal{D}(L)} = ||LP(z)^{-1}Lf||_X \le C(1+|z|)^m ||f||_{\mathcal{D}(L)},$$

for all $f \in \mathcal{D}(L)$ and $z \in \Sigma_\alpha$.

It follows that the two definitions are in fact equivalent.

We have

THEOREM 2.1. *Let us assume (P). If $k \ge m+3$, then (1.1), (1.2) has a unique k-integrated solution u for all $f \in C([0,\tau]; X)$ and $u_0 \in \mathcal{D}(L)$.*

Proof. We need to show that (2.3), (2.4) has a unique k-integrated solution. To accomplish this, we remark that if $v = v(t)$ is a strict solution to the problem

$$\frac{d}{dt}Tv(t) = -v(t) + \frac{t^{k-1}}{(k-1)!}w_0 + f_k(t), \qquad 0 \le t \le \tau, \tag{2.7}$$

$$Tv(0) = 0, \tag{2.8}$$

then such a solution satisfies the integral equation (2.2) and conversely.

To solve (2.7), (2.8) it suffices to apply Proposition 2.1 with the nonhomogeneous function $F(t)$ given by

$$F(t) = \frac{t^{k-1}}{(k-1)!}w_0 + f_k(t), \qquad 0 \le t \le \tau .$$

Since for each $j = 0, 1, ..., m+1$, one has

$$F^{(j)}(t) = \frac{t^{k-1-j}}{(k-1-j)!}w_0 + \int_0^t \frac{(t-s)^{k-1-j}}{(k-1-j)!}f(s)ds,$$

assumption $k \ge m+3$ implies that in fact $F^{(j)}(0) = 0, j = 0, 1, ..., m+1$, and this allows to use Proposition 2.1 to treat (2.7), (2.8). ■

REMARK 2.3. If the function $v(t)$ satisfying (2.7) is k-times continuously differentiable on $[0, \tau]$, then $v^{(k)}(t) = z(t)$ fulfils (2.3); in fact, deriving both the members of (2.2) we infer

$$Tv^{(j)}(t) = -v^{(j-1)}(t) + \frac{t^{k-j}}{(k-j)!}w_0 + \int_0^t \frac{(t-s)^{k-j}}{(k-j)!}f(s)ds, \qquad j = 1, 2, ..., k$$

and hence a further derivation gives the result. However, in general, no initial condition to $z(t)$ can be prescribed.

REMARK 2.4. To precise furthermore last affirmation contained in the remark above, we point out that even if no strict solution to (1.1) (1.2) could exist, according to Theorem 2.1 all possible solutions to (1.1) are to be sought among its integrated solutions.More precisely, denote by u_k the k-integrated solution to (1.1), (1.2) where $k \ge m+3$. Then by the definition it is clear that, whenever a strict solution u exists, one has $u = u_k^{(k)}$. However, in contrast to the case $M = I$, (see [2]), it can happen that $u_{m+3} \in C^{(\infty)}([0,\tau]; Y)$ but there does not exist a strict solution. See Example 2.2 below.

In fact, we illustrate Remarks 2.3 and 2.4 in the same time giving two very simple examples.

EXAMPLE 2.1. Let $f, g \in C([0,\tau]; \mathbb{C})$. Then it is easily seen that the algebraic-differential system

$$\frac{d}{dt}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}\right) = -\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \qquad 0 \le t \le \tau,$$

has the strict solution

$$u(t) = e^{-t}(u(0) + g(0)) - g(t) + \int_0^t e^{-(t-s)}(f(s) + g(s))ds\,,$$
$$v(t) = g(t), \qquad 0 \le t \le \tau\,.$$

We are clearly in the situation of assumption (P) with $m = 0, k = 3$.

In this case the 3-integrated solution $(x(t), y(t))$ to the problem is given by

$$y(t) = \int_0^t \frac{(t-s)^2}{2} g(s)ds\,,$$

while

$$(x(t + y(t))' = -x(t) + \frac{t^2}{2}(x(0) + y(0)) + \int_0^t \frac{(t-s)^2}{2} f(s)ds$$

leads to

$$((x(t) + y(t))^{(3)})' - (x(t) + y(t))^{(3)} + f(t) + g(t)\,,$$

that is,

$$(x(t) + y(t))^{(3)} = e^{-t}(x + y)^{(3)}(0) + \int_0^t e^{-(t-s)}(f(s) + g(s))ds\,.$$

Therefore, $u(t) = x^{(3)}(t), v(t) = y^{(3)}(t)$ solves uniquely the initial-value problem with given $u(0)$.

EXAMPLE 2.2. Given $f, g \in C([0,\tau];\mathbb{C}), u_0, v_0 \in \mathbb{C}$, the 4-integrated solution (x, y) to the problem

$$\frac{d}{dt}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}\right) = -\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \qquad 0 \le t \le \tau\,, \tag{2.9}$$

$$v(0) = v_0, \tag{2.10}$$

(observing that here $m = 1$, $k = 4$) is characterized by means of the relations

$$y'(t) = -x(t) + \frac{t^3}{6}v_0 + \int_0^t \frac{(t-s)^3}{6} f(s)ds,$$
$$y(t) = \int_0^t \frac{(t-s)^3}{6} g(s)ds,$$

so that the unique strict solution of (2.9), (2.10) exists iff $v_0 = g(0)$ and $g \in C^{(1)}([0,\tau];\mathbb{C})$; then it in fact coincides with $(x^{(4)}(t), y^{(4)}(t))$.

3. A Trotter-Kato type theorem

Suppose we are given a sequence of linear operators $L_n, M_n, n \in \mathbb{N}, L, M$, such that

(Q) (P) *holds for all pairs* (L_n, M_n) *and* $(L, M), n \in \mathbb{N}$, *with constants* a, m *and* K *independent of* n.

(R) $T_n = M_n L_n^{-1} \to T = ML^{-1}$ *strongly in* $\mathcal{L}(X)$ *as* $n \to \infty$.

It is proved very recently in Barbu and Favini [4] that (Q) and (R) imply that

$$L_n(zM_n + L_n)^{-1} \to L(zM + L)^{-1}$$

strongly in $\mathcal{L}(X)$ as $n \to \infty$ for all $z \in \Sigma_a$.

Let us consider the approximating problems

$$(3.1)_n \qquad \frac{d}{dt}(M_n u_n(t)) + L_n u_n(t) = f_n(t), 0 \le t \le \tau,$$

$$(3.2)_n \qquad M_n u_n(0) = w_{0n} (= M_n u_{0n}), u_{0n} \in \mathcal{D}(L_n)\,,$$

where $f_n \in C([0,\tau]; X)$. Denote by $u_n = u_n(\cdot), u = u(\cdot)$, the k-integrated solutions to $(3.1)_n$, $(3.2)_n$ and (1.1), (1.2), respectively, with $k \ge m+3$, whose existence has been established in section 2. We have

THEOREM 3.1. *Under* (Q) *and* (R), *if* $w_{0n} \to w_0$ *in* X *and* $f_n \to f$ *in* $C([0,\tau]; X)$ *as* $n \to \infty$, *then for all* $0 \le t \le \tau, L_n u_n(t) \to Lu(t)$ *in* X *as* $n \to \infty$.

Proof. Let w_n be the k-integrated solution to the problem

$$(3.3)_n \qquad \frac{d}{dt}(T_n w)(t) + w(t) = f_n(t), \qquad 0 \le t \le \tau,$$

$$(3.4)_n \qquad T_n w(0) = w_{0n}\,,$$

and thus $T_n w_n(0) = 0$ and

$$(3.5)_n \qquad \frac{d}{dt}(T_n w_n)(t) + w_n(t) = \frac{t^{k-1}}{(k-1)!} w_{0n} + \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} f_n(s)ds\,.$$

Assumption $k \ge m+3$ enables us to deduce that

$$F_n(t) = \frac{t^{k-1}}{(k-1)!} w_{0n} + \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} f_n(s)ds$$

is $(m+2)$-times continuously differentiable and (see notation in the proof of Theorem 2.1),

$$F_n^{(j)}(0) = F^{(j)}(0) = 0, j = 0, 1, ..., m+1,$$

$$F_n^{(j)} \to F^{(j)} \text{ uniformly on } [0,\tau] \text{ as } n \to \infty, j = 0, 1, ..., m+2\,.$$

In view of Theorem 4 in Barbu and Favini [4] we infer that $T_n w_n(t) \to Tv(t)$ in X as $n \to \infty$, where $v(\cdot)$ is the strict solution to (2.7), (2.8), thus concluding the proof of the theorem. ∎

EXAMPLE 3.1. Let us return to Example 2.1. Therefore we have two closed linear operators L, M acting in the complex Hilbert space H such that ($< , >$ and $|| \ ||$ denote the inner product and the norm in H generated by it)

$$Re < Lu, u > \geq c_0 ||u||^2, \qquad Re < L^* f, f > \geq c_0 ||f||^2, \tag{3.5}$$

for all $u \in \mathcal{D}(L), f \in \mathcal{D}(L^*)$, where c_0 is a positive constant. Moreover, we assume that $M \in \mathcal{L}(H)$ is selfadjoint and nonnegative. We also suppose to have a family of linear operators $L_n, M_n, n \in \mathbb{N}$, satisfying

$$Re < L_n u, u > \geq c_0 ||u||^2, \qquad Re < L_n^* f, f > \geq c_0 ||f||^2, \tag{3.6}$$

for all $u \in \mathcal{D}(L_n), f \in \mathcal{D}(L_n^*)$ and the same positive constant c_0 as in (3.5), independent of n,

(3.7) $M_n \in \mathcal{L}(H)$ selfadjoint and nonnegative for $n \in \mathbb{N}$,

(3.8) $M_n \to M$ and $L_n^{-1} \to L^{-1}$ strongly in $\mathcal{L}(H)$ as $n \to \infty$.

Since we know from Example 2.1 that for fixed $a > 0, P(z) = zM + L, P_n(z) = zM_n + L_n, n \in \mathbb{N}$, have bounded inverses, with

$$||P(z)^{-1}||_{\mathcal{L}(H)} \leq c_0^{-1}, \quad ||P_n(z)^{-1}||_{\mathcal{L}(H)} \leq c_0^{-1}, \quad z \in \Sigma_a, \quad n \in \mathbb{N},$$

the identity

$$L_n P_n(z)^{-1} = I - zM_n P_n(z)^{-1}$$

implies that

$$||L_n P_n(z)^{-1}||_{\mathcal{L}(H)} \leq 1 + |z| K_1 c_0^{-1},$$

where $K_1 = \sup_{n \in \mathbb{N}} ||M_n||_{\mathcal{L}(H)} < \infty$. Therefore Theorem 3.1 can be applied with $m = 1$.

4. A different approach

In this section we present a completely different approach by Laplace transform techniques similar to those applied in [2, 3] for integrated semigroups, which has been also used by Abdelaziz and Neubrander [1, Section 2] for the degenerate Cauchy problem. We obtain a slightly more general version of Theorem 2.1: L is not necessarily invertible and we win some regularity.

Let Y, X be Banach spaces and L be a closed linear operator from Y into X with domain $\mathcal{D}(L)$. Let $M \in \mathcal{L}(\mathcal{D}(L), X)$, where $\mathcal{D}(L)$ is considered with the graph norm.

THEOREM 4.1. *Assume that there exist $a \geq 0, 0 \leq \beta < 1$ such that $P(z) = zM + L$ is invertible for $Re z \geq a$ and*

$$||P(z)^{-1}||_{\mathcal{L}(X, \mathcal{D}(L))} \leq k(1 + |z|)^{m+\beta}, \qquad Re z \geq a, \tag{4.1}$$

where $m \in \{-1, 0, 1, ...\}$.

Then, given $u_0 \in \mathcal{D}(L)$, there exists a unique $(m+2)$-times integrated solution of (1.1), (1.2).

REMARK. If L is invertible, then the graph norm is equivalent to $||x||_{\mathcal{D}(L)} = ||Lx||_X$ and thus (4.1) is equivalent to (P).

For the proof we use the following result from [2, Proposition 3.1].

PROPOSITION 4.2. *Let $a \geq 0, 0 \leq \beta < 1, k \geq 0, m \in \{-1, 0, 1, ...\}$. Let Z be a Banach space and assume that $P : \{z \in \mathbb{C}; Re z > a\} \to Z$ is holomorphic satisfying*

$$||P(z)||_Z \leq C|z|^{m+\beta}, \qquad Re z > a.$$

Then there exists a continuous function $S : [0, \infty) \to Z$ such that

$$\text{(a)} \qquad \sup_{t \geq 0} ||e^{-wt} S(t)||_Z < \infty \qquad \textit{for all } w > a,$$

$$\text{(b)} \qquad S(0) = 0,$$

$$\text{(c)} \qquad P(z) = z^{m+2} \int_0^\infty e^{-zt} S(t) dt, \qquad Re\, z > a.$$

Proof of Theorem 4.1. By Proposition 4.2 there exists a continuous function $S : [0, \infty) \to \mathcal{L}(X, \mathcal{D}(L))$ such that

$$\text{(4.2)} \qquad \sup_{t \geq 0} ||e^{-wt} S(t)||_{\mathcal{L}(X, \mathcal{D}(L))} < \infty, \qquad w > a,$$

$$\text{(4.3)} \qquad S(0) = 0,$$

$$\text{(4.4)} \qquad P(\lambda)^{-1} = \lambda^{m+2} \int_0^\infty e^{-\lambda t} S(t) dt, \qquad \lambda > a.$$

Observe that $t \to LS(t)$ and $t \to MS(t)$ are continuous functions from $[0, \infty)$ into $\mathcal{L}(X)$ and

$$\sup_{t \geq 0} e^{-wt} ||LS(t)||_{\mathcal{L}(X)} < \infty, \qquad \sup_{t \geq 0} e^{-wt} ||MS(t)||_{\mathcal{L}(X)} < \infty, \qquad w > a.$$

Let $\lambda > a$. Denote by $I \in \mathcal{L}(X)$ the identity on X.

Then

$$\lambda^{m+3} \int_0^\infty e^{-\lambda t} \frac{t^{m+2}}{(m+2)!} dt I = I = (\lambda M + L) P(\lambda)^{-1}$$
$$= \lambda^{m+3} \int_0^\infty e^{-\lambda t} MS(t) dt + \lambda^{m+2} \int_0^\infty e^{-\lambda t} LS(t) dt$$
$$= \lambda^{m+3} \int_0^\infty e^{-\lambda t} \{MS(t) + \int_0^t LS(s) ds\} dt.$$

Hence

$$\int_0^\infty e^{-\lambda t} \left\{ \frac{t^{m+2}}{(m+2)!} I - MS(t) - \int_0^t LS(s) ds \right\} dt = 0$$

for all $\lambda > a$. It follows from the uniqueness theorem for Laplace transforms that

$$-\int_0^t LS(s)ds\}dt = MS(t) - \frac{t^{m+2}}{(m+2)!}I, \qquad t \geq 0. \tag{4.5}$$

Now let $f \in C([0,\tau];X), u_0 \in \mathcal{D}(L)$. Define

$$u(t) = S(t)Mu_0 + \int_0^t S(s)f(t-s)ds, \qquad 0 \leq t \leq \tau. \tag{4.6}$$

We show that u is an $(m+2)$-integrated solution, i.e., u satisfies (2.1) for $k = m+2$. Indeed, $u \in C([0,\tau];\mathcal{D}(L))$ and by (4.5),

$$\begin{aligned} Mu(t) &= MS(t)Mu_0 + \int_0^t MS(s)f(t-s)ds \\ &= -\int_0^t LS(s)Mu_0 ds + \frac{t^{m+2}}{(m+2)!}Mu_0 - \int_0^t \int_0^s LS(r)f(t-s)drds + f_{m+3}(t). \end{aligned}$$

Thus, in order to show (2.1), it suffices to show that

$$\int_0^t Lu(s)ds = \int_0^t LS(s)Mu_0 ds + \int_0^t \int_0^s LS(r)f(t-s)drds. \tag{4.7}$$

By (4.6) we have

$$\int_0^t Lu(s)ds = \int_0^t LS(s)Mu_0 ds + \int_0^t L \int_0^s S(r)f(s-r)drds.$$

Now (4.7) follows since by Fubini's theorem

$$\begin{aligned} \int_0^t L \int_0^s S(r)f(s-r)drds &= \int_0^t \int_r^t LS(r)f(s-r)dsdr \\ &= \int_0^t \int_s^{t-r} LS(r)f(s)dsdr = \int_0^t \int_0^r LS(t-r)f(s)dsdr \\ &= \int_0^t \int_s^t LS(t-r)f(s)drds = \int_0^t \int_0^{t-s} LS(r)f(s)drds \\ &= \int_0^t \int_0^s LS(r)f(t-s)drds. \end{aligned}$$

This finishes the proof of existence.

Before proving uniqueness, we establish some commutation properties. For $\lambda, \mu > a$, $\lambda \neq \mu$, one has

$$\frac{P(\lambda)^{-1} - P(\mu)^{-1}}{\mu - \lambda} = \frac{P(\lambda)^{-1}}{\mu - \lambda}\{\mu M + L - (\lambda M + L)\}P(\mu)^{-1} = P(\lambda)^{-1}MP(\mu)^{-1}.$$

Consequently,

$$P(\lambda)^{-1}MP(\mu)^{-1} = P(\mu)^{-1}MP(\lambda)^{-1}, \qquad \lambda, \mu > a. \tag{4.8}$$

Fix $\mu > a$. Then for all $\lambda > a$,

$$\lambda^{m+2}\int_0^\infty e^{-\lambda t}S(t)MP(\mu)^{-1}dt = P(\lambda)^{-1}MP(\mu)^{-1}$$
$$= P(\mu)^{-1}MP(\lambda)^{-1} = \lambda^{m+2}\int_0^\infty e^{-\lambda t}P(\mu)^{-1}MS(t)dt\,.$$

Thus, by the uniqueness theorem,

$$S(t)MP(\mu)^{-1} = P(\mu)^{-1}MS(t), \qquad \mu > a\,. \tag{4.9}$$

Let $y \in \mathcal{D}(L)$. Let $\mu > a$, $x = P(\mu)y$. Then by (4.9),

$$(\mu M + L)S(t)My = (\mu M + L)S(t)MP(\mu)^{-1}x = MS(t)x = MS(t)(\mu M + L)y\,.$$

Hence

$$LS(t)My = MS(t)Ly, \qquad y \in \mathcal{D}(L)\,. \tag{4.10}$$

It follows from (4.5) that

$$\frac{d}{dt}MS(t) = -LS(t) + \frac{t^{m+1}}{(m+1)!}I, \quad t > 0\,. \tag{4.11}$$

Now let $u \in C([0,\tau];\mathcal{D}(L))$ be an $(m+2)$-integrated solution. Then

$$\frac{d}{dt}Mu(t) = -Lu(t) + \frac{t^{m+1}}{(m+1)!}Mu_0 + f_{m+2}(t), \qquad t \in [0,\tau]. \tag{4.12}$$

Fix $0 < t \le \tau$. For $s \in [0,t]$ let $w(s) = MS(t-s)Mu(s)$. Then

$$w'(s) = LS(t-s)Mu(s) - \frac{(t-s)^{m+1}}{(m+1)!}Mu(s) + MS(t-s)\frac{d}{ds}(Mu(s)) \quad \text{(by (4.11))}$$
$$= LS(t-s)Mu(s) - \frac{(t-s)^{m+1}}{(m+1)!}Mu(s) - MS(t-s)Lu(s)$$
$$+ \frac{s^{m+1}}{(m+1)!}MS(t-s)Mu_0 + MS(t-s)f_{m+2}(s) \quad \text{(by (4.12))}$$
$$= -\frac{(t-s)^{m+1}}{(m+1)!}Mu(s) + \frac{s^{m+1}}{(m+1)!}MS(t-s)Mu_0$$
$$+ MS(t-s)f_{m+2}(s) \quad \text{(by (4.10))}\,.$$

Hence

$$0 = w(t) - w(0) = \int_0^t w'(s)ds = -\int_0^t \frac{(t-s)^{m+1}}{(m+1)!}Mu(s)ds$$
$$+ \int_0^t \frac{s^{m+1}}{(m+1)!}MS(t-s)Mu_0ds + \int_0^t MS(t-s)f_{m+2}(s)ds\,.$$

Differentiating $(m+2)$-times yields

$$(Mu)(t) = MS(t)Mu_0 + \int_0^t MS(s)f(t-s)ds\,. \tag{4.13}$$

If M is injective, this implies that u is given by (4.6) and uniqueness is proved. In order to prove uniqueness in the general case, let u_1, u_2 be two solutions of (2.1) for some $k \in \{0, 1, 2, ...\}$. Let $u = u_1 - u_2$. Then $u \in C([0,\tau]; \mathcal{D}(L))$ and

$$Mu(t) = -\int_0^t Lu(s)ds, t \in [0,\tau].$$

Moreover, it follows from (4.13) that $Mu(t) \equiv 0$. Hence $Lu(t) \equiv 0, t \in [0,\tau]$.

It follows that for $\lambda > a, (\lambda M + L)u(t) \equiv 0, t \in [0,\tau]$. Since $\lambda M + L$ is invertible one conclude that $u(t) \equiv 0$ on $[0,\tau]$. ■

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