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Local Integrated Semigroups: Evolution with Jumps of Regularity

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I. INTRODUCTION

Throughout this paper A denotes a closed operator on a Banach space X. Let $0 < \tau \leqslant \infty$. We consider the Cauchy problem

$$\left\{\begin{array}{ll} u \in C([0, \tau); D(A)) \cap C^1([0, \tau); X), & \\ u'(t) = Au(t), & t \in [0, \tau), \\ u(0) = x. & \end{array}\right\} \qquad C_0(\tau)$$

Here $D(A)$ is considered with the graph norm $\|x\|_A = \|x\| + \|Ax\|$.

Let u be a solution of $C_0(\tau)$, let $k \in \mathbb{N}_0$, and

$$v(t) = \int_0^t \frac{(t-s)^k}{k!} u(s)\, ds = \int_0^t \int_0^{t_1} \cdots \int_0^{t_k} v(t_{k+1})\, dt_{k+1}\, dt_k \cdots dt_1.$$

Then v is a solution of the problem

$$\left\{\begin{array}{ll} v \in C([0, \tau); D(A)) \cap C^1([0, \tau); X), & \\ v'(t) = Av(t) + (t^k/k!)x, & t \in [0, \tau), \\ v(0) = 0. & \end{array}\right\} \qquad C_{k+1}(\tau)$$

We call $C_{k+1}(\tau)$ the $(k+1)$-times integrated Cauchy problem.

DEFINITION 1.1. *Let* $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\tau > 0$. The Cauchy problem $C_k(\tau)$ is *well-posed* if for all $x \in X$ there exists a unique solution of $C_k(\tau)$.

If A is the generator of a C_0-semigroup, then for all $x \in D(A)$ there exists a unique solution of $C_0(\infty)$. The converse is not true (see [Na, A-II, Example 1.4]) unless the resolvent set is not empty. However, the following holds.

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THEOREM 1.2. *The problem $C_1(\tau)$ is well posed for some $\tau > 0$ if and only if A generates a C_0-semigroup.*

This is a modified version of a theorem due to van Casteren [vCa, Theorem 3.1]. It follows from Proposition 3.3 below and [Na, A-II, Corollary 1.2].

Thus well-posedness of $C_1(\tau)$ is characterized by the Hille–Yosida theorem.

Our goal is to characterize well-posedness of $C_{k+1}(\tau)$ by the resolvent for $k \in \mathbb{N}_0$ and $\tau > 0$. In our main result, we show in particular that well-posedness of $C_{k+1}(\tau)$ for some $k \in \mathbb{N}_0$ and some $\tau > 0$ is characterized by the fact that the resolvent set $\rho(A)$ of A contains an exponential region

$$E(\alpha, \beta) := \{x + iy : x \geqslant \beta, |y| \leqslant e^{\alpha x}\}$$

(where $\alpha > 0$, $\beta > 0$) on which $R(\lambda, A) := (\lambda - A)^{-1}$ is polynomially bounded.

This result should be compared with the characterization of those operators which generate an exponentially bounded k-times integrated semigroup for some $k \in \mathbb{N}$ by the fact that the resolvent set contains a semi-plane on which the resolvent is polynomially bounded (see [AK]).

The local problem has an interesting extension property: if $C_k(\tau)$ is well-posed then so is $C_{2k}(2\tau)$. Thus one can reach arbitrarily large times if one is ready to give up regularity. This result can be used to show that (under the additional assumption that $D(A)$ is dense) A generates a distribution semigroup (in the sense of Lions [Li]) if and only if $C_k(\tau)$ is well-posed for some $k \in \mathbb{N}$ and some $\tau > 0$. From our characterization theorem mentioned above one can now deduce Chazarain's theorem characterizing generators of distribution semigroups (see [Ch, Theorem 5.1]).

If $C_{k+1}(\tau)$ is well-posed, we obtain in the usual way an operator-valued function S on $[0, \tau)$ which governs the problem. We call it the *(local) k-times integrated semigroup* generated by A. Elementary spectral properties of the bounded operators $S(t)$ are used in order to compute the resolvent of the (unbounded) operator A (Section 2). This is a new approach which is very efficient in order to obtain complex characterizations.

A real characterization has been obtained by Tanaka and Okazawa [TO] (after previous work of Oharu [Oh1]). However, they define local integrated semigroups in a different way (by a functional equation corresponding to the integrated semigroup property).

Our characterization theorem covers global k-times integrated semigroups which are not necessarily exponentially bounded. Those have been considered before by Kellerman and Hieber [KH], Thieme [Th], and Lumer [Lu1, Lu2].

II. Characterization of Well-Posedness by the Resolvent

For $\alpha > 0$, $\beta > 0$, we define the exponential region $E(\alpha, \beta)$ by

$$E(\alpha, \beta) := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant \beta, |\operatorname{Im} \lambda| \leqslant e^{\alpha \operatorname{Re} \lambda}\}.$$

Let A be a closed operator on X. Well-posedness of the problem $C_{k+1}(\tau)$ is characterized by the following two theorems.

THEOREM 2.1. *Let $k \in \mathbb{N}$, $0 < \tau \leqslant \infty$. Assume that $C_{k+1}(\tau)$ is well-posed. Then for all $0 < \alpha < \tau/k$ there exist $\beta > 0$, $M \geqslant 0$ such that*

$$E(\alpha, \beta) \subset \rho(A) \quad \text{and} \quad \|R(\lambda, A)\| \leqslant M\, |\lambda|^k \; (\lambda \in E(\alpha, \beta)).$$

The converse theorem holds with some loss of regularity.

THEOREM 2.2. *Let $\alpha > 0$, $\beta > 0$, $M \geqslant 0$, $-1 < k \in \mathbb{R}$, and assume that $E(\alpha, \beta) \subset \rho(A)$ and*

$$\|R(\lambda, A)\| \leqslant M\, |\lambda|^k \qquad (\lambda \in E(\alpha, \beta)).$$

Let $\mathbb{N} \ni p > k+1$, $\tau = \alpha(p - (k+1))$. Then $C_{p+1}(\tau)$ is well-posed.

The proof of Theorem 2.2 is based on a contour argument which we give at the end of this section. Theorem 2.1 is more delicate. Assuming well-posedness of $C_{k+1}(\tau)$, we obtain an operator-valued function which governs the problem $C_{k+1}(\tau)$ (see Proposition 2.3). Its finite Laplace transform can be used to compute the resolvent of A.

PROPOSITION 2.3. *Let $k \in \mathbb{N}_0$, $0 < \tau \in \mathbb{R}$, and assume that $C_{k+1}(\tau)$ is well-posed. Then there exists a unique strongly continuous function S: $[0, \tau) \to \mathscr{L}(X)$ such that $\int_0^t S(s)x\, ds \in D(A)$ and*

$$A \int_0^t S(s)x\, ds = S(t)x - \frac{t^k}{k!}x \qquad (t \in [0, \tau), \textit{for all } x \in X).$$

We call S *the k-times integrated semigroup generated by A*. Thus, if $x \in X$ and if v is the solution of $C_{k+1}(\tau)$ then $S(t)x = v'(t)$, $0 \leqslant t < \tau$.

Proof of Proposition 2.3. It follows from the assumption that there exists a strongly continuous family of linear operators $S(t)$, $0 \leqslant t < \tau$, such that (2.1) holds. It remains to show that $S(t)$ is continuous, i.e., $S(t) \in \mathscr{L}(X)$ for $0 \leqslant t < \tau$.

For $x \in X$, let $V(t)x = \int_0^t S(s)\, x\, ds$ $(t \in [0, \tau))$ be the solution of $C_{k+1}(\tau)$. Consider the mapping $\Phi: X \to C([0, \tau); D(A))$, $\Phi(x) = V(\cdot)x$ where $D(A)$

is equipped with the graph norm $\|x\|_A$. The space $C([0, \tau); D(A))$ is a Fréchet space for the seminorms $p_n(v) = \sup_{0 \leqslant t \leqslant \tau - 1/n} \|v(t)\|_A$. Then Φ is linear and it follows from well-posedness that Φ has a closed graph. In fact, let $x_n \to x$ such that $v_n := \Phi(x_n) \to v$ in $C([0, \tau); D(A))$. Since $\int_0^t Av_n(s)\, ds = v_n(t) - (t^{k+1}/(k+1)!)\, x_n$ $(n \in \mathbb{N})$ it follows that $\int_0^t Av(s)\, ds = v(t) - (t^{k+1}/(k+1)!)x$, i.e., v is a solution of $C_{k+1}(\tau)$ and by uniqueness $v = \Phi(x)$. Hence for all $n \in \mathbb{N}$ there exists c_n such that

$$\|AV(t)x\| \leqslant c_n \|x\| \qquad \left(x \in X, 0 \leqslant t \leqslant \tau - \frac{1}{n}\right).$$

Since $S(t)x = AV(t)x + (t^k/k!)x$ $(x \in X)$ it follows that $S(t) \in \mathscr{L}(X)$ $(0 \leqslant t < \tau)$. ∎

In order to prove Theorem 2.1 we need some notation.

For $\lambda \in \mathbb{C}$, $t \geqslant 0$ let

$$g_\lambda(t) = \int_0^t e^{\lambda(t-s)} \frac{s^{k-1}}{(k-1)!}\, ds = \frac{e^{\lambda t}}{\lambda^k} + q_\lambda(t),$$

where

$$q_\lambda(t) = -\frac{1}{\lambda^k} - \frac{t}{\lambda^{k-1}} - \frac{t^2}{2!\, \lambda^{k-2}} - \cdots - \frac{t^{k-1}}{(k-1)!\, \lambda}.$$

Let

$$q(t) = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^{k-1}}{(k-1)!}.$$

Assume that A is the generator of a k-times integrated semigroup S on $[0, \tau)$. For $t \in [0, \tau)$ let $L_\lambda(t) = \int_0^t e^{-\lambda s} S(s)\, ds$ be the *finite Laplace transform of S.*

LEMMA 2.4. *One has $L_\lambda(t) \in \mathscr{L}(X, D(A))$ and*

$$(\lambda - A)\, L_\lambda(t)x = e^{-\lambda t}(g_\lambda(t) - S(t))x \tag{2.2}$$

for all $x \in X$, $t \in [0, \tau)$, $\lambda \in \mathbb{C}$.

Proof. Let $\lambda \in \mathbb{C}$, $t \geqslant 0$. Then

$$\begin{aligned} L_\lambda(t) &= \int_0^t e^{-\lambda s} \frac{d}{ds} \int_0^s S(r)\, dr\, ds \\ &= e^{-\lambda t} \int_0^t S(r)\, dr + \lambda \int_0^t e^{-\lambda s} \int_0^s S(r)\, dr\, ds. \end{aligned}$$

Since A is closed, it follows from (2.1) that $L_\lambda(t)x \in D(A)$ and

$$
\begin{aligned}
&(\lambda - A)\, L_\lambda(t)x \\
&\quad = \lambda L_\lambda(t)x - e^{-\lambda t}\left[S(t)x - \frac{t^k}{k!}x\right] - \lambda \int_0^t e^{-\lambda s}\left[S(s)x - \frac{s^k}{k!}x\right] ds \\
&\quad = -e^{-\lambda t}S(t)x + e^{-\lambda t}\frac{t^k}{k!}x + \lambda \int_0^t e^{-\lambda s}\frac{s^k}{k!}x\, ds \\
&\quad = -e^{-\lambda t}S(t)x + \int_0^t e^{-\lambda s}\frac{s^{k-1}}{(k-1)!}x\, ds \\
&\quad = -e^{-\lambda t}S(t)x + e^{-\lambda t}g_\lambda(t)x \\
&\quad = e^{-\lambda t}(g_\lambda(t) - S(t))x.
\end{aligned}
$$

For α, $\beta > 0$ let $\Lambda(\alpha, \beta) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant 1,\ \operatorname{Re}\lambda \geqslant \beta + \alpha \ln|\lambda|\}$. ∎

PROPOSITION 2.5. *Let $k \in \mathbb{N}$, $0 < \tau < \infty$. Assume that $C_{k+1}(\tau)$ is well-posed. Then for all $\alpha > k/\tau$ there exists $\beta > 0$, $M \geqslant 0$ such that $\Lambda(\alpha, \beta) \subset \rho(A)$ and*

$$\|R(\lambda, A)\| \leqslant M\,|\lambda|^k \qquad \textit{for all} \quad \lambda \in \Lambda(\alpha, \beta).$$

Proof. Let $0 < q < 1$, let $t = k/\alpha < \tau$, and $\beta = (1/t)\ln(\|S(t)\|/q + q(t))$.
Let $\lambda \in \Lambda(\alpha, \beta)$. Then $|\lambda| \geqslant 1$, hence $|q_\lambda(t)| \leqslant q(t)$. Moreover, $\operatorname{Re}\lambda \geqslant \beta + (k/t)\ln|\lambda|$; i.e., $e^{t\operatorname{Re}\lambda} \geqslant (\|S(t)\|/q + q(t))\,|\lambda|^k$. Hence

$$
\begin{aligned}
|g_\lambda(t)| &\geqslant e^{t\operatorname{Re}\lambda}\,|\lambda|^{-k} - |q_\lambda(t)| \\
&\geqslant e^{\operatorname{Re}\lambda}\,|\lambda|^{-k} - q(t) \\
&\geqslant \|S(t)\|/q.
\end{aligned}
$$

Thus

$$\|S(t)/g_\lambda(t)\| \leqslant q < 1 \tag{2.3}$$

and consequently, $(g_\lambda(t) - S(t))$ is invertible and

$$(g_\lambda(t) - S(t))^{-1} = \sum_{n=0}^{\infty} S(t)^n / g_\lambda(t)^{n+1}.$$

Moreover

$$\|(g_\lambda(t) - S(t))^{-1}\| \leqslant |g_\lambda(t)|^{-1}\,(1-q)^{-1}. \tag{2.4}$$

Let $R = e^{\lambda t} L_\lambda(t)(g_\lambda(t) - S(t))^{-1}$. Then by Lemma 2.4, $RX \subset D(A)$ and

$$(\lambda - A)\, Rx = x \qquad (x \in X). \tag{2.5}$$

We show that $(\lambda - A)$ is injective. Let $x \in D(A)$ be such that $Ax = \lambda x$. Then the solution of $C_{k+1}(\tau)$ is given by $v(r) = \int_0^r g_\lambda(s)\, x\, ds$.

Hence $S(t)x = g_\lambda(t)x$ and (2.3) implies that $x = 0$.

We have shown that $\lambda \in \rho(A)$ and $R(\lambda, A) = R = e^{\lambda t} L_\lambda(t)(g_\lambda(t) - S(t))^{-1}$. In order to prove polynomial growth, let $m \in \{0, 1, ..., k-1\}$. Then for $\lambda \in \Lambda(\alpha, \beta)$, $|\lambda^m e^{-\lambda t}| = |\lambda|^m e^{-\operatorname{Re} \lambda t} \leqslant |\lambda|^m e^{-\beta t} e^{-k \ln |\lambda|} = e^{-\beta t} |\lambda|^{m-k} \to 0$ for $\operatorname{Re} \lambda \to \infty$ and $\lambda \in \Lambda(\alpha, \beta)$. Consequently, $|e^{\lambda t} g_\lambda(t)^{-1} \lambda^{-k}| = |1 + q_\lambda(t)\, \lambda^k e^{-\lambda t}|^{-1} \to 1$ for $\operatorname{Re} \lambda \to \infty$, $\lambda \in \Lambda(\alpha, \beta)$. Thus, there exists $c_1 \geqslant 0$ such that $|e^{\lambda t} g_\lambda(t)^{-1}| \leqslant c_1 |\lambda|^k$ for all $\lambda \in \Lambda(\alpha, \beta)$. Since $\|L_\lambda(t)\|$ is bounded for $\lambda \in \Lambda(\alpha, \beta)$ we obtain from (2.4),

$$\begin{aligned}\|R(\lambda, A)\| &\leqslant \|L_\lambda(t)\| \cdot \|e^{\lambda t} \cdot (g_\lambda(t) - S(t))^{-1}\| \\ &\leqslant \text{const. } |e^{\lambda t} \cdot g_\lambda(t)^{-1}|\, (1-q)^{-1} \\ &\leqslant M\, |\lambda|^k \qquad (\lambda \in \Lambda(\alpha, \beta))\end{aligned}$$

for some $M \geqslant 0$. ∎

LEMMA 2.6. (a) *Let $\alpha' > 0$, $\beta' > 0$. For all $0 < \alpha < \alpha'$ there exists $\beta \geqslant \beta'$ such that $E(\alpha, \beta) \subset \Lambda(1/\alpha', \beta')$.*

(b) *Let $\alpha > 0$, $\beta \geqslant 1$. Then $\Lambda(1/\alpha, \beta) \subset E(\alpha, \beta)$.*

Recall that

$$\begin{aligned}E(\alpha, \beta) &= \{(x + iy) : x \geqslant \beta,\ |y| \leqslant e^{\alpha x}\}, \\ \Lambda(\alpha, \beta) &= \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant 1,\ \operatorname{Re} \lambda \geqslant \beta + \alpha \ln |\lambda|\}.\end{aligned}$$

Proof. (a) There exists $\beta \geqslant \max\{\beta', 1\}$ such that $1 \leqslant e^{-2\beta'\alpha'} e^{2(\alpha' - \alpha)x} - x^2 e^{-2\alpha x}$ $(x \geqslant \beta)$; i.e.,

$$e^{2\alpha x} \leqslant e^{-2\beta'\alpha'} e^{2\alpha' x} - x^2 \qquad (x \geqslant \beta). \tag{2.6}$$

Let $(x + iy) \in E(\alpha, \beta)$. Then $y^2 \leqslant e^{2\alpha x}$, $x \geqslant \beta$. By (2.6), $y^2 \leqslant e^{-2\beta'\alpha'} e^{2\alpha' x} - x^2$ and $x \geqslant \beta$. Hence $\ln(x^2 + y^2)^{1/2} \leqslant \alpha' x - \alpha'\beta'$ and $x \geqslant \beta$. Thus $x \geqslant \beta' + (1/\alpha') \ln(x^2 + y^2)^{1/2}$ and $x \geqslant 1$; i.e., $(x + iy) \in \Lambda(1/\alpha', \beta')$.

(b) Let $0 < \alpha$, $\beta > 1$. Let $(x + iy) \in \Lambda(1/\alpha, \beta)$. Then $x \geqslant 1$ and $x \geqslant \beta + (1/\alpha) \ln(x^2 + y^2)^{1/2}$. Hence $x \geqslant \beta$ and $y^2 \leqslant x^2 + y^2 \leqslant e^{2\alpha x} e^{-2\alpha\beta} \leqslant e^{2\alpha x}$. Thus $(x + iy) \in E(\alpha, \beta)$. ∎

Now Theorem 2.1 follows from Proposition 2.5 and Lemma 2.6.

For the proof of Theorem 2.2 we will use the following uniqueness theorem due to Ljubich.

THEOREM 2.7 (Ljubich). *Let A be an operator such that $[\lambda_0, \infty) \subset \rho(A)$ for some $\lambda_0 \in \mathbb{R}$ and*

$$\overline{\lim_{\lambda \to +\infty}} \frac{1}{\lambda} \ln \|R(\lambda, A)\| = 0.$$

If $u \in C([0, \tau]; X) \cap C^1((0, \tau]; X)$, where $\tau > 0$, such that

$$\begin{cases} u(t) \in D(A), & 0 < t \leqslant \tau; \\ u'(t) = Au(t), & 0 < t \leqslant \tau; \\ u(0) = 0, \end{cases}$$

then $u \equiv 0$.

The proof of [Pa, Theorem 1.2] also yields this slightly more general case.

Note that in the situation described in Theorem 2.2 the condition of Theorem 2.7 is fulfilled.

Finally we prove Theorem 2.2.

Proof of Theorem 2.2. Assume that $\alpha, \beta > 0$ such that $E(\alpha, \beta) \subset \rho(A)$ and $\|R(\lambda, A)\| \leqslant M |\lambda|^k$ for all $\lambda \in E(\alpha, \beta)$. We can assume that β is so large that $e^{2\alpha x} \geqslant x^2$ for all $x \geqslant \beta$. Let $\Gamma = \gamma_- \cup \gamma_0 \cup \gamma_+$ be the contour consisting of the paths $\gamma_\pm = x \pm i\sqrt{e^{2\alpha x} - x^2}$ $(x \geqslant \beta)$ and γ_0 the vertical line joining $\beta - i\sqrt{e^{2\alpha\beta} - \beta^2}$ and $\beta + i\sqrt{e^{2\alpha\beta} - \beta^2}$.

Let $p > k+1$, $0 < t < \tau := \alpha(p - (k+1))$. Since $|\gamma'_\pm(x)| \leqslant \text{const.}\, e^{\alpha x}$ $(x \geqslant \beta)$ one has for $\lambda = \gamma_\pm(x)$ $(x \geqslant \beta)$, $|\lambda| = e^{\alpha x}$, $|\gamma'_\pm(x)| \, \|e^{\lambda t} R(\lambda)\, \lambda^{-p}\| \leqslant \text{const.}\, e^{\alpha x} e^{t \operatorname{Re} \lambda} |\lambda|^{k-p} = \text{const.}\, e^{(t + \alpha(k+1-p))x}$ $(x \geqslant \beta)$. Thus $S_p(t) = (1/2\pi i) \int_\Gamma e^{\lambda t} R(\lambda, A)\, \lambda^{-p}\, d\lambda$ converges and defines a norm continuous function on $[0, \tau)$.

Moreover

$$\begin{aligned} \int_0^t S_p(s)\, ds &= \frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t} - 1}{\lambda} R(\lambda, A)\, \lambda^{-p}\, d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda, A)\, \lambda^{-p-1}\, d\lambda, \end{aligned}$$

by Cauchy's theorem, and

$$\begin{aligned} A \int_0^t S_p(s)\, ds &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} A R(\lambda, A)\, \lambda^{-p-1}\, d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda, A)\, \lambda^{-p}\, d\lambda - \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{-p-1}\, d\lambda \\ &= S_p(t) - \frac{t^p}{\Gamma(p+1)} I. \end{aligned}$$

Hence $\int_0^t S_p(s) x\, ds$ is a solution of $C_{p+1}(\tau)$ for $x \in X$.

Uniqueness follows from Ljubich's Theorem 2.7. ∎

III. Properties of the k-Times Integerated Semigroup

Let $k \in \mathbb{N}_0$, $0 < \tau \leqslant \infty$, we establish consequences from the well-posedness of $C_k(\tau)$. In the following it will be crucial that $\rho(A) \neq \emptyset$ which has been shown in Theorem 2.1.

Proposition 3.1. *Assume that $C_{k+1}(\tau)$ is well-posed and denote by S the k-times integrated semigroup generated by A. Then the following hold.*

(a) *$S(t)x = 0$ for all $t \in [0, \tau)$ implies $x = 0$ (i.e., S is non-degenerate).*

(b) *$R(\lambda, A)\, S(t) = S(t)\, R(\lambda, A)$ for all $\lambda \in \rho(A)$, $t \in [0, \tau)$.*

(c) *If $x \in D(A)$, then $S(t)x \in D(A)$ and $AS(t)x = S(t)\, Ax$.*

(d) *Let $x \in X$. Then $x \in D(A)$ if and only if there exists $y \in X$ such that $S(t)x = \int_0^t S(s)y\, ds + (t^k/k!)x$ for all $t \in [0, \tau)$. In that case $y = Ax$.*

(e) *$S(s)\, S(t) = S(t)\, S(s)$ for all $0 \leqslant s, t < \tau$.*

Proof. (a) This is immediate from the definition.

(b) Let $\lambda \in \rho(A)$. Let $x \in X$, $v(t) = R(\lambda, A) \int_0^t S(s)(\lambda - A)x\, ds$. Then by (2.1), $v(t) \in D(A)$ and $Av(t) = R(\lambda, A) A \int_0^t S(s)(\lambda - A)x\, ds = R(\lambda, A)\, S(t)(\lambda - A)x - (t^k/k!)\, R(\lambda, A)(\lambda - A)x = v'(t) - (t^k/k!)x$. Hence $v'(t) = S(t)x$ $(0 < t < \tau)$ which proves (b).

(c) This follows immediately from (b).

(d) Let $x \in D(A)$. Then by (2.1) and (b), $\int_0^t S(s)\, Ax\, ds = A \int_0^t S(s)x\, ds = S(t)x - (t^k/k!)x$. Conversely, assume that $x, y \in X$ such that $S(t)x = \int_0^t S(s)y\, ds + (t^k/k!)x$ $(t \in [0, \tau))$. Let $\lambda \in \rho(A)$. Then by (b), $\int_0^t S(s)\, R(\lambda, A)y\, ds = S(t)\, R(\lambda, A)x - (t^k/k!)\, R(\lambda, A)x = \int_0^t S(s)\, R(\lambda, A)x\, ds$ $(t \in [0, \tau))$ be the first part. It follows that $S(s)\, R(\lambda, A)y = S(s)\, AR(\lambda, A)x$ $(s \in [0, \tau))$ and consequently $R(\lambda, A)y = AR(\lambda, A)x$ by (a). Hence $x \in D(A)$ and $Ax = y$.

(e) Let $s \in [0, \tau)$ be fixed, $x \in X$. Then by (c) and (2.1), $A \int_0^t S(s)\, S(r)x\, dr = S(s)\, A \int_0^t S(r)x\, dr = S(s)\, S(t)x - (t^k/k!)\, S(s)x$. Since A is closed, it follows that $v(t) = \int_0^t S(s)\, S(r)x\, dr$ is the solution of $C_{k+1}(\tau)$ with x replaced by $S(s)x$. Hence $v'(t) = S(t)\, S(s)x$ $(t \in [0, \tau))$ which proves the claim. ∎

The following rescaling property is frequently useful. It is analogous to [ANS, Proposition 1.3] and its proof can be omitted.

Lemma 3.2. *Let $\tau, k \in \mathbb{N}$. Assume that $C_{k+1}(\tau)$ is well-posed; i.e., A generates a k-times integrated semigroup S on $[0, \tau)$. Let $r \in \mathbb{R}$. Then $A - rI$ generates the k-times integrated semigroup S_r on $[0, \tau)$ given by*

$$S_r(t) = e^{-rt} S(t) + \int_0^t e^{-rs} p_k(t - s)\, S(s)\, ds \qquad (0 \leqslant t \leqslant \tau),$$

where p_k is the polynomial of degree $(k-1)$ such that

$$\sum_{j=1}^{k} \binom{k}{j} r^j \mu^{-j} = \int_0^\infty e^{-\mu t} p_k(t)\, dt \qquad (\mu > 0).$$

Next we characterize well-posedness of $C_{k+1}(\tau)$ in terms of $C_0(\tau)$.

PROPOSITION 3.3. *Let $k \in \mathbb{N}_0$ and $0 < \tau \leqslant \infty$. The following are equivalent.*

(i) *$C_{k+1}(\tau)$ is well-posed;*

(ii) *$\rho(A) \neq \varnothing$ and for all $x \in D(A^{k+1})$ there exists a unique solution of $C_0(\tau)$.*

Proof. Assume that (i) holds. It follows from Theorem 2.1 that $\rho(A) \neq \varnothing$ (in Theorem 2.1 we supposed that $k \geqslant 1$, but if $C_1(\tau)$ is well-posed, then so is $C_2(\tau)$). For $x \in D(A^{k+1})$ define

$$T(t)x = S_{k+1}(t)\, A^{k+1}x + \frac{t^k}{k!} A^k x + \cdots + tAx + x \qquad (0 \leqslant t < \tau), \qquad (3.1)$$

where for $m \geqslant k$, S_m denotes the m-times integrated semigroup on $[0, \tau)$. Then T is a solution of $C_0(\tau)$ by (2.1). Uniqueness follows from Theorem 2.7.

For the converse, assume that (ii) holds. By Lemma 3.2 we can assume that $0 \in \rho(A)$ (considering $A - r$ otherwise). For $y \in D(A^{k+1})$ denote by $t \mapsto T(t)y : [0, \tau) \to X$ the solution of $C_0(\tau)$. Let for $x \in X$, $S_{k+1}(t)x = T(t)A^{-(k+1)}x - (t^k/k!)\, A^{-1}x - \cdots - tA^{-(k-1)}x - A^{-k}x$. Then $S_{k+1}(\cdot): [0, \tau) \to X$ is a solution of $C_{k+1}(\tau)$. Uniqueness follows from (ii). ∎

Remark. Densely defined operators satysfying (ii) of Proposition 3.3 have been characterized by real conditions (of Hille–Yosida type) by Oharu [Oh1].

Finally, we consider the inhomogeneous Cauchy problem. It is remarkable that under the assumption of well-posedness for very special inhomogeneities one obtains solutions for a much larger class.

In fact, on the basis of the preceding results one can proceed as in [Ar] and obtain the following results.

Let $k \in \mathbb{N}_0$, $\tau > 0$ and assume that $C_{k+1}(\tau)$ is well-posed.

Let $f \in C([0, \tau), X)$. We consider the problem

$$\left\{\begin{array}{ll} u \in C^1([0, \tau), X),\ u(t) \in D(A) & (t \in [0, \tau)), \\ u'(t) = Au(t) + f(t) & (t \in [0, \tau)), \\ u(0) = x. & \end{array}\right\} \qquad CP(f)$$

Let $v(t) = S(t)\,x + \int_0^t S(s)\,f(t-s)\,ds$ $(t \in [0, \tau))$. Then the following holds.

If there exists a solution u of $CP(f)$, then $v \in C^{k+1}([0, \tau); X)$ and $u = v^{(k)}$. (3.2)

If $v \in C^{k+1}([0, \tau); X)$, then $u = v^{(k)}$ is the solution of $CP(f)$. (3.3)

If $f \in C^{k+1}([0, \tau); X)$, and $x \in D(A)$, $u_1 := Ax + f(0) \in D(A)$; $u_2 := Au_1 + f'(0) \in D(A)$; $\cdots u_{m+1} := Au_m + f^{(m)}(0) \in D(A)$; $\cdots u_k := Au_{k-1} + f^{(k-1)}(0) \in D(A)$, then $CP(f)$ has a unique solution. (3.4)

The following converse of Proposition 2.3 will be useful later.

Remark 3.4. Let $\tau > 0$, $k \in \mathbb{N}$ and assume that there exists a strongly continuous function $S\colon [0, \tau) \to \mathscr{L}(X)$ commuting with A (i.e., $S(t)x \in D(A)$ and $AS(t)x = S(t)\,Ax$ for all $x \in D(A)$ $(0 \leqslant t < \tau)$) and such that $\int_0^t S(s)x\,ds \in D(A)$ for all $x \in X$, $0 \leqslant t < \tau$ and such that (2.1) holds. Then $C_{k+1}(\tau)$ is well-posed and S is the k-times integrated semigroup generated by A on $[0, \tau)$.

In fact, let $x \in X$. Then $v(t) = \int_0^t S(s)x\,ds$ is a solution of $C_{k+1}(\tau)$. Moreover, the proof of Proposition 2.5 is valid for the family S. Thus the hypothesis of Ljubich's theorem is satisfied and the solution of $C_{k+1}(\tau)$ is unique. One might also argue as in [Ar].

IV. Extension of Solutions

In this section we show that solutions given on a finite interval can always be extended if a loss of regularity is accepted. In fact, let $k \in \mathbb{N}$, $\tau > 0$ and assume that $C_{k+1}(\tau)$ is well-posed. We will show that $C_{2k+1}(2\tau)$ is well-posed. To make this more precise, denote by $S_k\colon [0, \tau) \to \mathscr{L}(X)$ the k-times integrated semigroup generated by A. It follows from the definitions that $C_{k+m+1}(\tau)$ is well-posed and the $(m+k)$-times integrated semigroup S_{k+m} generated by A on $[0, \tau)$ is given by

$$S_{k+m}(t) = \int_0^t \frac{(t-s)^{m-1}}{(m-1)!}\,S_k(s)\,ds \qquad (0 \leqslant t < \tau), \tag{4.1}$$

for all $m \in \mathbb{N}_0$. We will show that S_{k+m} can be extended to $[0, 2\tau)$ if $m \geqslant k$.

THEOREM 4.1. *Let $\tau_0 > 0$, $k \in \mathbb{N}$. Assume that $C_{k+1}(\tau_0)$ is well-posed. Then $C_{2k+1}(2\tau_0)$ is well-posed as well. In particular, for all $\tau' > 0$ there exists $k' \in \mathbb{N}$ such that $C_{k'}(\tau')$ is well-posed.*

Remark 4.2. If $C_{k+1}(\tau)$ is well-posed we have the resolvent estimate of Theorem 2.1. It follows from Ljubich's theorem (Theorem 2.7) that for every $m \in \mathbb{N}_0$ and every $\tau_1 > 0$ there exists at most one solution of $C_m(\tau_1)$.

Proof of Theorem 4.1. Let $\tau < \tau_0$. It suffices to show that there exists a solution of $C_{2k+1}(2\tau)$. Define S_{2k} on $[0, \tau]$ by (4.1) and on $(\tau, 2\tau]$ by

$$S_{2k}(t) := S_k(\tau)\, S_k(t-\tau) + \sum_{m=0}^{k-1} \frac{1}{m!} (\tau^m S_{2k-m}(t-\tau) + (t-\tau)^m\, S_{2k-m}(\tau)). \tag{4.2}$$

Then $S_{2k}\colon [0, 2\tau] \to \mathcal{L}(X)$ is strongly continuous. Let $x \in X$. We show that $v(t) = \int_0^t S_{2k}(s)x\, ds$ is a solution of $C_{2k+1}(2\tau)$.

Let $\tau < t \leqslant 2\tau$. Then

$$\begin{aligned}
A \int_0^t & S_{2k}(s)x\, ds \\
&= A \int_0^\tau S_{2k}(s)x\, ds + A \int_\tau^t S_{2k}(s)x\, ds \\
&= S_{2k}(\tau)x - \frac{\tau^{2k}}{(2k)!} x + A \int_0^{t-\tau} S_{2k}(s+\tau)x\, ds \\
&= S_{2k}(\tau)x - \frac{\tau^{2k}}{(2k)!} x + S_k(\tau)\, A \int_0^{t-\tau} S_k(s)x\, ds \\
&\quad + \sum_{m=0}^{k-1} \frac{1}{m!} \left(\tau^m A \int_0^{t-\tau} S_{2k-m}(s)x\, ds + \int_0^{t-\tau} s^m\, ds\, A S_{2k-m}(\tau)x \right) \\
&= S_{2k}(\tau)x - \frac{\tau^{2k}}{(2k)!} x + S_k(\tau)\, S_k(t-\tau)\, x - \frac{(t-\tau)^k}{k!} S_k(\tau)x \\
&\quad + \sum_{m=0}^{k-1} \left\{ \frac{1}{m!} \tau^m \left(S_{2k-m}(t-\tau) - \frac{(t-\tau)^{2k-m}}{(2k-m)!} \right) x \right. \\
&\quad + \frac{1}{(m+1)!} (t-\tau)^{m+1}\, S_{2k-m-1}(\tau)x \\
&\quad \left. - \frac{1}{(m+1)!} (t-\tau)^{m+1} \frac{\tau^{2k-m-1}}{(2k-m-1)!} x \right\} \\
&= S_k(\tau)\, S_k(t-\tau)x + \sum_{m=0}^{k-1} \frac{1}{m!} (\tau^m S_{2k-m}(t-\tau)x + (t-\tau)^m\, S_{2k-m}(\tau)x) \\
&\quad - \frac{\tau^{2k}}{(2k)!} x - \sum_{m=0}^{k-1} \frac{\tau^m}{m!} \frac{(t-\tau)^{2k-m}}{(2k-m)!} x - \sum_{m=0}^{k-1} \frac{(t-\tau)^{m+1}}{(m+1)!} \frac{\tau^{2k-m-1}}{(2k-m-1)!} x
\end{aligned}$$

$$= S_{2k}(t)x - \left\{\frac{\tau^{2k}}{(2k)!} + \sum_{m=0}^{k-1} \frac{\tau^m}{m!} \frac{(t-\tau)^{2k-m}}{(2k-m)!} + \sum_{l=k}^{2k-1} \frac{(t-\tau)^{2k-l}}{(2k-l)!} \frac{\tau^l}{l!}\right\} x$$

$$= S_{2k}(t)x - \sum_{m=0}^{2k} \frac{1}{(2k)!} \binom{2k}{m} \tau^m (t-\tau)^{2k-m} x$$

$$= S_{2k}(t)x - \frac{t^{2k}}{(2k)!} x.$$

This proves the claim. ∎

Remark 4.3. It follows from the proof of Theorem 4.1 and uniqueness of the solutions that formula (4.2) holds throughout, i.e., the following is true: Assume that $C_{k+1}(\tau)$ is well-posed (where $k \in \mathbb{N}$ and $\tau > 0$). Then the following functional equation holds:

$$S_k(s)\, S_k(t) = S_{2k}(t+s) - \sum_{m=0}^{k-1} \frac{1}{m!} (\tau^m S_{2k-m}(s) + s^m S_{2k-m}(t)) \tag{4.3}$$

for all $s, t \in [0, \tau)$, where S_l is the l-times integrated semigroup on $[0, \tau)$ generated by A if $l \geqslant k$ and S_{2k} is the $2k$-times integrated semigroup on $[0, 2\tau)$ generated by A.

We give some examples.

EXAMPLES 4.4. By a multiplication operator on $L^p(\Omega)$ we understand an operator A defined in the following way. We assume that Ω is a σ-finite measure space and let $1 \leqslant p \leqslant \infty$. Let $m: \Omega \to \mathbb{C}$ be measurable and let A be defined by $Af = mf$, $D(A) = \{f \in L^p : mf \in L^p\}$.

(a) Let A be a multiplication operator on L^p such that $E(\alpha, \beta) \subset \rho(A)$ where $\alpha > 0$, $\beta > 0$. Then $C_{k+1}(k\alpha)$ is well-posed for all $k \in \mathbb{N}$.

Proof. The spectrum of A is the essential image of m.

Let $k \in \mathbb{N}$, $\tau > 0$. Then one sees directly that $C_{k+1}(\tau)$ is well-posed if and only if

$$\sup_{\lambda \in \sigma(A)} \left| \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} e^{\lambda s}\, ds \right| < \infty \tag{4.4}$$

for all $t < \tau$ and in that case

$$(S(t)f)(x) = \left(\int_0^t \frac{(t-s)^{k-1}}{(k-1)!} e^{m(x)s}\, ds \right) \cdot f(x) \qquad (x \in \Omega) \tag{4.5}$$

defines the k-times integrated semigroup S on $[0, \tau)$. Since

$$\int_0^t \frac{(t-s)^{k-1}}{(k-1)!} e^{\lambda s}\, ds = \frac{e^{\lambda t}}{\lambda^k} - \frac{1}{\lambda^k} - \frac{1}{\lambda^{k-1}} - \cdots - \frac{1}{\lambda} \quad \text{if} \quad \lambda \neq 0,$$

(4.4) is equivalent to

$$\overline{\lim_{\lambda \in \sigma(A), \operatorname{Re}\lambda \to \infty}} \frac{e^{\operatorname{Re}\lambda t}}{|\lambda|^k} < \infty \qquad \text{for} \quad t < \tau. \tag{4.6}$$

Now assume $E(\alpha, \beta) \subset \rho(A)$. Let $\lambda \in \sigma(A)$, such that $\operatorname{Re}\lambda \geqslant \beta$. Then $|\operatorname{Im}\lambda| \geqslant e^{\alpha \operatorname{Re}\lambda}$. Hence for $t \leqslant \alpha$, $e^{\operatorname{Re}\lambda t} |\lambda|^{-k} \leqslant 1$. ∎

(b) Let A be a multiplication operator and suppose that $C_{k+1}(\tau)$ is well-posed where $k \in \mathbb{N}$ and $\tau > 0$. Then there exists $0 < \tau' \leqslant \tau$ such that $C_2(\tau')$ is well-posed.

Proof. By Theorem 2.1 there exist $\alpha > 0$, $\beta > 0$ such that $E(\alpha, \beta) \subset \rho(A)$. Now the claim follows from (a). ∎

Next we discuss the maximal interval of existence by two examples.

(c) Let $m(x) = x + ie^x$ $(x > 1)$ and let A be the multiplication operator defined by m on $L^p(1, \infty)$ $(1 \leqslant p \leqslant \infty)$. It follows from (4.6) that $C_2(1)$ is well-posed. Moreover the integrated semigroup S is given by $(S(t)f)(x) = (1/m(x))(e^{tm(x)} - 1) f(x)$ and exists for $t \leqslant 1$. The problem $C_2(\tau)$ is not well-posed for any $\tau > 1$. However, $\sup_{t<1} \|S(t)\| < \infty$.

(d) Let $m(x) = x + ix^{-1}e^x$ $(x > 1)$ and let A be the multiplication operator defined by m on $L^p(1, \infty)$ $(1 \leqslant p \leqslant \infty)$. Then $C_2(1)$ is well-posed. However, one has $\sup_{t<1} \|S(t)\| = \infty$, where S is the once integrated semigroup generated by A.

Next for $k \in \mathbb{N}$, $k \geqslant 2$, we give an example of an operator A such that $C_{k+1}(1)$ is well-posed but $C_k(\tau)$ is not for any $\tau > 0$. We use

LEMMA 4.5. *Let B be an operator on X. Consider the operator*

$$\mathscr{A} = \begin{pmatrix} B & B \\ 0 & B \end{pmatrix}$$

on $X \times X$ with domain $D(\mathscr{A}) = D(B) \times D(B)$. Let $k \in \mathbb{N}$, $\tau > 0$. If $C_{k+1}(\tau)$ is well-posed for B then $C_{k+2}(\tau)$ is well-posed for A. Moreover, if for $m \geqslant k$, S_m denotes the m-times integrated semigroup generated by B, then

$$V_{k+1}(t) = \begin{pmatrix} S_{k+1}(t) & tS_k(t) - (k+1)\, S_{k+1}(t) \\ 0 & S_{k+1}(t) \end{pmatrix} \tag{4.7}$$

is the $k+1$-times integrated semigroup by $\mathscr{A}$.

Proof. Let $\lambda \in \rho(B)$. Then $\lambda \in \rho(\mathscr{A})$ and

$$(\lambda - \mathscr{A})^{-1} = \begin{pmatrix} (\lambda - B)^{-1} & \lambda(\lambda - B)^{-2} - (\lambda - B)^{-1} \\ 0 & (\lambda - B)^{-1} \end{pmatrix}.$$

This shows that Ljubich's theorem holds for $\mathscr{A}$.

Define $V_{k+1}(t)$ by (4.7). Then it is easy to see that for $z \in X \times X$, $v(t) = \int_0^t V_{k+1}(s) z\, ds$ is a solution of $C_{k+1}(\tau)$ with inhomogeneity z. ∎

EXAMPLE 4.6. Let B be the multiplication operator defined by $m(x) = x + ie^x$ on $L^p(1, \infty)$. Then $(S(t)f)(x) = \int_0^t e^{sm(x)} f(x)\, ds$ defines the integrated semigroup generated by B on $[0, 1)$. Since e^{tm} is not bounded for any $t > 0$, S_1 is not strongly differentiable for any $t \in (0, 1)$. Define $\mathscr{A}_1 = \left(\begin{smallmatrix} B & B \\ 0 & B \end{smallmatrix}\right)$. Then $\mathscr{A}_1$ generates a twice integrated semigroup V_2 on $X_2 := X \times X$ given by (4.7) (for $k = 1$). Thus V_2 is not strongly differentiable on $(0, 1)$. Thus $C_2(1)$ is well-posed but $C_1(\tau)$ is not well-posed for any $\tau > 0$.

By induction, the construction of Lemma 4.7 yields an operator $\mathscr{A}_{k+1}$ on $X_{k+1} := X_k \times X_k$ which generates a $k+1$-times integrated semigroup on $(0, 1)$ which is not strongly differentiable for any $t \in (0, 1)$. Thus $C_k(\tau)$ is not well-posed for any $\tau > 0$.

V. GLOBAL INTEGRATED SEMIGROUPS

Theorems 2.1 and 2.2 give characterizations of well-posedness of $C_{k+1}(\infty)$; i.e., they characterize generators of global k-times integrated semigroups (up to the loss of regularity). The conditions formulated in these theorems do not imply that $\rho(A)$ contains a semi-plane.

EXAMPLE 5.1. Let $\alpha > 0$, $\beta > 0$, $l > 1$, and define

$$E(\alpha, \beta, l) := \{x + iy,\ x \geqslant \alpha,\ y \leqslant e^{\alpha x^l}\}.$$

Assume that $E(\alpha, \beta, l) \subset \rho(A)$ and $\|R(\lambda, A)\| \leqslant M\, |\lambda|^k$ for all $\lambda \in E(\alpha, \beta, l)$. It follows from Theorem 2.2 that $C_p(\infty)$ is well-posed if $p > k + 1$.

A concrete example is the multiplication operator A on $L^p(1, \infty)$ defined by $m(x) = x + ie^{x^2}$. Observe that $\rho(A)$ does not contain any semi-plane in this case.

However, if we suppose that $C_{k+1}(\infty)$ is exponentially well-posed, then $\rho(A)$ contains a semi-plane.

DEFINITION 5.2. Let $k \in \mathbb{N}_0$. We say that $C_{k+1}(\infty)$ is exponentially well-posed if $C_{k+1}(\infty)$ is well-posed and every solution v of $C_{k+1}(\infty)$ is exponentially bounded, i.e., there exist $M \geq 0$, $w \in \mathbb{R}$ such that $\|v(t)\| \leq Me^{wt}$ $(t \geq 0)$.

PROPOSITION 5.3. *Let $k \in \mathbb{N}$ and assume that $C_{k+1}(\infty)$ is exponentially well-posed. For $m \geq k$ we denote by S_m the m-times integrated semigroup generated by A. Then there exist $w \in \mathbb{R}$, $M \geq 0$ such that*

$$\|S_{k+1}(t)\| \leq Me^{wt} \qquad (t \geq 0). \tag{5.1}$$

Moreover, $\{\mathrm{Re}(\lambda) > w\} \subset \rho(A)$ *and*

$$R(\lambda, A) = \lambda^k \int_0^\infty e^{-\lambda t} S(t)\, dt \qquad (\mathrm{Re}\, \lambda > w). \tag{5.2}$$

In former articles properties (5.1) and (5.2) had been used to define integrated semigroups (cf. [ANS]).

The proof of Proposition 5.3 is based on the following *uniform exponential boundedness principle.*

PROPOSITION 5.4. *Let X, Y be Banach spaces and let $S: [0, \infty) \to \mathscr{L}(X, Y)$ be a function such that for all $x \in X$ there exist $M_x \geq 0$, $w_x \in \mathbb{R}$ depending on x, such that $\|S(t)x\| \leq M_x z^{w_x t}$ $(t \geq 0)$. Then there exist $M \geq 0$, $w \in \mathbb{R}$ such that*

$$\|S(t)\| \leq Me^{wt} \qquad (t \geq 0).$$

Proof. For $m \in \mathbb{N}$, the space

$$X_n := \{x \in X : \|S(t)x\| \leq ne^{nt} \|x\| \ (t \geq 0)\}$$

is closed. By hypothesis, $X = \bigcup_{n \in \mathbb{N}} X_n$. Then by Baire's theorem there exists $n_0 \in \mathbb{N}$ such that X_{n_0} has non-empty interior. Consequently, there exist $z \in X$, $\varepsilon > 0$, $M \geq 0$, $w \in \mathbb{R}$ such that

$$e^{-wt} \|S(t)x\| \leq M \qquad (t \geq 0),$$

whenever $\|x - z\| \leq \varepsilon$. This implies that for $\|y\| \leq 1$,

$$\begin{aligned} \varepsilon e^{-wt} \|S(t)y\| &\leq e^{-wt} \|S(t)(\varepsilon y + z)\| + e^{-wt} \|S(t)z\| \\ &\leq 2M \qquad (t \geq 0). \end{aligned}$$

Thus $\|S(t)\| \leq 2(M/\varepsilon)\, e^{wt}$ $(t \geq 0)$. ∎

Proof of Proposition 5.3. Property (5.1) is a direct consequence of Proposition 5.4. Let $Q = \lambda^k \int_0^\infty e^{-\lambda t} S_k(t)\, dt$ which exists for $\lambda > w$ (see [ANS]).

Integration by parts yields $Qx = \lambda^{k+1} \int_0^\infty e^{-\lambda t} S_{k+1}(t)x\,dt$ for all $x \in X$. It follows from (2.1) that $Qx \in D(A)$ and $(\lambda - A)\,Qx = \lambda^{k+2} \int_0^\infty e^{-\lambda t} S_k(t)x\,dt - \lambda^{k+1} \int_0^\infty e^{-\lambda t} AS_{k+1}(t)x\,dt = \lambda^{k+2} \int_0^\infty e^{-\lambda t} S_{k+1}(t)x\,d - \lambda^{k+1} \int_0^\infty e^{-\lambda t} S_k(t)x\,dt + \lambda^{k+1} \int_0^\infty e^{-\lambda t}(t^k/k!)x\,dt = x$. Since for $x \in D(A)$, $Q(\lambda - A)x = (\lambda - A)\,Qx$, it follows that $Q = (\lambda - A)^{-1}$. ∎

We conclude by the following characterization.

PROPOSITION 5.5. *Let $k \in \mathbb{N}_0$. The following are equivalent.*

(i) *$C_{k+1}(\infty)$ is exponentially well-posed;*

(ii)(a) *$\rho(A) \neq \varnothing$;*

(b) *for all $x \in D(A^{k+1})$ there exists a unique solution of $C_0(\infty)$; and this solution is exponentially bounded.*

Proof. By Lemma 3.2 we can assume that $0 \in \rho(A)$. Now it follows from Proposition 3.3 that $C_{k+1}(\infty)$ is well-posed if and only if (ii)(b) holds. It follows from formula (3.1) that the solutions of $C_{k+1}(\infty)$ are exponentially bounded if and only if the solutions of $C_0(\infty)$ for initial values in $D(A^{k+1})$ are exponentially bounded. ∎

VI. SOLUTION ON $D_\infty(A)$

Let A be a closed operator on X. Then $(D(A^m), \|\cdot\|_m)$ is a Banach space with the norm $\|x\|_m := \|x\| + \|Ax\| + \cdots + \|A^m x\|$ and $D_\infty := D_\infty(A) := \bigcap_{m \in \mathbb{N}} D(A^m)$ is a Fréchet space with respect to the family of seminorms $(\|\cdot\|_m)_{m \in \mathbb{N}}$. The restriction of A to D_∞ is a continuous linear operator. In this section we consider the Cauchy problem defined by A in the space D_∞.

THEOREM 6.1. *Let A be a closed densely defined linear operator. The following assertions are equivalent.*

(i) *There exist $\tau > 0$, $k \in \mathbb{N}$ such that $C_k(\tau)$ is well-posed.*

(ii) *One has $\rho(A) \neq \varnothing$ and the problem*

$$\left\{\begin{array}{ll} u \in C^\infty([0, \infty); D_\infty) & \\ u'(t) = Au(t) & (t \geqslant 0) \\ u(0) = x & \end{array}\right\} \tag{6.1}$$

has a unique solution for all $x \in D_\infty$.

(iii) *One has* $\rho(A) \neq \varnothing$ *and there exists* $\tau > 0$ *such that the problem*

$$\left.\begin{cases} u \in C^1([0,\tau]; D_\infty) \\ u'(t) = Au(t) \qquad (t \in [0,\tau]) \\ u(0) = x \end{cases}\right\} \tag{6.2}$$

has a unique solution for all $x \in D_\infty$.

Of course the spaces $C^1([0,\tau]; D_\infty)$ and $C^\infty([0,\infty); D_\infty)$ are understood with respect to the topology introduced above.

For the proof we need the following.

PROPOSITION 6.2. *Let* A *be a closed and densely defined linear operator on a Banach space* X *with nonempty resolvent set. Then* $D_\infty(A)$ *is dense in* X *and in* $(D(A^m), \|\cdot\|_m)$ *for all* $m \in \mathbb{N}$.

This is a consequence of an abstract version of the Mittag–Leffler Theorem (for proof and background we refer to Esterle [Es, Corollary 2.2]):

THEOREM 6.3 (Mittag–Leffler). *Let* (M_n, d_n), $n \in \mathbb{N}$, *be complete metric spaces and assume that* $\theta_n: M_{n+1} \to M_n$ *is continuous such that* $\theta_n M_{n+1}$ *is dense in* M_n *for all* $n \in \mathbb{N}$. *Then for all* $x \in M_1$ *and all* $\varepsilon > 0$ *there exists* $x_n \in M_n$ $(n \in \mathbb{N})$ *such that* $x_n = \theta_n x_{n+1}$ *and* $d_1(x, x_1) < \varepsilon$.

Proof of Proposition 6.2. Let $M_n = (D(A^n), \|\cdot\|_n)$ and let θ_n be the injection of M_{n+1} in M_n. Let $\lambda \in \rho(A)$. Then $(\lambda - A)^n$ is an isomorphism from $(D(A^n), \|\cdot\|_n)$ onto X which maps $D(A^{n+1})$ onto $D(A)$. Since by hypothesis $D(A)$ is dense in X, it follows that $D(A^{n+1})$ is dense in $(D(A^n), \|\cdot\|_n)$. Let $x \in X$, $\varepsilon > 0$. By Theorem 6.3 there exists $x_n \in M_n$ $(n \in \mathbb{N})$ such that $x_n = \theta_n x_{n+1}$ and $\|x - x_1\| < \varepsilon$. Thus $x_n = x_1$ for all n. Consequently, $x_1 \in D_\infty$. We have shown that D_∞ is dense in X. The last assertion follows from the first by replacing X by $(D(A^m), \|\cdot\|_m) =: X_m$ and A by the part of A in X_m. ∎

Proof of Theorem 6.1. (i) ⇒ (ii). Uniqueness follows from Theorem 2.7. Let $\tau > 0$ be arbitrary. Then by Theorem 4.1 there exists $k \in \mathbb{N}$ such that $C_{k+1}(\tau)$ is well-posed. For $m \geqslant k$ denote by S_m the m-times integrated semigroup generated by A on $[0, \tau)$. For $x \in D_\infty(A)$ let $T(t)x$ be given by (3.1). Since $S_{k+1}(t)$ commutes with A, it follows that $T(t)x \in D_\infty$ and $AT(t)x = T(t)Ax$. Consequently, $u(t) = T(t)x$ is a solution of (6.1).

(ii) ⇒ (iii). This is trivial.

(iii) ⇒ (i). Denote by $T(\cdot)x: [0, \tau] \to D_\infty$ the solution of (6.2) for every $x \in D_\infty$. It follows from the closed graph theorem that $T(t) \in \mathscr{L}(D_\infty, X)$ for all $t \in [0, \tau]$ (cf. Proposition 2.3). Moreover, T is strongly continuous by hypothesis. It follows from the Banach–Steinhaus theorem that T is equicontinuous.

In particular, there exist $k \in \mathbb{N}$, $c > 0$ such that

$$\|T(t)x\| \leqslant c\,\|x\|_k \qquad (0 \leqslant t \leqslant \tau,\ x \in D_\infty). \tag{6.3}$$

Since D_∞ is dense in $(D(A^k), \|\cdot\|_k)$, it follows that there exists $\tilde{T}\colon [0, \tau] \to \mathscr{L}(D(A^k), X)$ strongly continuous such that

$$\|\tilde{T}(t)x\| \leqslant c\,\|x\|_k \qquad (0 \leqslant t \leqslant \tau,\ x \in D_\infty) \tag{6.4}$$

and $\tilde{T}(t)x = T(t)x$ for $x \in D_\infty$, $0 \leqslant t \leqslant \tau$.

It follows from the uniqueness of the solutions of (6.2) that $AT(t)x = T(t)\,Ax$ $(x \in D_\infty)$. Consequently, $\tilde{T}(t)x \in D(A)$ and $A\tilde{T}(t)x = \tilde{T}(t)\,Ax$ whenever $x \in D(A^{k+1})$. For $x \in D_\infty$ one has

$$\int_0^t AT(s)x\,ds = T(t)x - x, \qquad t \in [0, \tau]. \tag{6.5}$$

Since D_∞ is dense in $(D(A^{k+1}), \|\cdot\|_{k+1})$ it follows that

$$\int_0^t A\tilde{T}(t)x\,ds = \tilde{T}(t)x - x, \qquad t \in [0, \tau], \tag{6.6}$$

whenever $x \in D(A^{k+1})$; i.e., $v(t) = \tilde{T}(t)x$ is a solution of $C_0(\tau')$ for all $0 < \tau' < \tau$. Let $u \in C^1([0, \tau); X) \cap C([0, \tau); D(A))$ such that $u'(t) = Au(t)$ $(0 \leqslant t < \tau')$, $u(0) = 0$. We show that $u \equiv 0$. Let $0 < t < \tau'$. Let $\mu \in \rho(A)$. Define $w(s) = \tilde{T}(t-s)\,R(\mu, A)^{k+1}\,u(s)$. Then $w'(s) = -A\tilde{T}(t-s)$ $R(\mu, A)^{k+1}\,u(s) + \tilde{T}(t-s)\,R(\mu, A)^{k+1}\,u'(s) = 0$ for all $s \in [0, t]$. Consequently $R(\mu, A)^{k+1}\,u(t) = w(t) = w(0)$. Hence $u(t) = 0$. We have shown that Proposition 3.3(ii) is satisfied. Thus $C_{k+1}(\tau)$ is well-posed by Proposition 3.3. ∎

As a consequence of Theorem 6.1 and its proof we obtain the following.

Assume that A is a closed densely defined linear operator such that $C_k(\tau)$ is well-posed for some $\tau > 0$, $k \in \mathbb{N}$. Then there exist a strongly continuous function

$$T\colon [0, \infty) \to \mathscr{L}(D_\infty(A))$$

satisfying $T(t+s) = T(t)\,T(s)$ $(s, t \geqslant 0)$; $T(0) = I$, $T(\cdot)x \in C^\infty([0, \infty); D_\infty(A))$ and $(d/dt)\,T(t)x = AT(t)x$ $(t \geqslant 0)$ for all $x \in D_\infty(A)$.

In particular, T_∞ is a strongly continuous semigroup on the Fréchet space $D_\infty(A)$ and $A_{|D_\infty}$ is its generator (see [Yo] for the theory of semigroups on Fréchet spaces). In the case where $C_{k+1}(\infty)$ is exponentially well-posed there exist a Banach space Z such that $D(A^k) \subsetneq Z \subsetneq X$ and A_Z generates a C_0-semigroup ([ANS], see also [Th]). Moreover there exists a Fréchet Z space which is maximal with respect to the property that A_Z generates a C_0-semigroup (see [dL]).

VII. Distribution Semigroups

In this section we show that a densely defined operator A generates a distribution semigroup if and only if $C_k(\tau)$ is well-posed for some $k \in \mathbb{N}$ and some $\tau > 0$; i.e., if A generates a local k-times integrated semigroup for some $k \in \mathbb{N}$.

Distribution semigroups were introduced by Lions [Li] in 1960 and have been studied in particular by Foias [Fo], Chazarain [Ch], Oharu [Oh1, Oh2], Balabane [Ba], and Balabane and Emami-Rad [BE1, BE2]; we refer to Fattorini's treatise [Fa, Chap. 8] for an introduction into the theory and further references.

For our purposes it will be suitable to follow closely the setting of Lions [Li].

Let Y be a Banach space. By $\mathscr{D} := \mathscr{D}(\mathbb{R})$ we denote the space of all the test functions on $\mathbb{R}$ with the usual Schwartz topology. Let $\mathscr{D}_0 := \{\phi \in \mathscr{D} : \phi(t) = 0 \text{ for } t \leqslant 0\}$. Then $\mathscr{D}_0$ is an algebra for convolution. Let Y be a Banach space. By $\mathscr{D}'(Y) := \mathscr{L}(\mathscr{D}; Y)$ we denote the space of all Y-valued distributions on $\mathbb{R}$, i.e., all continuous linear mappings from $\mathscr{D}$ into Y.

Let X be a Banach space.

Example. Let $T = (T(t))_{t \geqslant 0}$ be a C_0-semigroup on X. Let

$$G(\phi)x = \int_0^\infty \phi(t)\, T(t)x\, dt \qquad (x \in X) \qquad \text{for} \quad \phi \in \mathscr{D}.$$

Then $G \in \mathscr{D}'(\mathscr{L}(X))$ has the properties of the following definition.

Definition 7.1 (Lions [Li]). A *distribution semigroup* on X is a distribution $G \in \mathscr{D}'(\mathscr{L}(X))$ satisfying

$$\operatorname{supp} G \subset [0, \infty) \qquad (\text{i.e., } G(\phi) = 0 \text{ if } \operatorname{supp} \phi \subset (-\infty, 0)). \tag{7.1}$$

$$G(\phi * \psi) = G(\phi)\, G(\psi) \qquad (\phi, \psi \in \mathscr{D}_0); \tag{7.2}$$

let $\phi \in \mathscr{D}_0$, $x \in X$, $y = G(\phi)x$; then there exists $u \in C([0, \infty); X)$ satisfying $u(0) = y$ such that $G(\psi) = \int_0^\infty u(t)\, \psi(t)\, dt$ $(\psi \in \mathscr{D})$; (7.3)

$$\bigcup_{\phi \in \mathscr{D}_0} \operatorname{Range}(G(\phi)) \text{ is dense in } X; \tag{7.4}$$

$$\text{if } G(\psi)x = 0 \text{ for al } \psi \in \mathscr{D}_0, \text{ then } x = 0. \tag{7.5}$$

Let G be a distribution semigroup on X. The generator A of G is defined in the following way. Fix $(\rho_n) \subset \mathscr{D}_0$ an approximate unit (i.e., $\int_{\mathbb{R}} \rho_n(t)\,dt = 1$, $\operatorname{supp}(\rho_n) \subset (0, 1/n)$ $(n \in \mathbb{N})$). Define the operator A_0 on X by

$$D(A_0) = \{x \in X : \lim_{n \to \infty} G(\rho_n)x = x,\ G(-\rho_n')x \text{ converges}\},$$

$$A_0 x = \lim_{n \to \infty} G(-\rho_n')x.$$

It is shown by Lions [Li] that A_0 is closable, and by definition the closure A of A_0 is the generator of G. Then A has the following properties (see [Li]):

$$G(\phi)x \in D(A)$$

and

$$AG(\phi)x = -G(\phi')x - \phi(0)x \qquad \text{for all} \quad \phi \in \mathscr{D},\ x \in X; \tag{7.6}$$

$$AG(\phi)x = G(\phi)\,Ax \qquad \text{for all} \quad x \in D(A),\ \phi \in \mathscr{D}. \tag{7.7}$$

THEOREM 7.2. *Let A be a closed densely defined operator on X. The following are equivalent.*

(i) *There exist $k_0 \in \mathbb{N}$, $\tau > 0$ such that $C_{k_0}(\tau)$ is well-posed.*

(ii) *A is the generator of a distribution semigroup.*

Proof. (i)$\Rightarrow$(ii). By Theorem 4.1, for every $k \in \mathbb{N}$, $k \geqslant k_0$ there exists $\tau(k)$ such that $\tau(k+1) \geqslant \tau(k) > 0$, $\lim_{k \to \infty} \tau(k) = \infty$ and $C_{k+1}(\tau(k))$ is well-posed. Denote by S_k the k-times integrated semigroup generated by A on $[0, \tau(k))$. Then $S_k(t) = S_{k+m}^{(m)}(t)$ for all $t \in [0, \tau(k))$, $m \in \mathbb{N}$. Thus the following definition is independent of $k \in \mathbb{N}$.

Let $\phi \in \mathscr{D}(\mathbb{R})$. Choose $k \geqslant k_0$ such that $\operatorname{supp} \phi \subset (-\infty, \tau(k))$ and let

$$G(\phi) = (-1)^k \int_0^\infty \phi^{(k)}(t)\, S_k(t)\,dt.$$

The clearly $G \in \mathscr{D}'(\mathscr{L}(X))$ and (7.1) is satisfied.

Next we show that (7.6) holds for the given operator A. In fact, let $\phi \in \mathscr{D}$, $x \in X$. Let $k \in \mathbb{N}$ such that $\operatorname{supp} \phi \subset (-\infty, \tau(k))$. Then $AG(\phi) = (-1)^{k+1} \int_0^\infty \phi^{(k+1)}(t)\, S_{k+1}(t)x\,dt = (-1)^{k+1} \int_0^\infty \phi^{(k+1)}(t)(S_k(t)x - (t^k/k!)x)\,dt = G(\phi')x - \phi(0)$ (where we used (2.1)). Thus (7.6) holds. In particular,

$$G(\phi)x \in D_\infty(A) \qquad \text{for all} \quad x \in X,\ \phi \in \mathscr{D}_0 \tag{7.8}$$

and $A^m G(\phi)x = G((-1)^m \phi^{(m)})x$, $m \in \mathbb{N}$.

We consider the semigroup T on D_∞ introduced in Section 6; i.e., $T(t)x = S_k^{(k)}(t)x$ for $x \in D_\infty(A)$, $t < \tau(k)$. It follows from the definition that

$$G(\phi)x = \int_0^\infty \phi(t)\, T(t)x\, dt \qquad (\phi \in \mathscr{D}, x \in D_\infty(A)). \tag{7.9}$$

Let $\phi, \psi \in \mathscr{D}_0$. Then for $x \in D_\infty(A)$,

$$\begin{aligned} G(\phi)\, G(\psi)x &= \int_0^\infty \phi(t)\, T(t)\, G(\psi)x\, dt \\ &= \int_0^\infty \phi(t)\, T(t) \int_0^\infty \psi(s)\, T(s)x\, ds\, dt \\ &= \int_0^\infty \phi(t) \int_0^\infty \psi(s)\, T(t+s)x\, ds\, dt \\ &= \int_0^\infty \phi(t) \int_0^\infty \psi(s-t)\, T(s)x\, ds\, dt \\ &= \int_0^\infty (\phi * \psi)(t)\, T(t)x\, dt \\ &= G(\phi * \psi)x. \end{aligned}$$

Since $D_\infty(A)$ is dense in X it follows that (7.2) holds.

Property (7.3) is a direct consequence of (7.8) and (7.9).

It follows from (7.9) that $G(\rho_n)x \to x$ $(n \to \infty)$ for all $x \in D_\infty(A)$. Thus (7.4) follows from the fact that $D_\infty(A)$ is dense in X.

Finally, in order to show property (7.5), let $x \in X$ such that $G(\phi)x = 0$ for all $\phi \in \mathscr{D}_0$. Fix $k \in \mathbb{N}$, $k \geqslant k_0$. It follows from [Fa, Lemma 8.1.1] that $S_k(\cdot)x$ coincides with a polynomial of degree less than or equal to $k-1$ on $(0, \tau(k))$; i.e., there exists $x_m \in X$, $m = 0, 1, ..., k-1$, such that $S_k(t)x = \sum_{m=0}^{k-1} x_m t^m$ $(0 < t < \tau(k))$. Since $S_k(0) = 0$ we have $x_0 = 0$. It follows that $S_{k+1}(t)x = \sum_{m=1}^{k-1} x_m(t^{m+1}/(m+1)!)$ $(0 < t < \tau(k))$. It follows from (2.1) that $S_{k+1}(t)x \in D(A)$ and $A(\sum_{m=1}^{k-1} x_m(t^{m+1}/(m+1)!)) = \sum_{m=1}^{k-1} x_m t^m - x(t^k/k!)$ $(0 < t < \tau(k))$. Since A is closed and the function on the right hand side is C^k, we can differentiate k-times on both sides and conclude that $x_{k-1} \in D(A)$ and $Ax_{k-1} = -x$. Hence $A(\sum_{m=1}^{k-2} x_m(t^{m+1}/(m+1)!)) = \sum_{m=1}^{k-1} x_m t^m$. Differentiating $(k-1)$-times yields $x_{k-2} \in D(A)$ and $Ax_{k-2} = (k-1)!\, x_{k-1}$. Going on that way we obtain $x_m \in D(A)$ and $Ax_{m-1} = m!\, x_m$, $m = 1, ..., k-1$. Since $x_0 = 0$, it follows that $x_0 = x_1 = \cdots = x_{k-1} = 0$ and hence $x = 0$. We have shown that (7.5) holds. Let B be the generator of G. We have to show that $A = B$. Let B_0 be defined by

$$D(B_0) = \{x \in X : \lim_{n \to \infty} G(\rho_n)x = x, \lim_{n \to \infty} G(-\rho_n')x \text{ exists}\},$$

$B_0 x = \lim_{n \to \infty} G(-\rho_n')x$. Then by definition, $\bar{B} = B_0$.

Let $x \in D(B_0)$. Since by the above, $G(\rho_n)x \in D(A)$, and $AG(\rho_n)x = G(-\rho_n')x$ it follows from the closedness of A that $B_0 \subset A$. Thus $B \subset A$.

Conversely, let $x \in D_\infty(A)$. Then it follows from (7.9) that $G(\rho_n)x \to x$ $(n \to \infty)$. Hence $x \in D(B_0)$ and $B_0 x = Ax$. Since $D_\infty(A)$ is a core of A we conclude that $A \subset B$.

(ii) $\Rightarrow$ (i). Let G be the distribution semigroup generated by A. It follows from (7.6) that $G \in \mathscr{D}'(\mathscr{L}(X, D(A)))$, where $D(A)$ carries the graph norm. Let $\tau > 0$. It follows from the regularity theorem for distributions (see [Fa, Theorem 8.15]) that there exist a continuous function $S\colon [-\tau, \tau] \to \mathscr{L}(X, D(A))$ and $k \in \mathbb{N}$ such that

$$G(\phi) = (-1)^k \int_{-\tau}^{\tau} \phi^{(k)}(t)\, S(t)\, dt$$

for all $\phi \in \mathscr{D}$ with $\operatorname{supp} \phi \subset (-\tau, \tau)$. Since by (7.1), $\operatorname{supp} G \subset [0, \infty)$ one has $S(t) = 0$ for $t \leqslant 0$ (see [Fa, Remark 8.1.6]). It follows from (7.6) that

$$(-1)^k \int_0^{\tau} \phi^{(k)}(t)\, AS(t)x\, dt = -(-1)^k \int_0^{\tau} \phi^{(k+1)}(t)\, S(t)x\, dt \qquad (7.10)$$

for all $\phi \in \mathscr{D}$ with $\operatorname{supp} \phi \subset (-\tau, \tau)$, $x \in X$.

Integration by parts yields

$$\int_0^{\tau} \phi^{(k+1)}(t) \left(\int_0^t AS(s)x\, ds - S(t)x \right) dt = 0$$

for all $x \in X$, $\phi \in \mathscr{D}$ with $\operatorname{supp} \phi \subset (0, \tau)$.

It follows from [Fa, Lemma 8.1.1] that

$$\int_0^t AS(s)x\, ds - S(t)x = \sum_{p=0}^{k} t^p B_p x \qquad (x \in X),$$

where $B_p \in \mathscr{L}(X)$, $p = 0, 1, ..., k$.

Introducing this into (7.10) gives

$$(-1)^k \int_0^{\tau} \phi^{(k+1)}(t) \sum_{p=0}^{k} t^p B_p\, dt = \phi(0) I$$

for all $\phi \in \mathscr{D}$ with support in $(-\infty, \tau)$. For $p \in \{0, 1, ..., k\}$ one has

$$\int_0^{\tau} t^p \phi^{(k+1)}(t)\, dt = (-1)^{p+1}\, p!\, \phi^{(k-p)}(0).$$

It follows that $B_p = 0$ for $p < k$ and $B_k = -(1/k!)I$. Consequently

$$\int_0^t AS(s)\, ds = S(t) - \frac{t^k}{k!} I \qquad (0 \leqslant t < \tau).$$

Since by (7.7), $G(\phi)$ commutes with A, it follows that $AS(t)x = S(t)Ax$ $(x \in D(A),\ 0 \leqslant t < \tau)$. It follows from Remark 3.4 that $C_{k+1}(\tau)$ is well-posed and S is the k-times integrated semigroup on $[0, \tau)$ generated by A. ∎

From Theorems 2.1, 2.2, and 7.2 we obtain now the characterization theorem due to Chazarain [Ch].

COROLLARY 7.3. *Let A be a closed and densely defined operator. The following are equivalent.*

(i) *A is the generator of a distribution semigroup.*

(ii) *There exist $\alpha > 0$, $\beta > 0$, $k \in \mathbb{N}$, $M \geqslant 0$ such that $\Lambda(\alpha, \beta) \subset \rho(A)$ and*

$$\|R(\lambda, A)\| \leqslant M\, |\lambda|^k \qquad (\lambda \in \Lambda(\alpha, \beta)).$$

(iii) *There exist $\alpha > 0$, $\beta > 0$, $k \in \mathbb{N}$, $M \geqslant 0$ such that $E(\alpha, \beta) \subset \rho(A)$ and*

$$\|R(\lambda, A)\| \leqslant M\, |\lambda|^k \qquad (\lambda \in E(\alpha, \beta)).$$

Finally it follows from Theorems 7.2 and 6.1 that a densely defined closed operator A generates a distribution semigroup if and only if $A_{|D_\infty}$ generates a strongly continuous semigroup on D_∞ (in the sense of the theory of strongly continuous semigroups on Fréchet spaces, see [Yo]). This result is due to Ushijima [Us, Theorems 1, 2], see also Oharu [Oh2] and Chazarain [Ch, Theorem 6.6].

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