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Almost Periodic Solutions of First- and Second-Order Cauchy Problems

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Almost periodicity of solutions of first- and second-order Cauchy problems on the real line is proved under the assumption that the imaginary (resp. real) spectrum of the underlying operator is countable. Related results have been obtained by Ruess–Vũ and Basit. Our proof uses a new idea. It is based on a factorisation method which also gives a short proof (of the vector-valued version) of Loomis' classical theorem, saying that a bounded uniformly continuous function from $\mathbb{R}$ into a Banach space X with countable spectrum is almost periodic if $c_0 \not\subset X$. Our method can also be used for solutions on the half-line. This is done in a separate paper. © 1997 Academic Press

1. INTRODUCTION

A central subject in the theory of differential equations in Banach spaces is to find criteria for almost periodicity of solutions, see e.g. the monograph of Levitan–Zhikov [LZ] and recent articles by Ruess–Vũ [RV] and B. Basit [Bas1].

One interesting criterion is countability of the imaginary spectrum of the operator or the spectrum of the function. For example, by a central result of the theory [LZ, p. 92], a bounded uniformly continuous function of $\mathbb{R}$ into a Banach space X with countable spectrum is almost periodic whenever $c_0 \not\subset X$. The geometric condition on X is related to Kadets' theorem [LZ, p. 86] on primitives of almost periodic functions.

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The present paper is based on a new idea of proof. We consider the shift group $(S(t))_{t\in\mathbb{R}}$ on $BUC(\mathbb{R}, X)$ (the space of all bounded uniformly continuous functions from $\mathbb{R}$ into X) and the induced group $(\bar{S}(t))_{t\in\mathbb{R}}$ on the quotient space $BUC(\mathbb{R}, X)/AP(\mathbb{R}, X)$ with generator $\tilde{B}$ (where $AP(\mathbb{R}, X)$ denotes the space of all almost periodic functions from $\mathbb{R}$ into X). Then Kadets' result can be reformulated by saying that $c_0 \not\subset X$ if and only if $\tilde{B}$ has no point spectrum, and we obtain the above mentioned result from a well-known spectral property of bounded groups (Gelfand's theorem). The same argument also works in the situation considered by Ruess and Vũ [RV], where the geometric condition on X is replaced by an ergodicity condition on the function. This short factorisation proof given in Section 3 can also be applied to the inhomogeneous Cauchy problem (Section 4). In contrast to Levitan–Zhikov [LZ, Chap. 6, Theorem 5, p. 94] and Ruess–Vũ [RV, Theorem 4.4] we do not assume that the underlying operator generates a semigroup.

More technical problems have to be overcome in our result for the second order Cauchy problem (Theorem 4.5) which is based on the same technique. It implies, for example, that a bounded cosine function whose generator has countable spectrum is almost periodic whenever $c_0 \not\subset X$. In Section 5 we show that every bounded uniformly continuous solution of the first or second order Cauchy problem is almost periodic whenever the imaginary spectrum $\sigma(A)\cap i\mathbb{R}$ consists merely of poles. It is remarkable that here no geometric assumption on the space is needed.

In this paper we merely consider functions defined on the real line (except in Section 5 where also the Cauchy problem on the half-line is considered when the imaginary spectrum consists merely of poles). The factorisation technique also works on the half-line and we present the corresponding results in a separate paper [AB2]. It should be mentioned that those results are different since the half-line spectrum is much smaller than the line-spectrum, in general. Also the geometric condition $c_0 \not\subset X$ is typical for the line case and has no significance in the half-line case for the kind of results which interest us here.

2. PRELIMINARIES

Let A be a linear operator on a complex Banach space X. By $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ap}(A)$, $\rho(A)$ we denote the spectrum, point spectrum, approximate point spectrum and resolvent set, respectively, of A. For $\lambda\in\rho(A)$ we let $R(\lambda, A)=(\lambda-A)^{-1}$.

Assume in the following that A generates a bounded group U. We will use the following well-known result.

THEOREM 2.1 (Gelfand). *The following assertions hold*:

(a) $\sigma(A) \neq \varnothing$ *if* $X \neq 0$;

(b) $U(t) = I$ $(t \in \mathbb{R})$ *if* $\sigma(A) = \{0\}$;

(c) *every isolated point in* $\sigma(A)$ *is an eigenvalue.*

There are many different proofs of this theorem, by the theory of spectral subspaces (see [Arv] or [Da, Chap. 8]), or by Laplace transforms [AP, Theorem 3.11]. Gelfand actually proved the analogous result of Theorem 2.1 for isometric invertible operators (from which Theorem 2.1 follows by the weak spectral mapping theorem [Na, A-III Theorem 7.4, p. 91]). For a particularly elementary proof of this case see [AR, Theorem 1.1].

Next we recall some facts on the Arveson spectrum of U. For $f \in L^1(\mathbb{R})$ one defines the operator $f(U) \in \mathscr{L}(X)$ by

$$f(U)\, x = \int_{-\infty}^{+\infty} f(t)\, U(t)\, x\, dt \qquad (x \in X).$$

The *Arveson spectrum* of U is defined by

$$\begin{aligned}\operatorname{sp}(U) = \{\xi \in \mathbb{R};\ \forall \varepsilon > 0\ \exists f \in L^1(\mathbb{R}) \text{ such that}\\ \operatorname{supp} \bar{\mathscr{F}} f \subset (\xi - \varepsilon, \xi + \varepsilon) \text{ and } f(U) \neq 0\}.\end{aligned}$$

Here we let $(\bar{\mathscr{F}} f)(s) = \int_{-\infty}^{+\infty} e^{is} f(t)\, dt$. Then it is known that

$$i \operatorname{sp}(U) = \sigma(A) \tag{2.1}$$

(see [Da, Theorem 8.19, p. 213]).

For $y \in X$ denote by $X_y = \overline{\operatorname{span}}\{U(t)y \colon t \in \mathbb{R}\}$ the smallest closed subspace of X which is invariant under U. Let $U_y(t) = U(t)_{|X_y}$ and denote by A_y the generator of U_y on X_y.

The *Arveson spectrum of y with respect to U* is defined by

$$\begin{aligned}\operatorname{sp}^U(y) := \{\xi \in \mathbb{R} \colon \forall \varepsilon > 0\ \exists f \in L^1(\mathbb{R}) \text{ such that}\\ \operatorname{supp} \bar{\mathscr{F}} f \subset (\xi - \varepsilon, \xi + \varepsilon) \text{ and } f(U) \neq 0\}.\end{aligned} \tag{2.2}$$

Note that for $f \in L^1(\mathbb{R})$, $f(U)\, U(t) y = f_{-t}(U) y$, where $f_{-t}(s) = f(s-t)$. Since $(\bar{\mathscr{F}} f_{-t})(s) = e^{its} (\bar{\mathscr{F}} f)(s)$ it is clear that $\operatorname{sp}^U(y) = \operatorname{sp}(U_y)$. Thus, by (2.1),

$$i \operatorname{sp}^U(y) = \sigma(A_y). \tag{2.3}$$

Next we consider a special group. By $BUC(\mathbb{R}, X)$ we denote the Banach space of all bounded uniformly continuous functions on $\mathbb{R}$ with values in X with the uniform norm

$$\|u\| = \sup_{t \in \mathbb{R}} \|u(t)\|.$$

We consider the shift group S on $BUC(\mathbb{R}, X)$ given by

$$(S(t)\,u) = u(t+s) \qquad (t, s \in \mathbb{R}, u \in BUC(\mathbb{R}, X))$$

and denote by B the generator of S. The domain of B consists of all $u \in BUC(\mathbb{R}, X) \cap C^1(\mathbb{R}, X)$ such that $u' \in BUC(\mathbb{R}, X)$ and $Bu = u'$.

For $u \in BUC(\mathbb{R}, X)$ we denote by $\operatorname{sp}(u)$ the Arveson spectrum of u with respect to S. Thus

$$i \operatorname{sp}(u) = \sigma(B_u), \tag{2.4}$$

where B_u is the part of B in

$$BUC(\mathbb{R}, X)_u := \overline{\operatorname{span}}\{S(t)\,u : t \in \mathbb{R}\}.$$

Denote by $\mathscr{F}$ the Fourier transform

$$(\mathscr{F}f)(s) = \int_{-\infty}^{+\infty} e^{-ist} f(t)\,dt$$

($s \in \mathbb{R}$, $f \in L^1(\mathbb{R})$). Then $\mathscr{F}f = \overline{\mathscr{F}}\check{f}$, where $\check{f}(s) = f(-s)$. Thus, for $u \in B$, $f \in L^1(\mathbb{R})$,

$$(f(S)\,u)(t) = \int_{-\infty}^{+\infty} f(s)\,u(s+t)\,ds = (\check{f} * u)(t).$$

Now (2.2) becomes

$$\operatorname{sp}(u) = \{\xi \in \mathbb{R} : \forall \varepsilon > 0\ \exists f \in L^1(\mathbb{R}) \text{ such that}$$
$$\operatorname{supp} \mathscr{F}f \subset (\xi - \varepsilon, \xi + \varepsilon) \text{ and } f * u \neq 0\}.$$

This is sometimes called the *Beurling spectrum* of u. It coincides with the *Carleman spectrum* (by [P, Proposition 0.5, p. 22]), that is,

$$\operatorname{sp}(u) = \{\xi \in \mathbb{R} : \xi \text{ is not regular}\}, \tag{2.5}$$

where $\xi \in \mathbb{R}$ is called a *regular point* if the Carleman transform

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} u(t)\, dt & (\operatorname{Re} \lambda > 0) \\ -\int_0^\infty e^{\lambda t} u(-t)\, dt & (\operatorname{Re} \lambda < 0) \end{cases} \tag{2.6}$$

has a holomorphic extension to a neighborhood of $i\xi$.

The advantage of (2.5) is that it is usually easier to calculate than the Arveson or Beurling spectrum of u. For example, suppose that U is a bounded C_0-group on X generated by A. Let $y \in X$, and consider the function $u \in BUC(\mathbb{R}, X)$ given by $u(t) = U(t)y$. It is immediate from the definitions that

$$\operatorname{sp}(u) = \operatorname{sp}^U(y) \tag{2.7}$$

(see also [V, Theorem 3.4]). In addition,

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} U(t) y\, dt = R(\lambda, A) y & (\operatorname{Re} \lambda > 0) \\ -\int_0^\infty e^{\lambda t} U(-t) y\, dt = -R(-\lambda, -A) y = R(\lambda, A) y & (\operatorname{Re} \lambda < 0) \end{cases}.$$

The following lemma shows that in this case, in order that $i\xi$ is a regular point of $\hat{u}$, it suffices that a holomorphic extension of $\hat{u}$ from the right-half-plane should exist (see also [BNR1, Theorem 2.2] or [Ne, Lemma 5.3.3]).

LEMMA 2.2. *Let $y \in X$, $\eta \in \mathbb{R}$. Assume that $R(\lambda, A) y$ $(\operatorname{Re} \lambda > 0)$ has a holomorphic extension to a neighborhood of $i\eta$. Then $i\eta \notin \sigma(A_y)$; equivalently, $\eta \notin \operatorname{sp}^U(y)$.*

Proof. Let V be a neighborhood of $i\eta$ and let $h\colon V \to X$ be holomorphic such that $h(\lambda) = R(\lambda, A) y$ for $\lambda \in V$, $\operatorname{Re} \lambda > 0$. Then $R(\lambda, A) y = h(\lambda)$ also for $\lambda \in V$, $\operatorname{Re} \lambda < 0$. [In fact, $(\lambda - A)\, R(1, A)\, h(\lambda) = R(1, A) y$ for $\operatorname{Re} \lambda > 0$, $\lambda \in V$. Since $\lambda \mapsto (\lambda - A)\, R(1, A)\, h(\lambda)$ is holomorphic on V, the claim follows]. Thus η is not in the Carleman spectrum of $U(\cdot)\, x$. The claim follows from (2.7). ∎

For $\eta \in \mathbb{R}$, $x \in X$ we denote by $e_\eta \otimes x$ the periodic function

$$(e_\eta \otimes x)(t) = e^{i\eta t} x \qquad (t \in \mathbb{R}).$$

Linear combinations of such functions are called *trigonometric polynomials.* By

$$AP(\mathbb{R}, X) = \overline{\operatorname{span}}\{e_\eta \otimes x : \eta \in \mathbb{R}, x \in X\}$$

we denote the space of all almost periodic functions (where the closure is taken in $BUC(\mathbb{R}, X)$).

There are various different characterisations of $AP(\mathbb{R}, X)$ (see [LZ]). In particular, $u \in BUC(\mathbb{R}, X)$ is almost periodic if and only if the set $\{S(t)u : t \in \mathbb{R}\}$ is relatively compact in $BUC(\mathbb{R}, X)$. For $u \in AP(\mathbb{R}, X)$, $\eta \in \mathbb{R}$, we define the mean

$$\begin{aligned} M_\eta u &= \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{-i\eta s} S(s) u \, ds \\ &= \lim_{\tau \to \infty} \frac{1}{\tau} \int_{0}^{\tau} e^{-i\eta s} S(s) u \, ds \end{aligned}$$

which exists in $BUC(\mathbb{R}, X)$ (since it exists for trigonometric polynomials), cf. Lemma 2.4. It is immediate from semigroup properties that $S(t)\, M_\eta u = e^{i\eta t} M_\eta u$ $(t \in \mathbb{R})$, thus $M_\eta u = e_\eta \otimes (M_\eta u)(0)$.

The set

$$\operatorname{Freq}(u) = \{\eta \in \mathbb{R} : M_\eta u \neq 0\}$$

of all *frequencies* of u is countable [LZ, p. 23]. Clearly, if

$$p(t) = \sum_{j=1}^{n} e_{\eta_j} \otimes x_j$$

is a trigonometric polynomial with $\eta_j \neq \eta_k$ for $j \neq k$ and $x_j \neq 0$ for all $j = 1 \cdots n$, then $\operatorname{Freq}(p) = \{\eta_1, ..., \eta_n\}$. More generally, one has the following remarkable phenomenon of *spectral synthesis*

$$u \in \overline{\operatorname{span}}\{e_\eta \otimes (M_\eta u)(0) : \eta \in \operatorname{Freq}(u)\} \tag{2.8}$$

for all $u \in AP(\mathbb{R}, X)$, where the closure is taken in $BUC(\mathbb{R}, X)$ [LZ, p. 24]. Thus every almost periodic function can be uniformly approximated by trigonometric polynomials with the same frequencies. One can give the following operator theoretic description of the frequencies.

PROPOSITION 2.3. *Let* $u \in AP(\mathbb{R}, X)$. *Then*

(a) $i\operatorname{Freq}(u) = \sigma_p(B_u)$;

(b) $i\overline{\operatorname{Freq}(u)} = \operatorname{sp}(u)$.

Proof. (a) If $\eta \in \operatorname{Freq}(u)$, then $M_\eta(u) = e_\eta \otimes (M_\eta u)(0) \in D(B_u)$ and $B_u M_\eta(u) = i\eta M_\eta(u)$. Conversely, if $M_\eta(u) = 0$, then $M_\eta(g) = 0$ for all $g \in BUC(\mathbb{R}, X)_u$. Hence no such g can be eigenvector of B_u associated with $i\eta$.

(b) Let $\eta \notin \overline{\operatorname{Freq}(u)}$. Let $\varepsilon > 0$ such that $[\eta - \varepsilon, \eta + \varepsilon] \cap \operatorname{Freq}(u) = \emptyset$. Then for $f \in L^1(\mathbb{R})$ with $\operatorname{supp} \mathscr{F} f \subset (\eta - \varepsilon, \eta + \varepsilon)$ one has $f * (e_\lambda \otimes x) = (\mathscr{F} f)(\lambda)\, e_\lambda \otimes x = 0$ for all $\lambda \in \operatorname{Freq}(u)$, $x \in X$. It follows from (2.8) that $f * u = 0$. Thus $\eta \notin \operatorname{sp}(u)$. ∎

Finally, we mention a simple result from ergodic theory which is easy to prove.

LEMMA 2.4. *Assume that A generates a bounded group U on X. Let $x \in X$. Then $\lim_{t \to \infty} 1/t \int_0^t U(s)\, x\, ds$ exists if and only if $\lim_{t \to \infty} 1/2t \int_{-t}^t U(s)\, x\, ds$ exists and both limits coincide in that case.*

3. SPECTRAL CHARACTERISATION OF ALMOST PERIODIC FUNCTIONS

In this section we present the factorisation method to give new proofs of diverse vector-valued versions of Loomis' theorem.

As before we consider the shift group S on $BUC(\mathbb{R}, X)$ with generator B. Since S leaves $AP(\mathbb{R}, X)$ invariant there is an induced C_0-group on $Y := BUC(\mathbb{R}, X)/AP(\mathbb{R}, X)$ given by

$$\bar{S}(t)\, \pi(u) = \pi(\bar{S}(t)\, u)$$

($t \in \mathbb{R}$, $u \in B$) where $\pi: BUC(\mathbb{R}, X) \to BUC(\mathbb{R}, X)/AP(\mathbb{R}, X)$ denotes the quotient mapping. Its generator is denoted by $\tilde{B}$.

PROPOSITION 3.1. *The following are equivalent.*

(i) *Whenever $f \in AP(\mathbb{R}, X)$ and $F(t) = \int_0^t f(s)\, ds$ is bounded, then $F \in AP(\mathbb{R}, X)$;*

(ii) $0 \notin \sigma_p(\tilde{B})$;

(iii) *$\sigma_p(\tilde{B})$ is empty;*

(iv) *X does not contain c_0.*

Proof. (i) ⇒ (ii). Let $g \in BUC(\mathbb{R}, X)$ and suppose that $\pi(g) \in D(\tilde{B})$, $\tilde{B}(\pi(g)) = 0$. Then $S(t)\, g - g \in AP(\mathbb{R}, X)$. Hence

$$f := BR(1, B)\, g = \lim_{t \to 0} t^{-1} R(1, B)(S(t)\, g - g) \in AP(\mathbb{R}, X).$$

Hence $F(t)=\int_0^t f(s)\,ds=(R(1,B)\,g)(t)-(R(1,B)\,g)(0)$ is bounded. By (i), $F\in AP(\mathbb{R},X)$, so $R(1,B)\,g\in AP(\mathbb{R},X)$. Hence $R(1,\tilde{B})\,\pi(g)=\pi(R(1,B)\,g)=0$. Since $R(1,\tilde{B})$ is injective, it follows that $\pi(g)=0$.

(ii) $\Rightarrow$ (iii). Let $\xi\in\mathbb{R}$. Then $(Vu)(t)=e^{i\xi t}u(t)$ defines an isomorphism on $BUC(\mathbb{R},X)$. Since $AP(\mathbb{R},X)$ is invariant under V and V^{-1}, there is an induced isomorphism $\bar{V}$ on $BUC(\mathbb{R},X)$ given by $\bar{V}\pi(g)=\pi(Vg)$. Since $VS(t)\,V^{-1}=e^{-i\xi t}S(t)$, it follows that $\bar{V}\bar{S}(t)\,\bar{V}^{-1}=e^{-i\xi t}\bar{S}(t)$. Hence $0\in\sigma_p(\tilde{B})$ if and only if $i\xi\in\sigma_p(\tilde{B})$.

(iii) $\Rightarrow$ (i). Suppose that $f\in AP(\mathbb{R},X)$ such that F is bounded. Then $F\in D(B)$ and $BF=f$. Thus $\tilde{B}\pi(F)=0$. By (iii), $\pi(F)=0$; that is, $F\in AP(\mathbb{R},X)$.

(iv) $\Rightarrow$ (i). This is a result of Kadets [LZ, Theorem 2, p. 86].

(i) $\Rightarrow$ (iv). See the example [LZ, p. 81]. ∎

Now we obtain a very transparent operator theoretic proof of the following theorem [LZ, Chap. 6.4, Theorem 4] (the result is due to Loomis in the scalar case).

THEOREM 3.2. *Assume that X does not contain c_0. Let $u\in BUC(\mathbb{R},X)$ and assume that* $\mathrm{sp}(u)$ *is countable. Then $u\in AP(\mathbb{R},X)$.*

Proof. It follows from the definition of the Arveson spectrum that $\mathrm{sp}^{\bar{S}}(\pi(u))\subset\mathrm{sp}(u)$. In fact, let $\xi\in\mathbb{R}\backslash\mathrm{sp}(u)$. Then there exists $\varepsilon>0$ such that $f(S)u=\int_{-\infty}^{+\infty}f(t)\,S(t)u\,dt=0$ whenever $f\in L^1(\mathbb{R})$, $\mathrm{supp}\,\mathscr{F}f\subset(\xi-\varepsilon,\xi+\varepsilon)$. It follows that $f(\bar{S})\,\pi(u)=\pi(f(S)u)=0$. Hence $\xi\notin\mathrm{sp}^{\bar{S}}(\pi(u))$. Thus $\mathrm{sp}^{\bar{S}}(\pi(u))$ is countable. Now assume that $u\notin AP(\mathbb{R},X)$. Then $\bar{u}:=\pi(u)\neq 0$ and $Y_{\bar{u}}=\overline{\mathrm{span}}\{\bar{S}(t)\bar{u}:t\in\mathbb{R}\}\neq 0$. Then $\sigma(\tilde{B}_{\bar{u}})=i\,\mathrm{sp}^{S}(\bar{u})$ by (2.3). Thus $\sigma(\tilde{B}_{\bar{u}})$ is non-empty and countable. So it contains an isolated point $i\eta$. By Gelfand's theorem $i\eta\in\sigma_p(\tilde{B}_{\bar{u}})\subset\sigma_p(\tilde{B})$. This contradicts Proposition 3.1. ∎

Remark 3.3. One sees from the proof that it suffices that

$$\mathrm{sp}_{\mathrm{AP}}(\pi(u))=\{\xi\in\mathbb{R}:\forall\varepsilon>0\;\exists f\in L^1(\mathbb{R}),\text{ such that }\mathrm{supp}\,\mathscr{F}f\subset(\xi-\varepsilon,\xi+\varepsilon)\text{ and }f*u\notin AP(\mathbb{R},X)\}$$

is countable in order to conclude that $u\in AP(\mathbb{R},X)$ (still assuming that $c_0\not\subset X$). This is contained in [LZ, Chap. 6.4, Theorem 4].

More generally, one obtains the following.

THEOREM 3.4. *Let $\mathscr{E}$ and $\mathscr{G}$ be closed, translation-invariant subspaces of $BUC(\mathbb{R},X)$ and suppose that*

(a) $\mathscr{G}\subseteq\mathscr{E}$;

(b) *$\mathcal{G}$ contains all those constant functions which belong to $\mathcal{E}$;*

(c) *$\mathcal{E}$ and $\mathcal{G}$ are invariant by multiplication by $e^{i\xi \cdot}$ for all $\xi \in \mathbb{R}$;*

(d) *whenever $f \in \mathcal{G}$ and $F \in \mathcal{E}$, where $F(t) = \int_0^t f(s)\, ds$, then $F \in \mathcal{G}$.*

Let $u \in \mathcal{E}$ have countable reduced spectrum

$$\mathrm{sp}_{\mathcal{G}}(u) := \{\xi \in \mathbb{R} : \forall \varepsilon > 0\ \exists f \in L^1(\mathbb{R}) \text{ such that } \\ \operatorname{supp} \overline{\mathcal{F}} f \subset (\xi - \varepsilon, \xi + \varepsilon) \text{ and } f * u \notin \mathcal{G}\}.$$

Then $u \in \mathcal{G}$.

Proof. Consider the translation group S on $\mathcal{E}$, and the induced group $\bar{S}$ on $\bar{\mathcal{E}} = \mathcal{E}/\mathcal{G}$ with generator $\tilde{B}$. Then $\mathrm{sp}_{\mathcal{G}}(u) = \mathrm{sp}^{\bar{S}}(\pi(u)) = -i\sigma(\tilde{B}_{\bar{u}})$ where $\tilde{B}_{\bar{u}}$ is the restriction of $\tilde{B}$ to $\bar{\mathcal{E}}_{\bar{u}} = \overline{\mathrm{span}}\{\bar{S}(t)\bar{u} : t \in \mathbb{R}\}$. As in Proposition 3.1, it follows from (d) that $\sigma_p(\tilde{B}) = \varnothing$, and the proof is completed as in Theorem 2.2. ∎

We call $u \in BUC(\mathbb{R}, X)$ *totally ergodic* if

$$M_\eta u = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{i\eta s} S(s) u\, ds$$

exists in $BUC(\mathbb{R}, X)$ for all $\eta \in \mathbb{R}$.

Note that $M_\eta u \in \ker(B - i\eta)$, or equivalently, $S(t)\, M_\eta u = e^{i\eta t} M_\eta u$. Thus $M_\eta u = e_\eta \otimes (M_\eta u)(0)$.

Corollary 3.5. *Let $u \in BUC(\mathbb{R}, X)$ with countable reduced spectrum* $\mathrm{sp}_{\mathrm{AP}}(u)$. *Assume that*

(a) *$u(\mathbb{R})$ is relatively weakly compact in X; or*

(b) *u is totally ergodic.*

Then $u \in AP(\mathbb{R}, X)$.

Proof. In the case (a) we choose $\mathcal{E} = \{u \in BUC(\mathbb{R}, X) : u(\mathbb{R})$ is relatively weakly compact$\}$. It is shown in [LZ, Theorem 2, p. 86] that Theorem 3.4(d) is satisfied. In the case (b) we choose $\mathcal{E} = \{f \in BUC(\mathbb{R}, X) : f$ is totally ergodic$\}$. In order to show condition (d) of Theorem 3.4 to hold, let $f \in \mathcal{E}$ such that $F \in \mathcal{E}$. Then

$$(S(s)\, F - F)(t) = \int_0^{t+s} f(r)\, dr - \int_0^t f(r)\, dr \\ = \int_t^{t+s} f(r)\, dr = \int_0^s (S(r)\, f)(t)\, dr.$$

Thus $S(s)F - F = \int_0^s S(r) f\,dr \in AP(\mathbb{R}, X)$ for all $s \in \mathbb{R}$. Consequently,

$$M_0 F - F = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} (S(s)F - F)\,ds \in AP(\mathbb{R}, X).$$

Since $M_0 F \in AP(\mathbb{R}, X)$, it follows that $F \in AP(\mathbb{R}, X)$. In both cases we choose $\mathcal{G} = AP(\mathbb{R}, X)$ and apply Theorem 3.4. ∎

Part (a) of Corollary 3.5 is proved in [LZ, Theorem 4, p. 92], part (b) is due to Ruess–Vũ [RV, Section 3], see also [Bas1, Bas2]. The proofs given there are very different from ours, though.

4. ALMOST PERIODIC SOLUTIONS

The factorisation technique introduced in the previous section can be used to prove almost periodicity of solutions of some first- and second-order inhomogeneous Cauchy problem.

Throughout this section A denotes a closed, linear operator on a Banach space X. We first consider the Cauchy problem

$$(CP) \qquad \dot{u}(t) = Au(t) + \Phi(t) \qquad t \in \mathbb{R}$$

where $\Phi: \mathbb{R} \to X$ is continuous. By a *mild solution* of (CP) we understand a continuous function $u: \mathbb{R} \to X$ such that

$$\int_0^t u(s)\,ds \in D(A)$$

and

$$u(t) - u(0) = A \int_0^t u(s)\,ds + \int_0^t \Phi(s)\,ds \tag{4.1}$$

for all $t \in \mathbb{R}$.

Remark 4.1. (a) u is called a *classical solution*, if $u \in C^1(\mathbb{R}, X)$, $u(t) \in D(A)$ $(t \in \mathbb{R})$ and (CP) is satisfied. Integrating (CP) one sees that every classical solution is a mild solution. Conversely, if u is a mild solution and $u \in C^1(\mathbb{R}, X)$, then u is a classical solution.

(b) If A generates a C_0-semigroup $(T(t))_{t \geq 0}$, then it is easy to see that a continuous function $u: \mathbb{R} \to X$ is a mild solution of (CP) if and only if

$$u(t) = T(t-s)\,u(0) + \int_s^t T(t-r)\,\Phi(r)\,dr$$

whenever $t \geqslant s$, $t, s \in \mathbb{R}$. This is taken as a definition in [LZ, p. 93] and [RV]. One advantage of our approach is that A does not need to generate a C_0-semigroup.

We are interested in almost periodic solutions of (CP). In the homogeneous case they can be described by the eigenvectors of A.

PROPOSITION 4.2. *Let $u: \mathbb{R} \to X$. The following are equivalent.*

(i) *$u \in AP(\mathbb{R}, X)$ and u is a solution of (CP) with $\Phi \equiv 0$.*

(ii) *$u \in \overline{\operatorname{span}}\{e_\eta \otimes x : x \in D(A), Ax = i\eta x\}$, where the closure is taken in $BUC(\mathbb{R}, X)$.*

Proof. Since the uniform limit of solutions is a solution, (ii) implies (i). Assume now that $u \in AP(\mathbb{R}, X)$ is a solution of the homogeneous problem. Then $v(t) = e^{-\eta t}u(t)$ defines a solution of $\dot{v}(t) = (A - i\eta)\, v(t)$. Thus $1/2\tau\, (A - i\eta) \int_{-\tau}^{\tau} e^{-i\eta s} u(s)\, ds = 1/2\tau\, (e^{-i\eta\tau}u(\tau) - e^{+i\eta\tau}u(-\tau)) \to 0 \quad (\tau \to \infty)$. This implies that $(M_\eta(u))(0) \in D(A)$ and $(A - i\eta)\, M_\eta(u)(0) = 0$. Now the claim follows from (2.8). ∎

Next assume that Φ is bounded. Let $u \in BUC(\mathbb{R}, X)$ be a solution of (CP). Taking Laplace transforms in (4.1) one obtains that $\hat{u}(\lambda) \in D(A)$ and

$$\hat{u}(\lambda) = R(\lambda, A)\, u(0) + R(\lambda, A)\, \hat{\Phi}(\lambda) \tag{4.2}$$

$(\operatorname{Re} \lambda > 0,\ \lambda \in \rho(A))$ where $\hat{u}$ is defined by (2.6). Now assume that $\Phi = 0$. Then

$$\hat{u}(\lambda) = R(\lambda, A)\, u(0) \qquad (\operatorname{Re} \lambda \neq 0)$$

It follows from (2.5) that

$$i\, \operatorname{sp}(u) \subset \sigma(A) \cap i\boldsymbol{R}.$$

Thus, in the homogeneous case $\Phi = 0$, we obtain the following result immediately from Theorem 3.2 and Corollary 3.5.

THEOREM 4.3. *Assume that $\Phi \in AP(\mathbb{R}, X)$. Moreover, assume that $\sigma(A) \cap i\boldsymbol{R}$ is countable. Let $u \in BUC(\mathbb{R}, X)$ be a solution of (CP). Then $u \in Ap(\mathbb{R}, X)$ provided one of the following conditions is satisfied.*

(a) $c_0 \not\subset X$;

(b) *u is totally ergodic; or,*

(c) *$u(\mathbb{R})$ is relatively weakly compact.*

In the inhomogeneous case the idea of the proof consists in factoring out the inhomogeneity:

Proof. For $s \in \mathbb{R}$ let $u_s(t) = u(t+s)$ $(t \in \mathbb{R})$. Replacing u by u_s in (4.2) one obtains

$$\widehat{u_s}(\lambda) = R(\lambda, A)\, u(s) + R(\lambda, A)\, \widehat{\Phi_s}(\lambda) \tag{4.3}$$

$(\mathrm{Re}\lambda > 0,\ s \in \mathbb{R})$. Now, as in Section 3, consider the shift groups S on $BUC(\mathbb{R}, X)$ and $\bar{S}$ on $Y = BUC(\mathbb{R}, X)/AP(\mathbb{R}, X)$ with generators B and $\tilde{B}$, respectively. Note that $\widehat{u_s}(\lambda) = \int_0^\infty e^{-\lambda t} u(s+t)\, dt = (R(\lambda, B) u)(s)$, and similarly, $\widehat{\Phi_s}(\lambda) = (R(\lambda, B)\, \Phi)(s)$. Thus (4.3) becomes

$$R(\lambda, B) u = R(\lambda, A) \circ u + R(\lambda, A) \circ R(\lambda, B)\, \Phi. \tag{4.4}$$

Since $\Phi \in AP(\mathbb{R}, X)$, one has $\pi(R(\lambda, A) \circ R(\lambda, B)\, \Phi) = 0$. Thus

$$R(\lambda, \tilde{B})\, \pi(u) = \pi(R(\lambda, B) u) = \pi(R(\lambda, A) \circ u)$$

has a holomorphic extension to $i\eta$ whenever $i\eta \notin \sigma(A)$. It follows from Lemma 2.2 that $i\, \mathrm{sp}_{\mathrm{AP}}(u) = i\, \mathrm{sp}^{\bar{S}}(\pi(u)) \subset \sigma(A) \cap i\boldsymbol{R}$. Thus $\mathrm{sp}_{\mathrm{AP}}(u)$ is countable and the claim follows from Remark 3.3 and Corollary 3.5. ∎

In the case where A generates a C_0-semigroup, Theorem 4.3 is contained in [LZ, Theorem 5, p. 94] in the cases (a) and (c) and due to Ruess–Vũ [RV] in the case (b). The proof given there are based on harmonic analysis. Our proof, using Laplace transform and factorisation, works also for the second-order Cauchy problem, but needs further arguments.

Consider the *second-order Cauchy problem*

$$(CP_2) \qquad \begin{cases} \ddot{u}(t) = Au(t) + \Phi(t) & (t \in \mathbb{R}) \\ u(0) = x, \quad \dot{u}(0) = y, \end{cases}$$

where $\Phi\colon \mathbb{R} \to X$ is continuous and $x, y \in X$. By a *mild solution* of (CP_2) we understand a continuous function $u\colon \mathbb{R} \to X$ such that $\int_0^t (t-s)\, u(s)\, ds \in D(A)$ and

$$u(t) = x + ty + A \int_0^t (t-s)\, u(s)\, ds + \int_0^t (t-s)\, \Phi(s)\, ds \qquad (t \in \mathbb{R}). \tag{4.5}$$

Remark 4.4. If u is a *classical solution* of (CP_2) (*i.e.*, $u \in C^2(\mathbb{R}, X)$), $u(t) \in D(A)$ and (CP_2) holds for all $t \in \mathbb{R}$), then u is a mild solution (as can be seen by integrating (CP_2)). Conversely, if u is a mild solution and $u \in C^2(\mathbb{R}, X)$, then u is a classical solution.

In the following we will frequently use the notation $u_s = S(s)u$ for $u \in BUC(\mathbb{R}, X)$.

THEOREM 4.5. *Let $\Phi \in AP(\mathbb{R}, X)$. Assume that $\sigma(A) \cap (-\infty, 0]$ is countable. Let $u \in BUC(\mathbb{R}, X)$ be a mild solution of (CP_2). Then $u \in AP(\mathbb{R}, X)$ provided one of the conditions* (a), (b), (c) *of Theorem* 4.3 *satisfied.*

Proof. Replacing t by $t+s$ and by s in (4.5) and subtracting yields after a change of variable

$$u(t+s) - u(s) = ty + A\left\{\int_0^t (t-r)\, u_s(r)\, dr + tv(s)\right\}$$
$$+\int_0^t (t-r)\, \Phi_s(r)\, dr + t\Psi(s)$$

($t \in \mathbb{R}$, $s \in \mathbb{R}$) where $\Psi(s) = \int_0^s \Phi(r)\, dr$, $v(s) = \int_0^s u(r)\, dr$. Taking Laplace transforms we obtain

$$\widehat{u_s}(\lambda) = \frac{u(s)}{\lambda} + \frac{y}{\lambda^2} + A\left\{\frac{\widehat{u_s}(\lambda)}{\lambda^2} + \frac{v(s)}{\lambda^2}\right\}$$
$$+\frac{\widehat{\Phi_s}(\lambda)}{\lambda^2} + \frac{\Psi(s)}{\lambda^2} \qquad (\operatorname{Re}\lambda > 0),$$

where $u_s(t) = u(s+t)$. Thus if $\operatorname{Re}\lambda > 0$ and $\lambda^2 \in \rho(A)$, then

$$R(\lambda^2, A)\, \widehat{u_s}(\lambda) = \frac{1}{\lambda} R(\lambda^2, A)\, u(s) + \frac{R(\lambda^2, A)y}{\lambda^2}$$
$$+\frac{1}{\lambda^2} AR(\lambda^2, A)\, \widehat{u_s}(\lambda) + \frac{1}{\lambda^2} AR(\lambda^2, A)\, v(s)$$
$$+\frac{1}{\lambda^2} R(\lambda^2, A)\, \widehat{\Phi_s}(\lambda) + \frac{1}{\lambda^2} R(\lambda^2, A)\, \Psi(s).$$

Consequently,

$$\widehat{u_s}(\lambda) = \lambda R(\lambda^2, A)\, u(s) + R(\lambda^2, A)y + R(\lambda^2, A)\, \widehat{\Phi_s}(\lambda) + f_s(\lambda), \tag{4.6}$$

where $f_s(\lambda) = AR(\lambda^2, A)\, v(s) + R(\lambda^2, A)\, \Psi(s)$. Now let $\eta \in \mathbb{R}$ such that $-\eta^2 \in \rho(A)$. Let $r > 0$ such that $\lambda^2 \in \rho(A)$ whenever $|\lambda - i\eta| \leqslant 2r$ or $|\lambda + i\eta| \leqslant 2r$. We show that for $|i\eta - \lambda| < r$, the function $g(\lambda)$ given by $g(\lambda)(s) = f_s(\lambda)$ is in $BUC(\mathbb{R}, X)$ and $\lambda \mapsto g(\lambda)$ is holomorphic. Let $\lambda \in \mathbb{C}$, $|\lambda - i\eta| = 2r$.

First case. $\operatorname{Re}\lambda > 0$. Then

$$f_s(\lambda) = \widehat{u_s}(\lambda) - R(\lambda^2, A)\,\widehat{\Phi_s}(\lambda) - \lambda R(\lambda^2, A)\,u(s) - R(\lambda^2, A)\,y.$$

Hence $\|f_s(\lambda)\| \leqslant c_1/\operatorname{Re}\lambda + c_2$, where

$$c_1 = \|u\|_\infty + \|\Phi\|_\infty \sup_{\substack{|\mu - i\eta| = 2r \\ \operatorname{Re}\mu > 0}} \|R(\mu^2, A)\|,$$

$$c_2 = \sup_{\substack{|\mu - i\eta| = 2r \\ \operatorname{Re}\mu > 0}} \{\|\mu R(\mu^2, A)\|\;\|u\|_\infty + \|R(\mu^2, A)\,y\|\}$$

Second case. $\operatorname{Re}\lambda < 0$. Let $\lambda_1 = -\lambda$. Since $f_s(\lambda) = f_s(\lambda_1)$, $\operatorname{Re}\lambda_1 > 0$, $|\lambda_1 - (-i\eta)| = 2r$ the analogous estimate gives

$$\|f_s(\lambda)\| \leqslant \frac{c_1}{\operatorname{Re}\lambda_1} + c_2.$$

In both cases

$$\|f_s(\lambda)\| \leqslant \frac{c}{|\operatorname{Re}\lambda|} \qquad \text{for all} \quad s \in \mathbb{R} \qquad \text{if} \quad |\lambda - i\eta| = 2r,$$

where $c \in \mathbb{R}$ is such that

$$\frac{c_1}{|\operatorname{Re}\mu|} + c_2 \leqslant \frac{c}{|\operatorname{Re}\mu|} \qquad \text{whenever} \quad |\mu - i\eta| = 2r.$$

By the lemma below we conclude that

$$\|f_s(\lambda)\| \leqslant \frac{4}{3}\,\frac{c}{r} \qquad \text{whenever} \quad \lambda \in \mathbb{C},\ |\lambda - i\eta| \leqslant r, \quad s \in \mathbb{R}.$$

It is clear from the definition that $g(\lambda)$ is uniformly continuous for all $\lambda \in \bar{B}(i\eta, r)$. We have seen that $\|g(\lambda)\| \leqslant 4/3\ c/r$ for $\lambda \in \bar{B}(i\eta, r)$. Since for all $s \in \mathbb{R}$ the function $\lambda \mapsto g(\lambda)(s)$ is holomorphic in $B(i\eta, r)$ one concludes as in [BNR2, Proposition 3.2] or [Ne, Lemma 5.3.3] that $g: B(i\eta, r) \to BUC(\mathbb{R}, X)$ is holomorphic. Denote by S the translation group on $BUC(\mathbb{R}, X)$ as before and by B its generator. Then (4.8) can be rewritten as

$$\begin{aligned} R(\lambda, B)u &= \lambda R(\lambda^2, A) \circ u + R(\lambda^2, A)y + g(\lambda) \\ &\quad + R(\lambda^2, A) \circ R(\lambda, B)\,\Phi. \end{aligned}$$

Considering again the operator $\tilde{B}$ on $Y = BUC(\mathbb{R}, X)/AP(\mathbb{R}, X)$ we obtain

$$\begin{aligned} R(\lambda, \tilde{B})\, \pi(u) &= \pi(R(\lambda, B)u) \\ &= \pi(\lambda R(\lambda^2, A) \circ u + R(\lambda^2, A)y + g(\lambda)) \end{aligned}$$

since $R(\lambda^2, A) \circ R(\lambda, B)\, \Phi \in AP(\mathbb{R}, X)$ whenever $|\lambda - i\eta| < r$. Hence $R(\lambda, \tilde{B})$ $\pi(u)$ has a holomorphic extension to $B(i\eta, r)$ with values in Y. By Lemma 2.2 this implies that $\eta \notin \mathrm{sp}_{\mathrm{AP}}(u)$.

We have shown that $\mathrm{sp}_{\mathrm{AP}}(u) \subset \{\eta \in \mathbb{R}\colon -\eta^2 \in \sigma(A)\}$. Thus $\mathrm{sp}_{\mathrm{AP}}(u)$ is countable. It follows from Remark 3.3 (resp. Corollary 3.5) that $u \in AP(\mathbb{R}, X)$. ∎

Lemma 4.6. *Let $U \subset \mathbb{C}$ be an open neighborhood of $i\eta$. Assume that $\bar{B}(i\eta, 2r) \subset U$. Let $h\colon U \to X$ be holomorphic such that*

$$\|h(z)\| \leqslant \frac{c}{|\mathrm{Re}\, z|} \qquad \text{if} \quad |z - i\eta| = 2r, \quad \mathrm{Re}\, z \neq 0.$$

Then $\|h(z)\| \leqslant 4/3\ c/r$ for all $z \in \bar{B}(i\tau, r)$.

The proof of this lemma is contained in the proof of [BNR1, Theorem 2.2] or [Ne, Lemma 5.3.1].

Also in the case of the homogeneous second order Cauchy problem, almost periodic solutions are uniform limits of trigonometric polynomials with eigenvectors of A as coefficients.

Proposition 4.7. *Assume that $\Phi \equiv 0$. Let $u \in AP(\mathbb{R}, X)$ be a solution of (CP_2). Then*

$$u \in \overline{\mathrm{span}}\{e_\eta \otimes x \colon \eta \in \mathbb{R},\, x \in D(A),\, Ax + \eta^2 x = 0\}$$

Proof. Let $\eta \in \mathbb{R}$, $z = \lim_{\alpha \downarrow 0} \alpha \hat{u}(\alpha + i\eta) = (M_\eta\, u)(0)$. Since $\hat{u}(\lambda) = \lambda R(\lambda^2, A)x + R(\lambda^2, A)y$ ($\mathrm{Re}\, \lambda > 0$, $\lambda \in \rho(A)$), one has $((i\eta + \alpha)^2 - A)$ $\alpha \hat{u}(\alpha + i\eta) = \alpha(\alpha + i\eta)x + \alpha y \to 0$ $(\alpha \downarrow 0)$ and consequently $(-\eta^2 - A)$ $\alpha \hat{u}(\alpha + i\eta) \to 0$ $(\alpha \downarrow 0)$. Since A is closed, this implies that $z \in D(A)$ and $(-\eta^2 - A)\, z = 0$. Now the claim follows from (2.8). ∎

Next assume that for all $x \in X$ and $y = 0$ there exists a unique bounded solution $u(\cdot, x)$ of (CP_2). This is equivalent to saying that A generates a bounded cosine function. In fact, $C(t)x := u(t, x)$ defines a bounded operator on X. Moreover, $C\colon \mathbb{R} \to \mathscr{L}(X)$ is strongly continuous, bounded and

$$\lambda R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} C(t)x\, dt \qquad (\mathrm{Re}\, \lambda > 0,\, x \in X).$$

Then $C(0)=I$, $2C(t)\,C(s)=C(t+s)+C(t-s)$ $(t, s\in\mathbb{R})$. Since C is bounded, one has $\sigma(A)\subset(-\infty, 0]$.

We say that C is *almost periodic* if $C(\cdot)x\in AP(\mathbb{R}, X)$ for all $x\in X$. It follows from Theorem 4.5 that C is almost periodic whenever $c_0 \not\subseteq X$ and $\sigma(A)$ is countable. As in the case of bounded groups almost periodicity can be described by "complete point spectrum," i.e., totality of the eigenvectors.

PROPOSITION 4.8. *Let C be a bounded cosine function on a Banach space X. Then C is almost periodic if and only if*

$$\overline{span}\{w\in D(A)\colon \exists \eta\in\mathbb{R} \text{ such that } Aw=-\eta^2 w\})X. \tag{4.7}$$

Proof. If C is almost periodic, then by Proposition 4.7, $C(\cdot)x\in \overline{\text{span}}(e_\eta\otimes w\colon \eta\in\mathbb{R},\ w\in D(A),\ Aw=-\eta^2 w\}$, where the closure is taken in $BUC(\mathbb{R}, X)$. Hence $x=C(0)x\in\overline{\text{span}}\{w\in D(A)\colon \exists\eta\in\mathbb{R},\ Aw=-\eta^2 w\}$ in X. Conversely, let $x\in D(A)$ such that $Ax=-\eta^2 x$. Then $C(t)x=(\cos\eta t)x$. Thus if $x\in\text{span}\{w\in D(A)\colon \exists\eta\in\mathbb{R},\ Aw=-\eta^2 w\}=\colon Y$, then $C(\cdot)x$ is almost periodic. Since Y is dense, the claim follows. ∎

PROPOSITION 4.9. *Assume that A generates a bounded cosine function C on a Banach space X. Assume furthermore that*

(a) $c_0 \not\subset X$,

(b) $\sigma(A)$ *is countable, and*

(c) $0\notin\sigma(A)$.

Then, for all $x, y\in X$, the homogeneous problem (CP_2) has a unique solution, and this solution is almost periodic.

Proof. Let $S(t)y=\int_0^t C(s)y\,ds$. Then the solution of (CP_2) is given by $u(t)=C(t)x+S(t)y$. Since $C(t)x=x+A\int_0^t(t-s)\,C(s)x\,ds=x+A\int_0^t S(s)x\,ds$, $\int_0^t S(s)x\,ds=A^{-1}(C(t)x-x)$ is bounded. It follows from Taylor's formula that $\{S(t)\colon t\in\mathbb{R}\}$ is bounded as well. Thus each solution of (CP_2) is bounded. Since $d/dt\,S(t)y=C(t)y$ is bounded $S(\cdot)y\in BUC(\mathbb{R}, X)$ for all $y\in X$. Let $x\in D(A)$. Then $d/dt\,C(t)x=AS(t)x=S(t)\,Ax$ is bounded. Hence $C(\cdot)x$ is uniformly continuous. Since $D(A)$ is dense, it follows that $C(\cdot)x$ is uniformly continuous for all $x\in X$. Thus each solution is in $BUC(\mathbb{R}, X)$ and the claim follows from the previous results. ∎

EXAMPLE 4.10. Condition (a) cannot be omitted in Proposition 4.9. In fact, let $X=c:=\{(x_n)_{n\in\mathbb{N}}\subset\mathbb{C}\colon \lim_{n\to\infty}x_n$ exists$\}$. Define $A\in\mathscr{L}(X)$ by $Ax=(-\alpha_n^2 x_n)_{n\in\mathbb{N}}$ where $\alpha_n\in\mathbb{R}\backslash\{0\}$ such that $\alpha_n\neq\alpha_m$ for $n\neq m$, $\alpha=\lim_{n\to\infty}\alpha_n$ exists and $\alpha\neq 0$. Then A generates the bounded cosine function C given by $C(t)x=((\cos\alpha_n t)x_n)_{n\in\mathbb{N}}$. Let $e\in c$ be the constant-1-function. Then $u(t)=C(t)\,e=(\cos\alpha_n t)_{n\in\mathbb{N}}$ is not almost periodic (since it does not have relatively compact range or by Proposition 4.7). However, (b) and (c) are satisfied. Remark: the space c is isomorphic to c_0.

5. IMAGINARY SPECTRUM CONSISTING OF POLES

If the imaginary spectrum $\sigma(A)\cap i\mathbb{R}$ of the operator A consists only of poles then one obtains total ergodicity automatically. So our results can be simplified considerably in this case. This situation is frequent in applications. For instance, A may have compact resolvent or the essential spectral bound might be negative. We define

$$AP(\mathbb{R}_+, X)=\overline{\operatorname{span}}\{e_\eta\otimes x: \eta\in\mathbb{R}, x\in X\},$$

where $(e_\eta\otimes x)(t)=e^{i\eta t}x$ and the closure is understood in $BUC(\mathbb{R}_+, X)$. Every function $u\in AP(\mathbb{R}_+, X)$ has a unique extension to an almost periodic function on the line. By

$$AAP(\mathbb{R}_+, X)=C_0(\mathbb{R}_+, X)\oplus AP(\mathbb{R}_+, X)$$

we denote the space of all *asymptotically almost periodic functions*. This is a closed subspace of $BUC(\mathbb{R}_+, X)$. It is known that $u\in BUC(\mathbb{R}_+, X)$ is asymptotically almost periodic if and only if $\{u_\tau: \tau\geqslant 0\}$ is relatively compact in $BUC(\mathbb{R}_+, X)$ ([F, Chap. 9]). Let A be a linear closed operator on X. First we consider the Cauchy Problem on the half line

$$(CP_+)\qquad \begin{cases} \dot{u}(t)=Au(t) & (t\geqslant 0)\\ u(0)=x, \end{cases}$$

where $x\in X$. By a *weak solution* of (CP_+) we understand a continuous function $u=\mathbb{R}_+\to X$ such that $\int_0^t u(s)\,ds\in D(A)$ and

$$A\int_0^t u(s)\,ds=u(t)-x \qquad (t\geqslant 0).$$

Thus A generates a C_0-semigroup T if and only if (CP_+) has a unique solution for all $x\in X$. In that case the solution u is given by $u(t)=T(t)x$. First we consider a special case which can be proved directly.

PROPOSITION 5.1. *Let A be the generator of a bounded C_0-semigroup T. If A has compact resolvent, then T is asymptotically almost periodic (by this we mean that $T(\cdot)x\in AAP(\mathbb{R}_+, X)$ for all $x\in X$).*

Proof. (a) Let $x \in D(A)$. We show that $T(\cdot)x \in AAP(\mathbb{R}_+, X)$. For that, let $t_n \in \mathbb{R}_+$ ($n \in \mathbb{N}$). We have to show that there exists a subsequence $(t_{n_k})_{k \in \mathbb{N}}$ such that $T(t_{n_k} + \cdot)x$ converges in $BUC(\mathbb{R}_+, X)$. Let $\lambda \in \rho(A)$. Since $R(\lambda, A)$ is compact, there exists a subsequence $(t_{n_k})_{k \in \mathbb{N}}$ such that $y = \lim_{k \to \infty} T(t_{n_k})x = \lim_{k \to \infty} R(\lambda, A)\, T(t_{n_k})(I - A)x$ exists. Thus $T(t_{n_k} + \cdot)x$ converges to $T(\cdot)y$ in $BUC(\mathbb{R}_+, X)$.

(b) It follows from (a) and the density of $D(A)$ in X that $T(\cdot)x \in AAP(\mathbb{R}_+, X)$ for all $x \in X$. ∎

For individual solutions, this simple proof does not work any more.

THEOREM 5.2. *Assume that $\sigma(A) \cap i\mathbb{R}$ consists only of poles of the resolvent. Let $u \in BUC(\mathbb{R}_+, X)$ be a solution of (CP_+). Then $u \in AAP(\mathbb{R}_+, X)$.*

Proof. Since u is a solution one has $\hat{u}(\lambda) = R(\lambda, A)\, u(0)$ for all $\lambda > 0$, $\lambda \in \rho(A)$. Since by hypothesis $\sigma(A) \cap i\mathbb{R}$ consists of isolated points in the spectrum of A, $\sigma(A) \cap i\mathbb{R}$ is countable. Thus the sep $\mathrm{sp}_+(u)$ of all $\eta \in \mathbb{R}$ such that $\hat{u}$ does not have a holomorphic extension close to $i\eta$ is countable. So it follows from [AB2, Corollary 2.4] or [BNR2, Theorem 4.1] (see also [Ne, Theorem 5.3.5]) that $u \in AAP(\mathbb{R}_+, X)$, once we have shown that u is totally ergodic. For that, we have to show that $\alpha\widehat{u_s}(\alpha + i\eta)$ converges uniformly in $s \in \mathbb{R}_+$ as $\alpha \downarrow 0$ for all $\eta \in \mathbb{R}$. Since $\widehat{u_s}(\alpha + i\eta) = R(\alpha + i\eta)\, u(s)$, this is clear if $i\eta \in \rho(A)$. Assume that $i\eta \in \sigma(A)$. Denote by P the spectral projection associated with $i\eta$. Since $R(\lambda, A)(I - P)$ has a holomorphic extension close to $i\eta$, it follows that $\alpha R(\alpha + i\eta)(I - P)\, u(s)$ converges uniformly to 0 as $\alpha \downarrow 0$. It remains to show that $\alpha R(\alpha + i\eta)\, Pu(s)$ converges uniformly in $s \in \mathbb{R}_+$ as $\alpha \downarrow 0$. Since P commutes with the resolvent of A, also Pu is a mild solution of (CP_+). Thus, replacing X by PX, we can assume that P is the identity. But then A is bounded and $(A - i\eta)^m = 0$, $(A - i\eta)^{m-1} \neq 0$, where m is the order of the pole $i\eta$ (see, e.g., [Na, A-III. 3.6, p. 72]). It follows that

$$u(t) = e^{i\eta t} \sum_{k=0}^{m-1} \frac{t^k}{k!} (A - i\eta)^k x.$$

Since u is bounded, one concludes that $x \in \ker(A - i\eta)$. Thus $u(t) = e^{i\eta t}x$ $(t \geqslant 0)$. Consequently, $\alpha\widehat{u_s}(\alpha + i\eta) = e^{i\eta s}x$ for all $\alpha > 0$, $s \geqslant 0$, and the claim is proved. ∎

Remark 5.3. Let $u \in AAP(\mathbb{R}_+, X)$ be a solution of (CP_+). Let $u = u_0 + u_1$ where $u_0 \in C_0(\mathbb{R}_+, X)$, $u_1 \in AP(\mathbb{R}, X)$ (extended to $\mathbb{R}$). Then u_0 is a solution of (CP_+) and u_1 is a solution of the homogeneous problem (CP) on the line.

Proof. For $\eta \in \mathbb{R}$ let

$$x_\eta = (M_\eta u_1)(0) = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{-i\eta s} u_1(s)\, ds$$
$$= \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} e^{-i\eta s} u_1(s)\, ds$$
$$= \lim_{\tau \to \theta} \frac{1}{\tau} \int_0^{\tau} e^{-i\eta s} u(s)\, ds.$$

Since u is a solution, one has

$$\frac{1}{\tau}(A - i\eta) \int_0^{\tau} e^{-i\eta s} u(s)\, ds = \frac{1}{\tau}(e^{-i\eta\tau} u(\tau) - u(0)) \to 0 \qquad (\tau \to \infty).$$

Since A is closed, this implies that $x_\eta \in D(A)$ and $(A - i\eta)x_\eta = 0$. It follows that $u_1 \in \overline{\text{span}}\{e_\eta \otimes x : \eta \in \mathbb{R}, x \in D(A), Ax = i\eta x\}$ by spectral synthesis (2.8). Thus u_1 is a solution on the line. ∎

Using Theorem 4.3 in the homogeneous case, one obtains a similar result on the line:

THEOREM 5.4. *Assume that $\sigma(A) \cap i\mathbb{R}$ consists only of poles of the resolvent. Let $u \in BUC(\mathbb{R}, X)$ be a solution of the homogeneous Cauchy problem (CP) on the real line. Then $u \in AP(\mathbb{R}, X)$.*

The proof is completely analogous and can be omitted.

Finally, we consider the homogeneous second order Cauchy problem.

THEOREM 5.5. *Assume that $\sigma(A) \cap (-\infty, 0]$ consists only of poles of the resolvent. Let $u \in BUC(\mathbb{R}, X)$ be a solution of (CP_2) with $\Phi = 0$. Then $u \in AP(\mathbb{R}, X)$.*

Proof. In view of Theorem 4.5 it suffices to show that u is totally ergodic on $\mathbb{R}$.

(a) Let P_0 be the spectral projection with respect to 0. We show that $P_0 u$ is constant and so totally ergodic. Since $P_0 u$ is a solution, we can assume in this part of the proof that P_0 is the identity (replacing X by $P_0 X$ otherwise). Then A is a bounded operator and $A^m = 0$, where m is the order of the pole. Thus

$$u(t) = \sum_{k=0}^{m-1} \left\{ \frac{t^{2k}}{(2k)!} A^k x + \frac{t^{2k+1}}{(2k+1)!} A^k y \right\}$$

is a polynomial. Since u is bounded, it follows that $u(t) \equiv x$ $(t \in \mathbb{R})$.

(b) It remains to show that $(I-P_0)u$ is totally ergodic. Replacing X by $(I-P_0)X$ in this part of the proof, we can assume that $P_0=0$; that is, $0\in\rho(A)$. Let $v(s)=\int_0^s u(r)\,dr$. By (4.7), $\hat{u}$ has a holomorphic extension near 0. It follows from [Kor] or [AP, Remark 3.2] that $\sup_{s\geqslant 0}\|v(s)\|<\infty$. Replacing u by $t\mapsto u(-t)$ one sees that also $\sup_{s<0}\|v(s)\|<\infty$. Thus $v\in BUC(\mathbb{R},X)$. For $s\in\mathbb{R}$ we let $u_s(t)=u(t+s)$ $(t\in\mathbb{R})$. Let $\eta\in\mathbb{R}$; we have to show that $\alpha\widehat{u_s}(i\eta+\alpha)$ converge uniformly in $s\in\mathbb{R}$ as $\alpha\downarrow 0$. By (4.8) we have

$$\alpha\widehat{u_s}(i\eta+\alpha)=\alpha(i\eta+\alpha)\,R((i\eta+\alpha)^2,A)\,u(s)$$
$$+\alpha R((i\eta+\alpha)^2,A)y+\alpha AR((i\eta+\alpha)^2,A)\,v(s)$$

$(s\in\mathbb{R},\ \alpha>0)$.

Thus the limit exists uniformly in s as $\alpha\downarrow 0$ if $-\eta^2\in\rho(A)$. Now let $-\eta^2\in\sigma(A)$. Denote by Q the spectral projection with respect to $-\eta^2$. Then by the previous case, $(I-Q)u$ is uniformly ergodic in $i\eta$. It remains to show that Qu is uniformly ergodic in $i\eta$. Again, for this part, we may now assume that Q is the identity (replacing X by QX otherwise). Then A is bounded, $\sigma(A)=\{-\eta^2\}$ and $-\eta^2$ is a pole of the resolvent of A.

Consider the operator $B=\begin{pmatrix}0 & I\\ A & 0\end{pmatrix}$ on $X\times X$. Then

$$w(t):=\begin{pmatrix}u(t)\\ \dot{u}(t)\end{pmatrix}=e^{tB}\begin{pmatrix}x\\ y\end{pmatrix}\qquad (t\in\mathbb{R}).$$

Moreover, $\sigma(B)\subset\{\pm i\eta\}$ and

$$R(\lambda,B)=\begin{pmatrix}\lambda R(\lambda^2,A) & R(\lambda^2,A)\\ AR(\lambda^2,A) & \lambda R(\lambda^2,A)\end{pmatrix}$$

for $\lambda\notin\{\pm i\eta\}$. Thus $R(\cdot,B)$ has a pole in $\{\pm i\eta\}$. The function u is bounded by hypothesis. Hence $\ddot{u}(t)=Au(t)$ $(t\in\mathbb{R})$ is bounded as well. This implies that $\dot{u}$ is bounded. Thus $w\in BUC(\mathbb{R},X\times X)$. It follows from Theorem 5.3 that $w\in AP(\mathbb{R},X\times X)$, and in particular, that w is totally ergodic on $\mathbb{R}$. Thus u is totally ergodic on $\mathbb{R}$ as well. ∎

If A generates a cosine function then (CP_2) is well-posed; i.e., for every $x,y\in X$ there exists a unique mild solution u of (CP_2). In fact, u is given by

$$u(t)=C(t)\,x+S(t)y,$$

where $S(t)=\int_0^t C(s)\,ds$.

COROLLARY 5.6. *Let A be the generator of a bounded cosine function C. Assume that $\sigma(A)$ consists of poles only. Then $C(\cdot)$ is almost periodic. If in addition $0 \in \rho(A)$, then every solution of (CP_2) is almost periodic.*

Proof. If $0 \in \rho(A)$ it has been shown in the proof of Proposition 4.9 that each solution is in $BUC(\mathbb{R}, X)$. So the result follows from Theorem 5.4. If $0 \in \sigma(A)$ one uses a spectral projection and argues as before. ▮

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