Potential Analysis **5**: 207–230, 1996.

The Spectral Bound of Schrödinger Operators

WOLFGANG ARENDT[1] and CHARLES J. K. BATTY[2]
[1]*Laboratoire de Mathématiques, Université de Franche-Comté, Route de Gray, 25030 Besançon Cedex, France (e-mail: arendt@grenet.fr)**
[2]*St. John's College, Oxford OX1 3JP, England (e-mail: batty@vax.ox.ac.uk)*

(Received: 11 January 1994; accepted: 1 June 1994)

Abstract. Let $V : \mathbf{R}^N \to [0, \infty]$ be a measurable function, and $\lambda > 0$ be a parameter. We consider the behaviour of the spectral bound of the operator $\frac{1}{2}\Delta - \lambda V$ as a function of λ. In particular, we give a formula for the limiting value as $\lambda \to \infty$, in terms of the integrals of V over subsets of $\mathbf{R}^N$ on which the Laplacian with Dirichlet boundary conditions has prescribed values. We also consider the question whether this limiting value is attained for finite λ.

Mathematics Subject Classification (1991). 35J10.

Key words: Schrödinger semigroup, spectral bound, spectral function.

1. Introduction

Let $V : \mathbf{R}^N \to [0, \infty]$ be a measurable function, and $\lambda > 0$ be a parameter. Consider the Schrödinger semigroup $\{S_{\lambda V}(t) : t \geqslant 0\}$ on $L^p(\mathbf{R}^N)$ $(1 \leqslant p \leqslant \infty)$ associated with the formal operator $\frac{1}{2}\Delta - \lambda V$. The *spectral function* $s_V(\lambda)$ of V is the spectral bound of $\frac{1}{2}\Delta - \lambda V$, that is, the growth bound of the semigroup $S_{\lambda V}$; s_V is a convex, decreasing, function of λ. This paper is concerned with two questions about the behaviour of s_V:

(a) What is the value of $s_V(\infty) := \lim_{\lambda \to \infty} s_V(\lambda)$?
(b) Is s_V strictly decreasing?

It was shown in [6] that $s_V(\lambda) < 0$ for some (and hence all) $\lambda > 0$ if and only if $\int_E V = \infty$ for all Borel sets E such that $\mathbf{R}^N \setminus E$ is transient for Brownian motion, or equivalently, such that Δ_E, the Laplacian with Dirichlet boundary conditions on E, has strictly negative spectral bound. For general V, it suffices that $\int_E V = \infty$ for all closed sets E in this class; for V in $L^1_{\mathrm{loc}}(\mathbf{R}^N)$, it suffices that $\int_\Omega V = \infty$ for all open sets Ω in the class.

In Section 4, we shall show that

$$s_V(\infty) = \inf\left\{\sigma < 0 : \inf_{E \in \mathcal{F}_\sigma} \int_E V > 0\right\}. \tag{1.1}$$

* Current address: Mathematik V, Universität Ulm, D-89069 Ulm, Germany (e-mail: arendt@mathematik.uni-ulm.de)

Here, $\mathcal{F}_\sigma$ is the class of all Borel sets E such that the spectral bound of $\frac{1}{2}\Delta_E$ is greater than σ. Thus (1.1) is a quantitative version of the result in [6].

In Section 5, we shall give some partial answers to question (b). We shall show that there exists V in $L^\infty_{\text{loc}}(\mathbf{R}^N)$ such that s_V is eventually constant. On the other hand, in the case $N = 1$, we show that s_V is strictly decreasing if $V \in L^\infty(\mathbf{R})$ and $s_V(\infty) < 0$.

Information about the spectral bound of $\frac{1}{2}\Delta + \lambda V$, where V is non-negative, may be found in [25] and the references cited therein. The asymptotic behaviour of the semigroups generated by $\frac{1}{2}\Delta + \lambda m$, where m is a function which changes sign, was studied by Simon (see [32, B5]). Recently, there has been much interest in whether there is a principal eigenvalue, that is, a value of λ for which the spectral bound is zero and is an eigenvalue [8–13, 21, 27–28]. The techniques usually involve studying separately the effects of the positive and negative parts of m. In a separate paper [5], we shall apply the results of this paper to such questions.

2. Preliminaries

First, we establish some notation. We denote by $\mathbf{1}$ the function constantly equal to 1, and by $\mathbf{1}_E$ the characteristic function of a set E. For a subset E of $\mathbf{R}^N$, E^c will be the complement $\mathbf{R}^N \setminus E$. For x in $\mathbf{R}^N$ and $r > 0$, we put

$$B(x,r) = \{y \in \mathbf{R}^N : |y - x| < r\}.$$

All integrals over $\mathbf{R}^N$, or Borel subsets of $\mathbf{R}^N$, will be taken with respect to Lebesgue measure m; $\omega_N = m(B(0,1))$ will be the volume of the unit ball.

Throughout, we shall denote by $\{T(t) : t \geqslant 0\}$ the Gaussian semigroup defined on $L^p(\mathbf{R}^N)$ for $1 \leqslant p \leqslant \infty$ by:

$$T(t)f = f * p_t,$$

where p_t is the Gaussian kernel: $p_t(x) = (2\pi t)^{-N/2} \mathrm{e}^{-|x|^2/2t}$. We shall also need the functions ψ_t given by: $\psi_t(x) = \int_0^t p_s(x)\, \mathrm{d}s$. We shall denote the generator of T on $L^2(\mathbf{R}^N)$ by $\frac{1}{2}\Delta_2$. The Gaussian semigroup is also given by:

$$(T(t)f)(x) = \mathbf{E}^x \left[f(B(t)) \right],$$

where $\{B(t) : t \geqslant 0\}$ denotes Brownian motion on $\mathbf{R}^N$, and $\mathbf{E}^x$ is expectation with respect to the Wiener probability measure $\mathbf{P}^x$ corresponding to motion starting at x.

For a measurable function $V : \mathbf{R}^N \to [0, \infty]$, we shall denote the Schrödinger semigroups on $L^p(\mathbf{R}^N)$ by $\{S_V(t) : t \geqslant 0\}$. The semigroup may be defined by considering the quadratic form $\mathbf{a}_V$ on $L^2(\mathbf{R}^N)$ given by:

$$\begin{aligned} D(\mathbf{a}_V) &= \left\{ u \in W^{1,2}(\mathbf{R}^N) : \int_{\mathbf{R}^N} V u^2 < \infty \right\}, \\ \mathbf{a}_V(u) &= \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u|^2 + \int_{\mathbf{R}^N} V u^2. \end{aligned}$$

The associated positive-definite, self-adjoint, operator H_V defines a semigroup $\{e^{-tH_V}\}$ on $L^2(\mathbf{R}^N)$, which interpolates to provide a positive, contractive, semigroup S_V on the L^p-spaces (we do not distinguish notationally between the semigroups for different values of p). The Schrödinger semigroups may alternatively be defined by the Feynman–Kac formula:

$$(S_V(t)f)(x) = \mathbf{E}^x\left[\exp\left(-\int_0^t V(B(s))\,\mathrm{d}s\right) f(B(t))\right]$$

(see [23, 30]). If $V \notin L^1_{\mathrm{loc}}(\mathbf{R}^N)$, then $\mathbf{a}_V$ may not be densely-defined, and $S_V(t)$ may not converge to the identity as $t \downarrow 0$, but this does not affect our discussion (see [4]). Note that if $0 \leqslant V_1 \leqslant V_2$, then $T(t) \geqslant S_{V_1}(t) \geqslant S_{V_2}(t)$ as operators on $L^p(\mathbf{R}^N)$.

By duality and interpolation, $\|S_V(t)\|_{\mathcal{L}(L^2)} \leqslant \|S_V(t)\|_{\mathcal{L}(L^\infty)}$, where $\|\cdot\|_{\mathcal{L}(L^p)}$ denotes the L^p-operator norm. Simon [31] showed that the L^p-growth bound $\lim_{t\to\infty} t^{-1} \log \|S_V(t)\|_{\mathcal{L}(L^p)}$ is independent of p (see Lemma 3.2 below); it coincides with the supremum of the spectrum of the generator of S_V on any of the L^p-spaces (the spectrum is also independent of p [2, 19]). We shall denote this spectral bound by $s(\frac{1}{2}\Delta - V)$.

Let $\lambda > 0$ be a parameter. The *spectral function* s_V of V is defined by:

$$s_V(\lambda) = s(\tfrac{1}{2}\Delta - \lambda V).$$

By considering the case $p = 2$, one sees that

$$s_V(\lambda) = \sup\left\{-\frac{1}{2}\int_{\mathbf{R}}^N |\nabla u|^2 - \lambda \int_{\mathbf{R}}^N V u^2 : u \in W^{1,2}(\mathbf{R}^N),\right.$$
$$\left.\int_{\mathbf{R}}^N u^2 = 1\right\}. \tag{2.1}$$

By considering the case $p = \infty$, one sees that

$$s_V(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log\left\{\operatorname*{ess\,sup}_{x\in\mathbf{R}^N} \mathbf{E}^x\left[\exp\left(-\int_0^t V(B(s))\,\mathrm{d}s\right)\right]\right\}. \tag{2.2}$$

It follows easily from either (2.1) or (2.2) that s_V is a convex, decreasing, function of λ, and that,

$$\text{if } 0 \leqslant V_1 \leqslant V_2, \quad \text{then } 0 \geqslant s_{V_1}(\lambda) \geqslant s_{V_2}(\lambda).$$

If $s_V(\lambda)$ is a pole of the resolvent of $H_{\lambda V}$ on $L^2(\mathbf{R}^N)$, then $H_{\lambda V}$ has a positive normalised eigenvector u in $L^2(\mathbf{R}^N)$ (see [24, Corollary 1.4, p. 166]). In these circumstances, we say that $s_V(\lambda)$ is a *principal eigenvalue* and u is a *principal eigenvector*.

PROPOSITION 2.1. *Suppose that* $\liminf_{|x|\to\infty} V(x) > 0$ *and that* $s_V(\infty) := \lim_{\lambda\to\infty} s_V(\lambda) > -\infty$. *Then there exists* $\lambda_0 > 0$ *such that* $s_V(\lambda)$ *is a principal eigenvalue of* $H_{\lambda V}$ *whenever* $\lambda > \lambda_0$. *If* $V \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, *then the (normalised) principal eigenvector is unique and strictly positive (a.e.).*

Proof. There exist a compact set K and $\delta > 0$ such that $V \geqslant \delta$ a.e. in K^c. Let $\lambda_0 = -s_V(\infty)/\delta$. If $\lambda > \lambda_0$, then

$$s(\tfrac{1}{2}\Delta - \lambda(V + \delta \mathbf{1}_K)) \leqslant s(\tfrac{1}{2}\Delta - \lambda\delta\mathbf{1}) = -\lambda\delta < s_V(\lambda) = s(\tfrac{1}{2}\Delta - \lambda V).$$

Moreover, $\lambda\delta\mathbf{1}_K$ is a relatively compact perturbation of $H_{\lambda V}$. By [29, Corollary 2, p. 113], $s_V(\lambda)$ does not belong to the essential spectrum of $H_{\lambda V}$, so it is a principal eigenvalue.

If $V \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, then $S_{\lambda V}$ is irreducible [5] and the uniqueness and strict positivity of the principal eigenvector follow from [24, Proposition 3.5, p. 310].

Now, consider the case when E is a Borel subset of $\mathbf{R}^N$ and $V = \chi_{E^c}$, where

$$\chi_{E^c}(x) = \begin{cases} \infty & (x \in E^c) \\ 0 & (x \in E). \end{cases}$$

Then $\mathbf{a}_V$ is the form $\tilde{\mathbf{a}}_E$:

$$\begin{aligned} D(\tilde{\mathbf{a}}_E) &= W^{1,2}(\mathbf{R}^N) \cap L^2(E), \\ \tilde{\mathbf{a}}_E(u) &= -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2. \end{aligned}$$

The associated operator H_V is the *pseudo-Dirichlet Laplacian* $\frac{1}{2}\tilde{\Delta}_E$, and the semigroup S_V is the *pseudo-Dirichlet semigroup* $\tilde{T}_E$ given by

$$(\tilde{T}_E(t)f)(x) = \mathbf{E}^x[\mathbf{1}_{[B(s)\in E \text{ for almost all } s\leqslant t]} f(B(t))]$$

(see [4, Examples 5.6, 7.2]). We have

$$\begin{aligned} s_{\chi_{E^c}}(\lambda) &= s(\tfrac{1}{2}\tilde{\Delta}_E) \\ &= \sup\left\{ -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 : u \in W^{1,2}(\mathbf{R}^N), u = 0 \text{ a.e. in } E^c, \right. \\ &\qquad\qquad \left. \int_{\mathbf{R}^N} u^2 = 1 \right\} \qquad (2.3) \\ &= \lim_{t\to\infty} \frac{1}{t} \log\left\{ \operatorname*{ess\,sup}_{x\in\mathbf{R}^N} \mathbf{P}^x[B(s) \in E \text{ for almost all } s \leqslant t] \right\}. \qquad (2.4) \end{aligned}$$

We shall denote the *Dirichlet semigroup* on $L^p(E)$ by $\{T_E(t) : t \geqslant 0\}$. This semigroup is associated with the quadratic form $\mathbf{a}_E$ on $L^2(\mathbf{R}^N)$ given by:

$$\begin{aligned} D(\mathbf{a}_E) &= \{u \in W^{1,2}(\mathbf{R}^N) : \tilde{u} = 0 \text{ q.e. in } E^c\}, \\ \mathbf{a}_E(u) &= \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2, \end{aligned}$$

where $\tilde{u}$ is a quasi-continuous version of u, and 'q.e.' means 'quasi-everywhere', that is, except on a set of capacity zero (see [17, Sect. 3], [22, Proposition III.3.5]). For an open subset Ω of $\mathbf{R}^N$, $D(\mathbf{a}_\Omega) = W_0^{1,2}(\Omega)$ [15, 18]. The associated self-adjoint operator on $L^2(E)$ will be denoted by $\frac{1}{2}\Delta_E$ (the Laplacian with Dirichlet boundary conditions).

PROPOSITION 2.2. *Let E be the bounded Borel set in $\mathbf{R}^N$. Then $\frac{1}{2}\Delta_E$ has compact resolvent. In particular, $s(\frac{1}{2}\Delta_E)$ is a principal eigenvalue.*

Proof. Let B be an open ball containing E. Then T_E is dominated by T_B. Since $\frac{1}{2}\Delta_B$ has compact resolvent on $L^2(B)$, $\frac{1}{2}\Delta_E$ has compact resolvent on $L^2(F)$ by the Dodds–Fremlin Theorem [1, Theorem 16.20].

The Dirichlet semigroup may alternatively be given by the following formula:

$$(T_E(t)f)(x) = \mathbf{E}^x[1_{[B(s)\in E \text{ for all } s\leqslant t]} f(B(t))].$$

The growth bound of T_E is independent of p by Simon's argument (see Lemma 3.2), and is therefore given by:

$$s(\tfrac{1}{2}\Delta_E) = \sup\left\{-\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 : u \in W^{1,2}(\mathbf{R}^N), \tilde{u} = 0 \text{ q.e. in } E^c, \int_{\mathbf{R}^N} u^2 = 1\right\} \tag{2.5}$$

$$= \lim_{t\to\infty} \frac{1}{t} \log\left\{\operatorname*{ess\,sup}_{x\in\mathbf{R}^N} \mathbf{P}^x[B(s) \in E \text{ for all } s \leqslant t]\right\}. \tag{2.6}$$

Clearly, $s(\frac{1}{2}\Delta_E) \leqslant s(\frac{1}{2}\tilde{\Delta}_E)$. Equality holds if E is a regular open set [4, p. 444], [20, Theorem 2.1].

It was implicit in the proof of [6, Proposition 5.1] (see also Example 4.13 below) that, for any Borel set E, there is a Borel set F such that $E \subseteq F$, $F \setminus E$ is null, and $s(\frac{1}{2}\tilde{\Delta}_E) \leqslant s(\frac{1}{2}\Delta_F)$. Since $s(\frac{1}{2}\Delta_F) \leqslant s(\frac{1}{2}\tilde{\Delta}_F) = s(\frac{1}{2}\tilde{\Delta}_E)$, it follows that $s(\frac{1}{2}\tilde{\Delta}_E) = s(\frac{1}{2}\Delta_F)$.

3. The Class $\mathcal{F}_\sigma$

As in the introduction, $\mathcal{F}_\sigma$ will denote the class of all Borel sets E such that $s(\frac{1}{2}\Delta_E) > \sigma$, where $s(\frac{1}{2}\Delta_E)$ is given by (2.5) or (2.6). In this section, we give a few technical results concerning this class of sets. The first shows that the essential supremum in (2.6) can be replaced by the supremum.

LEMMA 3.1. *Let E be a Borel subset of $\mathbf{R}^N$. For $t > 0$,*

$$\|T_E(t)\|_{\mathcal{L}(L^\infty)} = \sup_{x\in\mathbf{R}^N} \mathbf{P}^x[B(s) \in E \textit{ for all } s \leqslant t].$$

Proof. Let $0 < \delta < t$, $x \in \mathbf{R}^N$. By the Markov property,

$$\begin{aligned}\mathbf{P}^x[B(s) \in E \text{ for all } s \leqslant t] &\leqslant \mathbf{P}^x[B(s) \in E \text{ whenever } \delta \leqslant s \leqslant t] \\ &= \int_{\mathbf{R}^N} p_\delta(y)\mathbf{P}^y[B(s) \in E \text{ for all } s \leqslant t-\delta]\,\mathrm{d}y \\ &= \int_{\mathbf{R}^N} p_\delta(y)(T_E(t-\delta)\mathbf{1})(y)\,\mathrm{d}y \\ &\leqslant \|T_E(t-\delta)\mathbf{1}\|_\infty \\ &= \|T_E(t-\delta)\|_{\mathcal{L}(L^\infty)}.\end{aligned}$$

Since T_E is dominated by the Gaussian semigroup, it is a holomorphic semigroup on $L^1(E)$ [26]. Hence $s \mapsto \|T_E(s)\|_{\mathcal{L}(L^1)} = \|T_E(s)\|_{\mathcal{L}(L^\infty)}$ is continuous for $s > 0$. Letting $\delta \downarrow 0$ gives the result.

The next lemma is essentially due to Simon [31, Theorem 1.3].

LEMMA 3.2. *Let $\sigma < 0$. There is a constant $c_{\sigma N}$ such that*

$$\|T_E(t)\mathbf{1}\|_\infty \leqslant c_{\sigma N} t^{N/2} \mathrm{e}^{\sigma t} \tag{3.1}$$

for all $t \geqslant 1$ and all Borel sets E with $s(\frac{1}{2}\Delta_E) \leqslant \sigma$. Hence

$$\mathbf{P}^x[B(s) \in E \textit{ for all } s \leqslant t] \leqslant c_{\sigma N} t^{N/2} \mathrm{e}^{\sigma t}$$

for all $x \in \mathbf{R}^N$, $t \geqslant 1$, and Borel sets E not in $\mathcal{F}_\sigma$.

Proof. Note first that $T(1)$ is a bounded operator from $L^2(\mathbf{R}^N)$ to $L^\infty(\mathbf{R}^N)$, with norm $\alpha := \|p_1\|_2 = N\omega_N\Gamma(N/2)(2\pi)^{-N}$. Let $a = 2(-\sigma)^{1/2}$. Let $t \geqslant 1$, E be a Borel set with $s(\frac{1}{2}\Delta_E) \leqslant \sigma$, $x_0 \in \mathbf{R}^N$, and $B = B(x_0, a(t+1))$. Since T_E is dominated by T, we obtain the following for almost all x in $B(x_0, a)$:

$$\begin{aligned}0 \leqslant (T_E(t)\mathbf{1})(x) &= (T_E(1)T_E(t-1)\mathbf{1}_B)(x) + (T_E(t)\mathbf{1}_{B^c})(x) \\ &\leqslant (T(1)T_E(t-1)\mathbf{1}_B)(x) + (T(t)\mathbf{1}_{B^c})(x) \\ &\leqslant \alpha\|T_E(t-1)\mathbf{1}_B\|_2 + \int_{B^c} p_t(x-y)\,\mathrm{d}y \\ &\leqslant \alpha\,\mathrm{e}^{\sigma(t-1)}\omega_N^{1/2}(a(t+1))^{N/2} + \int_{B(x,at)^c} p_t(x-y)\,\mathrm{d}y \\ &\leqslant \alpha\,\mathrm{e}^{-\sigma}\omega_N^{1/2}(2at)^{N/2}\mathrm{e}^{\sigma t} + \\ &\quad + \frac{N\omega_N t^{N/2}}{(2\pi)^{N/2}}\int_{at^{1/2}}^\infty \mathrm{e}^{-r^2/2} r^{N-1}\,\mathrm{d}r\end{aligned}$$

$$\leqslant \alpha\, e^{-\sigma}\omega_N^{1/2}(2a)^{N/2}t^{N/2}\, e^{\sigma t} + \\ + \frac{N\omega_N}{(2\pi)^{N/2}}\left(\int_0^\infty e^{-r^2/4}r^{N-1}\, dr\right) t^{N/2}\, e^{\sigma t}. \tag{3.2}$$

Since x_0 is arbitrary, it follows that $\|T_E(t)1\|_\infty$ is bounded by (3.2). The last statement now follows from Lemma 3.1.

The next lemma is analogous to the proof of [6, Proposition 4.10], but the argument is now more complicated. If K were allowed to depend on E, it would be possible to give a much simpler proof.

LEMMA 3.3. *Let $\sigma < \sigma' < 0$. There is a constant K (depending on N, σ, and σ') such that, for each E in $\mathcal{F}_{\sigma'}$, there exists x in E such that $E \cap \overline{B(x,K)} \in \mathcal{F}_\sigma$.*

Proof. Let $c_{\sigma N}$ be as in Lemma 3.2. Choose $t > 1$ such that

$$c_{\sigma N}t^{N/2}\, e^{\sigma t} < \tfrac{1}{2}e^{\sigma' t}.$$

Now, choose K such that

$$\mathbf{P}^0[|B(s)| > K \text{ for some } s \leqslant t] < \tfrac{1}{4}e^{\sigma' t}.$$

Let $E \in \mathcal{F}_{\sigma'}$. Then

$$\begin{aligned} e^{\sigma' t} < e^{s(\frac{1}{2}\Delta_E)t} &= \|T_E(t)\|_{\mathcal{L}(L^2)} \\ &\leqslant \|T_E(t)\|_{\mathcal{L}(\mathcal{L}^\infty)} = \|T_E(t)\mathbf{1}\|_\infty \\ &\qquad = \sup_{x\in E} \mathbf{P}^x[B(s) \in E \text{ for all } s \leqslant t]. \end{aligned}$$

Thus there exists x in $\mathbf{R}^N$ such that

$$\mathbf{P}^x[B(s) \in E \text{ for all } s \leqslant t] \geqslant \tfrac{3}{4}e^{\sigma' t}.$$

Hence

$$\mathbf{P}^x[B(s) \in E \cap \overline{B(x,K)} \text{ for all } s \leqslant t] \geqslant \tfrac{1}{2}e^{\sigma' t} > c_{\sigma N}t^{N/2}\, e^{\sigma t}.$$

It follows from Lemma 3.2 that $E \cap \overline{B(x,K)} \in \mathcal{F}_\sigma$.

4. The Value of $s_V(\infty)$

In this section, $V : \mathbf{R}^N \to [0,\infty]$ will be a measurable function, and we shall establish the equality (1.1), where $s_V(\infty) = \lim_{\lambda\to\infty} s_V(\lambda)$ and $s_V(\lambda)$ is defined by (2.1) or (2.2). However, we shall first establish a different formula for $s_V(\infty)$ under rather special conditions on V.

PROPOSITION 4.1. *Suppose that* $\liminf_{|x|\to\infty} V(x) > 0$, *and let* $E = \{x \in \mathbf{R}^N : V(x) = 0\}$. *Then* $s_V(\infty) = s(\frac{1}{2}\tilde{\Delta}_E)$. *Hence there is a Borel set* F *such that* $V = 0$ *a.e. in* F, $V > 0$ *a.e. in* F^c, *and* $s_V(\infty) = s(\frac{1}{2}\Delta_F)$.

Proof. Since $V = 0$ in E, it follows from either (2.1) and (2.3) or (2.2) and (2.4), that $s_V(\lambda) \geqslant s(\frac{1}{2}\tilde{\Delta}_E)$ for all $\lambda < \infty$, so $s_V(\infty) \geqslant s(\frac{1}{2}\tilde{\Delta}_E)$.

To prove the opposite inequality, we can assume that $s_V(\infty) > -\infty$. By Proposition 2.1, there exists λ_0 such that, for $n > \lambda_0$, H_{nV} has a (normalised) principal eigenvector u_n in $L^2(\mathbf{R}^N)$. Now

$$\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_n|^2 + n\int_{\mathbf{R}^N} V u_n^2 = -s_V(n).$$

It follows that $\sup_n \int_{\mathbf{R}^N} |\nabla u_n|^2 < \infty$ and $\int_{\mathbf{R}^N} V u_n^2 \to 0$ as $n \to \infty$. Thus (u_n) is a bounded sequence in $W^{1,2}(\mathbf{R}^N)$, which is compactly embedded in $L^2_{\text{loc}}(\mathbf{R}^N)$. Hence, there is a subsequence (u_{n_r}) which converges weakly in $W^{1,2}(\mathbf{R}^N)$ and strongly in $L^2_{\text{loc}}(\mathbf{R}^N)$ to a limit u.

By assumption, there exist a compact set K and $\delta > 0$ such that $V \geqslant \delta$ in K^c. Then E is contained in K and $\int_{K^c} u_n^2 \leqslant \delta^{-1}\int_{K^c} V u_n^2 \to 0$. It follows that $u = 0$ a.e. in K^c. Moreover, $\int_K (u_n - u)^2 \to 0$, so $\int_K V u^2 = \lim_{n\to\infty} \int_K V u_n^2 = 0$. Hence, $u = 0$ a.e. in $K \cap E^c$, so $u \in L^2(E) \cap W^{1,2}(\mathbf{R}^N)$. In addition, $\int_E u^2 = \int_K u^2 = \lim_{n\to\infty} \int_K u_n^2 = 1$.

For φ in $L^2(E) \cap W^{1,2}(\mathbf{R}^N)$,

$$\begin{aligned} -\frac{1}{2}\int_{\mathbf{R}^N} \nabla u_{n_r} \nabla\varphi &= -\frac{1}{2}\int_{\mathbf{R}^N} \nabla u_{n_r} \nabla\varphi - \int_{\mathbf{R}^N} V u_{n_r}\varphi \\ &= s_V(n_r)\int_{\mathbf{R}^N} u_{n_r}\varphi. \end{aligned}$$

Letting $r \to \infty$,

$$-\frac{1}{2}\int_{\mathbf{R}^N} \nabla u \nabla\varphi = s_V(\infty)\int_{\mathbf{R}^N} u\varphi.$$

Hence $u \in D(\frac{1}{2}\tilde{\Delta}_E)$ and $\frac{1}{2}\tilde{\Delta}_E u = s_V(\infty)u$. In particular, $s(\frac{1}{2}\tilde{\Delta}_E) \geqslant s_V(\infty)$.

The last statement follows from the remark at the end of Section 2.

REMARK 4.2. Suppose that $\liminf_{|x|\to\infty} V(x) > 0$ and that $E := \{x \in \mathbf{R}^N : V(x) = 0\}$ is not null. By Proposition 2.1, there exists λ_0 such that $H_{\lambda V}$ has a principal eigenvector u_λ whenever $\lambda > \lambda_0$. Moreover, $\frac{1}{2}\tilde{\Delta}_E$ has a principal eigenvector u. Suppose that $\tilde{T}_E$ is irreducible (for example, if E is open, connected and regular [14, Theorem 3.3.5]). Then u is unique, and the proof of Proposition 4.1 shows that $u_\lambda \to u$ strongly in $L^2(\mathbf{R}^N)$ and weakly in $W^{1,2}(\mathbf{R}^N)$ as $\lambda \to \infty$.

The next proposition establishes one inequality in (1.1).

PROPOSITION 4.3. *Let $\sigma < 0$, and suppose that*

$$\inf\left\{\int_E V : E \in \mathcal{F}_\sigma\right\} = 0. \tag{4.1}$$

Then $s_V(\lambda) \geqslant \sigma$ for all $\lambda > 0$.

Proof. Let E be any set in $\mathcal{F}_\sigma$. By Lemma 3.3, there is a bounded set F in $\mathcal{F}_\sigma$ such that $F \subseteq E$. By Proposition 2.2, the spectral bound $s(\frac{1}{2}\Delta_F)$ of $\frac{1}{2}\Delta_F$ is a principal eigenvalue [24, Proposition 3.5, p. 310], so there is a non-negative function u in $D(\frac{1}{2}\Delta_F)$ such that $\int_F u^2 = 1$ and $\frac{1}{2}\Delta_F u = s(\frac{1}{2}\Delta_F)u$. Then, for any fixed $t > 0$,

$$u = \mathrm{e}^{-\frac{1}{2}s(\Delta_F)t} T_F(t)u \leqslant \mathrm{e}^{-\sigma t} T(t)u = \mathrm{e}^{-\sigma t}(p_t * u).$$

It follows that $u(x) \leqslant \mathrm{e}^{-\sigma t}\|p_t\|_2$ for almost all x. Now

$$\begin{aligned} s_V(\lambda) &\geqslant -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 - \lambda\int_F V u^2 \\ &\geqslant \sigma - \lambda\,\mathrm{e}^{-2\sigma t}\|p_t\|_2^2 \int_E V. \end{aligned}$$

Taking the supremum over all E in $\mathcal{F}_\sigma$, it follows from (4.1) that $s(\lambda) \geqslant \sigma$.

COROLLARY 4.4. *Let Ω be a regular, bounded, open, subset of $\mathbf{R}^N$, and $\sigma > s(\frac{1}{2}\Delta_\Omega)$. Then*

$$\inf\{m(E \setminus \Omega) : E \in \mathcal{F}_\sigma\} > 0.$$

Proof. Let $V = \mathbf{1}_{\Omega^c}$. By Proposition 4.1, $s_V(\infty) < \sigma$. By Proposition 4.3, $\inf\{\int_E V : E \in \mathcal{F}_\sigma\} > 0$. Since $\int_E V = m(E \setminus \Omega)$, the result follows.

REMARK 4.5. Proposition 4.3 may alternatively be proved by the method used in [6, Proposition 4.8]. The argument given there shows that

$$T_E(t)\mathbf{1} - S_{\lambda V}(t)\mathbf{1} \leqslant \lambda \int_0^t T(t-s)V\mathbf{1}_E \,\mathrm{d}s = \lambda(V\mathbf{1}_E) * \psi_t. \tag{4.2}$$

If $s_V(\lambda) < \sigma$, then we can choose $t > 0$ and $\varepsilon > 0$ such that

$$\|S_{\lambda V}(t)\mathbf{1}\|_\infty < \varepsilon < \mathrm{e}^{\sigma t}.$$

Then (4.2) shows that

$$\inf_{E \in \mathcal{F}_\sigma} \|(V\mathbf{1}_E) * \psi_t\|_\infty > 0.$$

Moreover, integrating (4.2) over $E_t^*(1-\varepsilon) := \{x \in \mathbf{R}^N : (T_E(t)\mathbf{1})(x) > \varepsilon\}$ gives

$$(\varepsilon - \|S_{\lambda V}(t)\mathbf{1}\|_\infty) m(E_t^*(1-\varepsilon)) \leqslant \lambda t \int_E V.$$

A variant of [6, Lemma 4.7] shows that $\inf_{E\in\mathcal{F}_\sigma} m(E_t^*(1-\varepsilon)) > 0$, and Proposition 4.3 follows.

The proof of (4.2) in [6] used the Feynman–Kac formula. It may alternatively be obtained by first using the variation of constants formula to show that

$$S_{U_n}(t)1 - S_{V_m}(t)1 \leqslant \int_0^t T(t-s)V_m S_{U_n}(s)1\,\mathrm{d}s,$$

for U_n and V_m in $L^\infty(\mathbf{R}^N)_+$, and then letting $U_n \uparrow \chi_{E^c}$ and $V_m \uparrow V$, where $\chi_{E^c}(x) = 0$ if $x \in E$ and $\chi_{E^c}(x) = \infty$ if $x \in E^c$ (see [4, Section 5]).

Now we turn to the converse of Proposition 4.3. As in [6], this argument is less straightforward, using the strong Markov property of Brownian motion. The general strategy is the same as Propositions 4.9 and 4.10 of [6].

PROPOSITION 4.6. *Let $\sigma < 0$, $t > 0$, and suppose that*

$$\inf\{\|(V\mathbf{1}_E) * \psi_t\|_\infty : E \in \mathcal{F}_\sigma, E \textit{ closed}\} > 0. \tag{4.3}$$

Then $s_V(\infty) \leqslant \sigma$.

Proof. For $\alpha > 0$ and $\eta > 0$, let

$$E_{\alpha\eta} = \left\{y \in \mathbf{R}^N : \mathbf{P}^y\left[\int_0^t V(B(s))\,\mathrm{d}s \leqslant \alpha\right] \geqslant \eta\right\}.$$

By [6, Lemma 2.1], $E_{\alpha\eta}$ is closed. By [6, Lemma 4.2] applied to $V\mathbf{1}_{E_{\alpha\eta}}$,

$$\eta\mathbf{E}^x\left[\int_0^t (V\mathbf{1}_{E_{\alpha\eta}})(B(s))\,\mathrm{d}s\right] \leqslant \alpha$$

for all x in $\mathbf{R}^N$. Hence

$$\|(V\mathbf{1}_{E_{\alpha\eta}}) * \psi_t\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbf{R}^N} \mathbf{E}^x\left[\int_0^t (V\mathbf{1}_{E_{\alpha\eta}})(B(s))\,\mathrm{d}s\right] \leqslant \frac{\alpha}{\eta}.$$

It follows from (4.3) that there exists $\alpha > 0$ (depending on η) such that $E_{\alpha\eta} \notin \mathcal{F}_\sigma$.

For $t' > 1$, let $\gamma_{\sigma N}(t') = c_{\sigma N} t'^{N/2} \mathrm{e}^{\sigma t'}$, where $c_{\sigma N}$ is as in Lemma 3.2. Thus

$$\mathbf{P}^x[B(s) \in E_{\alpha\eta} \text{ for all } s \leqslant t'] \leqslant \gamma_{\sigma N}(t')$$

for all $t' > 1$ and all x in $\mathbf{R}^N$.

Let (K_n) be an increasing sequence of compact sets with union $E^c_{\alpha\eta}$, and let τ_n be the first hitting time of K_n. If $\tau_n < \infty$, then $B(\tau_n) \in E^c_{\alpha\eta}$. For any x in $\mathbf{R}^N$, the strong Markov property gives:

$$\begin{aligned}\mathbf{P}^x\left[\int_0^{t+t'} V(B(s))\,\mathrm{d}s > \alpha\right] &\geqslant \mathbf{P}^x\left[\tau_n < t', \int_{\tau_n}^{\tau_n+t} V(B(s))\,\mathrm{d}s > \alpha\right] \\ &= \mathbf{E}^x\left[\mathbf{1}_{[\tau_n<t']}\mathbf{P}^{B(\tau_n)}\left[\int_0^t V(B(s))\,\mathrm{d}s > \alpha\right]\right] \\ &\geqslant (1-\eta)\mathbf{P}^x\left[\tau_n < t'\right].\end{aligned}$$

Since $E_{\alpha\eta}$ is closed,

$$\lim_{n\to\infty} \mathbf{P}^x\left[\tau_n < t'\right] = \mathbf{P}^x\left[B(s) \in E^c_{\alpha\eta} \text{ for some } s \leqslant t'\right] \geqslant 1 - \gamma_{\sigma N}(t').$$

Hence

$$\mathbf{P}^x\left[\int_0^{t+t'} V(B(s))\,\mathrm{d}s \leqslant \alpha\right] \leqslant 1 - (1-\gamma_{\sigma N}(t'))(1-\eta) \leqslant \gamma_{\sigma N}(t') + \eta$$

for all x in $\mathbf{R}^N$.

Now, let $k \geqslant 1$, $m \geqslant 1$, and $n > 1$. Let $r_1, r_2, \ldots, r_{mn}$ be integers with $1 \leqslant r_1 < r_2 < \ldots < r_{mn} \leqslant (m+k)n$. Let $t_j = (r_j - r_{j-1})(t+t')$ $(j = 1, 2, \ldots, mn)$, where $r_0 = 1$. The Markov property gives:

$$\begin{aligned}&\mathbf{P}^x\left[\int_{(r_j-1)(t+t')}^{r_j(t+t')} V(B(s))\,\mathrm{d}s \leqslant \alpha \quad (j = 1, 2, \ldots, mn)\right] \\ &= \mathbf{E}^x\left[\mathbf{E}^{B(t_1)}\left[\mathbf{1}_{\left[\int_0^{t+t'} V(B(s))\,\mathrm{d}s\leqslant\alpha\right]}\mathbf{E}^{B(t_2)}\left[\mathbf{1}_{\left[\int_0^{t+t'} V(B(s))\,\mathrm{d}s\leqslant\alpha\right]}\right.\right.\right. \\ &\qquad \left.\left.\left.\ldots\, \mathbf{E}^{B(t_{mn})}\left[\mathbf{1}_{\left[\int_0^{t+t'} V(B(s))\,\mathrm{d}s\leqslant\alpha\right]}\right]\ldots\right]\right]\right] \\ &\leqslant (\gamma_{\sigma N}(t') + \eta)^{mn}.\end{aligned}$$

Hence

$$\begin{aligned}&\mathbf{P}^x\left[\int_0^{(m+k)n(t+t')} V(B(s))\,\mathrm{d}s \leqslant nk\alpha\right] \\ &\leqslant \mathbf{P}^x\left[\text{For some } 1 \leqslant r_1 < r_2 < \ldots r_{mn} \leqslant (m+k)n,\right. \\ &\qquad\qquad \left.\int_{(r_j-1)(t+t')}^{r_j(t+t')} V(B(s))\,\mathrm{d}s \leqslant \alpha\; (j = 1, 2, \ldots, mn)\right]\end{aligned}$$

$$\leqslant \binom{(m+k)n}{mn} (\gamma_{\sigma N}(t') + \eta)^{mn}$$

$$\leqslant \frac{c}{n^{1/2}} \left(\frac{m+k}{m}\right)^{1/2} \left(\frac{(m+k)^{m+k}}{m^m k^k}\right)^n (\gamma_{\sigma N}(t') + \eta)^{mn}$$

for some constant c, by Stirling's formula. Thus

$$\begin{aligned}
&(S_{\lambda V}((m+k)n(t+t'))\mathbf{1})(x) \\
&= \mathbf{E}^x \left[\exp\left(-\lambda \int_0^{(m+k)n(t+t')} V(B(s))\,\mathrm{d}s\right)\right] \\
&\leqslant \mathrm{e}^{-n\lambda\alpha k} + \frac{c}{n^{1/2}} \left(\frac{m+k}{m}\right)^{1/2} \left(\frac{(m+k)^{m+k}}{m^m k^k}\right)^n (\gamma_{\sigma N}(t') + \eta)^{mn}.
\end{aligned}$$

It follows that

$$\begin{aligned}
s_V(\lambda) &= \lim_{n\to\infty} \frac{\log \|S_{\lambda V}((m+k)n(t+t'))\mathbf{1}\|_\infty}{(m+k)n(t+t')} \\
&\leqslant \frac{1}{(m+k)(t+t')} \\
&\qquad \max\left\{-\lambda\alpha k, \log\left\{\frac{(m+k)^{m+k}}{m^m k^k}(\gamma_{\sigma N}(t') + \eta)^m\right\}\right\}, \qquad (4.4)
\end{aligned}$$

so

$$\begin{aligned}
s_V(\infty) &\leqslant \frac{(m+k)\log(m+k) - m\log m - k\log k + m\log(\gamma_{\sigma N}(t') + \eta)}{(m+k)(t+t')} \\
&= \frac{1}{t+t'}\left\{\log\frac{m+k}{m} + \frac{k\log m}{m+k} - \frac{k\log k}{m+k} + \right. \\
&\qquad \left. + \frac{m}{m+k}\log(\gamma_{\sigma N}(t') + \eta)\right\}.
\end{aligned}$$

Letting $\eta \to 0$ and $m \to \infty$ gives:

$$s_V(\infty) \leqslant \frac{\log \gamma_{\sigma N}(t')}{t+t'} = \frac{2\log c_{\sigma N} + N\log t' + 2\sigma t'}{2(t+t')}.$$

Letting $t' \to \infty$ gives:

$$s_V(\infty) \leqslant \sigma.$$

REMARK 4.7. It follows from (4.4) that if m, k, η, σ, and t' are chosen so that

$$\frac{(m+k)^{m+k}}{m^m k^k}(\gamma_\sigma N(t')+\eta)^m < 1,$$

then the right-hand derivative of s_V at $\lambda = 0$ satisfies:

$$s'_V(0+) \leqslant -\frac{k\eta}{(m+k)(t+t')} \inf_{E\in\mathcal{F}_\sigma} \|(V\mathbf{1}_E) * \psi_t\|_\infty \tag{4.5}$$

for all $t > 0$. We have not attempted to optimise either (3.1) or (4.5).

The next step is to convert the condition (4.3) into the condition $\inf_{E\in\mathcal{F}_\sigma} \int_E V > 0$. This is achieved in the following proposition, analogous to [6, Proposition 4.10].

PROPOSITION 4.8. *Let $\sigma < \sigma' < 0$, $t > 0$, and suppose that there is a constant $c > 0$ such that $\int_E V \geqslant c$ for all (closed) sets E in $\mathcal{F}_\sigma$. Then there is a constant $c' > 0$ such that $\|(V\mathbf{1}_{E'}) * \psi_t\|_\infty \geqslant c'$ for all (closed) sets E' in $\mathcal{F}_{\sigma'}$.*

Proof. Let K be as in Lemma 3.3, $\gamma = \inf_{|y|\leqslant K+1} \psi_t(y) > 0$, and $c' = c\gamma$. Let $E' \in \mathcal{F}_{\sigma'}$. By Lemma 3.3, there exists x_0 in E' such that $E := E' \cap \overline{B(x_0, K)} \in \mathcal{F}_\sigma$. By assumption, $\int_E V \geqslant c$. If $x \in B(x_0, 1)$, then

$$((V\mathbf{1}_{E'}) * \psi_t)(x) = \int_{E'} V(y)\psi_t(x-y)\,\mathrm{d}y \geqslant \int_E V(y)\gamma\,\mathrm{d}y \geqslant c'.$$

THEOREM 4.9. *Let $V : \mathbf{R}^N \to [0,\infty]$ be measurable, and $t > 0$. Then*

$$\begin{aligned} s_V(\infty) &= \inf\left\{\sigma < 0 : \inf_{E\in\mathcal{F}_\sigma} \int_E V > 0\right\} \\ &= \inf\left\{\sigma < 0 : \inf_{\substack{E\in\mathcal{F}_\sigma \\ E \text{ closed}}} \int_E V > 0\right\} \\ &= \inf\left\{\sigma < 0 : \inf_{E\in\mathcal{F}_\sigma} \|(V\mathbf{1}_E) * \psi_t\|_\infty > 0\right\} \\ &= \inf\left\{\sigma < 0 : \inf_{\substack{E\in\mathcal{F}_\sigma \\ E \text{ closed}}} \|(V\mathbf{1}_E) * \psi_t\|_\infty > 0\right\}. \end{aligned}$$

Proof. This follows immediately from Propositions 4.3, 4.6, and 4.8.

COROLLARY 4.10. *Let $V \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$. Then*

$$s_V(\infty) = \inf\left\{\sigma < 0 : \inf\left\{\int_\Omega V : \Omega \in \mathcal{F}_\sigma, \Omega \text{ open}\right\} > 0\right\}.$$

Proof. The proof is similar to [6, Corollary 3.8]. Let $E \in \mathcal{F}_\sigma$ and $\varepsilon > 0$. For each $n \geqslant 0$, there is an open set Ω_n such that $E_n := \{x \in E : n \leqslant |x| \leqslant n+1\} \subseteq \Omega_n$ and $\int_{\Omega_n} V < \int_{E_n} V + \varepsilon 2^{-n}$. Let $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$. Then $E \subseteq \Omega$, so $\Omega \in \mathcal{F}_\sigma$, Ω is open, and $\int_\Omega V < \int_E V + \varepsilon$. Thus $\inf\{\int_\Omega V : \Omega \in \mathcal{F}_\sigma, \Omega \text{ open}\} = \inf\{\int_E V : E \in \mathcal{F}_\sigma\}$.

In the one-dimensional case, a Borel set E belongs to $\mathcal{F}_\sigma$ if and only if E contains an interval of length greater than $\pi(-2\sigma)^{-1/2}$. Thus Theorem 4.9 reduces to the following.

COROLLARY 4.11. *Suppose that* $N = 1$. *Then*

$$s_V(\infty) = -\frac{\pi^2}{2d^2},$$

where $d = \inf\left\{\delta > 0 : \inf_{x \in \mathbf{R}} \int_x^{x+\delta} V > 0\right\}$.

For $V \in L^\infty(\mathbf{R}^N)$, it was shown in [3, Theorem 1.2, Proposition 1.4], [6, Proposition 4.19] that a condition involving the integrals of V over balls determines whether $s_V(\lambda) < 0$ or not. The following example shows that in the formulae for $s_V(\infty)$ given in Theorem 4.9, it is not possible to restrict attention to balls E if $N \geqslant 2$.

EXAMPLE 4.12. Suppose that $N \geqslant 2$. There exists V in $L^\infty(\mathbf{R}^N)_+$ such that

(i) For all $\delta > 0$, $\inf_{x \in \mathbf{R}^N} \int_{B(x,\delta)} V > 0$,

(ii) $s_V(\infty) > -\infty$.

We construct V as follows.

Let $I = (0,1)^N$, $\Gamma_n = \{x \in I : 2^n x \in \mathbf{Z}^N\}$ $(n \geqslant 0)$, and $\sigma < s(\frac{1}{2}\Delta_I)$. We claim that there exist open subsets Ω_n of I such that $\Omega_{n+1} \subseteq \Omega_n$, $\Gamma_n \subseteq \Omega_n^c$, and $s(\frac{1}{2}\Delta_{\Omega_n}) > \sigma$ for all $n \geqslant 0$. We take $\Omega_0 = I$. Given Ω_n such that $s(\frac{1}{2}\Delta_{\Omega_n}) > \sigma$, let

$$\Omega_{n+1}^\varepsilon = \Omega_n \setminus \bigcup_{x \in \Gamma_{n+1}} \overline{B(x,\varepsilon)}.$$

Now, $s(\frac{1}{2}\Delta_{\Omega_{n+1}^\varepsilon}) \to s(\frac{1}{2}\Delta_\Omega)$ as $\varepsilon \downarrow 0$ (this can easily be seen, either by variational arguments, or from properties of Brownian motion for $N \geqslant 2$). We may therefore choose $\Omega_{n+1} = \Omega_{n+1}^\varepsilon$ for $\varepsilon > 0$ sufficently small that $s(\frac{1}{2}\Delta_{\Omega_{n+1}}) > \sigma$. Thus we have constructed (Ω_n), by recursion.

Now let $e_1 = (1,0,0,\ldots,0) \in \mathbf{R}^N$, and put

$$V(x) = \begin{cases} 2^{-n} & \text{if } x - ne_1 \in \Omega_n;\ n = 0,1,2,\ldots \\ 1 & \text{otherwise.} \end{cases}$$

One sees that (i) holds. Moreover, if we take $E_n = \{x \in \mathbf{R}^N : x - ne_1 \in \Omega_n\}$, then $s(\frac{1}{2}\Delta_{E_n}) = s(\frac{1}{2}\Delta_{\Omega_n}) > \sigma$, so $E_n \in \mathcal{F}_\sigma$ for all n. However, $\int_{E_n} V \leqslant 2^{-n}$. It follows from Proposition 4.3 that $s_V(\infty) \geqslant \sigma$.

EXAMPLE 4.13. Let E be a Borel subset of $\mathbf{R}^N$, and

$$V(x) = \chi_{E^c}(x) = \begin{cases} \infty & (x \in E^c) \\ 0 & (x \in E). \end{cases}$$

The associated semigroup is the pseudo-Dirichlet semigroup $\widetilde{T}_E$. Theorem 4.9 gives

$$s(\tfrac{1}{2}\widetilde{\Delta}_E) = \sup\{s(\tfrac{1}{2}\Delta_F) : F \setminus E \text{ is null}\}.$$

Thus, there is a sequence (F_n) such that $F_n \setminus E$ is null, and $s(\frac{1}{2}\Delta_{F_n}) \uparrow s(\frac{1}{2}\widetilde{\Delta}_E)$. If $F = E \cup \bigcup_{n \geqslant 1} F_n$, then $E \subseteq F$, $F \setminus E$ is null, and $s(\frac{1}{2}\widetilde{\Delta}_E) = s(\frac{1}{2}\Delta_F)$. The existence of such a set F is implicit in the proof of [6, Proposition 5.1] (see also [4, Example 5.5] and the references cited therein).

Now suppose that $W : E \to [0, \infty]$ is measurable, and put

$$V(x) = \begin{cases} \infty & (x \in E^c) \\ W(x) & (x \in E). \end{cases}$$

Then $T_{\lambda V}$ is a Schrödinger semigroup on $L^p(E)$ with pseudo-Dirichlet boundary conditions and with potential λW. Theorem 4.9 says that the limit, as $\lambda \to \infty$, of the spectral bound of this semigroup is given by

$$\inf\left\{\sigma < 0 : \inf\left\{\int_F W : F \in \mathcal{F}_\sigma, E \setminus F \text{ is null}\right\} > 0\right\}.$$

5. Strict Monotonicity of s_V

We turn now to the question whether $s_V(\lambda)$ attains its limiting value $s_V(\infty)$ or whether on the contrary s_V is strictly decreasing. We begin by considering the situation of Proposition 4.1.

PROPOSITION 5.1. *Suppose that* $V \in L^1_{\text{loc}}(\mathbf{R}^N)$ *and* $\liminf_{|x|\to\infty} V(x) > 0$. *Then* s_V *is strictly decreasing.*

Proof. We may assume that $s_V(\infty) > -\infty$. By Proposition 2.1, there exists $\lambda_0 > 0$ such that $H_{\lambda V}$ has a (normalised) strictly positive principal eigenvector u_λ whenever $\lambda \geqslant \lambda_0$. Suppose that $\lambda_0 \leqslant \lambda_1 < \lambda_2$. Then

$$s_V(\lambda_1) \geqslant -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_{\lambda_2}|^2 - \lambda_1 \int_{\mathbf{R}^N} V u_{\lambda_2}^2$$

$$\begin{aligned} &> -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_{\lambda_2}|^2 - \lambda_2 \int_{\mathbf{R}^N} V u_{\lambda_2}^2 \\ &= s_V(\lambda_2). \end{aligned}$$

Thus s_V is strictly decreasing on $[\lambda_0, \infty)$. Since s_V is convex, it follows that it is strictly decreasing on $[0, \infty)$.

Next, we give a partial result, similar to Proposition 4.3, for bounded V.

PROPOSITION 5.2. *Suppose that* $V \in L^\infty(\mathbf{R}^N)_+$ *and that there exist* $\sigma < 0$ *and a sequence* $(E_n)_{n\geqslant 1}$ *of Borel sets such that*

(1) $E_n \in \mathcal{F}_\sigma$,
(2) $\int_{E_n} V \to 0$ *as* $n \to \infty$.

Then, for any $\lambda > -\sigma/\|V\|_\infty$,

$$s_V(\lambda) \geqslant \limsup_{n\to\infty} s(\tfrac{1}{2}\Delta - \lambda\|V\|_\infty \mathbf{1}_{E_n^c}).$$

Proof. We may assume that $\|V\|_\infty = 1$. Fix $\lambda > -\sigma$. Let $E_{nr} = E \cap B(0, r)$. As $r \to \infty$,

$$s(\tfrac{1}{2}\Delta_{E_{nr}}) \uparrow s(\tfrac{1}{2}\Delta_E), \qquad s(\tfrac{1}{2}\Delta - \lambda \mathbf{1}_{E_{nr}^c}) \uparrow s(\tfrac{1}{2}\Delta - \lambda \mathbf{1}_{E_n^c}).$$

Thus we may choose r (depending on n) such that, if $F_n = E_{nr}$, then $F_n \in \mathcal{F}_\sigma$ and

$$s_n := s(\tfrac{1}{2}\Delta - \lambda \mathbf{1}_{F_n^c}) > s(\tfrac{1}{2}\Delta - \lambda \mathbf{1}_{E_n^c}) - 2^{-n}.$$

If $W_n = \mathbf{1}_{F_n^c}$, then $s_{W_n}(\infty) \geqslant s(\frac{1}{2}\Delta_{F_n}) > \sigma$. By Proposition 2.1, $\frac{1}{2}\Delta_2 - \lambda W_n$ has a (normalised) principal eigenvector u_n. As in Proposition 4.3, $u_n \in L^\infty(\mathbf{R}^N)$, $\|u_n\|_\infty \leqslant \mathrm{e}^{-\sigma t}\|p_t\|_2$, and

$$\begin{aligned} s_V(\lambda) &\geqslant -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_n|^2 - \lambda \int_{\mathbf{R}^N} V u_n^2 \\ &\geqslant -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_n|^2 - \lambda \int_{F_n^c} u_n^2 - \lambda \int_{F_n} V u_n^2 \\ &\geqslant s_n - \lambda\, \mathrm{e}^{-2\sigma t}\|p_t\|_2^2 \int_{E_n} V. \end{aligned}$$

Letting $n \to \infty$, the result follows.

COROLLARY 5.3. *Suppose that* $N = 1$, $V \in L^\infty(\mathbf{R})_+$, *and* $s_V(\infty) < 0$. *Then* $s_V(\lambda) > s_V(\infty)$ *for all* $0 < \lambda < \infty$. *Hence,* s_V *is strictly decreasing.*

Proof. Let $\delta = \pi(-2s_V(\infty))^{-1/2}$. By Corollary 4.11, there exist x_n in $\mathbf{R}$ and $\delta_n > 0$ such that $\delta_n \uparrow \delta$ and $\int_{x_n}^{x_n+\delta_n} V \to 0$ as $n \to \infty$. Let $E_n = (x_n, x_n + \delta)$.

Since V is bounded, it follows that $\int_{E_n} V \to 0$. Moreover, $s(\frac{1}{2}\Delta_{E_n}) = s_V(\infty)$ for all n. It now follows from Propositions 5.2 and 5.1 that

$$s_V(\lambda) \geqslant s(\tfrac{1}{2}\Delta - \lambda\|V\|_\infty \mathbf{1}_{(0,\delta)^c}) > s(\tfrac{1}{2}\Delta_{(0,\delta)}) = s_V(\infty)$$

whenever $-s_V(\infty)/\|V\|_\infty < \lambda < \infty$.

We do not know whether Corollary 5.3 remains true if $N \geqslant 2$ (we shall return to this question in Remark 5.8). However, we shall now show that it may be false if V is in $L^\infty_{\text{loc}}(\mathbf{R}^N)$ for any $N \geqslant 1$. This depends on a 'decoupling' construction, which shows that given two or more potentials V_j, one can, by moving the potentials far apart, construct a potential whose spectral function behaves in a similar way to the maximum of the spectral functions s_{V_j}.

In what follows, the first coordinate of a point x in $\mathbf{R}^N$ will be denoted by ξ, and e_1 will denote the point $(1, 0, 0, \ldots, 0)$ in $\mathbf{R}^N$.

PROPOSITION 5.4. *Let* $V_1, V_2 \in L^\infty(\mathbf{R}^N)_+$, $a, b \in \mathbf{R}$, *and* $\varepsilon > 0$. *There exist* W *in* $L^\infty(\mathbf{R}^N)_+$ *and* $c > a - b$ *such that*

(1) $\|W\|_\infty \leqslant \max(\|V_1\|_\infty, \|V_2\|_\infty)$,
(2) $W(x) = V_1(x)$ *if* $\xi \leqslant a$,
(3) $W(x + ce_1) = V_2(x)$ *if* $\xi \geqslant b$,
(4) $s_W(1) \leqslant \max(s_{V_1}(1), s_{V_2}(1)) + \varepsilon$.

Proof. For $n \geqslant 1$, let $c_n = a + n + 2 - b$, and define W_n as follows:

$$W_n(x) = \begin{cases} V_1(x) & \text{if } \xi \leqslant a \\ \max(\|V_1\|_\infty, \|V_2\|_\infty) & \text{if } a < \xi < a + n + 2 \\ V_2(x - c_n e_1) & \text{if } \xi \geqslant a + n + 2. \end{cases}$$

We will show that $\limsup_{n\to\infty} s_{W_n}(1) \leqslant \max(s_{V_1}(1), s_{V_2}(1))$. It then follows that we may take $W = W_n$ and $c = c_n$ for n sufficiently large.

There exists u_n in $C_c^\infty(\mathbf{R}^N)$ such that $\int_{\mathbf{R}^N} u_n^2 = 1$ and

$$-\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_n|^2 - \int_{\mathbf{R}^N} W_n u_n^2 \geqslant s_{W_n}(1) - 2^{-n}. \tag{5.1}$$

Let $\gamma = \max(\|V_1\|_\infty, \|V_2\|_\infty) + 2$. Then $s_{W_n}(1) \geqslant 2 - \gamma$, so

$$\int_{\mathbf{R}^N} \left(\tfrac{1}{2}|\nabla u_n|^2 + u_n^2\right) \leqslant \gamma.$$

Hence there exists $r_n \in \{1, 2, \ldots, n\}$ such that

$$\int_{a+r_n-1 \leqslant \xi \leqslant a+r_n+2} \left(\tfrac{1}{2}|\nabla u_n|^2 + u_n^2\right) \leqslant \frac{3\gamma}{n}.$$

Let $\varphi\colon \mathbf{R} \to [0,1]$ be a fixed C^∞-function such that

$$\varphi(t) = \begin{cases} 1 & \text{if } t \leqslant -1 \\ 0 & \text{if } 0 \leqslant t \leqslant 1 \\ 1 & \text{if } t \geqslant 2. \end{cases}$$

Let

$$v_n(x) = \varphi(\xi - (a + r_n))u_n(x).$$

As $n \to \infty$,

$$\int_{\mathbf{R}^N} (u_n - v_n)^2 \to 0, \quad \int_{\mathbf{R}^N} |\nabla u_n - \nabla v_n|^2 \to 0,$$

$$\int_{\mathbf{R}^N} W_n(u_n^2 - v_n^2) \to 0. \tag{5.2}$$

Let

$$\begin{aligned} v_{n1} &= v_n \, \mathbf{1}_{[\xi \leqslant a + r_n]}, \\ v_{n2} &= v_n \, \mathbf{1}_{[\xi \geqslant a + r_n]}. \end{aligned}$$

Since $W_n \geqslant V_1$ on $\operatorname{supp} v_{n1}$,

$$\begin{aligned} -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla v_{n1}|^2 - \int_{\mathbf{R}^N} W_n v_{n1}^2 &\leqslant s_{V_1}(1) \int_{\mathbf{R}^N} v_{n1}^2 \\ &= s_{V_1}(1) \int_{\xi \leqslant a + r_n} v_n^2. \end{aligned} \tag{5.3}$$

Since $W_n \geqslant \widetilde{V}_{n2}$ on $\operatorname{supp} v_{n2}$, where $\widetilde{V}_{n2}(x) = V_2(x - c_n e_1)$,

$$\begin{aligned} -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla v_{n2}|^2 - \int_{\mathbf{R}^N} W_n v_{n2}^2 &\leqslant s_{\widetilde{V}_{n2}}(1) \int_{\mathbf{R}^N} v_{n2}^2 \\ &= s_{V_2}(1) \int_{\xi \geqslant a + r_n} v_n^2. \end{aligned} \tag{5.4}$$

Adding (5.3) and (5.4) gives

$$-\frac{1}{2}\int_{\mathbf{R}^N} |\nabla v_n|^2 - \int_{\mathbf{R}^N} W_n v_n^2 \leqslant \max(s_{V_1}(1), s_{V_2}(1)) \int_{\mathbf{R}^N} v_n^2.$$

Letting $n \to \infty$, using (5.1) and (5.2), gives

$$\limsup_{n \to \infty} s_{W_n}(1) \leqslant \max(s_{V_1}(1), s_{V_2}(1)).$$

PROPOSITION 5.5. *Let $(V_n)_{n\geqslant 1}$ be a sequence in $L^\infty(\mathbf{R}^N)_+$, $(K_n)_{n\geqslant 1}$ be a sequence of compact sets in $\mathbf{R}^N$, and $(\sigma_n)_{n\geqslant 1}$ be a sequence of real numbers such that $s_{V_n}(1) < \sigma_n \leqslant \sigma_{n+1}$ for all $n\geqslant 1$. There exist sequences $(a_n)_{n\geqslant 0}$ and $(c_n)_{n\geqslant 1}$ in $\mathbf{R}$, and $(W_n)_{n\geqslant 1}$ in $L^\infty(\mathbf{R}^N)_+$ such that, for all $n\geqslant 1$,*

(1) $a_{n-1} < \xi + c_n < a_n - 1$ *for all x in K_n,*
(2) $\|W_n\|_\infty \leqslant \max_{1\leqslant j\leqslant n} \|V_j\|_\infty$,
(3) $W_{n+1}(x) = W_n(x)$ *if* $\xi < a_n$,
(4) $W_n(x + c_n e_1) = V_n(x)$ *if* $x \in K_n$,
(5) $s_{W_n}(1) < \sigma_n$.

Proof. Choose a_0 and a_1 such that $a_0 < \xi < a_1 - 1$ for all x in K_1. Let $c_1 = 0$, $W_1 = V_1$. Then (1), (2), (4) and (5) are satisfied for $n = 1$.

Let $m\geqslant 0$. Suppose that a_m and W_m have been chosen so that (1), (2), (4) and (5) hold for $n = 0, 1, \ldots, m$ and (3) holds for $n = 0, 1, \ldots, m-1$. Choose b_m such that $\xi > b_m$ for all x in K_{m+1}. By Proposition 5.4, there exist W_{m+1} in $L^\infty(\mathbf{R}^N)_+$ and $c_{m+1} > a_m - b_m$ such that (2), (4) and (5) hold for $n = m + 1$ and (3) holds for $n = m$. Choose a_{m+1} such that $\xi + c_{m+1} < a_{m+1} - 1$ for all x in K_{m+1}. Then (1) holds for $n = m + 1$.

By recursion, the proof is complete.

PROPOSITION 5.6. *Let $(V_n)_{n\geqslant 1}$ be a sequence in $L^\infty(\mathbf{R}^N)_+$ and σ be a real number such that $s_{V_n}(1) < \sigma$ for all $n\geqslant 1$. There exists W in $L^\infty_{\mathrm{loc}}(\mathbf{R}^N)_+$ such that $s_W(1) \leqslant \sigma$ and $s_W(\infty)\geqslant \limsup_{n\to\infty} s_{V_n}(\infty)$. If $\sup_n \|V_n\|_\infty < \infty$, then W may be chosen to be in $L^\infty(\mathbf{R}^N)$.*

Proof. There exist u_n in $C_c^\infty(\mathbf{R}^N)$ such that $\int_{\mathbf{R}^N} u_n^2 = 1$ and

$$-\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_n|^2 - n\int_{\mathbf{R}^N} V_n u_n \geqslant s_{V_n}(n) - \frac{1}{n}.$$

Let $K_n = \operatorname{supp} u_n$ and $\sigma_n = \sigma$. Let (a_n), (c_n), and (W_n) be as in Proposition 5.5, and let $W(x) = \lim_{n\to\infty} W_n(x)$ (this exists by (3) of Proposition 5.5, since $a_n > a_{n-1} + 1$, so $a_n \to \infty$ as $n \to \infty$). If $u \in C_c^\infty(\mathbf{R}^N)$ and $\int_{\mathbf{R}^N} u^2 = 1$, then for all sufficiently large n, $W = W_n$ on $\operatorname{supp} u$, so

$$-\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 - \int_{\mathbf{R}^N} W u^2 \leqslant s_{W_n}(1) < \sigma.$$

Hence $s_W(1) \leqslant \sigma$.

Let $\tilde{u}_n(x) = u_n(x - c_n e_1)$. Then

$$\begin{aligned} s_W(n) &\geqslant -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla \tilde{u}_n|^2 - n\int_{\mathbf{R}^N} W\tilde{u}_n^2 \\ &= -\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_n|^2 - n\int_{\mathbf{R}^N} V_n u_n^2 \end{aligned}$$

$$\geqslant s_{V_n}(n) - \frac{1}{n}$$
$$\geqslant s_{V_n}(\infty) - \frac{1}{n},$$

where we used (4) of Proposition 5.5 in the second line. Letting $n \to \infty$, it follows that $s_W(\infty) \geqslant \limsup_{n\to\infty} s_{V_n}(\infty)$.

EXAMPLE 5.7. Let $N \geqslant 1$. There exists V in $L^\infty_{\mathrm{loc}}(\mathbf{R}^N)_+$ such that $s_V(\lambda) = s_V(1) < 0$ for all $\lambda \geqslant 1$. We construct V as follows.

Let $\sigma = s(\frac{1}{2}\Delta_B)$, where $B = B(0,1)$. Let $B_n = B(0, 1-2^{-n})$, so $s(\frac{1}{2}\Delta_{B_n}) = \sigma(1-2^{-n})^{-2}$. By Proposition 4.1, $s_{1_{B_n^c}}(\lambda) \to \sigma(1-2^{-n})^{-2}$ as $\lambda \to \infty$, so we may choose $\beta_n > 0$ such that $s_{1_{B_n^c}}(\beta_n) < \sigma$. Now Proposition 5.6 may be applied with $V_n = \beta_n 1_{B_n^c}$. It shows that there exists V in $L^\infty_{\mathrm{loc}}(\mathbf{R}^N)_+$ such that $s_V(1) \leqslant \sigma$ and $s_V(\infty) \geqslant \limsup_{n\to\infty} s(\frac{1}{2}\Delta_{B_n}) = \sigma$.

REMARK 5.8. The potential V constructed in Example 5.7 is necessarily unbounded, since $\beta_n \to \infty$. Although we do not know whether Corollary 5.3 is valid if $N > 1$, we can illuminate the question as follows.

For $\sigma < 0$ and $\lambda > 0$, let

$$\nu(\sigma, \lambda) = \inf\{s(\tfrac{1}{2}\Delta - \lambda 1_{\Omega^c}) : \Omega \in \mathcal{F}_\sigma, \Omega \text{ bounded open}\}.$$

Clearly, $\nu(\sigma,\lambda) \geqslant \max(\sigma, -\lambda)$.

Consider the hypothesis

$$\text{For all } \sigma < 0 \text{ and } \lambda > 0, \nu(\sigma,\lambda) > \sigma. \tag{5.5}$$

We do not know whether (5.5) is true. Suppose for the moment that it is. For any $\alpha > 0$,

$$\alpha\nu(\sigma, \alpha\lambda) = \nu(\alpha\sigma, \lambda),$$

and ν is continuous with respect to λ. Given $\sigma < 0$ and $\lambda > 0$, it now follows from (5.5) that by taking $\sigma_1 = \alpha\sigma$ and letting $\alpha \downarrow 1$, we can find $\sigma_1 < \sigma$ such that $\nu(\sigma_1, \lambda) > \sigma$. Let $V \in L^\infty(\mathbf{R}^N)_+$, and suppose that $s_V(\infty) < 0$. By Corollary 4.10, there is a sequence (Ω_n) of bounded open sets such that $s(\frac{1}{2}\Delta_{\Omega_n}) \to s_V(\infty)$ and $\int_{\Omega_n} V \to 0$. Given $\lambda > 0$, as above there exists $\sigma_1 < s_V(\infty)$ such that $\nu(\sigma_1, \lambda\|V\|_\infty) > s_V(\infty)$. Then, for all large n, $\Omega_n \in \mathcal{F}_{\sigma_1}$, so

$$s(\tfrac{1}{2}\Delta - \lambda\|V\|_\infty 1_{\Omega_n^c}) \geqslant \nu(\sigma_1, \lambda\|V\|_\infty) > s_V(\infty).$$

By Proposition 5.2, $s_V(\lambda) > s_V(\infty)$.

On the other hand, suppose that (5.5) is false (for some $N \geqslant 2$). Then there exist $\sigma < 0$, $\lambda_0 > 0$ and a sequence (Ω_n) of bounded open sets such that $s(\frac{1}{2}\Delta_{\Omega_n}) \downarrow \sigma$ and $s(\frac{1}{2}\Delta - \lambda_0 1_{\Omega_n^c}) \to \sigma$. Choose $0 < \alpha_n < 1$ such that

$$\sigma - \frac{1}{n} < \frac{s(\frac{1}{2}\Delta_{\Omega_n})}{\alpha_n^2} < \sigma, \qquad s(\tfrac{1}{2}\Delta - \lambda_0 1_{\Omega_n^c}) < \alpha_n^2 \sigma.$$

Let $V_n = \alpha_n^{-2}\lambda_0 \mathbf{1}_{\alpha_n \Omega_n^c}$. Then

$$\begin{aligned} s_{V_n}(1) &= \frac{s(\frac{1}{2}\Delta - \lambda_0 \mathbf{1}_{\Omega_n^c})}{\alpha_n^2} < \sigma, \\ s_{V_n}(n) \geqslant s(\tfrac{1}{2}\Delta_{\alpha_n\Omega_n}) &= \frac{s(\frac{1}{2}\Delta_{\Omega_n})}{\alpha_n^2} > \sigma - \frac{1}{n}, \\ \sup_n \|V_n\|_\infty &= \sup_n \frac{\lambda_0}{\alpha_n^2} < \infty. \end{aligned}$$

It then follows from Proposition 5.6 that there exists V in $L^\infty(\mathbf{R}^N)_+$ such that $s_V(\lambda) = \sigma$ for all $\lambda \geqslant 1$.

Thus, given $N \geqslant 2$, the question whether Corollary 5.3 holds is equivalent to the question whether (5.5) is true.

6. Generalisations

6.1. ELLIPTIC OPERATORS

It should be clear to the reader that all the results of this paper remain true if the operator $\frac{1}{2}\Delta$ is replaced throughout by any symmetric, strongly elliptic, operator H on $\mathbf{R}^N$:

$$H = \sum_{i,j=1}^{N} D_i(a_{ij} D_j),$$

where $a_{ij} = a_{ji} \in L^\infty(\mathbf{R}^N)$ and there is a constant $\gamma > 0$ such that $\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geqslant \gamma \sum_{i=1}^N \xi_i^2$ for all x in $\mathbf{R}^N$ and ξ in $\mathbf{R}^N$. The quadratic form

$$\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2$$

is replaced by the form

$$\sum_{i,j=1}^{N} \int_{\mathbf{R}^N} a_{ij}(D_i u)(D_j u),$$

Brownian motion is replaced by the diffusion process associated with H [17], and the class $\mathcal{F}_\sigma$ is defined in terms of H instead of $\frac{1}{2}\Delta$. Since the heat kernel satisfies Gaussian bounds [14], Simon's argument (see Lemma 3.2) shows that the spectral bound of the semigroups on $L^p(\mathbf{R}^N)$ is independent of p; in fact, the spectrum is independent of p [2].

Now, suppose that

$$H = \sum_{i,j=1}^{N} D_i(a_{ij} D_j) + \sum_{i=1}^{N} b_i D_i + c$$

is non-symmetric and strongly elliptic, and the coefficients belong to $L^\infty(\mathbf{R}^N)$. If the coefficients are sufficiently smooth and c is sufficiently negative, then H is associated with interpolating semigroups on $L^p(\mathbf{R}^N)$ for each $1 \leqslant p \leqslant \infty$, and with a diffusion process [22]. Then Theorem 4.9 remains valid, provided that the spectral function $s_V(\lambda)$ is now interpreted as the spectral bound of the appropriate semigroup on $L^\infty(\mathbf{R}^N)$ (it is no longer clear that the spectral bound is independent of p). The quadratic form technique involved in the proof of Proposition 4.3 is no longer appropriate, but it can be replaced by the argument outlined in Remark 4.5.

6.2. SINGULAR POTENTIALS

Another generalisation is to allow singular potentials. Thus the function V can be replaced by a positive measure μ (defined on Borel subsets of $\mathbf{R}^N$, but not necessarily σ-finite) such that $\mu(E) = 0$ for all polar sets E (for details of this case, see, for example, [7, 33, 35]). The quadratic form $\mathbf{a}_V$ is replaced by the form $\mathbf{a}_\mu$, given by

$$\begin{aligned} D(\mathbf{a}_\mu) &= \left\{u \in W^{1,2}(\mathbf{R}^N) : \textstyle\int_{\mathbf{R}^N} \widetilde{u}^2 \,\mathrm{d}\mu < \infty\right\}, \\ \mathbf{a}_\mu(u) &= \tfrac{1}{2} \textstyle\int_{\mathbf{R}^N} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbf{R}^N} \widetilde{u}^2 \,\mathrm{d}\mu. \end{aligned}$$

where $\widetilde{u}$ is a quasi-continuous version of u. Then

$$\begin{aligned} s_\mu(\lambda) &= \sup\left\{-\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 \,\mathrm{d}x - \lambda \int_{\mathbf{R}^N} \widetilde{u}^2 \,\mathrm{d}\mu : u \in W^{1,2}(\mathbf{R}^N),\right. \\ &\qquad\qquad \left. \int_{\mathbf{R}^N} u^2 \,\mathrm{d}x = 1\right\} \\ &= \lim_{t\to\infty} \frac{1}{t} \log\left\{\operatorname*{ess\,sup}_{x\in\mathbf{R}^N} \mathbf{E}^x\left[\exp(-A_\mu(t))\right]\right\}, \end{aligned}$$

where $\{A_\mu(t) : t \geqslant 0\}$ is the additive process associated with μ (see [34, Section 4]). Theorem 4.9 remains valid in this context in the form

$$s_\mu(\infty) = \inf\{\sigma < 0 : \inf\{\mu(E) : E \in \mathcal{F}_\sigma\} > 0\}.$$

To prove this version of Theorem 4.9, one has to establish the identity

$$\mathbf{E}^x\left[A_\mu(t)\right] = \int_{\mathbf{R}^N} \psi_t(x-y)\,\mathrm{d}\mu(y). \tag{6.1}$$

If μ is a Dynkin measure, this is given by [16, Theorem 8.4]. The general case follows by various approximation arguments. Once (6.1) is established, the necessary modifications to the proof of Theorem 4.9 are fairly routine.

Suppose that $V : \mathbf{R}^N \to [0, \infty]$ is measurable. If μ is defined by

$$\mu(E) = \int_E V \, dx,$$

then we return to the situation considered in Section 4.

Next, suppose that Ω is an open subset of $\mathbf{R}^N$ and $V : \Omega \to [0, \infty]$ is measurable. Define μ by

$$\mu(E) = \begin{cases} \int_E V \, dx & \text{if } E \setminus \Omega \text{ is polar}, \\ \infty & \text{otherwise}. \end{cases}$$

Then the semigroups being considered are the Schrödinger semigroups on $L^p(\Omega)$ with potential λV and with Dirichlet boundary conditions on Ω. Formally, the generator is $\frac{1}{2}\Delta_\Omega - \lambda V$, where Δ_Ω is the Laplacian with Dirichlet boundary conditions on Ω. Let $s_{V\Omega}(\lambda)$ be the spectral bound of the generator. Once again, this quantity is independent of p. The result which Theorem 4.9 gives is

$$\lim_{\lambda \to \infty} s_{V\Omega}(\lambda) = \inf \left\{ \sigma < 0 : \inf \left\{ \int_E V \, dx : E \in \mathcal{F}_\sigma, E \setminus \Omega \text{ is polar} \right\} > 0 \right\}.$$

Acknowledgements

The first author is grateful to the Science and Engineering Research Council (UK) for financial support during a visit to Oxford during which this work was carried out.

References

1. Aliprantis, C. D. and Burkinshaw, O.: *Positive Operators*, Academic Press, New York, 1985.
2. Arendt, W.: 'Gaussian estimates and interpolation of the spectrum in L^p', *Diff. Int. Equations* **7** (1994), 1153–1168.
3. Arendt, W. and Batty, C. J. K.: 'Exponential stability of a diffusion equation with absorption', *Diff. Int. Equations* **6** (1993), 1009–1024.
4. Arendt, W. and Batty, C. J. K.: 'Absorption semigroups and Dirichlet boundary conditions', *Math. Ann.* **295** (1993), 427–448.
5. Arendt, and Batty, C. J. K.: 'The spectral function and principal eigenvalues for Schrödinger semigroups', *Potential Anal.*, to appear.
6. Batty, C. J. K.: 'Asymptotic stability of Schrödinger semigroups: path integral methods', *Math. Ann.* **292** (1992), 457–492.
7. Baxter, J., Dal Maso, G. and Mosco, U.: 'Stopping times and Γ-convergence', *Trans. Amer. Math. Soc.* **303** (1987), 1–38.
8. Brown, K. J., Cosner, C. and Fleckinger, J.: 'Principal eigenvalues for problems with indefinite weight functions', *Proc. Amer. Math. Soc.* **109** (1990), 147–155.
9. Brown, K. J., Daners, D. and López-Gómez, L.: 'Change of stability for Schrödinger semigroups', Preprint, 1993.
10. Brown, K. J. and Tertikas, A.: 'The existence of principal eigenvalues for problems with indefinite weight function on $\mathbb{R}^k$', *Proc. Royal Soc. Edinburgh* **123A** (1993), 561–569.
11. Daners, D.: 'Principal eigenvalues for some periodic-parabolic operators on $\mathbb{R}^N$ and related topics', Preprint, 1993.

12. Daners, D. and Koch Medina, P.: 'Superconvexity of the evolution operator and parabolic eigenvalue problems on $\mathbb{R}^N$', *Diff. Int. Equations* 7 (1994), 235–255.
13. Daners, D. and Koch Medina, P.: 'Exponential stability, change of stability and eigenvalue problems for linear time-periodic parabolic equations on $\mathbb{R}^N$', *Diff. Int. Equations* 7 (1994), 1265–1284.
14. Davies, E. B.: *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, Cambridge, 1989.
15. Deny, J.: 'Les potentiels d'énergie finie', Acta Math. **82** (1950), 107–183.
16. Dynkin, E. B.: *Markov Processes I*, Springer, Berlin, 1965.
17. Fukushima, M.: *Dirichlet Forms and Markov Processes*, Kodansha, Tokyo, 1980.
18. Hedberg, L. I.: 'Spectral synthesis in Sobolev spaces and uniqueness of solutions of the Dirichlet problem', *Acta Math.* **147** (1981), 237–264.
19. Hempel, R. and Voigt, J.: 'The spectrum of a Schrödinger operator in $L_p(\mathbf{R}^\nu)$ is p-independent', *Comm. Math. Phys.* **104** (1986), 243–250.
20. Herbst, I. W. and Zhao, Z.: 'Sobolev spaces, Kac regularity and the Feynman–Kac formula', *Sem. Stochastic Processes* Princeton, 1987, *Prog. Prob. Stat.* **15**, Birkhäuser, Basel, 1988, 171–191.
21. Hess, P. and Kato, T.: 'On some linear and nonlinear eigenvalue problems with an indefinite weight function', *Comm. Partial Diff. Equations* **5** (1980), 999–1030.
22. Ma, Z. M. and Röckner, M.: *Introduction to the Theory of Non-Symmetric Dirichlet Forms*, Springer, Berlin, 1992.
23. McKean, H. P.: '$-\Delta$ plus a bad potential', *J. Math. Phys.* **18** (1977), 1277–1279.
24. Nagel, R. (ed.): 'One-parameter semigroups of positive operators', *Lecture Notes Math. 1184* Springer, Berlin, 1986.
25. Nussbaum, R. D. and Pinchover, Y.: 'On variational principles for the generalized principal eigenvalue of second order elliptic operators and some applications', *J. Anal. Math.* **59** (1992), 161–177.
26. Ouhabaz, E.-M.: 'Gaussian estimates and holomorphy of semigroups', *Proc. Amer. Math. Soc.* **123** (1995), 1465–1474.
27. Pinchover, Y.: 'Criticality and ground states for second order elliptic equations', *J. Differential Equations* **80** (1989), 237–250.
28. Pinchover, Y.: 'On criticality and ground states for second order elliptic equations II', *J. Differential Equations* **87** (1990), 353–364.
29. Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, New York, 1978.
30. Simon, B.: *Functional Integration and Quantum Physics*, Academic Press, New York, 1979.
31. Simon, B.: 'Brownian motion, L^p properties of Schrödinger operators, and the localization of binding', *J. Funct. Anal.* **35** (1980), 215–229.
32. Simon, B.: 'Schrödinger semigroups', *Bull. Amer. Math. Soc.* **7** (1982), 447–526.
33. Stollmann, P. and Voigt, J.: 'Perturbations of Dirichlet forms by measures', Preprint, 1992.
34. Sturm, K.-T.: 'Measures charging no polar sets and additive functionals of Brownian motion', *Forum Math.* **4** (1992), 257–297.
35. Sturm, K.-T.: 'Schrödinger operators with arbitrary nonnegative potentials', in: B. W. Schulze and M. Demuth (eds) *Operator Calculus and Spectral Theory*, Birkhäuser, Basel, 1992, 291–306.