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RESEARCH ARTICLE

A Complex Tauberian Theorem and Mean Ergodic Semigroups

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0. Introduction

Let X be a Banach space and $f \in L^1_{\rm loc}([0,\infty);X)$ such that the Laplace transform $\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt$ exists for $\lambda > 0$. Let $f_\infty \in X$. We say that $f(t)$ is *Abel convergent* (*A-convergent*, for short) to f_∞ as $t \to \infty$ if $A - \lim\limits_{t\to\infty} f(t) := \lim\limits_{\lambda\downarrow 0} \lambda \widehat{f}(\lambda) = f_\infty$; and f is *Cesaro convergent* (*C-convergent* for short) if $C - \lim\limits_{t\to\infty} f(t) := \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t f(s)ds = f_\infty$. If $\lim\limits_{t\to\infty} f(t) = f_\infty$, then $C - \lim\limits_{t\to\infty} f(t) = f_\infty$; and if $C - \lim\limits_{t\to\infty} f(t) = f_\infty$, then $A - \lim\limits_{t\to\infty} f(t) = f_\infty$. The converse implications are false, in general. Additional conditions which allow the inverse implication are called Tauberian conditions, and the corresponding statements Tauberian theorems.

Here we are interested in deducing C-convergence from A-convergence. A known condition is that f is bounded (see [11] or [2]). We weaken this assumption but impose conditions on $\widehat{f}$. For example we show the following.

Theorem 0.1. *Assume that the following conditions are satisfied.*

(a) $\|f(t)\| = 0(t) \qquad (t \to \infty)$;

(b) *there exists an open set $\Omega \subset \mathbb{C}$ containing $i\mathbb{R}$ such that $\widehat{f}$ has a holomorphic extension to Ω.*

Then

$$C - \lim_{t\to\infty} f(t) = 0.$$

Note that condition (b) implies that $A - \lim_{t\to\infty} f(t) = 0$. Neither of the conditions (a) or (b) can be omitted. However, it is shown that condition (b) can be considerably relaxed.

If in Theorem 0.1 instead of (a) one assumes that f is bounded, one can actually conclude that $\lim\limits_{t\to\infty} f(t) = 0$. A simple proof of this has been given by Korevaar [6] (but it follows from much older work of Ingham [5]) and the result has been generalized in [1], [2] and [3]. We use similar arguments based on Cauchy's theorem.

Applying the results to C_0-semigroups we obtain the following ergodic theorem.

Theorem 0.2. *Let A be the generator of a C_0-semigroup $T = (T(t))_{t\geq 0}$ satisfying*

$$\|T(t)\| = 0(t) \qquad (t \to \infty). \tag{0.1}$$

Assume that
 a) $0 \in \rho(A)$
 b) $\sigma(A) \cap i\mathbb{R}$ *consists of poles of the resolvent of order 1.*

Then T is uniformly C-ergodic.

An example of a C_0-semigroup satisfying (0.1) (without being bounded in general) is obtained by considering a matrix operator

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix},$$

where A_1, A_2 are generators of contraction semigroups and B is a bounded operator. Such systems are considered in Section 3 and can be applied to investigate the asymptotic behavior of solutions of the inhomogeneous Cauchy problem with periodic inhomogeneity (Section 4).

1. A Tauberian Theorem

Let $f \in L^1_{\text{loc}}([0,\infty); X)$ where X is a Banach space. We assume that

$$M := \limsup_{t\to\infty} \frac{1}{t} \|f(t)\| < \infty. \tag{1.1}$$

Then $\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt$ exists for $\operatorname{Re}\lambda > 0$ and defines a holomorphic function on $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$. In the following we let $F(t) = \int_0^t f(s)ds$.

Proposition 1.1. *Assume (in addition to (1.1)):*
 a) $\widehat{f}$ *has a continuous extension to* $(\mathbb{C}_+ \cup i[-R,R])\backslash\{0\}$, *where* $R > 0$;
 b) $\lim\limits_{\substack{\lambda\to 0\\ \lambda\in\mathbb{C}_+}} \lambda\widehat{f}(\lambda) = 0$.

Then

$$\limsup_{t\to\infty} \|\frac{1}{t}\int_0^t f(s)ds\| \leq \frac{4M}{R}.$$

Proof. Let $M_1 > M$. We have to show that

$$\limsup_{t\to\infty} \frac{1}{t} \|F(t)\| \leq 4M_1/R. \tag{1.2}$$

For this, we can assume that $\|f(t)\| \leq M_1 t \quad (t > 0)$. In fact, by (1.1) there exists $t_0 \geq 0$ such that $\|f(t)\| \leq M_1 t$ for all $t \geq t_0$. Let $f_1 = 1_{[0,t_0]} \cdot f$ and replace f by $\tilde{f} = f - f_1$. Since $\widehat{f_1}$ is entire, $\tilde{f}$ satisfies a) and b). Moreover, $\|\tilde{f}(t)\| \leq M_1 t \quad (t > 0)$ by construction. Finally, $\limsup\limits_{t\to\infty} \frac{1}{t} \|F(t)\| = \limsup\limits_{t\to\infty} \frac{1}{t} \|\int_0^t \tilde{f}(s)ds\|$.

By hypothesis, there exists $N(t) \downarrow 0 \quad (t \to \infty)$ such that

$$\|\widehat{f}(\lambda)\| \leq N(t) \cdot \frac{1}{|\lambda|} \text{ for all } |\lambda| \leq \frac{1}{t} \tag{1.3}$$

where $t \geq \frac{1}{R}$.
Let $g := \widehat{f}$ and for $t > 0$, $g_t(\lambda) := \int_0^t e^{-\lambda s} f(s) ds \quad (\lambda \in \mathbb{C})$. Then g_t is an entire function and $g_t(0) = F(t)$.

Let $\tau > \frac{1}{R}$. By Cauchy's theorem we have for $t > \tau$,

$$F(t) = \frac{1}{2\pi i} \int_{|\lambda|=R} e^{\lambda t} g_t(\lambda) \Big(1 + \lambda^2/R^2\Big)^2 \frac{d\lambda}{\lambda},$$
$$0 = \frac{-1}{2\pi i} \int_{\Gamma} e^{\lambda t} g(\lambda) \Big(1 + \lambda^2/R^2\Big)^2 \frac{d\lambda}{\lambda},$$

where Γ is the oriented contour consisting of the segments $[-Ri, -\frac{i}{\tau}]$, $[-\frac{i}{\tau}, -\frac{i}{t}]$, the semicircle $[\operatorname{Re}\lambda > 0, |\lambda| = \frac{1}{t}]$, the segments $[\frac{i}{t}, \frac{i}{\tau}]$, $[\frac{i}{\tau}, iR]$ and the semicircle $[\operatorname{Re}\lambda > 0, |\lambda| = R]$.

Adding up the two terms we obtain

$$\begin{aligned}
F(t) = & \frac{1}{2\pi i} \int_{\substack{|\lambda|=R \\ \operatorname{Re}\lambda>0}} e^{\lambda t}\Big(g_t(\lambda) - g(\lambda)\Big)\Big(1 + \frac{\lambda^2}{R^2}\Big)^2 \frac{d\lambda}{\lambda} \\
& + \frac{1}{2\pi i} \int_{\substack{|\lambda|=R \\ \operatorname{Re}\lambda<0}} e^{\lambda t} g_t(\lambda)\Big(1 + \frac{\lambda^2}{R^2}\Big)^2 \frac{d\lambda}{\lambda} \\
& - \frac{1}{2\pi i} \int_{\substack{\operatorname{Re}\lambda=0 \\ \frac{1}{\tau}\leq|\lambda|\leq R}} e^{\lambda t} g(\lambda)\Big(1 + \frac{\lambda^2}{R^2}\Big)^2 \frac{d\lambda}{\lambda} \\
& - \frac{1}{2\pi i} \int_{\substack{\operatorname{Re}\lambda=0 \\ \frac{1}{t}\leq|\lambda|\leq \frac{1}{\tau}}} e^{\lambda t} g(\lambda)\Big(1 + \frac{\lambda^2}{R^2}\Big)^2 \frac{d\lambda}{\lambda} \\
& - \frac{1}{2\pi i} \int_{\substack{\operatorname{Re}\lambda>0 \\ |\lambda|=\frac{1}{t}}} e^{\lambda t} g(\lambda)\Big(1 + \frac{\lambda^2}{R^2}\Big)^2 \frac{d\lambda}{\lambda} \\
=: & \; I_1(t) + I_2(t) - I_3(t,\tau) - I_4(t,\tau) - I_5(t).
\end{aligned}$$

We estimate the different integrals:

I_1: Let $\lambda = Re^{i\theta}$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then

$$\begin{aligned}
&\|e^{\lambda t}\Big(g_t(\lambda) - g(\lambda)\Big)\| = \|\int_0^\infty f(t+s)\, e^{-\lambda s}\, ds\| \\
&\leq M_1 \int_0^\infty (t+s) e^{-\operatorname{Re}\lambda s}\, ds = M_1\Big(\frac{t}{R\cdot\cos\theta} + \frac{1}{R^2(\cos\theta)^2}\Big); \\
&|\Big(1 + \frac{\lambda^2}{R^2}\Big)^2| = 4(\cos\theta)^2; \quad \frac{1}{|\lambda|} = \frac{1}{R}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|I_1(t)\| &\leq \frac{1}{2\pi} \cdot M_1\Big(\frac{t}{R} + \frac{1}{R^2}\Big) \cdot 4 \cdot \frac{1}{R} \cdot \pi \cdot R \\
&= 2M_1\Big(\frac{t}{R} + \frac{1}{R^2}\Big).
\end{aligned}$$

I_2: Let $\lambda = Re^{i\theta}$, $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Then

$$\begin{aligned}
\|e^{\lambda t} g_t(\lambda)\| &= \|e^{\lambda t} \int_0^t e^{-\lambda s} f(s) ds\| \le e^{\operatorname{Re}\lambda \cdot t} M_1 \int_0^t e^{-\operatorname{Re}\lambda s} \cdot s\, ds \\
&= M_1 \frac{1}{(-\operatorname{Re}\lambda)}[t + \frac{1}{(-\operatorname{Re}\lambda)}(e^{\operatorname{Re}\lambda t} - 1)] \le M_1 t/(-\operatorname{Re}\lambda) \\
&= \frac{M_1}{R} \cdot \frac{t}{(-\cos\theta)};
\end{aligned}$$

$$|(1+\lambda^2/R^2)^2| = 4(\cos\theta)^2;\ 1/|\lambda| = 1/R.$$

Hence

$$\|I_2(t)\| \le \frac{1}{2\pi}\ \frac{M_1}{R}\, t\, 4\, \frac{1}{R}\, \pi\, R = 2\,\frac{M_1}{R}\, t.$$

I_3: $\|1/2\pi i \int_{1/\tau}^{R} e^{ist} g(is)(1 - s^2/R^2)^2\ ds/s\|$

$$\le 1/2\pi \log(R\tau) \cdot \limsup_{1/\tau \le s \le R} \|\widehat{f}(is)\|.$$

Hence

$$\|I_3(t,\tau)\| \le 1/\pi \log(R\tau) \cdot \limsup_{1/\tau \le |s| \le R} \|\widehat{f}(is)\|.$$

I_4: $\|1/2\pi i \int_{1/t}^{1/\tau} e^{ist} g(is)\Big(1 - \frac{s^2}{R^2}\Big)^2 \frac{ds}{s}\|$

$$\le 1/2\pi \int_1^\infty N(\tau)\, t/r\, dr/r = t\, N(\tau)/2\pi.$$

Hence

$$\|I_4(t,\tau)\| \le 1/\pi\, N(\tau) \cdot t.$$

I_5: $\lambda = 1/t\, e^{i\theta}$, $\theta \in (-\pi/2, \pi/2)$. Then
$|e^{\lambda t}| \le e$; $\|g(\lambda)\| \le N(t) 1/|\lambda| = N(t) \cdot t$; $|(1+\lambda^2/R^2)^2| \le 4$; $|1/\lambda| = t$.
Hence

$$\|I_5(t)\| \le 1/2\pi \cdot e \cdot N(t) \cdot t \cdot 4 \cdot t\, \pi\, 1/t = 2eN(t)t.$$

Summing we conclude that

$$\limsup_{t\to\infty} \|F(t)/t\| \le 4M_1/R + 1/\pi N(\tau).$$

Letting $\tau \to \infty$ gives (1.2). ■

Theorem 1.2. *Let $f_\infty \in X$. Assume that f satisfies the following conditions (in addition to (1.1)):*

a) *There exist $\delta > 0$, $\eta_k \in \mathbb{R}$, $|\eta_k| \ge \delta$ such that $\widehat{f}$ has a continuous extension to $\overline{\mathbb{C}}_+ \backslash (\{i\eta_k : k \in \mathbb{N}\} \cup \{0\})$ and each $i\eta_k$ is a pole of order 1 of $\widehat{f}$;*

b) $\lim_{\substack{\lambda\to 0\\ \lambda \in \mathbb{C}_+}} \lambda \widehat{f}(\lambda) = f_\infty$.

Then

$$C - \lim_{t\to\infty} f(t) = f_\infty.$$

A special case, where (b) is satisfied, is when 0 is a pole of order 1. In that case $f_\infty = \operatorname{Res}(\widehat{f}, 0)$.

Corollary 1.3. *Assume that* $f \in L^1_{\mathrm{loc}}([0,\infty); X)$ *satisfies:*

a) $\|f(t)\| = 0(t) \quad (t \to \infty)$;

b) *every point on* $i\mathbb{R}$ *is regular or a pole of order 1 of* $\widehat{f}$.

Then

$$C - \lim_{t\to\infty} f(t) = \operatorname{Res}(\widehat{f}, 0).$$

Theorem 0.1 of the introduction is an immediate consequence of Corollary 1.3.

Proof of Theorem 1.2. a) One may assume that $f_\infty = 0$ considering $f(t) - f_\infty$ instead of f otherwise.

b) Let $R \notin \{|\eta_k| : k \in \mathbb{N}\}$, $R \geq \delta$. We will show that

$$\limsup_{t\to\infty} \|F(t)\|/t \leq 4M/R, \tag{1.4}$$

where

$$M := \limsup_{t\to\infty} \|f(t)\|/t.$$

Let $a_k := \operatorname{Res}(\widehat{f}, i\eta_k)$. Then $\widehat{f}(\lambda) - \frac{a_k}{(\lambda - i\eta_k)}$ has a holomorphic extension to a neighborhood of $i\eta_k$. Let $f_k(t) := a_k\, e^{i\eta_k t}$. Then $\widehat{f_k}(\lambda) = a_k/(\lambda - i\eta_k) \quad (\operatorname{Re}\lambda > 0)$. The function $h(t) = \sum_{|\eta_k|\leq R} f_k(t)$ is bounded, $\widehat{h}$ is holomorphic in 0, $(f-h)\widehat{\ }$ has a continuous extension to $(\mathbb{C}_+ \cup i[-R,R]) \setminus \{0\}$ and $\lim_{\substack{\lambda\to 0\\ \lambda\in\mathbb{C}_+}} \lambda (f-h)\widehat{\ }(\lambda) = f_\infty = 0$. Moreover, $\limsup_{t\to\infty} \|f(t) - h(t)\|/t = M$. Since $C - \lim_{t\to\infty} h(t) = 0$, it follows from Proposition 1.1 that

$$\limsup_{t\to\infty} \|\frac{1}{t} F(t)\| = \limsup_{t\to\infty} \|\frac{1}{t} \int_0^t (f(s) - h(s))ds\| \leq 4M/R.$$

Since R can be chosen arbitrarily large, we conclude that $C - \lim_{t\to\infty} f(t) = 0$. ∎

We deduce a Tauberian theorem for power series. Let $a_n \in X$, $(n \in \mathbb{N}_0)$ satisfy

$$\|a_n\| = 0(n) \quad (n \to \infty). \tag{1.5}$$

Then $p(z) := \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < 1$. If

$$C - \lim_{n\to\infty} a_n := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n = a_\infty$$

(for some $a_\infty \in X$), then it is easy to see that

$$A - \lim_{n\to\infty} a_n = \lim_{x\uparrow 1} (1-x)p(x) = a_\infty,$$

the converse being false, in general.

Note, in particular, if 1 is a pole of order 1 of p, then $A - \lim_{n\to\infty} a_n = \operatorname{Res}(p, 1)$.

Let $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$.

Corollary 1.4. *Let $a_\infty \in X$. Assume that (besides (1.5)) the following conditions are satisfied:*

a) *Every point on $\Gamma \setminus \{1\}$ is either regular or a pole of order 1 of p;*
b) *there exists $\delta > 0$ such that every $z \in \Gamma \setminus \{1\}$ satisfying $|z-1| < \delta$ is regular;*
c) $\lim\limits_{\substack{z\to 1\\ |z|<1}} (1-z)\,p(z) = a_\infty$.

Then

$$C - \lim_{n\to\infty} a_n = a_\infty.$$

Proof. Let $f(t) := a_n$ for $t \in [n, n+1)$. Then f satisfies (1.1) and $\widehat{f}(\lambda) = h(\lambda)p(e^{-\lambda})$ $(\mathrm{Re}\,\lambda > 0)$ where $h(\lambda) := (1-e^{-\lambda})/\lambda$ is an entire function. Since $C - \lim\limits_{n\to\infty} a_n = \lim\limits_{N\to\infty} \frac{1}{N}\int_0^N f(t)\,dt$, the conclusion follows from Theorem 1.2.∎

We give several comments on the preceding results.

Example 1.5. Let $R > 0$. Define $f(t) = te^{iRt}$. Then $M = 1$ in (1.1). Moreover, $\widehat{f}(\lambda) = (\lambda - iR)^{-2}$ $(\mathrm{Re}\ \lambda > 0)$. The Cesaro means are given by

$$\frac{1}{t}\int_0^t f(s)ds = \frac{1}{iR}\,e^{iRt} + \frac{1}{tR^2}\left(e^{iRt} - 1\right).$$

Hence

$$\limsup_{t\to\infty} \frac{1}{t}\ \|\int_0^t f(s)ds\| = \frac{1}{R}.$$

This shows that the estimate in Proposition 1.1 cannot be essentially improved. Condition b) of Theorem 1.2 is satisfied and $A - \lim\limits_{t\to\infty} f(t) = 0$. However a) does not hold since iR is a pole of order 2. And in fact, f is not C-convergent. This shows that the condition on the order of the poles in Theorem 1.2 cannot be weakened.

It is easy to see that condition (1.1) implies that every pole of $\widehat{f}$ on $i\mathbb{R}$ is at most of order 2. In the following we show that (under additional hypotheses) the poles are of order 1 if f is C-convergent.

Corollary 1.6. *Assume that f satisfies (1.1) and that every point on $i\mathbb{R}$ is regular besides a finite number of poles. Then f is C-convergent if and only if all poles on $i\mathbb{R}$ are of order 1.*

Proof. It is clear from Corollary 1.3 that the condition is sufficient. To prove the converse assume that $\widehat{f}$ has a finite number of poles of order 2 $\{i\eta_1, \ldots, i\eta_k\}$ on $i\mathbb{R}$. If 0 is a pole of order 2, then f is not $C-$ergodic. Hence we may assume that $\eta_j \neq 0$ $(j = 1, \ldots, k)$. Let $a_1, \ldots, a_k \in X$; $h(t) = \sum\limits_{j=1}^{k} a_j\, t\, e^{i\eta_j t}$.

Then $\widehat{h}(\lambda) = \sum\limits_{j=1}^{k} a_j(\lambda - i\eta_j)^{-2}$, Re $\lambda > 0$. Hence, for a suitable choice of $a_1, \ldots, a_k \in X \setminus \{0\}$, the function $(f-h)\widehat{\ }$ has merely regular points or first order poles on $i\mathbb{R}$. It follows from Corollary 1.3 that $f - h$ is C-convergent. However

$$\frac{1}{t}\int_0^t h(s)ds + \sum_{j=1}^{k} \frac{i}{\eta_j}\, a_j e^{i\eta_j t} = \frac{1}{t}\sum_{j=1}^{k} \frac{a_j}{\eta_j^2}\,(e^{i\eta_j t}-1) \longrightarrow 0 \quad (t\to\infty).$$

Since $\sum_{j=1}^{k} \frac{i}{\eta_j} a_j\, e^{i\eta_j t}$ does not converge for $t\to\infty$, it follows that h is not C-convergent. Consequently, f is not C-convergent either. ∎

Remark 1.7. For power series a stronger result than Corollary 1.6 can be proved in an elementary way. Let $p(z) := \sum_{k=0}^{\infty} a_n\, z^n$ be a power series with coefficients $a_n \in X$ (not necessarily satisfying (1.5)). Assume that the radius of convergence is 1 and that every point on Γ is regular or a pole. Then $(a_n)_{n\in\mathbb{N}_0}$ is C-convergent if and only if every pole on Γ is of order ≤ 1.

This can be proved in a similar way as Corollary 1.6 using that $\|a_n\| \to 0 \quad (n\to\infty)$ if p has a holomorphic extension to a neighborhood of $\{z\in\mathbb{C} : |z|\leq 1\}$.

If we are merely interested in boundedness of the Cesaro means, the proof of Proposition 1.1 shows that b) can be replaced by

$$\sup_{\lambda\in B(0,\delta)_+} \|\lambda\widehat{f}(\lambda)\| < \infty \tag{1.6}$$

for some $\delta > 0$ in order to deduce that

$$\limsup_{t\to\infty} \|\frac{1}{t}\int_0^t f(s)ds\| < \infty, \tag{1.7}$$

where $B(0,\delta) = \{z\in\mathbb{C} : |z| < \delta\}, \quad B(0,\delta)_+ = B(0,\delta)\cap\mathbb{C}_+$.
Condition (1.6) will be adequate for our application to semigroups in Theorem 2.1. In Proposition 1.9, we shall give a weaker condition which suffices for (1.7). We note first a necessary condition.

Lemma 1.8. *Let $f\in L^1_{\mathrm{loc}}([0,\infty);X)$ satisfying (1.7). Then for every $R>0$ there exists $M\geq 0$ such that*

$$\|\lambda\widehat{f}(\lambda)\| \leq \frac{M}{(\cos\,\theta)^2} \quad \textit{where} \tag{1.8}$$

$$\lambda = |\lambda|\, e^{i\theta}, \quad \theta\in(-\frac{\pi}{2},\frac{\pi}{2}), \quad |\lambda|\leq R.$$

Proof. We can assume that $M = \sup_{t>0} \frac{1}{t}\|\int_0^t f(s)ds\| < \infty$ (cf. proof of Prop. 1.1). Then

$$\begin{aligned}\|\lambda\widehat{f}(\lambda)\| &= \|\lambda^2\int_0^\infty e^{-\lambda t}\, F(t)dt\| \\ &= \|\lambda^2\int_0^\infty e^{-\lambda t}\, t\,\frac{1}{t}\, F(t)dt\| \leq |\lambda|^2 M\int_0^\infty e^{-\mathrm{Re}\,\lambda t}\, t\, dt \\ &= \frac{|\lambda|^2\cdot M}{(\mathrm{Re}\,\lambda)^2} = \frac{M}{(\cos\,\theta)^2}.\end{aligned}$$

∎

We do not know whether (1.8) for one (or all) $R > 0$ is sufficient for (1.7). However, we have the following.

Proposition 1.9. *Let* $f \in L^1_{\text{loc}}\big([0,\infty);X\big)$ *such that*

a) $\|f(t)\| = 0(t) \quad (t \to \infty)$;

b) *there exist* $R > 0,\ c \geq 0,\ \epsilon > 0$, *such that*

$$\|\lambda \widehat{f}(\lambda)\| \leq \frac{c}{(\cos\,\theta)^{1-\epsilon}} \qquad \textit{for all } \lambda = |\lambda|\,e^{i\theta} \in B(0,R)_+.$$

Then

$$\limsup_{t\to\infty}\ \frac{1}{t}\|\int_0^t f(s)ds\| < \infty.$$

Proof. We keep the notations of the proof of Proposition 1.1. Again we may assume that $\|f(t)\| \leq M_1 t \quad (t > 0)$.
Let $t > \frac{1}{R}$. Denote by Γ_t the contour

$$\{\lambda \in \mathbb{C} : |\lambda| = 1,\ \text{Re}\ \lambda > \frac{1}{t}\} \cup \{\lambda \in \mathbb{C} : \text{Re}\ \lambda = \frac{1}{t},\ |\lambda| \leq R\}.$$

Then, defining F, g and g_t as before,

$$F(t) = g_t(0) = 1/2\pi i \int_{|\lambda|=R} e^{\lambda t} g_t(\lambda)\Big(1 + \lambda^2/R^2\Big)^2\ \frac{d\lambda}{\lambda},$$

$$0 = -1/2\pi i \int_{\Gamma_t} e^{\lambda t}\, g(\lambda)\Big(1 + \lambda^2/R^2\Big)^2\ \frac{d\lambda}{\lambda}.$$

Hence

$$\begin{aligned} F(t) &= 1/2\pi i \int_{\substack{|\lambda|=R \\ \text{Re}\ \lambda > 1/t}} e^{\lambda t}\Big(g_t(\lambda) - g(\lambda)\Big)\Big(1 + \frac{\lambda^2}{R^2}\Big)^2\ \frac{d\lambda}{\lambda} \\ &\quad - 1/2\pi i \int_{\substack{|\lambda|=R \\ \text{Re}\ \lambda < 0}} e^{\lambda t}\ g_t(\lambda)\Big(1 + \lambda^2/R^2\Big)^2\ \frac{d\lambda}{\lambda} \\ &\quad - 1/2\pi i \int_{\substack{|\lambda|=R \\ 0 < \text{Re}\ \lambda < 1/t}} e^{\lambda t}\, g_t(\lambda)\Big(1 + \lambda^2/R^2\Big)^2\ \frac{d\lambda}{\lambda} \\ &\quad - 1/2\pi i \int_{\substack{|\lambda| \leq R \\ \text{Re}\ \lambda = 1/t}} e^{\lambda t}\, g(\lambda)\Big(1 + \lambda^2/R^2\Big)^2\ \frac{d\lambda}{\lambda} \\ &=: I_1(t) - I_2(t) - I_3(t) - I_4(t). \end{aligned}$$

As in the proof of Proposition 1.1 we obtain

$$1/t\big(\|I_1(t)\| + \|I_2(t)\|\big) \leq 4M_1/R + 2M_1/tR^2.$$

The remaining integrals are estimated as follows.

$I_3(t)$: Let $\lambda = R\,e^{i\theta},\quad 0 < \cos\theta < 1/Rt$. Then $|e^{\lambda t}| = e^{R\cos\,\theta\cdot t} \leq e$;

$$\begin{aligned} \|g_t(\lambda)\| = \|\int_0^t e^{-\lambda s} f(s)ds\| &\leq M_1 \int_0^t e^{-\text{Re}\ \lambda\cdot s}\cdot s\,ds \\ &\leq M_1/R^2(\cos\,\theta)^2; \end{aligned}$$

$|(1+\lambda^2/R^2)^2| = 4(\cos\theta)^2, \quad 1/|\lambda| = 1/R$. Thus

$$1/t\ \|I_3(t)\| \le 1/t \cdot 1/2\pi \cdot e \cdot 4M_1/R^2 \cdot 1/R \cdot 2R \text{ arc sin } 1/Rt$$
$$= 4M_1 e/(\pi t\, R^2) \cdot \text{arc sin } 1/Rt.$$

$I_4(t)$: Let $\lambda = \frac{1}{t} + is$. Then $|\lambda| = (1/t^2 + s^2)^{1/2}$ and

$$\|g(\lambda)\| \le c/|\lambda| \cdot (|\lambda|/\text{Re }\lambda)^{1-\epsilon} = c|\lambda|^{-\epsilon}\ (\text{Re }\lambda)^{\epsilon-1}.$$

Thus

$$\|g(\lambda)\|/|\lambda| \le c|\lambda|^{-1-\epsilon}\ (\text{Re }\lambda)^{\epsilon-1} = c\,t^{1-\epsilon}(1/t^2+s^2)^{-(1+\epsilon)/2}$$
$$= c\,t^2\big(1+t^2 s^2\big)^{-(1+\epsilon)/2};$$

$|(1+\lambda^2/R^2)^2| \le 4; \quad |e^{\lambda t}| = e.$

Hence

$$\|I_4(t)\| \le 1/2\pi \cdot 4ec \cdot \int_{-R}^{R} t^2 \cdot (1+t^2 s^2)^{-(1+\epsilon)/2} ds$$
$$= 1/\pi \cdot 2 \cdot e \cdot c \cdot t \cdot \int_{-R\cdot t}^{Rt} (1+r^2)^{-(1+\epsilon)/2}\, dr$$
$$\le \text{const} \cdot t.$$

Adding up one sees that

$$\limsup_{t\to\infty}\ 1/t\|F(t)\| < \infty. \qquad \blacksquare$$

2. Ergodic C_0-semigroups

Let $T = (T(t))_{t\ge 0}$ be a C_0-semigroup with generator A. By $\rho(A)$ we denote the resolvent set of A and by $R(\lambda, A) = (\lambda - A)^{-1}$ $(\lambda \in \rho(A))$ its resolvent. Then T is called *Abel-ergodic* (briefly *A-ergodic*) if $(0,\infty) \subset \rho(A)$ and $A - \lim\limits_{t\to\infty} T(t) := \lim\limits_{\lambda\downarrow 0} \lambda R(\lambda, A) = P$ exists strongly. In that case one has $X = N(A) \oplus \overline{R(A)}$ (where $N(A) := \{x \in D(A),\ Ax = 0\}$, $R(A) = \{Ax : x \in D(A)\}$ and P is the projection onto $N(A)$ along this decomposition).
It is well-known that T is A-ergodic if and only if

$$(2.1) \qquad (0,\infty) \subset \rho(A), \quad \sup_{0<\lambda\le 1} \|\Big(\lambda R(\lambda, A)\Big)\| < \infty \text{ and}$$

$$(2.2) \qquad N(A) \text{ separates } N(A').$$

Note that (2.2) follows from (2.1) if X is reflexive. We refer to [2] and [7] for these results. The semigroup is called *Cesaro ergodic* (or also *C-ergodic* or *mean-ergodic* or simply *ergodic*) if $C - \lim\limits_{t\to\infty} T(t) = \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t T(s)ds$ exists strongly. Of course, in that case, T is A-ergodic as well and $C - \lim\limits_{t\to\infty} T(t) = A - \lim\limits_{t\to\infty} T(t)$. We are interested in the converse implication.

Theorem 2.1. *Assume that*

a) $\|T(t)\| = 0(t) \quad (t \to \infty)$;

b) *There exists* $\delta > 0$ *such that* $\sup_{\lambda \in B(0,\delta)_+} \|\lambda R(\lambda, A)\| < \infty$;

c) $N(A)$ *separates* $N(A')$;

d) *Every* $\lambda \in \sigma(A) \cap i\mathbb{R} \setminus \{0\}$ *is a pole of order 1 of the resolvent of* A.

Then T *is* $C-$*ergodic.*

Remark 2.2.

1) If T is bounded, conditions (b) and (d) are far too strong. However, it will be shown below that they cannot be omitted if $\|T(t)\|$ growth like t.
2) It is easy to see that (b) implies that there exists $\theta \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(\theta) \cap B(0,\delta) \subset \rho(A)$ and $\sup_{\lambda \in \Sigma(\theta) \cap B(0,\delta)} \|\lambda R(\lambda, A)\| < \infty$ where $\Sigma(\theta) = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}$.
3) It is not difficult to see that the formally weaker condition $\|\lambda R(\lambda, A)\| \leq c/(\cos \theta)^{1-\epsilon}$ for $\lambda = |\lambda| e^{i\theta} \in B(0,\delta)_+$ implies condition (b) of Theorem 2.1 (cf. Proposition 1.9) in the case of resolvents.

Proof. It follows from (a) that $[\text{Re } \lambda > 0] \subset \rho(A)$ and from (b) and (c) that T is $A-$ergodic. It follows from the remark preceding Lemma 1.8 (or Proposition 1.9) that $\sup_{t \geq 1} \|\frac{1}{t} \int_0^t T(s)\, ds\| < \infty$. Since $T(t)x = x$ on $N(A)$ and $N(A) \oplus \overline{R(A)} = X$, it suffices to show that $C - \lim_{t \to \infty} T(t)x = 0$ for $x \in R(A)$. Let $x = Ay$ where $y \in D(A)$. Then $\lambda R(\lambda, A)x = \lambda^2 R(\lambda, A)y - \lambda y \quad (\text{Re } \lambda > 0)$. So the function $f(t) := T(t)x$ satisfies the hypotheses of Theorem 1.2 (with $f_\infty = 0$). Consequently, $C - \lim_{t \to \infty} T(t)x = 0$. ∎

Lemma 2.3. *Assume that* $i\eta \in i\mathbb{R}$ *is a pole of order* $k \geq 2$ *of the resolvent. Then* T *is not* C*-ergodic.*

Proof. There exists $x \in D(A^k)$ such that $(A - i\eta)^{k-1}x \neq 0$, $(A - i\eta)^k x = 0$ (see e.g. [Na, A-III. 3-6]). Hence $e^{-i\eta t}T(t)x = x + t(A - i\eta)x + \frac{t^2}{2!}(A - i\eta)^2 x + \cdots + t^{k-1}/(k-1)!(A - i\eta)^{k-1}x$. This implies that $T(t)x$ is not C-convergent. ∎

Theorem 2.4. *Assume that* $\|T(t)\| = 0(t) \quad (t \to \infty)$ *and that* $\sigma(A) \cap i\mathbb{R}$ *consists of poles of the resolvent only. Then the following are equivalent:*

(i) T *is* C*-ergodic;*

(ii) *Every point in* $\sigma(A) \cap i\mathbb{R}$ *is a pole of order 1;*

(iii) T *is uniformly* C*-ergodic; i.e.* $\frac{1}{t} \int_0^t T(s)ds$ *converges in the operator norm.*

Proof. (i) implies (ii) by Lemma 2.3. Assume (ii). We are going to show (iii). For that we can assume that $0 \in \rho(A)$ (considering otherwise the restriction of T to $(I-P)X$, where P is the residue at 0). Let $M = \limsup_{t \to \infty} \|T(t)\|/t$. Let $R > 0$ such that $\pm iR \notin \sigma(A)$. For $x \in X$, $\|x\| \leq 1$, let $f(t) := T(t)x$. Then by the proof of Theorem 1.2 $\limsup_{t \to \infty} \frac{1}{t} \|\int_0^t T(s)x\, ds\| \leq 4M/R$, uniformly in x. Since R can be chosen arbitrarily large, it follows that $\lim_{t \to \infty} \frac{1}{t} \|\int_0^t T(s)\, ds\| = 0$. ∎

We continue with some comments and examples.

Remark 2.5.

a) If $\|T(t)\| = 0(t) \quad (t \to \infty)$ and $0 \in \rho(A)$ or 0 is a pole of the resolvent of order 1, then $\sup_{t\geq 0} \frac{1}{t} \| \int_0^t T(s)\, ds\| < \infty$. This can be seen in a direct way (instead of applying Proposition 1.1). In fact, one can assume $0 \in \rho(A)$ (cf. proof of Theorem 2.4). Then $\frac{1}{t} \int_0^t T(s)ds = \frac{1}{t}(A^{-1}T(t) - A^{-1})$.

b) If $\|T(t)\| = o(t) \quad (t \to \infty)$ and $0 \in \rho(A)$, the above argument shows that without any further hypotheses T is uniformly C-ergodic.

The following example shows that the growth condition $\|T(t)\| = 0(t) \quad (t \to \infty)$ is essential in the results of this section.

Example 2.6. There exists a C_0-semigroup T satisfying $\|T(t)\| = 0(t^2) \quad (t \to \infty)$, $\sigma(A) \cap i\mathbb{R} = \emptyset$, but $\sup_{t>0} \frac{1}{t} \| \int_0^t T(s)\, ds\| = \infty$.

Let $H = \oplus_{n\in\mathbb{N}} \ell^2(3)$ and let $A = (A_n)_{n\in\mathbb{N}}$ with maximal domain, where

$$A_n = \begin{pmatrix} \lambda_n & 1 & 0 \\ 0 & \lambda_n & 1 \\ 0 & 0 & \lambda_n \end{pmatrix}.$$

A generates the C_0-semigroup $T(t) = (T_n(t))_{n\in\mathbb{N}}$, where

$$T_n(t) = e^{\lambda_n t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Now let $\lambda_n = -\frac{1}{n^2} + in$.
It is easy to see that

$$\|\frac{1}{n^2} \int_0^{n^2} T(s)\, ds\| \geq \|\frac{1}{n^2} \int_0^{n^2} T_n(s) ds\| \to \infty \quad (n \to \infty).$$

However, $\sigma(A) = \{-\frac{1}{n^2} + in : n \in \mathbb{N}\}$ so that $\sigma(A) \cap i\mathbb{R} = \emptyset$.

Finally, we mention, that in Theorem 2.4 in general one has no stronger convergence than in the sense of Cesaro. In fact, in [1] an example is given where $\|T(t)\| = 0(t) \quad (t \to \infty)$, $\sigma(A) \cap i\mathbb{R} = \emptyset$, but $T(t)$ does not converge strongly.

3. Application to Triangular Systems

Let A_1 and A_2 be the generators of bounded C_0-semigroups $T_1 = (T_1(t))_{t\geq 0}$ and $T_2 = (T_2(t))_{t\geq 0}$ on X_1 and X_2, respectively. Let $B \in \mathcal{L}(X_2, X_1)$ and consider the operator

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \tag{3.1}$$

with domain $D(A) = D(A_1) \times D(A_2)$ on $X = X_1 \times X_2$. Such operators have been studied recently by Nagel [9], and Phong [10]. The resolvent of A is given by

$$R(\lambda, A) = \begin{pmatrix} R(\lambda, A_1) & R(\lambda, A_1)\, B\, R(\lambda, A_2) \\ 0 & R(\lambda, A_2) \end{pmatrix}$$

($\lambda \in \rho(A_1) \cap \rho(A_2)$). Since $R(\lambda, A_1)B\,R(\lambda, A_2)$ is the Laplace transform of $T_{12}(t) = \int_0^t T_1(t-s)B\,T_2(s)\,ds$, the operator A generates the C_0-semigroup

$$T(t) = \begin{pmatrix} T_1(t) & T_{12}(t) \\ 0 & T_2(t) \end{pmatrix}$$

(cf. [9]). Clearly $\|T(t)\| = 0(t) \quad (t \to \infty)$, but T is not bounded in general.

Example 3.1. Assume that $X_1 = X_2$; $A_1 = A_2$, $B = I$. Then $T_{12}(t) = t\,T_1(t)$.

Now we obtain the following results for ergodicity of T.

Theorem 3.2. *Assume that every $i\eta \in i\mathbb{R}$ is a pole of order $k_i(\eta)$ of $R(\lambda, A_i)$ and $k_1(\eta) + k_2(\eta) \leq 1$ for all $\eta \in \mathbb{R}$. Then T is uniformly $C-$ergodic.*

Here a pole of order 0 is by definition a regular point.

Proof. The hypothesis implies that every point on $i\mathbb{R}$ is a pole of $R(\lambda, A)$ of order ≤ 1. So the claim follows from Theorem 2.4. ∎

Assume that S is a bounded holomorphic C_0-semigroup with generator B. Then the following are equivalent:

(i) S is $A-$ergodic;
(ii) S is $C-$ergodic;
(iii) $\lim_{t\to\infty} S(t)$ exists strongly.

In fact, there exists $M \geq 0$ such that $\|t\,B\,S(t)\| \leq M \quad (t > 0)$. Hence $\lim_{t\to\infty} S(t)x = 0$ for $x \in R(B)$ and so for $x \in \overline{R(B)}$. Thus, if S is $A-$ergodic, i.e. if $X = N(B) \oplus \overline{R(B)}$, then $\lim_{t\to\infty} S(t) = P$ strongly, where P denotes the projection onto $N(B)$ along this decomposition.

This clarifies the asymptotic behavior if T_1 and T_2 are both bounded holomorphic C_0-semigroups. If merely one of them is holomorphic one can apply the results of Section 2.

Theorem 3.3. *Assume that A_1 (or A_2) generates a bounded holomorphic $A-$ergodic C_0-semigroup and that $R(\lambda, A_2)$ (resp. $R(\lambda, A_1)$) has merely poles of order ≤ 1 on $i\mathbb{R}$ and $0 \in \rho(A_2)$ (resp. $0 \in \rho(A_1)$). Then T is $C-$ergodic.*

Proof. We merely consider the first case. Since T_1 is a bounded holomorphic C_0-semigroup one has $\sigma(A_1) \cap i\mathbb{R} \subset \{0\}$ and $\sup_{\mathrm{Re}\,\lambda > 0} \|\lambda\,R(\lambda, A_1)\| < \infty$. Thus A satisfies the hypotheses of Theorem 2.1. ∎

It is interesting that by applying our results to the system (3.1) (with $A_1 = A_2$) one may recover results for a given semigroup T_1 as is shown in the following theorem. Alternatively, one may apply the Tauberian theorem of [1] (see also [2]).

Theorem 3.4. *Let A_1 be the generator of a bounded C_0-semigroup T_1 and assume that $i\mathbb{R} \subset \rho(A_1)$. Then* $\lim_{t\to\infty} \int_0^t T_1(s)\,ds = R(0, A_1)$ *in the operator norm.*

Proof. Let $S_1(t) = \int_0^t T_1(s)\,ds$. Consider the system (3.1) with $A_2 = A_1$ and $B = I$. Then $T(t)_{12} = t\,T_1(t)$ (see Example 3.1) and $\int_0^t T_{12}(s)\,ds = t\,S_1(t) - \int_0^t S_1(s)\,ds$. It follows from Theorem 2.4 that

$$\|S_1(t) - 1/t \int_0^t S_1(s)\,ds\| \to \infty \quad (t \to \infty).$$

But $S_1(t)/t = 1/t(A_1^{-1}\,T_1(t) - A_1^{-1}) \to 0 \quad (t \to \infty)$. Hence

$$1/t \int_0^t S_1(s)\,ds = 1/t(A_1^{-1}\,S_1(t) - t\,A_1^{-1}) \to -A_1^{-1} = R(0, A_1)$$

in norm. Thus $\lim_{t\to\infty} S_1(t) = R(0, A_1)$ in norm. ∎

Remark 3.5. Theorem 3.4 is no longer true if $0 \in \rho(A_1)$ but $\sigma(A_1) \cap i\mathbb{R} \neq \emptyset$ as the example $X = \mathbb{C}$, $A_1 = i$ shows. However, the following holds.
Let A be the generator of a bounded semigroup T and $S(t) = \int_0^t T(s)\,ds$. If $0 \in \rho(A)$, then $R(0, A) = C - \lim_{t\to\infty} S(t)$ strongly. In fact, $C - \lim_{t\to\infty} S(t)x = C - \lim_{t\to\infty} T(t)\,A^{-1}x - A^{-1}x = A - \lim_{t\to\infty} T(t)A^{-1}x - A^{-1}x$ (since T is bounded) $= \lim_{\lambda\downarrow 0} \lambda R(\lambda)A^{-1}x - A^{-1}x = -A^{-1}x \quad (x \in X)$.

4. Periodic Inhomogeneities

Our last application concerns the asymptotic behavior of solutions of the inhomogeneous Cauchy problem with periodic inhomogeneity.

Let A_1 be the generator of a bounded C_0-semigroup $T_1 = (T_1(t))_{t\geq 0}$ on X_1. Let $\tau > 0$. We consider the space

$$X_2 := C_\tau(\mathbb{R}, X_1) = \{f : \mathbb{R} \to X_1 \text{ continuous } : f(s+\tau) = f(s) \text{ for all } s \in \mathbb{R}\}$$

as well as the subspace $\widetilde{X}_2 := \{f \in X_2 : M_0 f = 0\}$ where $M_0 f = 1/\tau \int_0^\tau f(s)\,ds$ is the mean of f.

We consider the inhomogeneous Cauchy problem

$$(CP) \qquad \begin{cases} u'(t) = Au(t) + f(t) & (t \geq 0) \\ u(0) = x \end{cases}$$

with $f \in X_2$, $x \in X_1$. The function

$$u(t) = T_1(t)x + \int_0^t T_1(t-s)\,f(s)\,ds$$

$(t \geq 0)$ is called the *mild solution* of (CP).
For $f \in X_2$ we define the first moment $M_1 f \in X_1$ of f by $M_1 f = 1/\tau \int_0^\tau t f(t)\,dt$.

Theorem 4.1. *Assume that*

a) $2\pi\, i/\tau \cdot \mathbb{Z} \setminus \{0\} \subset \rho(A_1)$;

b) *every point on* $\sigma(A_1) \cap i\mathbb{R}$ *is a pole of order 1 of the resolvent of* A_1. *Let* $P = \text{Res}\,(R(\lambda, A_1), 0)$.

Then

$$C - \lim_{t\to\infty} u(t) = P(x - M_1 f)$$

whenever $x \in X_1,\ f \in \widetilde{X}_2$.
Moreover, if $P = 0$, *i.e. if* $0 \in \rho(A_1)$, *then*

$$C - \lim_{t\to\infty} u(t) = R(0, A_1) M_0 f$$

for all $f \in X_2$.

Remark 4.2.

1. Since 0 is a pole of order ≤ 1 of the resolvent of A_1, T_1 is $A-$ergodic and $A - \lim_{t\to\infty} T_1(t) = P \quad (= \text{Res}\,(R(\lambda, A_1), 0))$. Since T_1 is bounded, it follows that $C - \lim_{t\to\infty} T_1(t) = P$.
2. Assume that $P \neq 0$. Let $x = Px \neq 0$ and $f(t) \equiv x$. Then $u(t) := (1+t)x$ is not $C-$convergent.

Theorem 4.3. *Assume that* T_1 *is a bounded holomorphic,* $C-$*ergodic semigroup. Let* $P := C - \lim_{t\to\infty} T_1(t)$. *Then*

$$C - \lim_{t\to\infty} u(t) = P(x - M_1 f)$$

for all $x \in X_1,\ f \in \widetilde{X}_2$.

For the proofs we consider the C_0-semigroup T_2 on X_2 given by

$$\big(T_2(t)f\big)(s) = f(t+s) \qquad (f \in X_2,\ t \ge 0,\ s \in \mathbb{R})$$

and denote by A_2 its generator.

It is easy to see that $\sigma(A_2) = \{2\pi i\, \frac{n}{\tau} : n \in \mathbb{Z}\}$ and $\big(R(\lambda, A_2)f\big)(s) = (1 - e^{-\lambda\tau})^{-1} \int_0^\tau e^{-\lambda t} f(s+t)dt$. Each value $2\pi i\, n/\tau$ is a pole of order 1 of $R(\lambda, A_2)$ and $C - \lim_{t\to\infty} T_2(t) = A - \lim_{t\to\infty} T_2(t) = Q_0$, where $(Q_0 f)(s) \equiv \frac{1}{\tau} \int_0^\tau f(t)dt$. So $N(A_2) = \{f \in X_2 : f \equiv \text{constant}\}$ and $\widetilde{X}_2 = (I - Q_0) X_2$. We denote by $\widetilde{T}_2$ the restriction of T_2 to $\widetilde{X}_2$ and by $\widetilde{A}_2$ its generator. Then $0 \in \rho(\widetilde{A}_2)$.

Let $B \in \mathcal{L}(X_2, X_1)$ be given by $Bf = f(0)$ and consider the operator

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix},$$

on $X = X_1 \times X_2$ which generates the semigroup

$$T(t) = \begin{pmatrix} T_1(t) & T_{12}(t) \\ 0 & T_2(t) \end{pmatrix},$$

where $T_{12}(t)f = \int_0^t T_1(t-s)B\,T_2(s)f\,ds = \int_0^t T_1(t-s)\,f(s)\,ds$, so that $u(t) = T_1(t)x + T_{12}(t)f = (T(t)(x,f))_1$. Denote by $\widetilde{T}$ the restriction of T to $\widetilde{X} := X_1 \oplus \widetilde{X}_2$.
It is easy to see that $R(0,\widetilde{A}_2)$ is given by

$$(R(0,\widetilde{A}_2)f)(s) = -1/\tau \int_0^\tau t\,f(s+t)\,dt \qquad (s \in \mathbb{R})$$

for all $f \in \widetilde{X}_2$. Thus

$$BR(0,\widetilde{A}_2) = -M_1. \tag{4.1}$$

Proof of Theorem 4. 1. It follows from Theorem 3.2 that $\widetilde{T}$ is $C-$ergodic. Moreover,

$$\begin{aligned} C - \lim_{t\to\infty} \widetilde{T}(t)_{12} &= A - \lim_{t\to\infty} (\widetilde{T}(t))_{12} = \lim_{\lambda\downarrow 0} \lambda R(\lambda, A_1)BR(\lambda,\widetilde{A}_2) \\ &= PBR(0,\widetilde{A}_2) = -PM_1. \end{aligned}$$

This proves the first claim.
If $0 \in \rho(A_1)$, then Theorem 3.2 implies that T is $C-$ergodic. Then

$$\begin{aligned} C - \lim_{t\to\infty} T(t)_{12} &= A - \lim_{t\to\infty} T_{21}(t) = \lim_{\lambda\downarrow 0} R(\lambda, A_1)B\lambda R(\lambda, A_2) \\ &= R(0,A_1)B\,Q_0 = R(0,A_1)M_0. \end{aligned}$$

■

Theorem 4.3 follows in the same way from Theorem 3.3.

Remark 4.4. In a different context, the analogous semigroup T on $X_1 \oplus X_2$, $X_2 = UCB(\mathbb{R}, X_1)$, has been used by Phong [10] in order to investigate the inhomogeneous Cauchy problem related to A_1.

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