Operator Theory:
Advances and Applications, Vol. 75

PRINCIPAL EIGENVALUES AND PERTURBATION

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Dedicated to Professor A.C. Zaanen on the occasion of his 80th birthday.

INTRODUCTION. In a classical article, Hess and Kato [HK] study the problem

$$\begin{cases} Au + \lambda mu = 0 \\ 0 \leq u \in D(A), \quad u \neq 0, \end{cases} \tag{0.1}$$

where A is a strongly elliptic operator on a bounded open set Ω of $\mathbf{R}^n$ with Dirichlet boundary conditions and m is a continuous bounded function on Ω. They show that there exists a unique $\lambda > 0$ such that the problem (0.1) has a solution.

This result can be reformulated by saying that there is a unique $\lambda > 0$ such that the spectral bound $s(A + \lambda m)$ of the operator $A + \lambda m$ on $C_0(\Omega)$ is 0.

The motivation of Hess and Kato was to investigate bifurcation of a nonlinear problem and (0.1) is obtained by linearization.

There is also another reason to study the spectral bound : in many cases it determines the asymptotic behavior of the semigroup. In particular, if $s(A) = 0$, then, under suitable hypotheses, the semigroup generated by A converges to a rank-1-projection. We show a typical result of this sort in the first section.

In the second section we assume that A generates a positive semigroup such that $s(A) < 0$ and consider a positive compact perturbation $B : D(A) \to E$. We show that there exists a unique $\lambda > 0$ such that $s(A + \lambda B) = 0$. Continuity properties of the spectral bound play a role in this context. They are presented in the appendix.

In Section 3 we consider the case where m is no longer positive. A beautiful theorem due to Kato [K2] says that spectral bound $s(A + \lambda m)$ and type $\omega(A + \lambda m)$ are convex functions of $\lambda \in \mathbf{R}$ if $E = C_0(\Omega)$ and $m \in C^b(\Omega)$ or $E = L^p(\Omega)$ and $m \in L^\infty(\Omega)$.

We extend this result to the case where A is the generator of a positive C_0−semigroup on an arbitrary Banach lattice and m is in the centre of E.

In this context the lattice structure, and Kakutani's theorem in particular, play an important role. First of all, it allows one to define multiplication operators abstractly (in the form of the centre, see Section 3 and Zaanen [Z1], [Z2]). Secondly, in the proof of the convexity theorem we use an interesting approximation property in Banach lattices due to B. Walsh [W]. We give the proof of this apparently not very well-known result in Appendix B. Thus our presentation of the convexity theorem is selfcontained.

1. PERRON-FROBENIUS THEORY AND PRINCIPAL EIGENVALUES.

The aim of Perron-Frobenius theory is to deduce asymptotic behavior from the location of the spectrum. We describe one particular case.

Let E be a complex Banach lattice, e.g., $E = L^p$, $1 \le p < \infty$, or $E = C_0(\Omega)$, the space of all continuous functions on a locally compact space Ω which vanish at infinity.

Let A be the generator of a positive semigroup $T = (T(t))_{t \ge 0}$ on E (by this we mean a C_0−semigroup throughout). We denote by $\sigma(A)$ the spectrum of A and by

$$s(A) = \sup \{\mathrm{Re}\, \lambda : \lambda \in \sigma(A)\} \tag{1.1}$$

the **spectral bound** of A. Then $s(A) < \infty$, and if $s(A) > -\infty$, then $s(A) \in \sigma(A)$. Moreover, $R(\mu, A) := (\mu - A)^{-1} \ge 0$ whenever $\mu > s(A)$ and conversely, if $\mu \in \rho(A) \cap \mathbf{R}$ such that $R(\mu, A) \ge 0$, then $\mu > s(A)$.

The spectral bound is the analogue of the spectral radius for unbounded operators. This is made precise by the following formula

$$r(R(\mu, A)) = \frac{1}{\mu - s(A)} \quad (\mu > s(A)), \tag{1.2}$$

where $r(B)$ denotes the spectral radius of B. In particular, $s(A) = -\infty$ if and only if $R(\mu, A)$ is quasi-nilpotent. See [N] for these results. We use the following notation. An element $u \in E_+$ is called a **quasi-interior point** if $\varphi \in E'_+$, $\langle u, \varphi \rangle = 0$ implies $\varphi = 0$. Thus, u is quasi-interior if and only if the principal ideal $E_u = \{f \in E : \exists\, n \in \mathbf{N} \quad |f| \leq nu\}$ is dense in E. If $E = L^p$ $(1 \leq p < \infty)$ this is equivalent to $u > 0$ a.e. A positive linear form $\varphi \in E'_+$ is **strictly positive** if $\langle f, \varphi \rangle > 0$ for all $f \in E_+, f \neq 0$. We write $\varphi \gg 0$.

An operator $B \in \mathcal{L}(E)_+$ is **strictly positive** (we write $B \gg 0$) if Bf is a quasi-interior point for all $f \in E_+$, $f \neq 0$.

We say that the semigroup T is **irreducible** if $R(\mu, A) \gg 0$ for all $\mu > s(A)$.

The following definition is central for our purposes.

DEFINITION 1.1. We say that 0 is a **principal eigenvalue** if
(a) there exists $\epsilon > 0$ such that $\{\lambda \in \sigma(A) : \operatorname{Re} \lambda > -\epsilon\} = \{0\}$ and
(b) 0 is a pole of the resolvent.

Note that $s(A) = 0$ whenever 0 is a principal eigenvalue. If 0 is a principal eigenvalue and T satisfies a regularity condition, then T has a very specific asymptotic behavior : it converges to a rank-1-projection.

We say that T is **eventually norm continuous** if there exists $t_0 > 0$ such that $\lim_{t \downarrow 0} \|T(t_0 + t) - T(t_0)\| = 0$; e.g., a holomorphic semigroup satisfies this condition.

THEOREM 1.2. *Let T be irreducible and eventually norm continuous. Assume that 0 is a principal eigenvalue. Then there exists a unique quasi-interior point $u_0 \in D(A)$ such that $Au_0 = 0$, $\|u_0\| = 1$ and a unique $0 \ll \varphi_0 \in D(A')$ satisfying $A'\varphi_0 = 0$ and $\langle u_0, \varphi_0 \rangle = 1$. Moreover there exist $\delta > 0, M \geq 0$ such that*

$$\|T(t) - \varphi_0 \otimes u_0\| \leq Me^{-\delta t} \quad (t \geq 0). \tag{1.2}$$

We call u_0 the **principal eigenvector** *of A and φ_0 the* **principal eigenfunctional** *of A'. By $P = \varphi_0 \otimes u_0$ we mean the rank-1-operator $Pf = \langle f, \varphi_0 \rangle u_0$.*

REMARK 1.3. Note that 0 is the only eigenvalue with a positive eigenvector. In fact, assume that $0 \leq v \in D(A)$, $Av = \mu v$. Then $\mu\langle v, \varphi_0 \rangle = \langle Av, \varphi_0 \rangle = \langle v, A'\varphi_0 \rangle = 0$. Hence $\mu = 0$ or $\langle v, \varphi_0 \rangle = 0$ (which implies $v = 0$ since $\varphi_0 \gg 0$).

Let $f \in D(A)$. Then $u(t) = T(t)f$ is the unique solution of the problem

$$(P) \qquad \begin{cases} u \in C^1([0,\infty); E); \\ u(t) \in D(A) \quad (t \geq 0); \\ u'(t) = Au(t) \quad (t \geq 0); \\ u(0) = f. \end{cases}$$

Note that in the situation of Theorem 1.2 $T(t)u_0 \equiv u_0$. The estimate (1.2) implies that every solution converges to the stationary point u_0, i.e.

$$\lim_{t\to\infty} T(t)f = \langle f, \varphi_0 \rangle u_0.$$

Next we give a criterion for 0 to be a principal eigenvalue.

THEOREM 1.4. *Assume that T is irreducible and holomorphic. Assume that $s(A) = 0$. If*
(a) $R(\mu, A)$ is compact for (some) $\mu \in \rho(A)$ or
(b) $s(A - B) < 0$ for some compact operator $B : D(A) \to E$,
then 0 is a principal eigenvalue.

Here we consider $D(A)$ as a Banach space with the graph norm $\|f\|_A := \|f\| + \|Af\|$. Then $R(\mu, A)$ is an isomorphism from E onto $D(A)$ ($\mu > s(A)$). To say that $B : D(A) \to E$ is compact is equivalent to $BR(\mu, A) : E \to E$ being compact for one (equivalently all) $\mu \in \rho(A)$.

Note that, if $B : D(A) \to E$ is compact, then by a result of Desch-Schappacher [DS1] $A - B$ generates a holomorphic semigroup.

Theorems 1.2 and 1.4 are variants of the Perron-Frobenius theory developed in [N]. An essential argument is the cyclicity of the boundary spectrum, a result which is due to G. Greiner and is analogous to results of Lotz for bounded positive operators and of Perron-Frobenius for positive matrices (see [N] and [S] for details and bibliographical notes).
We give the proofs (based on [N]) in order to be complete.

PROOF OF THEOREM 1.2. It follows from [N, C-III, Prop. 3.5, p. 310] that 0 is a pole of order 1 and that the residue is of the form $P = \varphi_0 \otimes u_0$ with φ_0, u_0 as in the statement of the theorem. By [N, A-III, Theorem 3.3], P is the spectral projection corresponding to $\{0\}$. Let A_1 be the generator of $T_1(t) = T(t)_{|F}$ with $F = (I - P)E$. Then $s(A_1) < 0$. Since T_1 is eventually norm continuous one has $s(A_1) = \omega(A_1)$ (the type of T_1). Thus, if δ is such that $s(A_1) < -\delta < 0$, then there exists $M \geq 0$ such that $\|T_1(t)\| \leq Me^{-\delta t} \quad (t \geq 0)$. This implies (1.2). $\diamondsuit$

PROOF OF THEOREM 1.4. Let $\epsilon > 0$ be arbitrary in the first case and $s(A - B) < -\epsilon < 0$ in the second. It follows from [K1, IV §5.6, p. 242-244] that the set $H = \{\mu \in \sigma(A) : \text{Re } \mu \geq -\epsilon\}$ consists of isolated points which are poles of the resolvent of A. Since T is eventually norm-continuous the set H is compact [N, A-II, Theorem 1.20, p. 38]. Thus H is finite. By cyclicity [N, C-III, Theorem 3.12, p. 315] $\sigma(A) \cap i\mathbf{R}$ is unbounded or reduced to $\{0\}$. Since H is compact, it follows that $\sigma(A) \cap i\mathbf{R} = \{0\}$. Hence 0 is a principal eigenvalue. $\diamondsuit$

REMARK. Theorem 1.4 remains true if T is merely eventually norm-continuous. In that case $A - B$ is not necessarily the generator of a semigroup in the case (b), see Desch-Schappacher [DS2].

2. POSITIVE COMPACT PERTURBATION.

Let A be the generator of a positive irreducible holomorphic semigroup on a Banach lattice E. We consider perturbations of the form $A + \lambda B$ $(\lambda \geq 0)$, where $B : D(A) \to E$ is linear, compact ($D(A)$ being considered with the graph norm) and positive (i.e. $0 \leq f \in D(A)$ implies $Bf \geq 0$).

Then $A + \lambda B$ generates a holomorphic semigroup for all $\lambda > 0$ by a result of Desch-Schappacher [DS1] (see also [KS]). This semigroup is positive and irreducible (see proof of Theorem 2.4).

We assume throughout that $B \neq 0$. Under these conditions we show the following.

THEOREM 2.1. *Assume in addition to the hypotheses made above that $s(A) < 0$. Then there exists a unique $\lambda_0 > 0$ such that $s(A + \lambda_0 B)$ is a principal eigenvalue.*

COROLLARY 2.2. *Under the hypotheses of Theorem 2.1 there exists a* *unique* $\lambda_0 > 0$ *such that the problem*

$$\begin{cases} 0 \le u \in D(A), & \|u\| = 1 \\ Au + \lambda_0 Bu = 0 \end{cases}$$

has a solution. Moreover, this solution u is unique and u is a quasi interior point.

PROOF. This follows from Theorems 2.1, 1.2 and Remark 1.3. ◇

The proof of Theorem 2.1 consists of several steps.

We investigate the function

$$s : [0, \infty) \to [s(A), \infty)$$
$$\lambda \mapsto s(A + \lambda B).$$

Since B is positive, the function s is increasing :

$$0 \le \lambda_1 \le \lambda_2 \text{ implies } s(\lambda_1) \le s(\lambda_2). \tag{2.1}$$

PROOF. Let $\mu >$ max $\{s(\lambda_1), s(\lambda_2)\}$. Then

$$R(\mu, A + \lambda_2 B) - R(\mu, A + \lambda_1 B) = R(\mu, A + \lambda_2 B)(\lambda_2 - \lambda_1) B R(\mu, A + \lambda_1 B) \ge 0.$$

Hence $0 \le R(\mu, A + \lambda_1 B) \le R(\mu, A + \lambda_2 B)$ and by (1.2),

$$\frac{1}{\mu - s(A + \lambda_1 B)} = r(R(\mu, A + \lambda_1 B)) \le r(R(\mu, A + \lambda_2 B)) = \frac{1}{\mu - s(A + \lambda_2 B)}.$$ ◇

Next we show that

$$\lim_{\lambda \to \infty} s(A + \lambda B) = \infty. \tag{2.2}$$

This depends heavily on irreducibility (see Remark 2.7 below). In fact, we use the following deep theorem due to de Pagter [P] :

THEOREM 2.3. *Let K be a positive compact operator on E. If K is irreducible, then* $r(K) > 0$.

REMARK. A positive operator $K \in \mathcal{L}(E)$ is called **irreducible** if $(e^{tK})_{t \ge 0}$ is irreducible, i.e. if $R(\mu, K) \gg 0$ for $\mu > r(K)$. From the Neumann series one sees that K is irreducible whenever $K \gg 0$.

PROOF OF (2.2). Assume that there exists μ_0 such that $s(A + \lambda B) \le \mu_0$ for all $\lambda > 0$. Let $\mu > \mu_0$. Then for $\lambda > 0$,

$$R(\mu, A + \lambda B) = R(\mu, A) + \lambda R(\mu, A + \lambda B) B R(\mu, A) \ge \lambda R(\mu, A) B R(\mu, A)$$

(see the proof of (2.1)). Hence

$$\frac{1}{\mu-\mu_0} \geq \frac{1}{\mu - s(A+\lambda B)} = r(R(\mu, A+\lambda B)) \geq \lambda r(K)$$

for all $\lambda > 0$, where $K = R(\mu, A)BR(\mu, A)$. Consequently, $r(K) = 0$. However, K is compact and strictly positive, and thus irreducible. This is impossible by de Pagter's theorem. ◇

Next we show that

(2.3) s is continuous on $[0, \infty)$.

PROOF. We know that s is increasing and $s(0) = s(A) < 0$. Let $\lambda_0 := \sup \{\lambda \geq 0 : s(\lambda) = s(A)\}$. It follows from Proposition A1 a) in Appendix A that $\overline{\lim_{\lambda \downarrow \lambda_0}} s(\lambda) \leq s(\lambda_0) = s(A)$. Thus s is continuous at λ_0 and on $[0, \lambda_0]$. If $\lambda > \lambda_0$, $s(A+\lambda B) > s(A)$. Thus $s(A+\lambda B)$ is an isolated point in the spectrum of $A+\lambda B$. So continuity in λ follows from Proposition A1 b). ◇

It follows from (2.2), (2.3) and the assumption $s(A) < 0$ that there exists $\lambda_0 > 0$ such that $s(A+\lambda_0 B) = 0$. Then 0 is a principal eigenvalue of $A+\lambda_0 B$ by Theorem 1.4.

Finally, it follows from Proposition A2 that

(2.4) $s(\lambda_1) < s(\lambda_2)$ whenever $\lambda_0 < \lambda_1 < \lambda_2$.

This shows uniqueness of λ_0 and the proof of Theorem 2.1 is finished.

More generally, our arguments show the following.

THEOREM 2.4. *Let A be a resolvent positive densely defined operator (see Appendix A) such that*

(a) $s(A) < 0$;

(b) $\sup_{\mu \geq 0} \|\mu R(\mu, A)\| =: M < \infty$;

(c) $R(\mu, A) \gg 0 \quad (\mu > 0)$.

Let $B : D(A) \to E$ be positive, compact and $\neq 0$.

Then $A+\lambda B$ is resolvent positive for all $\lambda \geq 0$. The function $s(\lambda) = s(A+\lambda B)$ is continuous on $[0,\infty)$, strictly increasing and $\lim_{\lambda\to\infty} s(\lambda) = \infty$. In particular, there exists a unique $\lambda_0 > 0$ such that $s(A+\lambda_0 B) = 0$.

PROOF. a) We show that $A+\lambda B$ is resolvent positive for all $\lambda \geq 0$. In fact, an argument given by Desch-Schappacher [DS1] can be adapted to the situation considered here. We can assume $\lambda = 1$. Since $s(A) < 0$, one may consider the equivalent norm $\|x\|_{D(A)} := \|Ax\|$ on $D(A)$. It follows from (b) that

$$\lim_{\mu\to\infty} \|R(\mu,A)x\|_{D(A)} = \lim_{\mu\to\infty} \|AR(\mu,A)x\| = \lim_{\mu\to\infty} \|\mu R(\mu,A)x - x\| = 0 \quad (x \in E).$$

Since

$$\sup_{\mu\geq 0} \|R(\mu,A)\|_{\mathcal{L}(E,D(A))} = \sup_{\mu\geq 0} \|AR(\mu,A)\| = \sup_{\mu\geq 0} \|\mu R(\mu,A) - I\| \leq M+1 < \infty,$$

it follows that $\lim_{\mu\to\infty} \|R(\mu,A)x\|_{D(A)} = 0$ uniformly for x in compact subsets of E. Since $B : D(A) \to E$ is compact, we conclude that

$$\|R(\mu,A)B\|_{\mathcal{L}(D(A))} \to 0 \qquad (\mu \to \infty).$$

Hence there exists $\mu_0 > 0$ such that $\|R(\mu,A)B\|_{\mathcal{L}(D(A))} \leq \frac{1}{2}$ whenever $\mu \geq \mu_0$. This implies that $(I - R(\mu,A)B)$ is invertible in $\mathcal{L}(D(A))$ and

$$(I - R(\mu,A)B)^{-1} = \sum_{n=0}^{\infty} (R(\mu,A)B)^n \geq 0.$$

Hence

$$(\mu - (A+B))^{-1} = [(\mu - A)(I - R(\mu,A)B)]^{-1} = (I - R(\mu,A)B)^{-1} R(\mu,A)$$

exists and is positive for all $\mu \geq \mu_0$.

b) Note that

$$R(\mu, A+B) = R(\mu,A) + \sum_{n=1}^{\infty} (R(\mu,A)B)^n R(\mu,A) \geq R(\mu,A) \gg 0 \qquad (\mu > \mu_0).$$

The remaining arguments are the same as above. ◇

REMARK 2.5. The proof also shows that

$$\overline{\lim_{\mu\to\infty}} \|\mu R(\mu;\, A+\lambda B)\| < \infty \qquad (\lambda > 0).$$

REMARK 2.6. Also in the situation of Theorem 2.4 we can conclude that there exists a unique $\lambda_0 > 0$ such that the problem

$$\begin{cases} (A+\lambda_0 B)u = 0 \\ u \in D(A), u \geq 0, \|u\| = 1 \end{cases} \tag{2.5}$$

has a solution. And, as before, this solution u is unique and u is a quasi interior point. However, we do not know whether, in the general situation considered in Theorem 2.4, 0 is a principal eigenvalue. As before, the boundary spectrum $\sigma(A+\lambda_0 B) \cap i\mathbf{R}$ is cyclic (by the proofs given in [N]). But it is not clear whether $\sigma(A+\lambda_0 B) \cap i\mathbf{R}$ is bounded (which is needed to conclude that $\sigma(A+\lambda_0 B) \cap i\mathbf{R} = \{0\}$).

REMARK 2.7. In Theorem 2.1 and 2.4 irreducibility is essential. In fact, if $E = \mathbf{R}^2$, $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $s(A+\lambda B) = -1$ for all $\lambda > 0$.

3. KATO'S CONVEXITY THEOREM.

If B is a multiplication operator, then the function $\lambda \mapsto s(A+\lambda B)$ is continuous even if no assumption of compactness and on the sign of B is made. This follows from the following theorem due to T. Kato [K2].

THEOREM 3.1. *Let A be the generator of a positive semigroup on $E = L^p$ $(1 \leq p < \infty)$ or $E = C_0(\Omega)$ (Ω locally compact).*

Then the functions

$$m \mapsto s(A+m) \quad \textit{and}$$
$$m \mapsto \omega(A+m)$$

from L^∞ (resp. $C^b(\Omega)$) into $[-\infty, \infty)$ are convex.

Here we identify $m \in L^\infty$ (resp. $m \in C^b(\Omega)$) with the multiplication operator $f \mapsto mf$ on L^p (resp. $C_0(\Omega)$). Moreover, we use the following definition of convexity :

A function s defined on a vector space Z with values in $[-\infty, \infty)$ is called **convex** if either $s(m) = -\infty$ for all $m \in Z$ or $s(m) > -\infty$ for all $m \in Z$ and $s(\theta m_1 + (1-\theta)m_2) \leq \theta s(m_1) + (1-\theta)\, s(m_2)$ whenever $0 \leq \theta \leq 1$, $m_1, m_2 \in Z$.

The case $C_0(\Omega)$ is of particular interest. It seems to play a special role concerning de Pagter's theorem. In fact, in this case, it is very easy to see that every positive irreducible operator on $C_0(\Omega)$ has strictly positive spectral radius even without compactness assumptions (cf. [S, V. 6.1], [N, B-III. Prop. 3.5a] and the proof below). Thus, as a consequence of Kato's theorem, one obtains the same conclusions as in Theorem 2.4 in the case where $E = C_0(\Omega)$ and B is a positive multiplication operator :

COROLLARY 3.2. *Let A be the generator of a positive irreducible semigroup on $C_0(\Omega)$, Ω locally compact. Assume that $s(A) < 0$. Let $0 \leq m \in C^b(\Omega)$, $m \neq 0$. Then there exists a unique $\lambda > 0$ such that $s(A + \lambda m) = 0$.*

PROOF. a) If K is a strictly positive operator on $C_0(\Omega)$, then $r(K) > 0$. In fact, let $f \in C_0(\Omega)$ be of compact support such that $f \geq 0$, $\|f\| = 1$. Since $Kf \gg 0$, there exists $c > 0$ such that $Kf \geq cf$. Consequently, $K^n f \geq c^n f$ $(n \in \mathbf{N})$, and so $\|K^n\| \geq c^n$. It follows that $r(K) \geq c$.

b) Now the proof of (2.2) shows that $\lim_{\lambda \to \infty} s(A + \lambda m) = \infty$. Since the function $\lambda \mapsto s(A + \lambda m)$ is convex and $s(A) < 0$, the claim follows. $\Diamond$

It has been pointed out by Kato, that Theorem 3.1 remains true on similar spaces where multiplication operators can be defined. The purpose of this section is to show that, indeed, the theorem remains true on arbitrary Banach lattices if multiplication operators are replaced by operators in the centre.

Let E be an arbitrary real Banach lattice. By

$$Z(E) := \{T \in \mathcal{L}(E) : \exists\, c \geq 0 \quad |Tx| \leq c|x| \quad (x \in E)\}$$

we denote **the centre of** E. It is remarkable that $Z(E)$ can be described purely in terms of orthogonality in the lattice sense :

$$Z(E) = \{T : E \to E \text{ linear } : |x| \wedge |y| = 0 \Rightarrow |Tx| \wedge |y| = 0\},$$

see Zaanen [Z2] for proofs and further results.

For our purposes the following theorem is important. It is a consequence of Kakutani's representation theorem (see Zaanen [Z2] and Meyer-Nieberg [MN]).

THEOREM 3.3. *The space $Z(E)$ is a closed subalgebra of $\mathcal{L}(E)$. Moreover, there exists a compact space K and a lattice and algebra isomorphism $\phi : C(K) \to Z(E)$.*

In the concrete cases considered in Theorem 3.1 the centre consists precisely of all multiplication operators :

EXAMPLE 3.4. a) Let Ω be locally compact, $E = C_0(\Omega)$. Then $Z(E)$ is isomorphic to $C^b(\Omega)$ identifying elements of $C^b(\Omega)$ with multiplication operators on E.

The space $C^b(\Omega)$ being isomorphic (as Banach lattice and algebra) to the space $C(\beta\Omega)$, this yields a proof of Theorem 3.3 in this concrete case.

b) Let (Ω, Σ, μ) be a σ–finite measure space and $E = L^p(\Omega)$, $1 \leq p \leq \infty$. Then $Z(E)$ is isomorphic to L^∞ (by action as multiplication operators). Again L^∞ is isomorphic to a space $C(K)$, K compact.

The results reported in Example 3.4 are due to Zaanen [Z1], see also Zaanen [Z2].

Now we can formulate the following more general version of Kato's theorem :

THEOREM 3.5. *Let A be the generator of a positive semigroup on a Banach lattice E. Then the functions*

$$M \mapsto s(A+M) \quad \textit{and}$$
$$M \mapsto \omega(A+M)$$

from $Z(E)$ into $[-\infty, \infty)$ are convex.

For the proof we use the following defintion. Let $-\infty \leq a < b \leq \infty$; $\mathcal{L}(E)_+ = \{Q \in \mathcal{L}(E) : Q \geq 0\}$.

DEFINITION 3.6. (a) A function $q : (a,b) \to [0,\infty)$ is **superconvex** if $\log q$ is convex.

(b) A function $Q : (a,b) \to \mathcal{L}(E)_+$ is **superconvex** if $\langle Q(\cdot)x, x'\rangle$ is superconvex for all $x \in E_+$, $x' \in E'_+$.

The following properties are immediately clear.

LEMMA 3.7. *(a) Let $Q_1, Q_2 : (a,b) \to \mathcal{L}(E)_+$ be superconvex and let $\alpha, \beta \geq 0$. Then $\alpha Q_1 + \beta Q_2$ is superconvex.*

(b) Let $Q_n : (a,b) \to \mathcal{L}(E)_+$ be superconvex. Assume that $Q_n(\lambda)$ converges to $Q(\lambda)$ in the weak operator topology for all $\lambda \in (a,b)$. Then $Q(\lambda)$ is superconvex.

It is clear that the product of two numerical-valued superconvex functions is super-convex. Walsh's approximation theorem (Theorem B.1) allows one to extend this to operator-valued functions :

PROPOSITION 3.8. *Let $Q_1, Q_2 : (a,b) \to \mathcal{L}(E)_+$ be superconvex. Then $\lambda \mapsto Q_1(\lambda)\, Q_2(\lambda)$ is superconvex.*

PROOF. (a) Let $y \in E_+, y' \in E'_+$. Then for all $x \in E_+$, $x' \in E'_+$, $\lambda \mapsto \langle Q_1(\lambda)(y' \otimes y)\, Q_2(\lambda)x, x'\rangle = \langle Q_2(\lambda)x, y'\rangle\, \langle Q_1(\lambda)y, x'\rangle$ is superconvex.

(b) By Theorem B1 there exists a net R_α in P (see Appendix B for the definition) which converges strongly to the identity. It follows from (a) and Lemma 3.7 that $Q_1(\cdot)\, R_\alpha\, Q_2(\cdot)$ is superconvex for all α. Hence the limit $Q_1(\cdot)\, Q_2(\cdot)$ is superconvex by Lemma 3.7. $\diamondsuit$

COROLLARY 3.9. *If $Q : (a,b) \to \mathcal{L}(E)_+$ is super-convex, then the spectral radius $r(Q(\cdot))$ is superconvex.*

PROOF. Let $B_+ = \{x \in E_+ : \|x\| \le 1\}$, $B'_+ = \{x' \in E'_+ : \|x'\| \le 1\}$. By Proposition 3.8, the function $\lambda \mapsto \langle Q(\lambda)^n x, x'\rangle$ is superconvex for all $x \in B_+$, $x' \in B'_+$. Hence

$$\begin{aligned}\lambda \mapsto r(Q(\lambda)) &= \lim_{n\to\infty} \|Q(\lambda)^n\|^{1/n} \\ &= \lim_{n\to\infty} \Big(\sup_{\substack{x \in B_+ \\ x' \in B'_+}} \langle Q(\lambda)^n\, x, x'\rangle \Big)^{1/n}\end{aligned}$$

is superconvex. $\diamondsuit$

PROOF OF THEOREM 3.5. Let $M \in Z(E)$.

(a) We show that the function $\lambda \mapsto e^{\lambda M} : \mathbf{R} \to \mathcal{L}(E)$ is superconvex. Let $x \in E_+$, $x' \in E'_+$. There exists an algebra isomorphism ϕ from $Z(E)$ onto $C(K)$. In particular, $e^{\phi(N)} = \phi(e^N) \quad (N \in Z(E))$.

By the Riesz representation theorem there exists a positive Borel measure μ on K such that

$$\langle Nx, x'\rangle = \int_K \phi(N) d\mu \quad (N \in Z(E)).$$

Let $\lambda_1, \lambda_2 \in \mathbf{R}, 0 < \theta < 1, M_1 = \lambda_1 M, M_2 = \lambda_2 M$. Then by Hölder's inequality,

$$\begin{aligned}\langle e^{\theta M_1 + (1-\theta)M_2}\, x, x'\rangle &= \int \phi(e^{\theta M_1 + (1-\theta)M_2})\, d\mu \\ &= \int (e^{\phi(M_1)})^{\theta}.(e^{\phi(M_2)})^{1-\theta}\, d\mu \\ &\leq (\int e^{\phi(M_1)}\, d\mu)^{\theta}.(\int e^{\phi(M_2)}\, d\mu)^{1-\theta} \\ &= (\langle e^{M_1}\, x, x'\rangle)^{\theta}(\langle e^{M_2}\, x, x'\rangle)^{1-\theta}.\end{aligned}$$

(b) Let $t > 0$, $M \in Z(E)$. By (a) the function $\lambda \mapsto e^{\lambda t M}$ is superconvex. It follows from Lemma 3.7 and Proposition 3.8 that also the function $\lambda \mapsto e^{t(A+\lambda M)} = s - \lim_{n\to\infty} (e^{t/nA}\, e^{\lambda t/nM})^n$ is superconvex.

(c) It follows from (b) and Corollary 3.9 that the function $\lambda \mapsto r(e^{(A+\lambda M)}) = e^{\omega(A+\lambda M)}$ is superconvex. Hence $\omega(A + \lambda M)$ is convex in λ.

(d) Let $\lambda_0 > 0$. We show that $s(A+\lambda M)$ is convex in $\lambda \in (-\lambda_0, \lambda_0)$. Let $w > \omega(A)+\lambda_0 \|M\|$. There exists $c > 0$ such that $\|e^{t(A+\lambda M)}\| \leq c\, e^{wt}$ $(t \geq 0)$ for all $\lambda \in (-\lambda_0, \lambda_0)$.

It follows from (b) and Lemma 3.7 that

$$\lambda \mapsto R(\mu, A + \lambda M) = \int_0^{\infty} e^{-\mu t}\, e^{t(A+\lambda M)}\, dt$$

is super convex on $(-\lambda_0, \lambda_0)$ whenever $\mu > w$.

By Corollary 3.9, $r(R(\mu, A + \lambda M))$ is super convex and a fortiori convex in $\lambda \in (-\lambda_0, \lambda_0)$. Consequently, $\mu^2\, r(R(\mu, A+\lambda M)) - \mu = \frac{\mu^2}{\mu - s(A+\lambda M)} - \mu = s(A + \lambda M)(1 - \frac{s(A+\lambda M)}{\mu})^{-1}$ is convex in $\lambda \in (-\lambda_0, \lambda_0)$ for all $\mu > w$. Letting $\mu \to \infty$ one concludes that $s(A + \lambda M)$ is convex.

e) Let $M_1, M_2 \in Z(E)$, $0 \leq \theta \leq 1$. It follows from (d) that $\lambda \mapsto s(A + B(\lambda))$ is convex where $B(\lambda) = M_2 + \lambda(M_1 - M_2)$. In particular,

$$\begin{aligned}&s(A + \theta M_1 + (1-\theta)M_2) = s(A + B(\theta\, 1 + (1-\theta)0)) \\ &\leq \theta s(A + B(1)) + (1-\theta)\, s(A + B(0)) \\ &= \theta s(A + M_1) + (1-\theta)\, s(A + M_2).\end{aligned}$$

This proves convexity of $M \mapsto s(A + M)$.
In the same way (c) implies convexity of $M \mapsto \omega(A + M)$. $\diamond$

REMARK : As pointed out by Kato [K2, p. 268], it follows from Theorem B1 that (weak) superconvexity as defined here (Definition 3.6 (b)) is equivalent to (strong) superconvexity in the sense of [K2, p. 26]. This simplifies the proof (but is restricted to Banach lattices). Otherwise, the arguments given here are those of Kato besides (a) in the proof of Theorem 3.5 which establishes the link with the abstract setting.

For positive perturbations which are not multiplication operators, the function $\lambda \mapsto s(A + \lambda B)$ is not convex, in general :

EXAMPLE 3.9. Let $E = \mathbf{R}^2$, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $s(A + \lambda B) = \sqrt{2 + \lambda}$ is a concave function. $\diamond$

REMARK 3.10. We have considered a real Banach lattice E. Of course, the spectrum is understood with respect to the corresponding complexifications. However, for a resolvent positive operator A on E one always has $s(A) = \inf \{\mu \in \mathbf{R} : (\lambda - A)^{-1}$ exists and $(\lambda - A)^{-1} \geq 0$ for all $\lambda > \mu\}$; this expression involves only the real space and real operators.

APPENDIX A. THE SPECTRAL BOUND OF RESOLVENT POSITIVE OPERATORS

Let A be a Banach lattice. An operator A on E is called **resolvent positive** if there exists $\lambda_0 \in \mathbf{R}$ such that $(\lambda_0, \infty) \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \lambda_0$. The spectral bound of such an operator can be defined as in Section 1 and has the same properties (see [A]). We need the following continuity property of the spectral bound.

PROPOSITION A1. *Let A_n, A be resolvent positive operators. Assume that there exists $\mu >$ sup $(\{s(A_n) : n \in \mathbf{N}\} \cup \{s(A)\})$ such that $\lim_{n\to\infty} \|R(\mu, A_n) - R(\mu, A)\| = 0$. Then*

a) $\overline{\lim_{n\to\infty}}\, s(A_n) \leq s(A)$ and,

b) if $s(A)$ is an isolated point in $\sigma(A)$, then $\lim_{n\to\infty} s(A_n) = s(A)$.

PROOF. a) This follows from the upper continuity of the spectral radius [Au, Théorème 3, p. 6] and (1.2).

b) The assertion follows from a) and (1.2) together with the corresponding result for bounded operators [Au, Théorème 4, p. 8]. $\diamond$

Next we formulate a result on strict monotonicity of the spectral bound. The argument is the same as in [AB, Theorem 1.3].

PROPOSITION A2. *Let A_1, A_2 be resolvent positive operators with dense domain such that*

$$0 \ll R(\lambda, A_1) \leq R(\lambda, A_2) \text{ for } \lambda > \max \{s(A_1), s(A_2)\}.$$

Assume that
(a) $A_1 \neq A_2$ and
(b) $s(A_i)$ is a pole of the resolvent of A_i, $i = 1, 2$.

Then $s(A_1) < s(A_2)$.

APPENDIX B. WALSH'S APPROXIMATION PROPERTY AND SUPERCONVEXITY.

Let E be a real Banach lattice. By P we denote the cone of all positive operators on E which are of the form $\sum_{i=1}^{n} x_i' \otimes x_i$ (i.e. $x \mapsto \sum_{i=1}^{n} \langle x, x_i' \rangle \, x_i$) with $x_i \in E_+$, $x_i' \in E_+'$ $i = 1, \ldots, n$.

The following approximation property is due to B. Walsh [W].

THEOREM B1. *The identity I is in the closure of P with respect to the strong operator topology.*

For completeness we include Walsh's proof. It depends on

LEMMA B2. *The assertion of Theorem B1 holds iff for every operator $R = \sum_{i=1}^{n} x_i' \otimes x_i$, $R \geq 0$ implies $\sum_{i=1}^{n} \langle x_i, x_i' \rangle \geq 0$.*

PROOF. The dual space of $\mathcal{L}_s(E)$ is $E' \otimes E$, the duality being defined by $\langle T, x' \otimes x \rangle = \langle Tx, x' \rangle$ (see [S1, p. 139]).

a) Assume that $I \notin \overline{P}$. Then there exists $R \in E' \otimes E$ such that $\langle p, R \rangle \geq 0$ for all $p \in P$ but $\langle I, R \rangle < 0$. Write $R = \sum_{j=1}^{n} x_j' \otimes x_j$. Then $0 > \langle I, R \rangle = \sum_{j=1}^{n} \langle x_j, x_j' \rangle$.
On the other hand let $p = y' \otimes y$ with $y' \in E_+'$, $y \in E_+$. Then $p \in P$. Thus $0 \leq \langle p, R \rangle =$

$\sum_{j=1}^{n} \langle px_j, x'_j \rangle = \sum_{j=1}^{n} \langle y, x'_j \rangle \langle x_j, y' \rangle = \langle Ry, y' \rangle$. Since $y' \in E'_+$, $y \in E_+$ are arbitrary this implies that R is a positive operator.

b) Conversely, suppose that $R = \sum_{j=1}^{n} x'_j \otimes x_j \geq 0$. Then $\langle Ry, y' \rangle \geq 0$ for all $y \in E_+$, $y' \in E'_+$. Thus $0 \leq \langle p, R \rangle$ for all $p \in P$. Hence, if $I \in \overline{P}$, then it follows that $0 \leq \langle I, R \rangle = \sum_{j=1}^{n} \langle x'_j, x_j \rangle$. ◇

PROOF OF THEOREM B1. a) One easily verifies that the theorem holds if $E = C(K)$, K compact ; see [S2, IV. Theorem 2.4, p. 239].

b) Let E be arbitrary. Let $R = \sum_{j=1}^{n} x'_j \otimes x_j$ be positive. By Lemma B1 it suffices to show that $\sum_{j=1}^{n} \langle x'_j, x_j \rangle \geq 0$. Let $u = \sum_{j=1}^{n} |x_j|$. Then R leaves E_u invariant. By Kakutani's theorem E_u is isomorphic to a space $C(K)$. Thus we can assume that $E = C(K)$. Now the claim follows from a) and the other implication of Lemma B2. ◇

REMARK. Walsh [W] actually shows that Theorem B1 holds in an ordered Banach space E with normal and generating cone if and only if E' is a lattice.

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AMS classification: 47D07, 47B65

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