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Integral representations of resolvents and semigroups

Wolfgang Arendt and Alexander V. Bukhvalov

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Abstract. The goal of this paper is to find out under which conditions the resolvent $R(\lambda, A)$ of the generator A of a positive semigroup $T = (T(t))_{t \geq 0}$ on the space $L^p(\Omega)$ $(1 \leq p \leq \infty)$ is an integral operator. For this purpose we investigate the integral representability of some integrals of operator-valued functions (Theorems 2.1 and 2.10).

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Introduction

Let $\Omega \subset \mathbb{R}^N$ be open and $1 \leq p \leq \infty$. An operator U on $L^p(\Omega)$ is called an integral operator if there exists a measurable function $K\colon \Omega \times \Omega \to \mathbb{C}$ such that

$$(Uf)(y) = \int_\Omega K(y, x) f(x)\, dx \quad y\text{-a.e.}$$

for all $f \in L^p(\Omega)$.

Using classical methods, boundary value problems are frequently solved by the use of Green's function so that the solution is obtained by an integral operator. On the other hand, modern variational methods yield easily weak solutions in much more generality. But these methods give no information on the solution operator.

The purpose of the present paper is to investigate under which conditions the resolvent $R(\lambda, A)$ of the generator A of a positive semigroup $T = (T(t))_{t \geq 0}$ on $L^p(\Omega)$ $(1 \leq p < \infty)$ is an integral operator.

The following question arises naturally. Assume that a function $S\colon [a,b] \to \mathscr{L}(L^p(\Omega))$ is strongly continuous (where $[a, b]$ is a compact interval) and let $Q = \int_a^b S(t)\, dt$. If $S(t)$ is integral for all $t \in [a, b]$, does it follow that Q is an integral operator? We show in Section 2 that this is actually true if S is positive in addition (but not in general). Consequently, if any operator $T(t)$ is integral for $t > 0$, then $R(\lambda, A) := (\lambda - A)^{-1}$ is integral if λ is sufficiently large.

We give two different proofs for this result which both are of independent interest. The first consists in showing that the kernels of $S(t)$ can be chosen measurable on $[a, b] \times \Omega \times \Omega$. This is not obvious. We use a criterion due to A.V. Bukhvalov [Bu1] (see also [Bu3, KA, MN]) which characterizes integral operators by the property to transform dominated convergent sequences into almost everywhere convergent sequences.

Our second proof is based on the following intermediate result. Denote by $\mathscr{K}^r(L^p)$ the space of all operators which are dominated by a positive compact integral operator (those operators are compact and integral themselves). We show that $Q \in \mathscr{K}^r(L^p)$ whenever $S(t) \in \mathscr{K}^r(L^p)$ for all $t \in [a, b]$.

This is an order theoretical version of a recent result of J. Voigt [Vo] (after previous work by L. Weis [W]) saying that Q is compact whenever $T(t)$ is compact for all $t \in [a, b]$).

The space $\mathscr{K}^r$ has good permanence properties. Using a result in [A1] we show that if $R(\lambda_0, A)^k \in \mathscr{K}^r$ for some $\lambda_0 \in \varrho(A)$ such that $R(\lambda_0, A) \geq 0$, then $R(\lambda, A) \in \mathscr{K}^r$ for all $\lambda \in \varrho(A)$ (so that all resolvents are integral). A very convenient sufficient condition for an operator U on L^p to be integral is that $UL^p \subset L^\infty$ $(1 \leq p < \infty)$. If Ω is bounded, this even implies $U \in \mathscr{K}^r$. This can be frequently applied to elliptic operators for which one can show that $D(A^k) \subset L^\infty$ for some k by Sobolev imbedding theorems and elliptic regularity; or, more generally, by logarithmic Sobolev inequalties (we refer to the books by Davies [Da] or Robinson [Ro]).

For bounded open Ω with non-regular boundary such results may fail if Neumann conditions are imposed. But in fact, one always has local regularity, i.e. $D(A^k) \subset C(\Omega)$ for some $k \in \mathbb{N}$. It is shown that this suffices to ensure an integral representation of the resolvent.

Finally, we would like to mention that, even though we have restricted ourselves to L^p-spaces, our results are still valid for very wide classes of Banach function spaces which include Orlicz spaces known for their importance in the investigation of general elliptic problems.

1. Integral operators

In this section we put together known results on integral operators as they are needed later.

In the following (X, μ) and (Y, ν) are σ-finite measure spaces. Let $1 \leq p, q \leq \infty$ and let $F = L^p(X, \mu)$, $G = L^q(Y, \nu)$.

Definition 1.1. A linear operator $T: F \to G$ is an *integral operator* if there exists a measurable function $K: Y \times X \to \mathbb{C}$ such that

(a) $K(y, \cdot) f(\cdot) \in L^1(X, \mu)$ y-a.e. for all $f \in F$ and
(b) $(Tf)(y) = \int K(y, x) f(x) d\mu(x)$ y-a.e. for all $f \in F$.

Remark 1.2. It follows from the closed graph theorem that every integral operator is bounded. In fact, let $f_n \to f$ in F and $Tf_n \to g$ in G. Then there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of (f_n) which converges a.e. and such that $|f_{n_k}| \leq h$ $(k \in \mathbb{N})$ for some $h \in F$. Now it follows from Lebesgue's theorem that $(Tf)(y) = \int K(y,x)f(x)\,d\mu(x) = \lim_{k\to\infty}(Tf_{n_k})(y)$ a.e. Thus $Tf(y) = g(y)$ a.e.

We denote by $I(F, G)$ the space of all integral operators and let $I(F) := I(F, F)$.

The following sufficient condition due to Kantorovich-Vulikh (cf. [KA]) is well-known.

Theorem 1.3. *Any bounded operator from $L^p(X,\mu)$ $(1 \leq p < \infty)$ into $L^\infty(Y,\nu)$ is an integral operator.*

For completeness we include an easy proof for $p = 1$ and sketch the proof for $p > 1$ (cf. the proofs in [KA] due to A. Bukhvalov).

Proof. *a*) Let $p = 1$. If $K \in L^\infty(Y \times X)$, then it follows from Fubini's theorem that

$$(1.1) \qquad (S_K f)(y) = \int K(y,x)f(x)\,d\mu(x)$$

defines an integral operator such that $\|S_K\| \leq \|K\|_\infty$.

Conversely, let S be a bounded operator acting from $L^1(X,\mu)$ to $L^\infty(Y,\nu)$. Consider the space H of all functions of the form

$$\Big(\sum_{i=1}^{n} f_i \otimes g_i\Big)(y,x) = \sum_{i=1}^{n} g_i(y)f_i(x)$$

with $f_i \in L^1(X,\mu)$, $g_i \in L^1(Y,\nu)$, $|f_i| \wedge |f_j| = 0$ for $i \neq j$; $i, j = 1, \ldots, n$; $n \in \mathbb{N}$, which is obviously dense in $L^1(Y \times X)$. Letting

$$\Phi_S\Big(\sum_{i=1}^{n} f_i \otimes g_i\Big) := \sum_{i=1}^{n} \langle Sf_i, g_i\rangle$$

defines a functional on H such that $\|\Phi_S\| \leq \|S\|$. Thus, Φ_S has a continuous extension to the whole space $L^1(Y \times X)$ whose dual is identified with $L^\infty(Y \times X)$. Hence, there exists $K \in L^\infty(Y \times X)$ such that $\|K\|_\infty \leq \|S\|$ and $\langle Sf, g\rangle = \int\int K(y,x)f(x)\,d\mu(x)g(y)\,d\nu(y)$. This implies that $S_K = S$. It is clear now that $K \in L^\infty(Y \times X) \to S_K \in \mathscr{L}(L^1(X), L^\infty(Y))$ is an isometric isomorphism.

b) Let $1 < p < \infty$, $1/p + 1/p + 1/p' = 1$. The space

$$L^\infty[L^{p'}] := \{K \mid K\colon Y \times X \to \mathbb{C} \text{ measurable, } \operatorname*{ess\,sup}_{y\in Y} (\textstyle\int |K(y,x)|^{p'} d\mu(x))^{1/p'} < \infty\},$$

which is a Banach space (for the obvious norm), is the dual space of

$$L^1[L^p] := \{K \mid K\colon Y \times X \to \mathbb{C} \text{ measurable, } \textstyle\int (\int |K(y,x)|^p d\mu(x))^{1/p} d\nu(y) < \infty\}.$$

One shows as in *a*) that $K \to S_K : L^\infty[L^{p'}] \to \mathscr{L}(L^p, L^\infty)$ is an isometric isomorphism, where S_K is given by the formula (1.1). □

Corollary 1.4. *Let* $T \in \mathscr{L}(L^p(X, \mu))$, $1 \le p < \infty$, *such that* $TL^p \subset L^\infty$. *Then* T *is an integral operator.*

However, if (X, μ) is not purely atomic, there always exist (many) operators on L^p which are not integral operators (for example the identity is not). The following criterion is due to A.V. Bukhvalov [Bu1], see also [MN, Th. 3.3.11], [KA, Bu3].

Theorem 1.5. *Let* $1 \le p \le \infty$ *and let* $T: L^p(X, \mu) \to L^q(Y, \nu)$ *be linear. Then* T *is an integral operator if the following is satisfied:*

$$\left.\begin{array}{l} f_n, f \in L^p \\ |f_n(x)| \le f(x) \quad x\text{-a.e.} \\ f_n \to 0 \quad x\text{-a.e.} \end{array}\right\} \Rightarrow \quad (Tf_n)(y) \to 0 \quad y\text{-a.e.} \tag{1.2}$$

It follows immediately from Lebesgue's theorem that condition (1.2) is necessary. If $1 \le p < \infty$ we can change in (1.2) the condition of $f_n \to 0$ x-a.e. to that of $\|f_n\| \to 0$ in L^p.

Remark 1.6. Theorem 1.5 remains true if we replace $G = L^q$ by $G = L^0(Y, \nu)$, the vector lattice of all measurable functions on Y, and define integral operators from F into L^0 exactly as in Definition 1.1.

On the basis of this criterion we prove the following extension of Corollary 1.4.

Proposition 1.7. *Let* $T \in \mathscr{L}(L^p(X, \mu))$, $1 \le p < \infty$. *If* $T(L^p(X, \mu)) \subset L^\infty_{\text{loc}}(X, \mu)$, *then* T *is an integral operator.*

Here we define $L^\infty_{\text{loc}}(X, \mu) := \{f | f: X \to \mathbb{C} \text{ measurable}, f|_{B_n} \in L^\infty(X, \mu) \text{ for all } n \in \mathbb{N}\}$, where $(B_n)_{n \in \mathbb{N}}$ is a fixed increasing sequence of measurable subsets of X such that $X \setminus \bigcup_{n \in \mathbb{N}} B_n$ is negligible. The definition of $L^\infty_{\text{loc}}(X, \mu)$ does depend on the choice of this sequence.

Proof. L^∞_{loc} is a Fréchet space for the seminorms $p_n(f) = \|f|_{B_n}\|_\infty$. It follows from the closed graph theorem that T is continuous as a mapping from L^p into L^∞_{loc}. Thus there exist constants $C_n \ge 0$ such that $p_n(Tf) \le C_n \|f\|_p$ for all $f \in L^p$, i.e.

$$|(Tf)(x)| \le C_n \|f\|_p \quad (x \in B_n) \tag{1.3}$$

for all $f \in L^p$, $n \in \mathbb{N}$. This clearly implies (1.1). □

Permanence properties of integral operators are not good: $I(F, G)$ is not closed in $\mathscr{L}(F, G)$, and $I(F)$ does not form an ideal in $\mathscr{L}(F)$. For that reason we introduce the smaller class of regular integral operators.

Definition 1.8. (a) A linear mapping $T: F \to G$ is *regular* if it satisfies the following equivalent conditions.

(i) T is a linear combination of positive operators;
(ii) there exists a positive operator $S \in \mathscr{L}(F, G)$ such that $|Tf| \leq S|f|$ for all $f \in F$;
(iii) T is *order bounded*, i.e. for all $f \in F_+$ there exists $g \in G_+$ such that $|h| \leq f$ implies $|Th| \leq g$.

(b) By $\mathscr{L}^r(F, G)$ we denote the space of all regular operators and by $I^r(F, G) := \mathscr{L}^r(F, G) \cap I(F, G)$ the space of all regular integral operators.

The space $\mathscr{L}^r(F, G)$ is a Banach lattice, the modulus $|T|$ of $T \in \mathscr{L}^r(F, G)$ being given by $|T|f = \sup_{|g| \leq f} |Tg|$ and the norm by $\|T\|_r := \| |T| \|$. Moreover, $\mathscr{L}^r(F) := \mathscr{L}^r(F, F)$ is a Banach algebra. If $p = 1$ or $q = \infty$ then $\mathscr{L}^r(F, G) = \mathscr{L}(F, G)$ and $\|T\| = \|T\|_r$ for all $T \in \mathscr{L}^r(F, G)$, whereas $\mathscr{L}^r(L^p)$ is not even dense in $\mathscr{L}(L^p)$ in the operator norm if $1 < p < \infty$. All these facts are standard, see [S, Chapter IV] (see also [KA, MN, Z]); the very last property is shown in [AV].

Proposition 1.9.

(a) *Let $T \in I(F, G)$ be represented by the kernel K. Then T is regular if and only if for all $f \in F$, $\int |K(\cdot, x)| f(x)\, d\mu(x) \in E$. In that case $|T|$ is represented by the kernel $|K|$.*

(b) *Let $S, T \in \mathscr{L}^r(F, G)$, $|S| \leq |T|$. If $T \in I(F, G)$, then $S \in I(F, G)$.*

(c) *Let $S_\alpha \in I^r(F, G)$, $S_\alpha \leq S \in \mathscr{L}^r(F, G)$. Then $\sup_\alpha S_\alpha \in I^r(F, G)$.*

(d) *$I^r(F, G)$ is closed in $\mathscr{L}^r(F, G)$.*

(e) *If $1 \leq p < \infty$, then $I^r(L^p)$ is an algebraic ideal in $\mathscr{L}^r(L^p)$.*

For (a)–(d) we refer to Schaefer [S, Chapter IV], for example.

Proof of (e). Let $T \in I^r(L^p)$, $S \in \mathscr{L}^r(L^p)$. We verify that TS and ST satisfy (1.2). Let $|f_n| \leq f$, $\|f_n\| \to 0$. Since S is regular, $|Sf_n| \leq |S|f$ and $\|Sf_n\| \to 0$. Consequently, $(TSf_n)(y) \to 0$ a.e. by (1.2). Hence $TS \in I(F)$. In order to show that $ST \in I(F)$, let $g_n = \sup_{k \geq n} |Tf_k|$. Since $(Tf_n)(y) \to 0$ a.e. one has $g_n \downarrow 0$ a.e. Thus $\|g_n\| \to 0$. This implies $\| |S| g_n \| \to 0$. Because of monotonicity one deduces $(|S| g_n)(y) \to 0$ a.e. Since $|(STf_n)(y)| \leq (|S| |Tf_n|)(y) \leq (|S| g_n)(y)$ a.e., this implies that ST satisfies (1.2). □

Properties (b) and (c) say that $I^r(F, G)$ is a band in $\mathscr{L}^r(F, G)$. It turns out to be the band generated by the order continuous finite rank operators. More precisely, let $F_n' = F'$ if $p < \infty$ and $F_n' = L^1$ if $p = \infty$ (so that F_n' is the space of all order continuous linear forms of F). For $f' \in F_n'$, $g \in G$ denote by $f' \otimes g$ the operator given by

$$(f' \otimes g)(f) = \int f(y) f'(y)\, d\mu(y) \cdot g$$

and by $F'_n \otimes G$ the space of all linear combinations of such operators. Then the following holds (see [S, IV. 9.8]).

Theorem 1.10. $(F'_n \otimes G)^{dd} = I^r(F, G)$.

Here for $M \subset \mathscr{L}^r(F, G)$ we let $M^d = \{T \in \mathscr{L}^r(F, G): |T| \wedge |S| = 0 \ \forall S \in M\}$, so that M^{dd} is the band generated by M in $\mathscr{L}^r(F, G)$.

Definition 1.11. Let $1 \leq p < \infty$. By $\mathscr{K}^r(F, G)$ we denote the space of all operators $T \in I^r(F, G)$ such that $|T|$ is compact.

Proposition 1.12. *Let* $1 \leq p, q < \infty$.

(a) *The space* $\mathscr{K}^r(F, G)$ *coincides with the closure of* $F' \otimes G$ *in* $\mathscr{L}^r(F, G)$.

(b) $\mathscr{K}^r(F)$ *is an algebraic ideal and lattice ideal in* $\mathscr{L}^r(F)$.

For the proof see [S, Theorem 10.3]

It is clear that $\mathscr{K}^r(L^1) = \mathscr{K}(L^1)$; however, if $1 < p < \infty$ there exists a positive compact operator $T \neq 0$ such that $T \wedge S = 0$ for all positive integral operators S on L^p (see [A2]).

The following is a convenient sufficient condition for T to belong to $\mathscr{K}^r(F)$.

Proposition 1.13. *Let* $F = L^p(X, \mu)$, $1 < p < \infty$, *and let* $T \in \mathscr{L}(F)$. *Assume that there exists* $u \in F_+$ *such that* $TF \subset F_u$. *Then* $T \in \mathscr{K}^r(F)$.

Here $F_u := \{f \in F: |f| \leq mu$ for some $m \in \mathbb{N}\}$ denotes the principal ideal in F generated by u.

Proof. It follows from the hypothesis that T is order bounded and $|T| F \subset F_u$. So we can assume $T \geq 0$. By Kakutani's theorem there exists a compact extremely disconnected space Z and a bijective linear mapping $j: E_u \to C(Z)$ such that $j \geq 0$, $j^{-1} \geq 0$ and $j(u) = \mathbf{1}_Z$. It follows from the closed graph theorem that $j \circ T$ is continuous, so there exists $C \geq 0$ such that

$$|Tf| \leq C \|f\| u$$

for all $f \in E$. This clearly implies (1.2), so that $|T| \in I(F)$. It remains to show that T is compact. Let $(f_n)_{n \in \mathbb{N}} \subset F$, $\|f\| \leq 1$. Since F is reflexive we can assume that f_n converges weakly to f, say. Let $a \in K$ and $\varphi = (j \circ T)'(\delta_a) \in E' = L^{p'}$. Then $\langle f_n, \varphi \rangle \to \langle f, \varphi \rangle$, i.e. $(j(Tf_n))(a) \to (jTf_n)(a)$. Let $h_m = \sup_{n \geq m} |Tf_n - Tf|$. Then $j(h_m) = \sup_{n \geq m} |(j \circ T) f_n - (j \circ T) f| \downarrow 0$ in $C(Z)$ and so $h_m(x) \downarrow 0$ $(m \to \infty)$ x-a.e. in X. It follows from Lebesgue's theorem that $\|Tf_n - Tf\|_p \to 0$ $(n \to \infty)$. □

Remark. If $p = 2$ then T satisfies the condition of Proposition 1.13 if and only if T is a Hilbert-Schmidt operator.

Proposition 1.14. *Let $F = L^1(X, \mu)$ and $T \in \mathscr{L}(F)$. Assume that there exists $u \in F_+$ such that $TF \subset F_u$. Then $T \in I^r(F)$ and $T^2 \in \mathscr{K}^r(F)$.*

Proof. The proof of Proposition 1.13 also shows $|T| \in I^r(F)$ in this case. Since order intervals are compact in L^1, the operator $|T|$ is weakly compact and so $|T|^2$ is compact since L^1 has the Dunford-Pettis property. So $|T^2| \le |T|^2 \in \mathscr{K}(F) \cap I^r(F)$. Consequently $T^2 \in \mathscr{K}^r(F)$. □

Integral operators also behave badly with respect to interpolation:

Example 1.15. Let T_p on $L^p(\mathbb{R})$, $1 \le p \le 2$ be the truncated Fourier transform, i.e.

$$(T_p f)(x) = \int_{-\infty}^{+\infty} e^{-ixy} f(x)\,dx\,1_{(0,1)}(y).$$

Then $T_p \in \mathscr{L}(L^p(\mathbb{R}))$. For $p = 1$ one has $(|T_1|f)(x) = \int_{-\infty}^{+\infty} f(x)\,dx\,1_{(0,1)}(y)$, i.e. $|T_1|$ is of rank one and $T_1 \in \mathscr{K}^r(L^1(\mathbb{R}))$. However, T_p is not an integral operator for any $p \in (1, 2]$.

2. Integrals of integral operators

Let $F = L^p(X, \mu)$, $G = L^q(Y, \nu)$, $1 \le p, q \le \infty$, where (X, μ) and (Y, ν) are σ-finite measure spaces. Let $J \subset \mathbb{R}$ be an interval. The following is the main result of this section.

Theorem 2.1. *Let $T: J \to I^r(F, G)$ be a function such that*

(a) *$T(\cdot)f: J \to G$ is measurable for all $f \in F$*

(b) *$\|T(t)\|_r \le \kappa(t)$ $(t \in J)$ for some $\kappa \in L^1(J, \mathbb{R})$.*

Then $\quad Qf = \int_J T(t) f\,dt \quad (f \in F)$

defines an operator $Q \in I^r(F, G)$.

We will give two different proofs of this theorem based on the two criteria given by Theorem 1.5 and Theorem 1.10 respectively.

Let $K(t, \cdot, \cdot): Y \times X \to \mathbb{C}$ be the kernel of $T(t)$. One would expect that $K_1(t, y, x) := \int_J K(t, y, x)\,dt$ is the kernel of Q. However, K need not be measurable in t. The first method of proof consists in replacing K by a kernel K which is measurable on $J \times Y \times X$ and such that for a.e. t the operator $T(t)$ is represented by the kernel $\tilde{K}(t, \cdot, \cdot)$. The following Lemma 2.2 goes back to the 1930s.

Lemma 2.2. *Let $u: J \to G = L^q(Y, \nu)$ be a (Bochner) integrable function. Then there exists a measurable function $\Phi: J \times Y \to \mathbb{C}$ such that*

$$\text{for a.e. } t \in J \quad u(t)(y) = \Phi(t, y) \quad y\text{-a.e.}$$

Moreover, for almost all $t \in J$ the function $\Phi(t, \cdot)$ is integrable and $(\int_J u(t)\,dt(y) = \int \Phi(t, y)\,dt$ y-a.e.

Proof. We can assume that $q = 1$ (otherwise take $\varrho \in L^{p'}(Y, \nu)$ such that $\varrho(y) > 0$ for all $y \in Y$, then $G \subset L^1(Y, \varrho d\nu)$ and we may replace G by $L^1(Y, \varrho d\nu)$). There exists a sequence of functions $u_n : J \to G$ integrable and countably valued such that $\int_J \|u(t) - u_n(t)\|_G\,dt \to 0$ and $\|u(t) - u_n(t)\|_G \to 0$ t-a.e. (see [HP, Def. 3.7.3]). Writing $u_n = \sum_{k=1}^{\infty} 1_{J_k} g_k$ with $J_k \subset J$ measurable, pairwise disjoint and $g_k \in G$ one sees that the function $\varphi_n(t, y) := (u_n(t))(y)$ is measurable on $J \times Y$ with values in $\mathbb{C}$. Since $\|\varphi_n - \varphi_m\|_{L^1(J \times Y)} = \int_J \|u_n(t) - u_m(t)\|\,dt \to 0\ (n, m \to \infty)$, the sequence $(\varphi_n)_{n \in \mathbb{N}}$ is Cauchy in $L^1(J \times Y)$. Let $\Phi \in L^1(J \times Y)$ be its limit. It follows from Fubini's theorem that $u_n(t) = \varphi_n(t, \cdot) \to \Phi(t, \cdot)$ in $G = L^1(Y)$ t-a.e. Hence $u(t) = \Phi(t, \cdot)$ in $L^1(Y)$ t-a.e. This proves the first claim.

The second follows since for all $g \in G' = L^\infty(Y)$,

$$\langle \int_J u(t)\,dt, g \rangle = \int_J \langle u(t), g \rangle\,dt = \int_Y \int_J \Phi(t, y)\,dt\,g(y)\,d\nu(y)$$

by Fubini's theorem. □

Lemma 2.3. (a) *Let $\psi_i : J \times Y \to \mathbb{C}$ be measurable; $i = 1, 2$. Then $\psi_1(t, y) = \psi_2(t, y)$ (t, y)-a.e. if and only if for a.e. t*

$$\psi_1(t, y) = \psi_2(t, y) \quad y\text{-a.e.}$$

(b) *Let $\psi_i : J \times Y \to \mathbb{C}$ be measurable $(i \in \mathbb{N})$. Then $\psi_i(t, y) \to 0$ $(i \to \infty)$ (t, y)-a.e. if and only if t-a.e. $\psi_i(t, y) \to 0$ $(i \to \infty)$ y-a.e.*

Proof. Truncate ψ_i in order to obtain integrable functions and use Fubini's theorem. □

Lemma 2.4. *Let $L : Y \times X \to \mathbb{C}$ be measurable. Then*

$$(Tf)(y) = \int_X L(y, x) f(x)\,d\mu(x)$$

defines a regular integral operator if and only if

$$\int_Y \int_X |L(y, x)| f(x)\,d\mu(x)\,g(y)\,d\nu(y) < \infty$$

for all $0 \le f \in L^p(X, \mu)$, $0 \le g \in L^{q'}(Y, \nu)$, $1/q + 1/q' = 1$.

This follows from Fubini's theorem.

Lemma 2.5. *Let $L : J \times Y \to \mathbb{C}$ be measure such that $L(t, \cdot)$ is integrable t-a.e. Then the function $L_1(t) = \int_Y L(t, y)\,d\nu(y)$ is measurable on J.*

This is a consequence of Fubini's theorem.

Proof of Theorem 2.1. By Lemma 2.2, for all $f \in F$ there exists the unique measurable function $\Phi_f : J \times Y \to \mathbb{C}$ such that

$$t\text{-a.e.} \quad (T(t)f)(y) = \Phi_f(t, y) \quad y\text{-a.e.}$$

Define $W : F \to L^0(J \times Y)$ by $Wf = \Phi_f$. It is clear that W is linear. We show that W is an integral operator, using criterion (1.2) (cf. Remark 1.5). Let $f_n, f \in F$, $|f_n| \le f$ such that $\|f_n\|_F \to 0$ $(n \to \infty)$. Since $T(t)$ is an integral operator, it follows that t-a.e. $(Wf_n)(t, y) = \Phi_{f_n}(t, y) \to 0$ y-a.e. Thus by Lemma 2.3(b) W satisfies (1.2). By the criterion there exists a measurable kernel $K : J \times Y \times X \to \mathbb{C}$ such that $x \to \tilde{K}(t, y, x)f(x)$ is integrable (t, y)-a.e. and $(Wf)(t, y) = \int_X \tilde{K}(t, y, x)f(x)\,d\mu(x)$. It follows from Lemma 2.3(a) that for all $f \in F$,

$$t\text{-a.e.} \quad (T(t)f(y) = \int_X \tilde{K}(t, y, x)f(x)\,d\mu(x) \quad y\text{-a.e.};$$

i.e. t-a.e. $\tilde{K}(t, \cdot, \cdot)$ is a kernel for $T(t)$. Then $|\tilde{K}(t, \cdot, \cdot)|$ is a kernel for $|T(t)|$ (t-a.e.). Let $L(y, x) = \int_J |\tilde{K}(t, x, y)|\,dt$. We show that L defines an operator $Q_1 \in I^r(F, G)$ using Lemma 2.5. Let $0 \le f \in F$, $\|f\| \le 1$, $0 \le g \in L^{q'}$, $\|g\| \le 1$. Then by hypothesis (b)

$$\begin{aligned}
&\int_Y \int_X L(y, x)f(x)\,d\mu(x)\,g(y)\,d\nu(y) \\
&= \int_J \int_Y \int_X |\tilde{K}(t, y, x)|\,f(x)\,d\mu(x)\,d\nu(y) \\
&= \int_J \langle |T(t)|f, g\rangle\,dt \le \int_J \kappa(t)\,dt < \infty.
\end{aligned}$$

Let $K_1(y, x) = \int_J \tilde{K}(t, y, x)\,dt$. Then $|K_1(y, x)| \le L(y, x)$. It follows that K_1 defines an integral operator Q_1. By Fubini's theorem

$$\begin{aligned}
\langle Q_1 f, g\rangle &= \int_Y \int_X K_1(y, x)f(x)\,d\mu(x)\,g(y)\,d\nu(y) \\
&= \int_J \int_Y \int_X \tilde{K}(t, y, x)f(x)\,d\mu(x)\,g(y)\,d\nu(y)\,dt \\
&= \int_J \langle T(t)f, g\rangle\,dt = \langle Qf, g\rangle.
\end{aligned}$$

Thus $Q = Q_1$. □

We would like to emphasize that one cannot replace condition (b) in Theorem 2.1 by the weaker condition

(b′) $\quad \|T(t)\| \le \kappa(t)$ for some $\kappa \in L^1(J, \mathbb{R})$

(which still implies that $Q \in \mathscr{L}(F, G)$). The following example shows that under this weaker condition it can happen that Q is neither integral nor regular, even though $T(t) \in \mathscr{K}^r(F, G)$.

Example 2.6. Let $F = G = L^2(\mathbb{R})$, $J = [1, \infty)$, $1 < \alpha < 3/2$. Define $T(t) \in \mathscr{L}(F)$ by $T(t)f = 1_{[0,1]}\mathscr{F}S(t)f$, where $\mathscr{F} : L^2 \to L^2$ is the Fourier transform and $S : J \to \mathscr{L}(F)$

is defined by $(S(t)f)(x) = 1/x^\alpha 1_{[1,\infty)}(x)f(x)$. It follows from Lebesgue's theorem that $S(\cdot)f: J \to F$ is continuous ($f \in F$). Moreover, $\|T(t)\| \leq \|S(t)\| = 1/t^\alpha =: \kappa(t)$. Since $\kappa \in L^1(J, \mathbb{R})$, $T(\cdot)f \in L^1(J, F)$ for all $f \in F$ and

$$Qf = \int_1^\infty T(t) f dt$$

defines an operator $Q \in \mathscr{L}(F)$. It is easy to see that $Qf = 1_{(0,1)} \mathscr{F} R$ where $(Rf)(x) = 1_{[1,\infty)}(x) x^{-\alpha}(x-1) f(x)$. Thus $Q \notin I(F)$ and $Q \notin \mathscr{L}^r(F)$. One has $T(t) \in \mathscr{K}^r(F)$; in fact, $|T(t)| = g_t \otimes 1_{[0,1]}$, where $g_t(x) = 1/x^\alpha 1_{[t,\infty)}(x)$. However, $Q \notin I(F)$ and $Q \notin \mathscr{L}^r(F)$. In fact, it is easy to see that $Qf = 1_{(0,1)} \mathscr{F} R$, where $(Rf)(x) = 1_{[1,\infty)}(x) x^{-\alpha}(x-1) f(x)$. The restriction Q_0 of Q to $L^1 \cap L^2$ is an integral operator given by the kernel

$$K(y, x) = 1_{[1,\infty)}(x) x^{-\alpha}(x-1) e^{-ixy} 1_{(0,1)}(y).$$

This kernel does not defines an integral operator on $L^2(\mathbb{R})$. In fact, the formula $f(x) = 1_{[1,\infty)}(x) x^{\alpha-2}$ defines a function $f \in L^2(\mathbb{R})$ (since $\alpha < 3/2$). But $x \to K(y, x) f(x)$ is not integrable for any $y \in \mathbb{R}$. Thus, $Q \notin I(F)$.

The modulus $|Q_0|$ of Q_0 is given by the kernel

$$|K(y, x)| = 1_{[1,\infty)}(x) x^{-\alpha}(x-1) 1_{(0,1)}(y).$$

Thus, $|Q_0|$ has no extension to L^2. This implies that Q is not regular.

Our second approach is based on the criterion given in Theorem 1.10 and avoids measure theoretic arguments. However, we have to assume that F is separable; i.e. we will assume in the following that $F = L^p(X, \mu)$, $1 \leq p < \infty$ and that (X, μ) is a separable measure space. In the remainder of this section we assume that $T: J \to \mathscr{L}^r(F, G)$ is a function such that

(2.1) $\quad T(\cdot)f: J \to G$ is measurable for all $f \in F$,

and we will add further assumptions.

Proposition 2.7. *The function* $|T(\cdot)|f: J \to G$ *is measurable for all* $f \in F$.

For the proof we use

Lemma 2.8. *Let* $\varphi_1, \varphi_2: J \to G$ *be measurable. Then* $|\varphi|$, $\varphi_1 \wedge \varphi_2$ *and* $\varphi_1 \vee \varphi_2$ *are measurable as well.*

Proof. By definition, measurability of φ_1 means that φ_1 can be approximated by a sequence of functions of the form

$$g(t) = \sum_{k=1}^\infty 1_{J_k}(t) g_k \quad (t \in J),$$

where the sets $J_k \subset J$ are measurable, pairwise disjoint and $g_k \in G$. It follows that $|\varphi_1|$ can be approximated by the functions $|g(t)| = \sum_{k=1}^{\infty} 1_{J_k}(t)|g_k|$ and so $|\varphi_1|$ is measurable. Since $\varphi_1 \vee \varphi_2 = \frac{1}{2}(\varphi_1 + \varphi_2 + |\varphi_1 - \varphi_2|)$ and $\varphi_1 \wedge \varphi_2 = -((-\varphi_1) \vee (-\varphi_2))$ the proof is complete. □

Proof of Proposition 2.7. Let $f \in F_+$. Since F is separable, there exists a dense sequence $(f_n)_{n\in\mathbb{N}}$ in $D(f) := \{g \in F: |g| \leq f\}$. Hence $|T(t)|f = \sup_{g\in D(f)} |T(t)g| = \sup_{n\in\mathbb{N}} |T(t)f_n| = \lim_{n\to\infty} |T(t)f_1| \vee \ldots \vee |T(t)f_n|$. So it follows from Lemma 2.8 that $|T(\cdot)|f$ is measurable. Since $F = \operatorname{span}(F_+)$ the proof is finished. □

Now we assume in addition that

(2.2) $\quad \|T(t)\|_r \leq \kappa(t) \quad (t \in J)$, where $\kappa \in L^1(J, R)$.

Let $\quad Qf = \int_J T(t) f\,dt \quad (f \in F)$.

Proposition 2.9. *The operator Q is regular.*

Proof. It follows from Proposition 2.7 and (2.2) that $|T(\cdot)|f \in L^1(J, G)$ for all $f \in F$. Hence $Rf := \int_J |T(t)|f\,dt$ $(f \in F)$ defines a positive operator $R \in \mathscr{L}(F, G)$. Clearly, $|Qf| \leq R|f|$ for all $f \in F$. □

The following result will be of interest for applications to semigroups given in Section 4.

Theorem 2.10. *Let $F = L^p(X, \mu)$, $G = L^q(Y, \nu)$, $1 \leq p < \infty$, $1 \leq q \leq \infty$. Assume that (2.1) and (2.2) are satisfied. If $T(t) \in \mathscr{K}^r(F, G)$ for all $t \in J$, then $Q \in \mathscr{K}^r(F, G)$.*

This theorem is the order theoretical analog of the following result due to J. Voigt [Vo] (after previous work by L. Weis [W]).

Theorem 2.11 (Voigt). *Let E_1, E_2 be Banach spaces and let $S: J \to \mathscr{L}(E_1, E_2)$ satisfy*

(a) *$S(\cdot)f \in L^1(J, E_2)$ for all $f \in E_1$;*
(b) *$S(t)$ is compact for all $t \in J$;*
(c) *$\|S(t)\| \leq \kappa(t) \quad (t \in J)$ for some $\kappa \in L^1(J, \mathbb{R})$.*

Then $Rf = \int_J S(t) f\,dt \quad (f \in E_1)$ defines a compact operator $R: E_1 \to E_2$.

Since $\mathscr{K}^r(F, G) = \mathscr{K}(F, G)$ (the compact operators) if $p = 1$ or $q = \infty$, Theorem 2.10 follows from Voigt's result in that case. In the other cases Theorem 2.10 can be deduced from Voigt's result and Theorem 2.1 (see Remark 2.12). Here we do the converse, we give a direct proof of Theorem 2.10 (for separable F) and then deduce from it Theorem 2.1 with help of the second criterion (Theorem 1.10).

Proof of Theorem 2.10. We assume that $1 < p < \infty$. For measurable subsets $X_1, \ldots, X_n$ of X such that $X_i \cap X_j = \emptyset$ $(i \neq j)$ and $0 < \mu(X_i) < \infty$ $(i = 1, \ldots, n)$ we consider the conditional expectation

$$P = \sum_{j=1}^{n} f_j' \otimes f_j,$$

where $f_j' = \frac{1}{\mu(X_j)} 1_{X_j}$ and $f_j = 1_{A_j}$. Then $0 \leq P \in \mathscr{L}(F)$, $\|P\| \leq 1$ and the adjoint $P' \in \mathscr{L}(F')$ of P is given by $P' = \sum_{j=1}^{n} f_j \otimes f_j'$. Since (X, μ) is separable, there exists a sequence $(P_n)_{n \in \mathbb{N}}$ of such operators satisfying

$$(2.6) \qquad \lim_{n \to \infty} P_n f = f \text{ in } F$$

for all $f \in F$, and, since $p > 1$, also

$$(2.7) \qquad \lim_{n \to \infty} P_n' f' = f' \text{ in } F' \text{ for all } f' \in F'.$$

We show that

$$(2.8) \qquad \|SP_n - S\|_r \to 0 \quad (n \to \infty)$$

for all $S \in \mathscr{K}^r(F, G)$.

a) Let $S = f' \otimes g \in F' \otimes G$. Then

$$\|SP_n - S\|_r = \|(P_n' f' - f') \otimes g\|_r = \|P_n' f' - f'\| \|g\| \to 0 \quad (n \to \infty).$$

b) It follows from *a*) that (2.8) holds for $S \in F' \otimes G$.

c) Let $S \in \mathscr{K}^r(F, G)$ and let $\varepsilon > 0$. There exists $S_1 \in F' \otimes G$ such that $\|S_1 - S\|_r < \varepsilon$. Hence

$$\limsup_{n \to \infty} \|SP_n - S\|_r$$
$$\leq \limsup_{n \to \infty} \{\|(S - S_1)P_n\|_r + \|S_1 P_n - S_1\|_r + \|S_1 - S\|_r\} \leq 2\varepsilon$$

by *b*).

Next we show that $T: J \to \mathscr{K}^r(F, G)$ is measurable. In fact, it follows from (2.8) that $\|T(t)P_n - P_n\|_r \to 0$ $(n \to \infty)$. So it suffices to show that $T(t)P_n$ is measurable. Since $P_n \in F' \otimes G$, it is sufficient to prove that $t \to T(t)(f' \otimes f) = f' \otimes T(t)f$: $J \to \mathscr{L}^r(F, G)$ is measurable for all $f' \in F'$, $f \in F$. The mapping $T(\cdot)f: J \to G$ is measurable by hypothesis, and the mapping $\Phi: U \to f' \otimes U$ acting from F to $\mathscr{L}^r(F, G)$ is continuous. Consequently, $T(\cdot)(f' \otimes f) = \Phi \circ T(\cdot)f$ is measurable from J into $\mathscr{L}^r(F, G)$.

It follows from (2.2) that $T \in L^1(J, \mathscr{K}^r(F, G))$. Hence, from the properties of Bochner integral we derive that $Q = \int_J T(t)\,dt \in \mathscr{K}^r(F, G)$. $\square$

Second Proof of Theorem 2.1. We deduce Theorem 2.1 from Theorem 2.10. Since the measure spaces are σ-finite, there exists functions $u' \in F'$, $v \in G$ such that $u'(x) > 0$ (μ-a.e.) and $v(y) > 0$ (ν-a.e.). Let $U = u' \otimes v$. Then $I^r(F, G) = (F' \otimes G)^{dd} = U^{dd}$. In particular, for $0 \leq S \in \mathscr{L}(F, G)$ one has $S \in I(F, G)$ if and only if $Sf = \sup_{n \in \mathbb{N}} (S \wedge nU)f$ for all $0 \leq f \in F$. It follows from Proposition 2.7 that $(|T(t)| \wedge nU)f = \{\frac{1}{2}(|T(t)| + nU) - ||T(t)| - nU|\}f$ is measurable for all $f \in F$. Since $|T(t)| \wedge nU \in \mathscr{K}^r(F, G)$, it follows from Theorem 2.10 that $R_n := \int_J |T(t)| \wedge nU\,dt \in \mathscr{K}^r(F, G)$. We claim that

$$(2.9) \qquad R = \sup_{n \in \mathbb{N}} R_n,$$

where $Rf = \int_J |T(t)| f\,dt$. Let $0 \leq f \in F$ and

$$0 \leq g' \in G_n' = \begin{cases} G' & \text{if } q < \infty, \\ L^1(Y, \nu) & \text{if } q = \infty. \end{cases}$$

Then by the Beppo-Levi theorem

$$\begin{aligned} \langle Rf, g' \rangle &= \int_J \langle |T(t)| f, g' \rangle\,dt = \int \sup_{n \in \mathbb{N}} \langle (|T(t)| \wedge nU)f, g' \rangle\,dt \\ &= \sup_{n \in \mathbb{N}} \int \langle (|T(t)| \wedge nU)f, g' \rangle\,dt = \sup_{n \in \mathbb{N}} \langle R_n f, g' \rangle \\ &= \langle (\sup_{n \in \mathbb{N}} R_n)f, g' \rangle. \end{aligned}$$

This proves (2.9) and so $R \in (F_n' \otimes G)^{dd} = I^r(F, G)$. Since Q is dominated by R, the proof is complete. □

Remark 2.12. We gave a direct proof of Theorem 2.10 in the case where $1 < p < \infty$ and (X, μ) separable. As mentioned before, if $p = 1$ or $q = \infty$ Theorem 2.10 follows directly from Voigt's result. If $1 \leq p \leq \infty$ and $q < \infty$, then Theorem 2.10 can be deduced from Theorem 2.1 and Voigt's result. In fact, it suffices to show that $|T(\cdot)|f: J \to G$ is measurable for all $f \in F$. But, by the proof of Theorem 2.1, the function $(t, y) \to \int ||K(t, y, x)| f(x)\,d\mu(x)$ is measurable. Since G has order continuous norm this implies that

$$t \to \int |\tilde{K}(t, \cdot, x)| f(x)\,d\mu(x) = |T(t)| f \in G$$

is measurable (see, for example, [Bu2]).

Thus, Theorem 2.10 actually remains true without the assumption that (X, μ) is separable.

3. Analytic dependence

Frequently, a family of integral operators is given which depends analytically on a parameter. We show that the kernels can be chosen in such a way that they depend analytically on z as well. More precisely, we show the following.

Theorem 3.1. *Let (X, μ), (Y, ν) be σ-finite measure spaces; we suppose that (X, μ) is separable. Let $F = L^p(X, \mu)$, $1 \le p < \infty$, and $G = L^q(Y, \nu)$, $1 \le q \le \infty$. Let $D \subset \mathbb{C}$ be open and $T: D \to \mathscr{L}^r(F, G)$ be a function such that*

(a) *$T(\cdot)f: D \to G$ is analytic for all $f \in F$;*
(b) *for all $z_0 \in D$ there exists $r > 0$ and $\kappa \in L^1([0, 2\pi], \mathbb{R})$ such that $\bar{B}(z_0, r) \subset D$ and $\| T(z_0 + re^{i\theta})\|_{\mathscr{L}^r} \le \kappa(\theta) \quad (\theta \in [0, 2\pi])$.*

Then there exists a measurable function $K: D \times Y \times X \to \mathbb{C}$ such that

1. *$K(\cdot, y, x): D \to \mathbb{C}$ is analytic for all $y \in Y$, $x \in X$.*
2. *$T(z)$ is represented by $K(z, \cdot, \cdot) \quad (z \in D)$.*

Remark. 1. It is well-known that (a) is equaivalent to

(a′) $\quad \langle T(\cdot)f, \varphi\rangle : D \to \mathbb{C}$ is analytic for all $f \in$ F, $\varphi \in G'$.

2. In (b) we let $\bar{B}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \le r\}$.
3. Condition (b) follows from (a) if $q = \infty$ or $p = 1$.

Proposition 3.2. *Under the hypotheses of Theorem 3.1, the function $T: D \to \mathscr{L}^r(F, G)$ is analytic.*

Proof. Let Let $z_0 \in D$. We show that T is holomorphic at z_0. We can assume $z_0 = 0$. Choose $r > 0$ from the condition (b) of Theorem 3.1. Then

$$T'(0)f = \frac{1}{2\pi i} \int_{|t|=r} T(\xi) f \frac{d\xi}{\xi^2}$$

defines a regular operator $T'(0) \in \mathscr{L}^r(F, G)$ (by Proposition 2.9). Let $f \in F$. Then

$$\begin{aligned}
&\frac{1}{z}(T(z) - T(0))f - T'(0)f \\
&= \frac{1}{2\pi i} \int_{|\xi|=r} T(\xi) f \left\{ \frac{1}{z}\left(\frac{1}{\xi - z} - \frac{1}{\xi}\right) - \frac{1}{\xi^2}\right\} d\xi \\
&= \frac{1}{2\pi i} \int_{|\xi|=r} T(\xi) f \frac{1}{\xi} \left\{ \frac{1}{\xi - z} - \frac{1}{\xi}\right\} d\xi .
\end{aligned}$$

Consequently, for $f \in F_+$,

$$\begin{aligned}
&\left| \frac{1}{z}(T(z) - T(0)) - T'(0) \right| f \\
&\le \frac{1}{2\pi} \int_0^{2\pi} |T(re^{i\theta})| f \left| \frac{1}{re^{i\theta} - z} - \frac{1}{re^{i\theta}} \right| d\theta .
\end{aligned}$$

Hence,

$$\left\| \frac{1}{z}(T(z)-T(0))-T'(0) \right\|_r$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} \kappa(\theta)\,d\theta \sup_{\theta\in[0,2\pi]} \left| \frac{1}{re^{i\theta}-z} - \frac{1}{re^{i\theta}} \right| \to 0 \quad (z\to 0)\,. \quad \square$$

Remark 3.3. For Proposition 3.2 it suffices that $T(z)\in \mathscr{L}^r(F,G)$ for all $z\in D$ (instead of $\Gamma^r(F,G)$).

Lemma 3.4. *Let (W,ϱ) be a σ-finite measure space, $D\subset\mathbb{C}$ open and let $u: D \to E := L^1(W,\varrho)$ be a holomorphic function. Then there exists a measurable function $g: D\times W\to\mathbb{C}$ such that*

(a) *$g(\cdot,x): D\to\mathbb{C}$ is holomorphic for all $x\in W$;*
(b) *for all $z\in D$ one has $u(z)(x)=g(z,x)$ x-a.e.*

Proof. Within this proof we say that an open set $D_1\subset D$ has property (P) if there exists $g: D_1\times W\to\mathbb{C}$ satisfying (a) and (b) with D replaced by D_1.

1. Let $\bar{B}(z_0,r):=\{z\in\mathbb{C}: |z-z_0|\leq r\}\subset D$. Then $B(z_0,r)$ has (P). In fact, there exist $a_n\in E$ such that $\sum_{n=0}^{\infty}\|a_n\|r^n<\infty$ and $u(t)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ in L^1 for all $z\in\bar{B}(z_0,r)$. In particular, the series $\sum_{n=0}^{\infty}|a_n|r^n$ converges in L^1. Consequently, $\sum_{n=0}^{\infty}|a_n(x)|r^n<\infty$ for all $x\notin N$ where $N\subset W$ is of measure 0. Let $g(z,x)=\sum_{n=0}^{\infty}a_n(x)(z-z_0)^n$ if $x\notin N$ and $g(z,x)=0$ if $x\in N$. Then $g: B(z_0,r)\times W\to\mathbb{C}$ satisfies (a) and (b) on $B(z_0,r)$.

2. Suppose that D_1 and D_2 satisfy (P) with functions $g_i: D_i\times W\to\mathbb{C}$ $(i=1,2)$. It suffices to show that for a.e. $x\in W$ $g_1(z,x)=g_2(z,x)$ for all $z\in D_1\cap D_2$. We can assume that $D_1\cap D_2\neq\emptyset$. Then $D_1\cap D_2$ contains a countable set D_0 with a limit point. It follows from (b) that for a.e. $x\in W$ one has $g_1(z,x)=g_2(z,x)$ for all $z\in D_0$. Now the identity theorem for analytic functions implies that for a.e. $x\in W$ one has $g_1(z,x)=g_2(z,x)$ for all $z\in D_1\cap D_2$.

3. It follows from **2** that (P) is preserved by finite unions.

4. Write $D=\bigcup_{n\in\mathbb{N}}D_n$, where $D_n\subset D$ is open, relatively compact and $\bar{D}_n\subset D_{n+1}$. It follows from **1** and **2** that each D_n satisfies (P). Let $g_n: D_n\times W\to\mathbb{C}$ be the corresponding function. The argument given in **2** shows that there exists a set $N_n\subset W$ of measure 0 such that $g_n(z,x)=g_k(z,x)$ for all $z\in D_k$ if $x\notin N_n$, $k=1,\dots,n$. The set $N=\bigcup_{n\in\mathbb{N}}N_n$ is of measure 0. Let $g(z,x)=g_n(z,x)$ for $z\in D_n$, $x\notin N$ and $g(z,x)=0$ for all $z\in D$ if $x\in N$. Then g satisfies (a) and (b). $\square$

Proof of Theorem 3.1. **1.** We can assume that $p=\infty$ and $q=1$. In fact, otherwise, let $h\in L^p(X,\mu)$ such that $h(x)>0$ for all $x\in X$ and let $k\in L^{q'}(Y,\nu)$ such that $k(y)>0$ y-a.e. Consider the mapping $j: L^q(Y,\nu)\to L^1(Y,k\,d\nu)$, $j(g)=g$, and replace T by $\tilde{T}(z)f=j(T(z)(hf))$.

2. Since $p = \infty$, $q = 1$, the space $I^r(F, G) = I^r(L^\infty(X, \mu), L^1(Y, \nu))$ is isometrically isomorphic to $L^1(Y \times X, \nu \times \mu)$, where the isomorphism is given by $K \in L^1(Y \times X) \to T_K \in I^r(F, G)$, $(T_K f)(y) = \int K(y, x) f(x) d\mu(x)$ y-a.e. Now the claim follows from Lemma 3.4. □

4. Integral representations of semigroups and resolvents

In this section (X, μ) is a σ-finite, separable measure space and we write for short $L^p = L^p(X, \mu)$ $(1 \leq p < \infty)$. For most applications it suffices to take X an open subset of $\mathbb{R}^N$ and μ Lebesgue measure.

Let A be an operator on $E := L^p$. Our goal is to investigate under which condition the resolvent of A consists of integral operators. Because of the better permanence properties we restrict ourselves to regular integral operators. Therefore, it is natural to introduce

$$\varrho_0(A) := \{\lambda \in \varrho(A) : R(\lambda, A) \in \mathscr{L}^r(L^p)\},$$

the *order resolvent set* of A. It can actually happen that $\varrho_0(A) \neq \varrho(A)$ (see [A1] and below), but $\varrho_0(A) = \varrho(A)$ if $p = 1$ or ∞. By $\sigma_0(A) := \mathbb{C} \setminus \varrho_0(A)$ we denote the *order spectrum*. This notion has been introduced by Schaefer [S2] for bounded operators.

Proposition 4.1. *The order-resolvent set $\varrho_0(A)$ is open and the function $\lambda \to R(\lambda, A)$ from $\varrho_0(A)$ into $\mathscr{L}^r(L^p)$ is analytic.*

Proof. Let $\lambda_0 \in \varrho_0(A)$. If $|\lambda - \lambda_0| \, \|R(\lambda_0, A)\|_r < 1$ then writing $(\lambda - A) = (I - (\lambda_0 - \lambda) R(\lambda_0, A))(\lambda_0 - A)$ one sees that $R(\lambda, A) = R(\lambda_0, A) \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^n \in \mathscr{L}^r(L^p)$. □

Proposition 4.2. *Let A be an operator on L^p, $1 \leq p < \infty$, such that $R(\xi, A) \in I^r(L^p)$ for some $\xi \in \varrho(A)$. Then $R(\lambda, A) \in I^r(L^p)$ for all $\lambda \in \varrho_0(A)$. Moreover, $R(\lambda, A)$ can be represented by a kernel $K(\lambda, y, x)$ which depends analytically on $\lambda \in \varrho_0(A)$.*

Proof. The first claim follows from the resolvent equation and the fact that $I^r(L^p)$ is an algebraic ideal in $\mathscr{L}^r(L^p)$. The analytical dependence of the kernels follows from Theorem 3.1. □

The *spectral bound* of A is defined by

$$s(A) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}.$$

The operator A is called *resolvent positive* if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \varrho(A)$ and $R(\lambda, A) \geq 0$ $(\lambda > \omega)$. In that case, $R(\lambda, A) \geq 0$ for all $\lambda > s(A)$, and $R(\lambda, A) \in \mathscr{L}^r(L^p)$, $|R(\lambda, A)| \leq R(\mathrm{Re}\,\lambda, A)$ whenever $\mathrm{Re}\,\lambda > s(A)$. Moreover, if $s(A) > -\infty$, then $s(A) \in \sigma(A)$ (see [A3]).

Now assume that A is a resolvent positive operator on L^p and assume that $1 \leq p < \infty$. Then A generates a positive integrated semigroup S on L^p; i.e. $S: [0, \infty)$

$\to \mathscr{L}(L^p)$ is strongly continuous, exponentially bounded, $0 = S(0) \le S(s) \le S(t)$ for $0 \le s \le t$ and

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} dS(t) \quad (\operatorname{Re}\lambda > s(A)),$$

(see [A3]). For example, if A generates a positive C_0-semigroup $T = (T(t))_{t \ge 0}$, then A is resolvent positive and $S(t) = \int_0^t T(s)\,ds$.

Theorem 4.3. *Let A be a resolvent positive operator on $E := L^p$ $(1 \le p < \infty)$ and let S be the integrated semigroup generated by A. The following are equivalent:*

(*i*) *There exists $\lambda > s(A)$ such that $R(\lambda, A) \in I^r(E)$;*
(*ii*) *$S(t) \in I^r(E)$ for all $t > 0$;*
(*iii*) *$R(\lambda, A) \in I^r(E)$ for all $\lambda \in \varrho_0(A)$.*

Proof. (*i*) $\Rightarrow$ (*ii*) *a*) Assume that $s(A) < 0$ and $\lambda = 0$, so that $R(0, A) \in I^r(E)$. Then $0 \le S(t) \le R(0, A)$ for all $t \ge 0$. This implies (*ii*).

b) Let $\lambda > s(A)$ be arbitrary. Then $B := A - \lambda$ is resolvent positive, $s(B) < 0$, $R(\lambda, A) = R(0, B)$. Denote by S_λ the integrated semigroup generated by B. It follows from *a*) that $S_\lambda(t) \in I^r(E)$ $(t \ge 0)$. Since $S(t) = e^{\lambda t} S_\lambda(t) - \lambda \int_0^t e^{\lambda s} S_\lambda(s)\,ds$ $(t \ge 0)$, one concludes from Theorem 2.1 that $S(t) \in I^r(E)$ for all $t \ge 0$.

(*ii*) $\Rightarrow$ (*iii*) Let $\lambda > \max\{s(A), 0\}$. Then $R(\lambda, A) = \lambda \int_0^\infty e^{-\lambda t} S(t)\,dt \in \mathscr{L}^r(E)$ by Theorem 2.1. Let $\xi \in \varrho_0(A)$. Then $R(\xi, A) = R(\lambda, A)\,(I + (\lambda - \xi) R(\xi, A)) \in I^r(E)$ since $I^r(E)$ is an ideal in $\mathscr{L}^r(E)$. □

Corollary 4.4. *Assume that A generates a positive C_0-semigroup $T = (T(t))_{t \ge 0}$ on L^p $(1 \le p < \infty)$. If $T(t) \in I^r(L^p)$ $(t \ge 0)$, then $R(\lambda, A) \in I^r(L^p)$ for all $\lambda \in \varrho_0(A)$ (and, in particular, for* $\operatorname{Re}\lambda > s(A)$*).*

If $T(t) \in \mathscr{K}^r(L^p)$, we will see actually that $R(\lambda, A) \in \mathscr{K}^r(L^p)$ for all $\lambda \in \varrho(A)$ (Corollary 4.8). We first show that $\varrho(A) = \varrho_0(A)$ in that case.

Theorem 4.5. *Let B be an operator on $E = L^p$ $(1 \le p < \infty)$. Assume that there exist $\xi_1, \xi_2 \in \varrho(B)$, $k \in \mathbb{N}$ such that*

1. $R(\xi_1, B) \in \mathscr{L}^r(E)$;
2. $R(\xi_2, B)^k \in \mathscr{K}^r(E)$.

Then $R(\lambda, B) \in \mathscr{L}^r(E)$ and $R(\lambda, B)^k \in \mathscr{K}^r(E)$ for all $\lambda \in \varrho(B)$.

In particular, in the situation of Theorem 4.5 one has $\varrho(B) = \varrho_0(B)$. This is not always the case. There exists a compact positive operator $U \in \mathscr{L}(L^2(0,1))$ such that $\varrho(U) \ne \varrho_0(U)$ (and $\sigma_0(U)$ is even uncountable). However, the following is shown in [A1, Corollary 2.9] (see also [MN, Corollary 4.5.7]).

Theorem 4.6. *Let* $U \in \mathscr{L}^r(L^p)$, $1 \le p < \infty$. *If* $U^k \in \mathscr{K}^r(L^p)$ *for some* $k \in \mathbb{N}$, *then* $R(\lambda, U) \in \mathscr{L}^r(L^p)$ *for all* $\lambda \in \varrho(U)$.

Proof of Theorem 4.5. a) By the resolvent equation we have

$$R(\lambda, B)^k = R(\xi, B)^k [I + (\xi - \lambda) R(\lambda, B)]^k \tag{4.1}$$

for all $\lambda, \xi \in \varrho(B)$. Letting $\lambda = \xi_1$, $\xi = \xi_2$ one sees that $R(\xi_1, B)^k \in \mathscr{K}^r(E)$ since $\mathscr{K}^r(E)$ is an algebraic ideal in $\mathscr{L}^r(E)$.

b) Let $\lambda \in \varrho(B)$. We show that $R(\lambda, B) \in \mathscr{L}^r(E)$. We can suppose that $\lambda \neq \xi_1$. Thus $\alpha := (\xi_1 - \lambda)^{-1} \in \varrho(R(\xi_1, B))$ and $(\alpha - R(\xi_1, B))^{-1} = (\xi_1 - \lambda)(\xi_1 - B) R(\lambda, B)$. Since $R(\xi_1, B) \in \mathscr{L}^r(E)$ and $R(\xi_1, B)^k \in \mathscr{K}^r(E)$, it follows from Theorem 4.6 that $(\alpha - R(\xi_1, B))^{-1} \in \mathscr{L}^r(E)$. Hence $R(\lambda, B) = \alpha R(\xi_1, B)(\alpha - R(\xi_1, B))^{-1} \in \mathscr{L}^r(E)$.

c) Choosing $\xi = \xi_1$ in (4.1) it follows now that $R(\lambda, B)^k \in \mathscr{K}^r(E)$ for all $\lambda \in \varrho(B)$. □

Now we obtain the following result which is analogous to Theorem 4.3.

Theorem 4.7. *Let A be a resolvent positive operator on* $E = L^p$ $(1 \le p < \infty)$ *and let S be the integrated semigroup generated by A. The following are equivalent:*

(*i*) $R(\lambda, A) \in \mathscr{K}^r(E)$ *for some* $\lambda \in \varrho(A)$;
(*ii*) $R(\lambda, A) \in \mathscr{K}^r(E)$ *for all* $\lambda \in \varrho(A)$;
(*iii*) $S(t) \in \mathscr{K}^r(E)$ *for all* $t > 0$.

Proof. The equivalence of (*i*) and (*ii*) follows from Theorem 4.5. The equivalence of (*ii*) and (*iii*) can be proved as Theorem 4.3 with help of Theorem 2.1. □

Corollary 4.8. *Let A be the generator of a positive* C_0*-semigroup* $T = (T(t))_{t \ge 0}$ *on* $E = L^p$ $(1 \le p < \infty)$. *If* $T(t) \in \mathscr{K}^r(E)$ $(t > 0)$, *then* $R(\lambda, A) \in \mathscr{K}^r(E)$ *for all* $\lambda \in \varrho(A)$.

Proof. For $\operatorname{Re}\lambda > s(A)$, $R(\lambda, A) = \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} T(t)\,dt$ in the sense of $\mathscr{L}^r(E)$. So $R(\lambda, A) \in \mathscr{K}^r(E)$ by Theorem 4.6. □

The converse of Corollary 4.8 is not true (see Example 4.15).

Proposition 4.9. *Let A be an operator on* $E = L^p$ $(1 \le p < \infty)$ *and let* $\omega \in \mathbb{R}$, $M \ge 0$. *Assume that*

(a) $[\omega, \infty) \subset \varrho(A)$;
(b) $R(\lambda, A) \in \mathscr{L}^r(E)$ $(\lambda \ge \omega)$;
(c) $M := \sup_{\lambda \ge \omega} \|\lambda R(\lambda, A)\|_r < \infty$;
(d) $R(\xi, A)^k \in \mathscr{K}^r(E)$ *for some* $\xi \in \varrho(A)$ *and some* $k \in \mathbb{N}$.

Then $R(\lambda, A) \in \mathscr{K}^r(E)$ *for all* $\lambda \in \varrho(A)$.

Proof. **1.** It follows from Proposition 4.6 that $R(\lambda, A) \in \mathscr{L}^r(E)$ for all $\lambda \in \varrho(A)$.
2. Let $\lambda_0 \in \varrho(A)$. We claim that

$$\lim_{\lambda \to \infty} \|(\lambda R(\lambda, A))^m R(\lambda_0, A) - R(\lambda_0, A)\|_r = 0 \tag{4.2}$$

for all $m \in \mathbb{N}$. In fact, (4.2) being trivial for $m = 0$ assume (4.2) to hold for $m \in \mathbb{N}$. By the resolvent equation we obtain

$$\begin{aligned} &\|(\lambda R(\lambda, A))^{m+1} R(\lambda_0, A) - R(\lambda_0, A)\|_r \\ &\leq \|\lambda R(\lambda, A)[(\lambda R(\lambda, A))^m R(\lambda_0, A) - R(\lambda_0, A)]\|_r \\ &\quad + \|\lambda R(\lambda, A) R(\lambda_0, A) - R(\lambda_0, A)\|_r \\ &\leq M\|(\lambda R(\lambda, A))^m R(\lambda_0, A) - R(\lambda_0, A)]\|_r \\ &\quad + \left\| \left(\frac{\lambda}{\lambda_0 - \lambda} - 1\right) R(\lambda_0, A) - \frac{\lambda}{\lambda - \lambda_0} R(\lambda, A) \right\|_r \to 0 \quad (\lambda \to \infty). \end{aligned}$$

3. Since $\mathscr{K}^r(E)$ is an ideal in $\mathscr{L}^r(E)$, and since by Theorem 4.5 $R(\lambda, A)^k \in \mathscr{K}^r(E)$ for all $\lambda \in \varrho(A)$ the claim follows from (4.2). □

Corollary 4.10. *Let A be the generator of a positive C_0-semigroup on E. If there exist $\xi \in \varrho(A)$, $k \in \mathbb{N}$ such that $R(\xi, A)^k \in \mathscr{K}^r(E)$, then $R(\lambda, A) \in \mathscr{K}^r(E)$ for all $\lambda \in \varrho(A)$.*

Corollary 4.11. *Let A be the generator of a positive C_0-semigroup on E. Assume that $R(\xi, A)^k \in I^r(E)$ for some $\xi > s(A)$ and some $k \in \mathbb{N}$. Then $R(\lambda, A) \in I^r(E)$* for all *$\lambda \in \varrho_0(A)$.*

Proof. Since $0 \leq R(\lambda, A) \leq R(\lambda_0, A)$ for $\lambda > \lambda_0$, it follows that $R(\lambda, A)^k \bullet I^r(E)$ for all $\lambda \leq \lambda_0$. Now the claim follows from (4.2) and the fact that $I^r(E)$ is closed ideal in $\mathscr{L}^r(E)$. □

Remark 4.12. The proof of Proposition 4.9 is analogous to [dP, Lemma 3.4]. The argument (due to de Pagter) shows that the generator B of a C_0-semigroup has compact resolvent, whenever there exists $k \in \mathbb{N}$ such that $R(\lambda, B)^k$ is compact for some $\lambda \in \varrho(B)$.

Theorem 4.13. *Let A be the generator of a positive C_0-semigroup $T = (T(t))_{t \geq 0}$ on $E = L^p(X, \mu)$, $1 \leq p < \infty$. Assume that there exists $u \in E_+$ such that*

$$D(A^\infty) \subset E_u.$$

Then $R(\lambda, A) \in \mathscr{K}^r(E)$ for all $\lambda \in \varrho(A)$.
If, in addition, T is differentiable, then also $T(t) \in \mathscr{K}^r(E)$ for all $t > 0$.

Here we let $D(A^\infty) = \bigcap_{n \in \mathbb{N}} D(A^n)$. This is a Fréchet space for the semi-norms $p_k(f) = \|f\| + \|Af\| + \ldots + \|A^k f\|$.

Proof. Consider the isomorphism $j: E_u \to C(Z)$ introduced in the proof of Proposition 1.13. It follows from the closed graph theorem that the restriction j_0 of j to $D(A^\infty)$ is continuous. Consequently, there exist $k \in \mathbb{N}$ and $c \geq 0$ such that

(4.3) $\quad \|j(f)\|_\infty \leq c p_k(f) \quad (f \in D(A^\infty))$.

The space $(D(A^k), p_k)$ is a Banach space and $D(A^\infty)$ is dense in $(D(A^k), p_k)$. Thus, it follows from (4.3) that $D(A^k) \subset E_u$ and (4.3) remains true for all $f \in D(A^k)$.

Let $\alpha > s(A)$. Then there exists a constant $c_1 \geq 0$ such that $p_k(R(\alpha, A)^k f) \leq c_1 \|f\|$ $(f \in E)$. Consequently, $\|j(R(\alpha, A)^k f)\|_\infty \leq c_2 \|f\|$ for all $f \in E$ (where $c_2 = c c_1$); i.e.

$$|R(\alpha, A)^k f| \leq c_2 \|f\| u \quad (f \in E).$$

It follows from Proposition 1.13 or 1.14 that $R(\alpha, A)^{2k} \in \mathscr{K}^r(E)$. Now Corollary 4.10 implies that $R(\lambda, A) \in \mathscr{K}^r(E)$ for all $\lambda \in \varrho(A)$.

If T is differentiable, then $T(t) \in \mathscr{L}(E, D(A^k))$ for all $t > 0$. It follows that $T(t) E \subset E_u$ for all $t > 0$. Now Proposition 1.13 or 1.14 imply that $T(2t) \in \mathscr{K}^r(E)$ $(t > 0)$. □

Corollary 4.14. *Assume that $\mu(X) < \infty$. Let $E = L^p(X, \mu)$ $(1 \leq p < \infty)$ and let $T = (T(t))_{t \geq 0}$ be a positive C_0-semigroup on E with the generator A. If*

$$D(A^\infty) \subset L^\infty(X),$$

then $R(\lambda, A) \in \mathscr{K}^r(E)$ for all $\lambda \in \varrho(A)$. If, in addition, T is differentiable, then $T(t) \in \mathscr{K}^r(E)$ for all $t > 0$.

If T is not differentiable it can happen that $T(t)$ is not an integral operator.

Example 4.15. Let $E = L^p(0, 1)$, $1 \leq p < \infty$, and let T be given by

$$(T(t)f)(x) = \begin{cases} f(x+t) & \text{if } x + t \leq 1 \\ 0 & \text{if } x + t > 1 . \end{cases}$$

Then $(R(\lambda, A)f)(x) = e^{\lambda x} \int_x^1 e^{-\lambda s} f(s)\, ds$ $(\lambda \in \mathbb{C})$. Thus, $D(A) = R(\lambda, A) E \subset L^\infty(0, 1)$. However, $T(t)$ is not an integral operator for $t < 1$.

The following simple case occurs very frequently in applications.

Theorem 4.16. *Let $\Omega \subset \mathbb{R}^N$ be open and let A be the generator of a positive semigroup T on $L^p(\Omega)$ $(1 \leq p < \infty)$. Assume that there exists $k \in \mathbb{N}$ such that*

$$D(A^k) \subset L^\infty(\Omega).$$

Then the following holds.

(a) *$R(\lambda, A)$ is an integral operator for all $\lambda \in \varrho_0(A)$.*
(b) *If Ω is bounded, then $R(\lambda, A) \in \mathscr{K}^r(L^p(\Omega))$ for all $\lambda \in \varrho(A)$.*
(c) *If T is holomorphic of angle $\alpha \in (0, \pi/2]$, then $T(z)$ is an integral operator and the kernel $K(z, y, x)$ can be chosen analytic in*

$$z \in \Sigma(\alpha) := \{re^{i\theta} : r > 0, |\theta| < \alpha\} \quad \textit{for all } x, y \in \Omega.$$

Proof. (a) Let $\lambda > s(A)$. Then $R(\lambda, A) \geq 0$ and $R(\lambda, A)^k L^p \subset L^\infty$. It follows from Theorem 1.3 that $R(\lambda, A)^k \in I^r(L^p(\Omega))$. Now we obtain from Proposition 4.9 that $R(\lambda, A) \in I^r(L^p(\Omega))$ for all $\lambda \in \varrho_0(A)$.

(b) Assume now that Ω is bounded. Let $\lambda > s(A)$. It follows from Corollary 4.14 that $R(\lambda, A) \in \mathscr{K}^r(L^p(\Omega))$ for all $\lambda \in \varrho(A)$.

(c) If T is holomorphic, then $T: \Sigma(\alpha) \to \mathscr{L}(L^p, D(A^k)) \subset \mathscr{L}(L^p, L^\infty)$ is a holomorphic function. So, the claim follows from Theorem 3.1. □

5. Applications to elliptic operators

In this section we illustrate how the abstract results can be applied to elliptic operators. As a general reference we use [Da].

Diverse realizations of elliptic operators via suitable boundary conditions generate positive holomorphic semigroups on $L^2(\Omega)$. Much effort has been done in order to show that

(5.1) $\quad D(A^k) \subset L^\infty$

for some $k \in \mathbb{N}$. Since the semigroup is holomorphic this immediately implies that it consists of integral operators. What our results give in addition is that the resolvents are integral operators.

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $a_{ij} \in L^1_{\text{loc}}(\Omega)$ be real such that $a_{ij} = a_{ji}$ $(i, j = 1, \ldots, N)$ and

(5.2) $$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad x\text{-a.e.}$$

for all $\xi \in \mathbb{R}^N$, where $\alpha > 0$.

a) **Dirichlet boundary conditions**

Consider the symmetric quadratic form

(5.3) $$Q(u) = \int \sum_{i,j=1}^{N} a_{ij}(x) D_i u D_j \bar{u} \, dx$$

with $D(Q) = C_c^\infty(\Omega)$. Let H be the operator on $L^2(\Omega)$ associated with the closure of Q. Then H is self-adjoint and $-H$ generates a positive holomorphic semigroup $(e^{-tH})_{t \geq 0}$ on $L^p(\Omega)$. It follows from the Beurling-Deny criterion that there exists positive contraction semigroups T_p on $L^p(\Omega)$ with generator A_p such that

$$T_p(t)f = T_q(t)f \quad (f \in L^p \cap L^q, t > 0),$$

$1 \leq p, q < \infty$ and $A_2 = -H$.

We call A_p a *strictly elliptic operator on $L^p(\Omega)$ with Dirichlet boundary conditions* (cf. [Da, pp. 9,10]).

b) **Neumann boundary conditions**
We assume in addition that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2 \quad x\text{-a.e.} \tag{5.4}$$

for all $\xi \in \mathbb{R}^N$, where $\beta > 0$. Then we define Q by (5.3) with domain $W^{1,2}(\Omega)$. Now Q is a closed form. Let H be the operator associated with Q. Again one finds interpolating positive contraction semigroups T_p on $L^p(\Omega)$ with generator B_p ($1 \leq p < \infty$) such that $B_2 = -H$. We say that B_p *is a uniformly elliptic operator with Neumann boundary conditions* (cf. [Da, Theorem 1.2.10]).

Theorem 5.1. *Let Ω be a region in $\mathbb{R}^N$ and let A_p be a strongly elliptic operator with Dirichlet boundary conditions or a uniformly elliptic operator with Neumann boundary conditions on $L^p(\Omega)$, $1 \leq p < \infty$. In the second case we assume, in addition, that Ω has the extension property* [Da, p. 46].

Then $R(\lambda, A_p)$ is an integral operator for all $\lambda \in \varrho_0(A_p)$) (and, in particular, if $\mathrm{Re}\,\lambda > 0$*).*

If Ω is bounded, then $R(\lambda, A_p) \in \mathscr{K}^r(L^p(\Omega))$ for all $\lambda \in \varrho(A_p)$ (i.e. $R(\lambda, A_p)$ is an integral operator and the modulus of the kernel defines a compact operator).

Proof. Let $k > N/2$, $\lambda > 0$. Then

$$R(\lambda, A_1)^k L^1(\Omega) \subset L^\infty(\Omega)\,. \tag{5.5}$$

This follows from [Da, Lemma 2.1.2 and Theorem 2.4.1] together with [Da, Theorem 2.3.6] in the case of Dirichlet boundary conditions and [Da, Theorem 2.4.4] in the case of Neumann boundary conditions. By interpolation it follows from (5.5) that

$$R(\lambda, A_p)^k L^p(\Omega) \subset L^\infty(\Omega)\,. \tag{5.6}$$

Now the claim follows from Theorem 4.16. □

Theorem 5.1 does not hold true, in general, for Neumann boundary conditions if the regularity assumption is omitted. In fact, it can happen, that Ω is bounded but A_2 does not have compact resolvent. Nevertheless, the first assertion remains true. We restrict ourselves to the Laplacian.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set and let A be the Neumann Laplacian on $L^2(\Omega)$. Then $R(\lambda, A)$ is an integral operator for all $\lambda \in \varrho_0(A)$.*

Proof. It is known that one still has local regularity in the sense that

$$D(A^k) \subset C(\Omega)$$

(the space of continuous functions on Ω) if $k > N/4$; see e.g., [Br, IX. 6.1]. Hence $R(\lambda, A)^k L^2(\Omega) \subset C(\Omega)$ for $\lambda > 0$, $k > N/4$. It follows from Proposition 1.7 that $R(\lambda, A)^k \in I^r(L^2(\Omega))$. Now the claim follows from Proposition 4.12. □

We conclude with an example of an operator of order larger than two.

Example 5.3 (the bi-Laplacian). Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let B be the Laplacian with Dirichlet boundary conditions. Let $A = -B^2$. Then A generates a contractive semigroup T on $L^2(\Omega)$ (by the spectral theorem). Since $0 \in \varrho(B)$ and $B^{-1} \in \mathscr{K}^r(L^2)$ it follows that $R(0, A) = -A^{-1} = -B^{-2} \in \mathscr{K}^r(L^2)$. Hence by Theorem 4.5 $R(\lambda, A) \in \mathscr{K}^r(L^2)$ for all $\lambda \in \varrho(A) = -\varrho(B)^2$. Thus, for all $\lambda \in -\varrho(B)^2$, the solution of

$$\begin{cases} \lambda u + \Delta^2 u = v \text{ on } \Omega \\ u|_{\partial\Omega} = (\Delta u)|_{\partial\Omega} = 0 \end{cases}$$

is given by $u(y) = \int_\Omega K(y, x) v(x) dx$, where $K(y, x)$ and $|K(y, x)|$ define compact operators on $L^2(\Omega)$.

6. Generalizations

As it has been noticed in the introduction our results are still true for very general classes of Banach function spaces. Here we give some hints concerning the main results leaving the rest to a reader.

First of all, both criteria for integral representability (Theorems 1.5 and 1.10) are true for general spaces of measurable functions (see [Bu1, Bu3, Z]). This gives the basis for generalizations, together with some basic facts about Banach function spaces, which the reader can find in [KA, Z].

We recall that a Banach space F of measurable functions on (X, μ) is said to be a Banach function space (BFS) provided

$$(f_1 \in L^0, f_2 \in F, |f_1| \le |f_2|) \Rightarrow (f_1 \in F, \|f_1\| \le \|f_2\|).$$

A BFS F has order continuous norm if

$$(f_n \downarrow 0) \Rightarrow (\|f_n\| \to 0),$$

and it is said that F satisfies the strong Fatou property provided

$$(f_n \uparrow, \sup \|f_n\| < \infty) \Rightarrow (\exists f \in F: f_n \uparrow f, \|f_n\| \uparrow \|f\|).$$

Theorem 6.1. *Theorem 2.1 holds true if F and G are BFS, and G has order continuous norm or satisfies the strong Fatou property.*

Theorem 6.2. *Proposition 2.9 holds true if F and G are BFS, and G has order continuous norm or satisfies the strong Fatou property.*

This means, for example, that Proposition 2.9 holds without any separability assumptions. The proof is based on some lattice theoretic arguments.

Theorem 6.3. *Theorem 2.10 holds true if F and G are BFS, and G has order continuous norm.*

In Section 3 we, really, need no special assumptions concerning the spaces. Theorem 3.1 holds true for arbitrary BFS F and G without any separability conditions (cf. Remark 6.2).

Certainly, all this means that the results of Sections 4 and 5 could be correspondingly generalized.

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References

[AB] Aliprantis, C.D., Burkinshaw, O.: Positive Operators. Acad. Press, London 1985
[A1] Arendt, W.: On the o-spectrum of regular operators and the spectrum of measures. Math Z. **178** (1981), No. 2, 271–287
[A2] –: Order properties of convolution operators (1980) (Unpublished manuscript)
[A3] –: Resolvent positive operators. Proc. London Math. Soc. **54** (1987), 321–349
[ABa] –, Batty, C.: Absorption semigroups and Dirichlet boundary conditions. Math. Ann., to appear
[AV] –, Voigt, J.: Approximation of multipliers by regular operators. Indag. Mathem., N.S. **2** (1991), No. 2, 159–170
[Br] Brézis, H.: Analyse Fonctionelle. Masson, 1983
[Bu1] Bukhvalov, A.V.: On integral representation of linear operators. Zap. Nauchn. Sem. LOMI **47** (1974), 5–14; English transl.: J. Soviet. Math. **9** (1978), No. 2, 129–137
[Bu2] –: Integral operators and represention of completely linear functionals on spaces with a mixed norm. Sibirsk. Mat. Zh. **16** (1975), 483–493; English transl.: Siberian Math. J. **16** (1975), 368–376
[Bu3] –: Application of methods of the theory of order-bounded operators to the theory of operators in L^p-spaces. Uspekhi Mat. Nauk **38** (1983), No. 6, 37–83; English transl.: Russian Math. Surveys **38** (1983), No. 6, 43–98
[Da] Davies, E.B.: Heat Kernels and Spectral Theory. Cambridge University Press, vol. 92, Cambridge 1989
[HP] Hille, E., Phillips, R.S.: Functional Analysis and Semigroups. Amer. Math. Soc. Coll. Publ., vol. 31, Providence (R.I.) 1957
[KA] Kantorovich, L.V., Akilov, G.P.: Functional Analysis, 2nd rev. ed. "Nauka", Moscow 1977 (Russian); English transl.: Pergamon Press, Oxford 1982
[Lo] Lozanovskiĭ, G.Ya.: On almost integral operators in KB-spaces. Vestnik Leningrad. Univ. (1966), No. 7, 35–44 (Russian)
[MN] Meyer-Nieberg, P.: Banach Lattices. Springer, Berlin 1991
[dP] de Pagter, B.: A characterization of sun-reflexivity. Math. Ann. **283** (1989), 511–518
[Ro] Robinson, D.W.: Elliptic Operators and Lie Groups. Oxford Science Publications, Oxford 1991

[S1] Schaefer, H.H.: Banach Lattices and Positive Operators. Springer-Verlag, Berlin 1974
[S2] –: On the o-spectrum of order bounded linear operators. Math. Z. **154** (1977), 79–84
[Vo] Voigt, J.: On the convex compactness property for the strong operator topology. Preprint
[W] Weis, L.: A generalization of the Vidav-Jorgens perturbation theorem of semigroups and its applications to transport theory. J. Math. Anal. Appl. **29** (1988), 6–23
[Z] Zaanen, A.C.: Riesz Spaces II. North-Holland Publ. Co., Amsterdam 1983

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Wolfgang Arendt, Equipe de Mathématiques, U.A. C.N.R.S. 74-1, Université de Franche-Comté, 25030 Besançon Cedex, France
E-mail address: ARENDT@GRENET.FR.bitnet

Alexander V. Bukhvalov, Dept. of Mathematics, St.-Petersburg University of Economics and Finance, Sadovaya street 21, 191023 St.-Petersburg, Russia