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The Spectral Function and Principal Eigenvalues for Schrödinger Operators

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Abstract. Let $m \in L^1_{\text{loc}}(\mathbb{R}^N), 0 \neq m_+$ in Kato's class. We investigate the *spectral function* $\lambda \mapsto s(\Delta + \lambda m)$ where $s(\Delta + \lambda m)$ denotes the upper bound of the spectrum of the Schrödinger operator $\Delta + \lambda m$. In particular, we determine its derivative at 0. If m_- is sufficiently large, we show that there exists a unique $\lambda_1 > 0$ such that $s(\Delta + \lambda_1 m) = 0$. Under suitable conditions on m^+ it follows that 0 is an eigenvalue of $\Delta + \lambda_1 m$ with positive eigenfunction.

Key words: Principal eigenvalue, Schrödinger semigroup, exponential stability, spectral bound, Brownian motion.

Introduction

Let $m \in L^1_{\text{loc}}(\mathbb{R}^N)$ be such that m^+ is in Kato's class. For $\lambda > 0$ we consider the Schrödinger operator $\Delta + \lambda m$ on $L^p(\mathbb{R}^N)$. By $s(\Delta + \lambda m) = \sup\{\mu \colon \mu \in \sigma(\Delta + \lambda m)\}$ we denote the *spectral bound* of $(\Delta + \lambda m)$ (which is independent of $p \in [1, \infty)$). The function $\lambda \in [0, \infty) \mapsto s(\Delta + \lambda m)$ is convex, we call it the *spectral function* of m. The purpose of this paper is a systematic investigation of this function.

In Section 2 we consider the case when $m \geqslant 0$. Fefferman and Phong [19] and Schechter [33] have used Fourier analysis and L^2-methods to obtain some estimates for the spectral bound. We use L^∞-methods (the Feynman–Kac formula and Kashmin'skii's lemma) to obtain a different upper bound for the spectral function. For small values of λ, our estimate is quite sharp; in particular, it enables us to characterize when the derivative

$$\mathrm{d}/\mathrm{d}\lambda_{|\lambda=0} s(\Delta + \lambda m^+) \text{ is } 0.$$

Criteria for $s(\Delta - \lambda m^-) < 0$ $(\lambda > 0)$ had been given in [5], [9] and [7]. Using those and the results of Section 2, we are able to describe conditions under which there exists a unique $\lambda_1 > 0$ such that $s(\Delta + \lambda_1 m) = 0$ (Section 3). This is interesting for the asymptotic behavior of the semigroup $e^{t(\Delta + \lambda_1 m)}$. Under suitable assumptions we can show in addition that λ_1 is a principal eigenvalue. We also investigate when the spectral function has a strict minimum. This is related to the question when $s(\Delta - \lambda m^-)$ is strictly decreasing which has been treated in [7].

The motivation of this work comes from two sources. Of course the spectral behaviour of $\Delta + \lambda m$ is an important subject in mathematical physics and there are previous results of B. Simon [34] and [35] on $s(\Delta + \lambda m)$ for small $\lambda > 0$ (see also the survey article [36]).

The study of principal eigenvalues for elliptic operators was initiated by a famous article by P. Hess and T. Kato [21] who considered a bounded domain and Dirichlet boundary conditions. They were interested in a nonlinear eigenvalue problem and used the principal eigenvalue for the linearized problem in order to study bifurcation. More recently such problems were considered on $\mathbb{R}^N$ by Allegretto [2], and Brown *et al.*, [10–13]. Their motivation comes partly from a biological model (see, e.g. Fleming [20]). Similar questions for non-autonomous periodic parabolic problems on $\mathbb{R}^N$ are considered by Daners and Koch-Medina [14, 15] and Daners [16]. Principal eigenvalues for second order operators are also studied by different methods by Agmon [1], Pinchover [28, 29] and Nussbaum and Pinchover [26].

1. Preliminaries

For our purposes, the natural class of potentials is the class $\widehat{K}_N$ introduced by Voigt [40, Section 5]. It is the Banach space

$$\widehat{K}_N = \{m \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \colon m * (1_{B(0,1)} E_N) \in L^\infty(\mathbb{R}^N)\}$$

with norm $\|m\|_{\widehat{K}_N} = \|m * (1_{B(0,1)} E_N)\|_\infty$.

Here we denote by $B(x,R) = \{y \in \mathbb{R}^N \colon |x-y| < R\}$ the ball of center x and radius R and by 1_Ω the characteristic function of a set Ω. By E_N we denote the usual fundamental solution of $-\Delta = \delta$, i.e.

$$E_1(x) = -\tfrac{1}{2}|x|,$$

$$E_2(x) = -\frac{1}{2\pi} \ln|x|,$$

$$E_N(x) = (N-2)^{-1} 2^{-1} \pi^{-N/2} \Gamma\left(\frac{N}{2}\right) |x|^{2-N} \quad (N \geqslant 3).$$

The kernel of the Gaussian semigroup is denoted by $p_t(x) = (4\pi t)^{-N/2} \,\mathrm{e}^{-|x|^2/4t}$ and we define

$$\psi_t(x) = \int_0^t p_s(x)\,\mathrm{d}s = (4\pi)^{-N/2} |y|^{2-N} \int_{|y|^2/t}^\infty u^{(N/2)-2}\,\mathrm{e}^{-(u/4)}\,\mathrm{d}u$$

$$(t > 0,\ x \in \mathbb{R}^N).$$

PROPOSITION 1.1. *For* $m \in L^1_{\text{loc}}(\mathbb{R}^n)$ *the following are equivalent:*

(i) $m \in \widehat{K}_N$;
(ii) m *is relatively bounded with respect to* Δ_1, *the Laplacian on* $L^1(\mathbb{R}^N)$;
(iii) $|m| * \psi_t \in L^\infty(\mathbb{R}^N)$ *for some* $t > 0$;
(iv) $|m| * \psi_t \in L^\infty(\mathbb{R}^N)$ *for all* $t > 0$.

In this case, $c_N(m) := \inf_{t>0} \||m| * \psi_t\|_\infty = \inf_{R>0} \|(1_{B(0,R)} E_N) * |m|\|_\infty$.

Proof. The equivalence of (i), (ii) and (iii) follows from [40, 5.1 (a)]. Property (iii) follows trivially from (iv). We show that (ii) implies (iv). Let $t > 0$. Note that $\psi_t * f = \int_0^t \mathrm{e}^{s\Delta_1} f \, \mathrm{d}s \in D(\Delta_1)$ for all $f \in L^1(\mathbb{R}^N)$. Hence by assumption $|m|(\psi_t * f) \in L^1(\mathbb{R}^N)$ for all $f \in L^1(\mathbb{R}^N)$. It follows by duality that $\psi_t * |m| \in L^\infty(\mathbb{R}^N)$.

The last identity is given in [40, 5.1 (c)]. □

REMARK 1.2. One has $\|\psi_t * |m|\|_\infty = \||m| \int_0^t \mathrm{e}^{s\Delta_1} \, \mathrm{d}s\|_{\mathcal{L}(L^1)}$ (by duality). In particular, it follows from [40, 5.1 (b)] that

$$m \mapsto \||m| * \psi_1\|_\infty$$

defines an equivalent norm on $\widehat{K}_N$.

Kato's class in the sense of Simon [37, A2] is the closed subspace $K_N = \{m \in \widehat{K}_N : c_N(m) = 0\}$ of $\widehat{K}_N$.

By Δ_p we denote the Laplacian on $L^p (:= L^p(\mathbb{R}^N))$ (i.e. $D(\Delta_p) = \{f \in L^p : \Delta f \in L^p\}, \Delta_p f = \Delta f)$.

Let $m \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $m^+ \in \widehat{K}_N, c_N(m^+) < 1$ (see Proposition 1.1). Then one defines a C_0-semigroup $(\mathrm{e}^{t(\Delta_p + m)})_{t \geqslant 0}$ on $L^p(\mathbb{R}^N)$ $(1 \leqslant p < \infty)$ in the following way: denote by $s-\lim$ the limit in the strong operator topology in $\mathcal{L}(L^p)$. Let $m_k^+ = \inf\{k, m^+\}, m_k^- = \inf\{k, m^-\}$. Then

$$\mathrm{e}^{t(\Delta_p + m^+)} = s - \lim_{k \to \infty} \mathrm{e}^{t(\Delta_p + m_k^+)};$$

$$\mathrm{e}^{t(\Delta_p - m^+)} = s - \lim_{k \to \infty} \mathrm{e}^{t(\Delta_p - m_k^-)};$$

$$\mathrm{e}^{t(\Delta_p + m)} = s - \lim_{k \to \infty} \mathrm{e}^{t(\Delta_p + m^+ - m_k^-)} = s - \lim_{k \to \infty} \mathrm{e}^{t(\Delta_p - m^- + m_k^+)}.$$

By $\Delta_p + m$ we denote the generator of the semigroup $(\mathrm{e}^{t(\Delta_p + m)})_{t \geqslant 0}$. This notation is symbolic; i.e., in general one has $D(\Delta_p + m) \supset D(\Delta_p) \cap D(m)$ with strict inclusion. However, if $p = 1$, then $D(\Delta_1 + m) = D(\Delta_1) \cap D(m)$ (where $D(m) = \{f \in L^1 : fm \in L^1\}$) and $D(\Delta_1 + m) = D(\Delta_1)$ if $m^- \in L^\infty$. We refer to [40, 41] for all this.

It follows from the definition that $\mathrm{e}^{t(\Delta_p + m)} \geqslant 0$ for $t > 0$ (in the sense of positive, i.e. positivity preserving operators).

PROPOSITION 1.3. *Let* $1 \leqslant p \leqslant \infty$. *The semigroup* $(e^{t(\Delta_p+m)})_{t\geqslant 0}$ *on* $L^p(\mathbb{R}^N)$ *is irreducible (see* [25, p. 306] *for the defintion).*

Proof. Since $0 \leqslant e^{t(\Delta_p-m^-)} \leqslant e^{t(\Delta_p+m)}$ we can assume that $m^+ = 0$. Let $V = m^-, V_k = \inf\{V,k\})k \in \mathbb{N})$. Then by [41, § 3] (or [30, S.16, p. 373]) $e^{t(\Delta_p-(V-V_k))}$ converges strongly to $e^{t\Delta}$ as $k \to \infty$. Now assume that $\mathcal{J} \subset L^p(\mathbb{R}^N)$ is a closed ideal invariant under $e^{t(\Delta-V)}$. Since $e^{t(\Delta-(V-V_k))} \leqslant e^{kt}\, e^{t(\Delta-V)}$, it follows that $e^{t(\Delta-(V-V_k))}\mathcal{J} \subset \mathcal{J}(t \geqslant 0, k \in \mathbb{N})$. Letting $k \to \infty$, one obtains $e^{t\Delta}\mathcal{J} \subset \mathcal{J}(t \geqslant 0)$ and so $\mathcal{J} = 0$ or $\mathcal{J} = L^2(\mathbb{R}^N)$ since $(e^{t\Delta})_{t\geqslant 0}$ is irreducible.

REMARK 1.4. The proof shows that $(e^{t(\Delta_p+m)})_{t\geqslant 0}$ is an irreducible, positive C_0-semigroup whenever $m\colon \mathbb{R}^N \to R$ is measurable, $m^+ \in \widehat{K}_N, c_N(m^+) < 1$ and m^- is $e^{t\Delta_p}$-regular in the sense of Voigt [41, Definition 31]; this property is independent of $p \in [1,\infty)$, ([40], [41, 4.3]). There exists a regular $m_-\colon \mathbb{R}^N \to [0,\infty)$ which is nowhere integrable [38].

We define $\Delta_\infty + m$ as the adjoint of $\Delta_1 + m$. Thus $\Delta_\infty + m$ generates a weak*-continuous semigroup $(e^{t(\Delta_\infty+m)})_{t\geqslant 0}$ on $L^\infty(\mathbb{R}^N)$.

It has been shown by Hempel and Voigt [22] that the spectrum $\sigma(\Delta_p + m)$ is independent of $p \in [1,\infty]$.

2. The Spectral Function for Positive Potentials

Let $m \in \widehat{K}_N$. Then for $0 \leqslant \lambda < \lambda_0 := c_N(m^+)^{-1}$ the operator $\Delta_p + \lambda m$ is defined on $L_p(\mathbb{R}^N)(1 \leqslant p \leqslant \infty)$ and its spectrum is real and independent of p (see Section 1). By

$$s(\lambda) = s(\Delta_p + \lambda_m) = \sup\{\mu \in \sigma(\Delta_p + \lambda m)\}$$

we denote the *spectral bound* of $\Delta_p + \lambda m$ and we call $s\colon [0,\lambda_0) \mapsto \mathbb{R}$ the *spectral function* of m. In this section we investigate s in a neighborhood of 0; in particular we determine the derivative of s at 0 if m is positive.

PROPOSITION 2.1. *For* $0 \leqslant \lambda < \lambda_0$ *one has the variational formula*

$$s(\lambda) = \sup\left\{-\int |\nabla u|^2 + \lambda \int mu^2\colon u \in \mathcal{D}_1\right\}, \tag{2.1}$$

where $\mathcal{D}_1 = \{u \in \mathcal{D}(\mathbb{R}^N); \|u\|_{L^2} = 1\}$. *In particular,* s *is a convex function on* $[0,\lambda_0)$. *Here* $\mathcal{D}(\mathbb{R}^N)$ *denotes the space of all test functions.*

Proof. It follows from [40, § 5] that for $0 \leqslant \lambda < \lambda_0$ the operator $-(\Delta_2 + \lambda m)$ is associated with the closed lower bounded form $a_\lambda(u,v) = \int \nabla u \nabla v - \lambda \int muv$ with domain $D(a_\lambda) = H^1(\mathbb{R}^N) \cap Q(m^-)$, where $Q(m^-)$ is the form domain of m^-. Since $\mathcal{D}(\mathbb{R}^N)$ is a form core of a_λ [24, VI Lemma 4.6, p. 349], (2.1) is the usual variational formula. □

Assume that $m \in \widehat{K}_N$. Then $s(0) = 0$ and

$$s'(0+) = \lim_{\lambda \downarrow 0} \frac{s(\lambda)}{\lambda}$$

exists. In fact, formula (2.1) defines a convex function from $s\colon [0, \lambda_0) \to \mathbb{R}$. Our aim is to determine when $s'(0+) = 0$.

THEOREM 2.2. *Let* $0 \leqslant m \in \widehat{K}_N$. *The following assertions are equivalent.*

(a) $s'(0+) = 0$;
(b) $\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} \frac{1}{t}(\psi_t * m)(x) = 0$;
(c) $\lim_{R\to\infty} \sup_{x\in\mathbb{R}^N} \frac{1}{R^N} \int_{B(x,R)} m(y)\,\mathrm{d}y = 0$;
(d) $\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} (p_t * m)(x) = 0$.

REMARK. In condition (b) and (d) we can replace the supremum by the essential supremum; i.e., (b) is equivalent to $\lim_{t\to\infty} \frac{1}{t}\|\psi_t * m\|_\infty = 0$ and (d) is equivalent to $\lim_{t\to\infty} \|p_t * m\|_\infty = 0$. In fact, if $0 \leqslant m \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $q\colon \mathbb{R}^N \to [0,\infty)$ is continuous, then $m * q\colon \mathbb{R}^N \to [0,\infty]$ is lower semi-continuous.

The equivalence of (a) and (b) follows from the following identity.

PROPOSITION 2.3. *Let* $m \in \widehat{K}_N$ *such that* $m^- \in L^\infty$. *Then*

$$s'(0+) = \lim_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}^N} (m * \psi_t)(x). \tag{2.2}$$

Proof. Observe that $s(\Delta + \lambda(m+c)) = s(\Delta + \lambda m) + \lambda c$ and $((m+c)*\psi_t)(x) = (m * \psi_t)(x) + tc$. Thus, replacing m by $m + c$ if necessary, we can assume that $m \geqslant 0$.

We will use the Feynman–Kac formula (see [40, § 6]) which is usually associated to $\frac{1}{2}\Delta$ instead of Δ. Let $\widetilde{\psi}_t(x) = \int_0^t p_{s/2}(x)\,\mathrm{d}s = 2\psi_{t/2}(x)$. Then (2.2) is equivalent to

$$\lim_{\lambda\downarrow 0} \frac{s(\frac{1}{2}\Delta + \lambda m)}{\lambda} = \lim_{t\to\infty} \frac{1}{t}\|\widetilde{\psi}_t * m\|_\infty. \tag{2.3}$$

One has $(m * \widetilde{\psi}_t)(x) = E^x[\int_0^t m(B(s))\,\mathrm{d}s]$ $(x \in \mathbb{R}^N)$, see [9, p. 462]. Let $\alpha_t = \|m * \widetilde{\psi}_t\|_\infty$. Let $t > 0$, $0 < \lambda < \alpha_t^{-1}$. Then $\sup_x E^x[\exp \int_0^t \lambda m(B(s))\,\mathrm{d}s] = \alpha_t \lambda < 1$. It follows from Khasmin'skii's lemma [37, B.1.2, p. 461] that $\mathrm{e}^{\frac{1}{2}ts(2\lambda)} \leqslant$

$\|e^{t(\frac{1}{2}\Delta_\infty+\lambda m)}\|_{\mathcal{L}(L^\infty)} = \|e^{t(\frac{1}{2}\Delta_\infty+\lambda m)}\mathbf{1}\|_\infty = \sup_x E^x[\exp \lambda\int_0^t m(B(s))\,\mathrm{d}s] \leqslant \frac{1}{1-\lambda\alpha_t}$. Thus $s(\lambda) \leqslant \frac{1}{t}\log\frac{1}{1-\lambda\alpha_t}$. Taking the derivative at 0 with respect to λ on both sides yields $s'(0+) \leqslant \frac{\alpha_t}{t}$ $(t>0)$. Hence $s'(0+) \leqslant \underline{\lim}_{t\to\infty} \frac{\alpha_t}{t}$.

In order to show the reverse inequality let $0<\varepsilon<\lambda_0$, $w\in\mathbb{R}$ such that $s(\frac{1}{2}\Delta+\lambda m)\leqslant \lambda w$ for all $0\leqslant\lambda\leqslant\varepsilon$. Then by Jensen's inequality,

$$\overline{\lim_{t\to\infty}}\ \frac{1}{t}\sup_x E^x\left[\lambda\int_0^t m(B(s))\,\mathrm{d}s\right] \leqslant$$

$$\overline{\lim_{t\to\infty}}\ \frac{1}{t}\log\sup_x E^x\left[\exp\lambda\int_0^t m(B(s))\,\mathrm{d}s\right] =$$

$$\overline{\lim_{t\to\infty}}\ \frac{1}{t}\log\|e^{t(\frac{1}{2}\Delta_\infty+\lambda m)}\mathbf{1}\|_{L^\infty} =$$

$$\overline{\lim_{t\to\infty}}\ \frac{1}{t}\log\|e^{t(\frac{1}{2}\Delta_\infty+\lambda m)}\|_{\mathcal{L}(L^\infty)} = s(\tfrac{1}{2}\Delta+\lambda m)\leqslant\lambda w.$$

Hence $\overline{\lim}_{t\to\infty}\ \frac{1}{t}\|\widetilde{\psi}_t * m\|_\infty \leqslant w$. This shows that $\overline{\lim}_{t\to\infty}\ \frac{1}{t}\|\widetilde{\psi}_t * m\|_\infty \leqslant s'(0+)$. □

In order to prove the other equivalences of Theorem 2.2 we need some preparation.

Let $R>0$. By a cube of length R we mean a set of the form $x-Q_R$ where $x\in\mathbb{R}^N$ and $Q_R=\{y\in\mathbb{R}^N : 0\leqslant y_j\leqslant R, j=1,\dots,N\}$. Let $m\colon\mathbb{R}^N\to[0,\infty)$ be measurable. We let $q_R(m)=\sup\{\int_Q m(y)\,\mathrm{d}y : Q$ is a cube of length $R\}$, i.e., $q_R(m)=\|1_{Q_R} * m\|_\infty$.

LEMMA 2.4. *Let* $t>0, R>0$. *Then*

$$\|p_t * m\|_\infty \leqslant 2^N((4\pi t)^{-\frac{1}{2}}+R^{-1})^N\,.\,q_R(m). \tag{2.4}$$

Proof. Let $c=q_R(m)$. We have to show that $(p_t*m)(x)\leqslant c2^N((4\pi t)^{-\frac{1}{2}}+\frac{1}{R})^N$. Replacing m by $m(\,.\,+x)$ if necessary we can assume that $x=0$. For $n=(n_1,\dots,n_N)\in\mathbb{N}_0^N$ we let $n^2=\Sigma_{j=1}^N n_j^2$ and $Q(R,n)=\{y\in\mathbb{R}^N; Rn_j\leqslant |y_j|\leqslant(n_j+1)R\ (j=1\leqslant N)\}$. Since $Q(R,n)$ is the union of 2^N cubes of length R, one has $\int_{Q(R,n)} m(y)\,\mathrm{d}y\leqslant c2^N$. Hence

$$(p_t*m)(0) = \sum_{n\in\mathbb{N}_0^N}(4\pi t)^{-\frac{N}{2}}\int_{Q(R,n)} m(y)\,e^{-y^2/4t}\,\mathrm{d}y$$

$$\leqslant (4\pi t)^{-N/2}\sum_{n\in\mathbb{N}_0^N} e^{-n^2R^2/4t}\int_{Q(R,n)} m(y)\,\mathrm{d}y$$

$$\begin{aligned}
&\leqslant (4\pi t)^{-N/2} c \cdot 2^N \sum_{n \in \mathbb{N}_0^N} \mathrm{e}^{-n^2 R^2/4t} \\
&= (4\pi t)^{-N/2} c \cdot 2^N \left(\sum_{k=0}^{\infty} \mathrm{e}^{-R^2 k^2/4t} \right)^N \\
&\leqslant (4\pi t)^{-N/2} c \cdot 2^N \left(1 + \int_0^{\infty} \mathrm{e}^{-R^2 u^2/4t}\, \mathrm{d}u \right)^N \\
&= (4\pi t)^{-N/2} c 2^N (1 + \sqrt{4\pi t} R^{-1})^N. \qquad \square
\end{aligned}$$

LEMMA 2.5. *Let* $m: \mathbb{R}^N \to [0, \infty)$ *be measurable. The following are equivalent.*

(i) *For all* $t > 0$ *one has* $p_t * m \in L^\infty$;
(ii) *there exists* $t > 0$ *such that* $p_t * m \in L^\infty$;
(iii) $\sup_{R \geqslant 1} \sup_{x \in \mathbb{R}^N} R^{-N} \int_{B(x,R)} m(y)\, \mathrm{d}y < \infty$;
(iv) *there exists* $R > 0$ *such that* $\sup_{x \in \mathbb{R}^N} R^{-N} \int_{B(x,R)} m(y)\, \mathrm{d}y < \infty$.

Proof. (i) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (iv) are trivial.
(ii) $\Longrightarrow$ (iii) : Assume that $p_{t_0} * m \in L^\infty$. Then $p_t * m = p_{t-t_0} * (p_{t_0} * m) \in L^\infty$ and $\|p_t * m\|_\infty \leqslant \|p_{t_0} * m\|_\infty$ for all $t \geqslant t_0$. Let $R \geqslant t_0^{\frac{1}{2}}$. Let $t = R^2$. Then for $x \in \mathbb{R}^N$,

$$\begin{aligned}
R^{-N} \int_{B(x,R)} m(y)\, \mathrm{d}y &\leqslant \mathrm{e}^{\frac{1}{4}} t^{-N/2} \int_{B(x,t^{1/2})} \mathrm{e}^{-|x-y|^2/4t} m(y)\, \mathrm{d}y \\
&\leqslant (4\pi)^{N/2}\, \mathrm{e}^{\frac{1}{4}} (p_t * m)(x) \leqslant (4\pi)^{N/2}\, \mathrm{e}^{\frac{1}{4}} \|p_t * m\|_\infty. \qquad (2.5)
\end{aligned}$$

(iv) $\Longrightarrow$ (i) This follows from (2.4). $\square$

PROPOSITION 2.6. *Let* $0 \leqslant m \in \widehat{K}_N$. *Then*

(a) $p_t * m \in L^\infty$ *for all* $t > 0$;
(b) $\sup_{R \geqslant 1} \sup_{x \in \mathbb{R}^N} R^{-N} \int_{B(x,R)} m(y)\, \mathrm{d}y < \infty$
(c) $\|\psi_t * m\|_\infty \leqslant (1 + t) \|\psi_1 * m\|_\infty$.

Proof. It follows from the definition that m satisfies (iv) of Lemma 2.5 (see [40, p. 183]). Hence (a), (b) are valid. We show by induction that (c) is satisfied for $t \in [n, n+1)$. This is clear for $n = 0$ since $\|\psi_t * m\|_\infty$ is increasing in t. Assume that it is true for n. Let $t \in [n+1, n+2)$. Since $p_s * p_r = p_{s+r}$, one has

$p_{t-1} * \psi_1 = \psi_t - \psi_{t-1}$. Hence, $\|\psi_t * m\|_\infty = \|\psi_{t-1} * m + (p_{t-1} * \psi_1) * m\|_\infty \leqslant \|\psi_{t-1} * m\|_\infty + \|\psi_1 * m\|_\infty \leqslant (t+1)\|\psi_1 * m\|_\infty$ by the inductive assumption. □

Proof of Theorem 2.2. The equivalence of (a) and (b) follows from (2.2).

(b) $\Longrightarrow$ (c). First case: $N \geqslant 2$. Let $c = (4\pi)^{-N/2} \int_1^\infty u^{\frac{N}{2}-2} \mathrm{e}^{-u/4} \,\mathrm{d}u$. Then

$$\psi_t(x) = (4\pi)^{-N/2}|x|^{2-N} \int_{|x|^2/t}^\infty u^{\frac{N}{2}-2} \mathrm{e}^{-u/4} \,\mathrm{d}u \geqslant ct^{-N/2}t$$

whenever $|x| \leqslant t^{\frac{1}{2}}$. Hence for $t > 0, x \in \mathbb{R}^N$,

$$\begin{aligned} t^{-N/2} \int_{B(x,t^{1/2})} m(y) \,\mathrm{d}y &\leqslant c^{-1}\frac{1}{t} \int_{B(x,t^{1/2})} m(y)\psi_t(x-y) \,\mathrm{d}y \\ &\leqslant c^{-1}\frac{1}{t}(m * \psi_t)(x). \end{aligned}$$

Thus (b) implies (c).

Second case: $N = 1$. Then $\psi_t(x) \geqslant (4\pi)^{-1/2}|x| \int_{|x|^2/t}^4 u^{-3/2} \mathrm{e}^{-u/4} \,\mathrm{d}u \geqslant (4\pi)^{-1/2}|x| \,\mathrm{e}^{-1} \int_{|x|^2/t}^4 u^{-3/2} \,\mathrm{d}u = 2^{-1} \mathrm{e}^{-1}\pi^{-1/2}t^{1/2}$ for $|x| \leqslant t^{\frac{1}{2}}$. Hence $t^{-\frac{1}{2}} \int_{B(x,t^{1/2})} m(y) \,\mathrm{d}y \leqslant 2e\pi^{\frac{1}{2}} \frac{1}{t}(m * \psi_t)(x) \quad (t > 0, x \in \mathbb{R})$.

(c) $\Longrightarrow$ (d). By (2.4), $\overline{\lim_{t\to\infty}} \|p_t * m\|_\infty \leqslant 2^N R^{-N} q_R(m)$ for all $R > 0$. Thus (c) implies (d).

(d) $\Longrightarrow$ (b). Let $\varepsilon > 0$. By assumption, there exists $\tau > 0$ such that $\|p_t * m\|_\infty \leqslant \varepsilon$ for all $t \geqslant \tau$. Hence $\overline{\lim_{t\to\infty}} \frac{1}{t}\|\psi_t * m\|_\infty \leqslant \overline{\lim_{t\to\infty}} \sup_x \frac{1}{t} \int_0^\tau (p_s * m)(x) \,\mathrm{d}s + \overline{\lim_{t\to\infty}} \sup_x \frac{1}{t} \int_\tau^t (p_s * m)(x) \,\mathrm{d}s \leqslant \overline{\lim_{t\to\infty}} \frac{1}{t}(t-\tau)\varepsilon = \varepsilon$. □

COROLLARY 2.7. *The set of all* $0 \leqslant m \in \widehat{K}_N$ *such that* $\frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0+} s(\Delta + \lambda m) = 0$ *is a closed cone.*

Proof. It follows from Theorem 2.2 that the set in question is a cone. Let $0 \leqslant m \in \widehat{K}_N$, let m_1 be in the set, $\|m - m_1\|_{\widehat{K}_N} \leqslant \varepsilon$. Then $\frac{1}{t}\|\psi_t * m\|_\infty \leqslant \frac{1}{t}\|\psi_t * |m - m_1|\|_\infty + \frac{1}{t}\|\psi_t * m_1\|_\infty \leqslant 2\|\psi_1 * |m - m_1|\|_\infty + \frac{1}{t}\|\psi_t * m_1\|_\infty$ (by Prop. 2.6 c) $\leqslant \mathrm{const}\|m - m_1\|_{\widehat{K}_N} + \frac{1}{t}\|\psi_t * m_1\|_\infty$ by Remark 1.2. Hence $\overline{\lim_{t\to\infty}} \frac{1}{t}\|\psi_t * m\|_\infty \leqslant \mathrm{const}.\varepsilon$ □

Of special interest is the class

$$K_{N,0} := \{m \in \widehat{K}_N \colon m \text{ is } \Delta_1\text{–compact}\} \tag{2.6}$$

It has been shown by Voigt [40, 5.5] that $K_{N,0}$ coincides with the closure of $\mathcal{D}(\mathbb{R}^N)$ in $\widehat{K}_N$. In particular, $K_{N,0} \subset K_N$. Let $L_0^\infty(\mathbb{R}^N) = \{m \in L^\infty(\mathbb{R}^N)$: ess. $-\lim_{|x|\to\infty} |m(x)| = 0\}$.

EXAMPLE 2.8. (cf. [40, 5.6]). Let $\frac{N}{2} < p < \infty$ if $N \geqslant 2$, and $1 \leqslant p < \infty$ if $N = 1$. Then $L^p + L_0^\infty \subset K_{N,0}$. In fact, by Hölder's inequality $L^p \hookrightarrow \widehat{K}_N$. Since $\mathcal{D}(\mathbb{R}^N)$ is dense in L^p, it follows that $L^p \subset \overline{\mathcal{D}}^{K_N}$.

LEMMA 2.9. *Let* $m \in K_{N,0}$, $q \in L^1_{\text{loc}}(\mathbb{R}^N)$, *such that* $|q| \leqslant |m|$. *then* $q \in K_{N,0}$.

Proof. There exists $g \in L^\infty$ such that $gm = q$. Since by assumption, $m(1 - \Delta_1)^{-1}$ is compact, it follows that $q(1-\Delta_1)^{-1} = gm(1-\Delta_1)^{-1}$ is compact. □

COROLLARY 2.10. *If* $m \in K_{N,0}$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} s(\Delta + \lambda m) = 0.$$

Proof. (a) If $m \in L^p, N < p < \infty$, it follows from Theorem 2.2 (applying criterion (c)) that $\frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0+} s(\Delta + \lambda m) = 0$. By Corollary 2.7 the same remains true if $0 \leqslant m \in K_{N,0}$.

(b) Let $m \in K_{N,0}$. Since $\sigma_{\text{ess}}(\Delta_1) = \sigma_{\text{ess}}(\Delta_1 + \lambda m^-)$ it follows that $s(\Delta - \lambda m^-) \geqslant 0$ $(\lambda > 0)$. Hence by (a), $0 \leqslant \lim_{\lambda\downarrow 0} \frac{s(\Delta-\lambda m^-)}{\lambda} \leqslant \lim_{\lambda\downarrow 0} \frac{s(\Delta+\lambda m)}{\lambda} \leqslant \lim_{\lambda\downarrow 0} \frac{s(\Delta+\lambda m^+)}{\lambda} = 0$. We have shown that $\frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0+} s(\Delta + \lambda m) = 0$. The proof is finished by replacing m by $-m$. □

REMARKS 2.11. (a) If $N \geqslant 3$ and $m \in K_{N,0}$ a much stronger result than Corollary 2.10 is true: There actually exists $\lambda_1 > 0$ such that $s(\Delta+\lambda m) = 0$ for all $\lambda \in [-\lambda_1, \lambda_1]$. In fact, $s(\Delta+\lambda m) > 0$ implies that $s(\Delta+\lambda m) \in \sigma_{\text{ess}}(\Delta_1+\lambda m) = \sigma_{\text{ess}}(\Delta_2 + \lambda m)$ (cf. Remark 3.4). Now the claim follows from the Cwikel–Lieb–Rosenbljum bound [31, p. 101].
(b) If $N \geqslant 3$, $m \in L^{N/2-\varepsilon} \cap L^{N/2+\varepsilon}$ for some $\varepsilon > 0$, then Simon ([34], see also [35, Theorem 1.2], [36, Theorem 1.4], [37, Theorem B.5.2]) showed that for some $\lambda_1 > 0$ one has $s(\Delta + \lambda m) = 0$ for all $\lambda \in [0, \lambda_1]$ and he showed in addition that $\sup_{t\geqslant 0} \|e^{t(\Delta_p+\lambda m)}\|_{\mathcal{L}(L^p)} < \infty$ $(1 \leqslant p \leqslant \infty)$ for all $\lambda \in [0, \lambda_1)$.
The situation is much different for $N = 1, 2$:
(c) Let $N = 1$, $m \in L^1(\mathbb{R})$ such that $\int m > 0$. Then $s(\Delta+\lambda m) > 0$ for all $\lambda > 0$. In fact, let $u \in \mathcal{D}(\mathbb{R})$ such that $0 \leqslant u(x) \leqslant 1$ $(x \in \mathbb{R})$ and $u(x) = 1$ $(|x| < 1)$. Let $u_n(x) = u(\frac{x}{n})$. Then $0 \leqslant u_n \leqslant 1, u_n(x) \to 1, \int |\nabla u_n|^2 \to 0$ $(n \to \infty)$. Hence $\lim_{n\to\infty} (-\int |\nabla u_n|^2 + \lambda \int u_n^2 m) = \lambda \int m > 0$ and so $s(\Delta + \lambda m) > 0$ whenever $\lambda > 0$.

(d) Let $N = 2$ and $0 \leqslant m \in \widehat{K}_N$, m continuous, $m \neq 0$. Then $s(\Delta + \lambda m) > 0$ for all $\lambda > 0$. In fact, there exists $0 \leqslant q \leqslant m$, $0 \neq q \in \mathcal{D}(\mathbb{R}^N)$. By [31, Theorem XIII.11, p. 100], $s(\Delta + \lambda q) > 0$ $(\lambda > 0)$.
(e) Other related results for $N = 1, 2$ are [31, Theorem X.III.110, p. 338 and Notes p. 363] and [10, Theorem 3.2].

EXAMPLES 2.12. (a) If $m \in L^1(\mathbb{R}^N)$, then m satisfies condition (c) and (d) of Theorem 2.2. However $L^1(\mathbb{R}^N)$ is not contained in $\widehat{K}_N$. E.g. let $2 < a < N$, $m(x) = |x|^{-a} 1_{B(0,1)}(x)$. Then $m \in L^1(\mathbb{R}^N) \backslash \widehat{K}_N$.
(b) Let $0 \leqslant m \in L^\infty$ such that there exist $j_0 \in \{1, \ldots, N\}, -\infty < a < b < \infty$ such that supp $m \subset \Omega := \{x : a < x_{j0} < b\}$. Then $s'(0+) = 0$. This can be seen by applying criterion (c) of Theorem 2.2. On the other hand, it is easy to see that $1_\Omega \notin K_{N,0}$ if $N \geqslant 2$.
(c) On has $L^{N/2} \not\subset \widehat{K}_N$ for $N = 4$. In fact, it is easy to see that $m(x) = -|x|^{-2}(\log|x|)^{-1} 1_{B(0,1/2)}$ is in $L^2(\mathbb{R}^4)$ but $m \notin \widehat{K}_4$ (cf. [37, A4]).

The main point in this section is to characterize when $s'(0_+) = 0$ (Theorem 2.2), and that is what is needed in Section 3. However, our arguments allow us also to estimate $s(\Delta + \lambda m)$ for fixed λ by averages of m over balls. This is of independent interest and we want to give more details.

In the remainder of this section we assume that $0 \leqslant m \in \widehat{K}_N$ and let $\lambda_0 = c_N(m)^{-1}$. For convenience, we denote by

$$a_m(R) = \sup_{x \in \mathbb{R}^N} \frac{1}{R^N} \int_{B(x,R)} m(y)\,\mathrm{d}y \qquad (0 < R < \infty)$$

the upper bound of the averages of m over balls of radius R. It is not difficult to see that there exists a constant $\lambda > 0$ (depending only on the dimension N) such that

$$a_m(R_1) \geqslant k\, a_m(R_2) \quad \text{if} \quad 0 < R_1 \leqslant R_2. \tag{2.7}$$

THEOREM 2.13. *One has*

$$\frac{s(\lambda)}{\lambda} \leqslant c_1 a_m(R) \tag{2.8}$$

provided $0 < \lambda, R$ *satisfies one of the following conditions:*

$$\lambda \|\psi_{R^2} * m\|_\infty \leqslant \tfrac{1}{2} \quad \text{it} \tag{2.9a}$$

$$\lambda c_1' \int_0^R r\, a_m(r)\,\mathrm{d}r \leqslant \tfrac{1}{2}. \tag{2.9b}$$

Here $c_1, c_2' > 0$ *are constants which depend only on the dimension* N.

Proof. By the proof of Proposition 2.3 we have $s(\lambda) \leqslant \frac{1}{t}\log\{(1-\lambda\|\psi_t * m\|_\infty)^{-1}\}$ if $\lambda\|\psi_t * m\|_\infty < 1$. Lemma 2.4 implies that $\|p_s * m\|_\infty \leqslant$ const $a_m(s^{1/2})$. Hence (in view of (2.7)) $s(\lambda) \leqslant \frac{1}{t}\log[\{1-\lambda(\|\psi_\tau * m\|_\infty +$ const $(t-\tau)a_m(\tau^{1/2}))\}^{-1}]$ provided that $0<\tau<t$ and λ is sufficiently small. If $\lambda\|\psi_\tau * m\|_\infty \leqslant \frac{1}{2}$ we may choose $t>\tau$ such that λ const $(t-\tau)a_m(\tau^{1/2}) = \frac{1}{4}$ and deduce that there exists a constant $c_1 > 0$ (depending only on N) such that

$$s(\lambda) \leqslant c_1 \lambda a_m(R) \tag{2.10}$$

whenever $\lambda\|\psi_{R^2} * m\|_\infty \leqslant \frac{1}{2}$. By Proposition 1.1, given $0<\lambda<\frac{\lambda_0}{2} = (2\,c_N(m))^{-1}$, we always find $R>0$ such that $\lambda\|\psi_{R^2} * m\|_\infty \leqslant \frac{1}{2}$. Lemma 2.4 shows that

$$\begin{aligned}\|\psi_{R^2} * m\|_\infty &\leqslant \int_0^{R^2} \|p_t * m\|_\infty \,\mathrm{d}t \\ &\leqslant \text{const.} \int_0^{R^2} t^{-N/2} q_{t^{1/2}}(m)\,\mathrm{d}t \\ &\leqslant \text{const.} \int_0^{R} r\, a_m(r)\,\mathrm{d}r. \qquad \square\end{aligned}$$

The estimates given by Theorem 2.13 are quite sharp. This is shown by the following (much easier) lower estimate.

PROPOSITION 2.14. *For all* $0<\lambda<\lambda_0$ *there exists* $R_\lambda > 0$ *such that*

$$c_0 a_m(R_\lambda) \leqslant \frac{s(\lambda)}{\lambda}. \tag{2.11}$$

Here $c_0 > 0$ depends only on N.

Proof. Let $0 \leqslant \varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $\varphi(y) = 1$ if $|y| \leqslant 1$. Let $x \in \mathbb{R}^N$, $R>0$ and put $u(y) = \varphi(R^{-1}(y-x))$. Then (2.1) gives $-R^{N-2}\|\nabla\varphi\|^2 + \lambda\int_{B(x,R)} m(y)\,\mathrm{d}y \leqslant s(\lambda)R^N$. Choosing $c_0 = \|\varphi\|^2 + \|\nabla\varphi\|^2$ and $R = R_\lambda = s(\lambda)^{-1/2}$ gives (2.11). $\square$

REMARK. Since (2.1) defines a convex and hence continuous function on $(-\infty, \lambda_0)$ one has $\lim_{\lambda\to 0} s(\lambda) = 0$ and thus $\lim_{\lambda\to 0+} R_\lambda = \infty$. This gives another proof of (a) $\Longrightarrow$ (c) in Theorem 2.2.

Estimates similar to (2.8) but with L_1-averages (considered here) replaced by L_p-averages $(1 < p < \infty)$ are obtained by Fefferman–Phong (see [19]) and Schechter [33]. In fact, they show that

$$s(\lambda) \leqslant \sup_{R>0} \left\{ C_p \lambda \sup_{x \in \mathbb{R}^N} \left(\frac{1}{R^N} \int_{B(x,R)} m(y)^p \, \mathrm{d}y \right)^{1/p} - \frac{1}{R^2} \right\} \tag{2.12}$$

where C_P is a constant depending on p and N, see [19, Theorem 5, p. 145], [33, Corollary 3.3]. Schechter's proof of (2.12) showed that

$$1 \leqslant C_p' \lambda \sup_{r \leqslant s(\lambda)^{-\frac{1}{2}}} r^2 \sup_{x \in \mathbb{R}^N} \left(\frac{1}{R^N} \int_{B(x,R)} m(y)^p \, \mathrm{d}y \right)^{\frac{1}{p}}$$

[33, Theorem 3.2], from which it is easy to deduce that

$$s(\lambda) \leqslant C_p'' \lambda \sup_{x \in \mathbb{R}^N} \left(\frac{1}{R^N} \int_{B(x,R)} m(y)^p \, \mathrm{d}y \right)^{1/p} \tag{2.13}$$

provided that

$$C_p' \lambda \sup_{r \leqslant R} r^2 \sup_{x \in \mathbb{R}^N} \left(\frac{1}{R^N} \int_{B(x,R)} m(y)^p \, \mathrm{d}y \right)^{1/p} < 1.$$

The estimates (2.12) and (2.13) are sometimes infinite – there exist functions in K_N which are not in L^p_{loc} for any $p > 1$. On the other hand, there are some functions not in $\widehat{K}_N$ for which (2.12) and (2.13) are finite (see Example 4.3).

3. Potentials with Changing Sign

Throughout this section we assume that $m \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $0 \neq m^+ \in K_N$. Then $\lambda_0 = c_N(m^+)^{-1} = \infty$ (Section 2). By $s(\lambda) = s(\Delta + \lambda m) = \sup\{-\int |\nabla u|^2 + \lambda \int mu^2 \colon u \in \mathcal{D}_1\}$ $(\lambda \geqslant 0)$ we denote the spectral function. Since $m^+ \neq 0$, there exists $u \in D_1$ such that $\int mu^2 > 0$. Consequently, $\lim_{\lambda \to \infty} s(\lambda) = \infty$. Since s is convex, there are three different possible cases: $1. s(\lambda) > 0$ for all $\lambda > 0$; 2. there exists $\lambda_0 > 0$ such that $s(\lambda) = 0$ on $[0, \lambda_0]$; 3. there exists a unique $\lambda_1 > 0$ such that $s(\lambda_1) = 0$.

We are interested in finding conditions for the third case to occur. Since $s(\Delta - \lambda m^-) \leqslant s(\Delta + \lambda m)$ a necessary condition is that $s(\Delta - m^-) < 0$. We recall the results from [5], [9] and [7] characterizing this condition.

THEOREM 3.1. 1. *Let* $m^- \in L^1 + L^\infty$. *Then* $s(\Delta - m^-) < 0$ *if and only if there exists* $R > 0$ *such that*

$$\inf_{x\in\mathbb{R}^N} \int_{B(x,R)} m^-(y)\,\mathrm{d}y > 0.$$

2. *In general one has* $s(\Delta - m^-) < 0$ *if and only if* $\int_\Omega m^- = \infty$ *whenever* $\Omega \subset \mathbb{R}^N$ *is open such that* $s(\Delta_\Omega) = 0$. *Here* Δ_Ω *denotes the Dirichlet Laplacian on* $L^2(\Omega)$, *i.e.* $-\Delta_\Omega$ *is associated with the form* $a(u,v) = \int \nabla u\ \nabla v, Q(a) = H_0^1(\Omega)$.

Note that condition (b) in the following theorem has been investigated in Section 2.

THEOREM 3.2. *Let* $m \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *such that* $0 \neq m^+ \in K_N$. *Assume that*

(a) $s(\Delta - m^-) < 0$ *and*
(b) $\frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0_+} s(\Delta + \lambda m^+) = 0$.

Then there exists a unique $\lambda_1 > 0$ *such that* $s(\Delta + \lambda_1 m) = 0$.

Proof. By convexity, (a) is equivalent to $\frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0_+} s(\Delta - \lambda m^-) < 0$. It follows from the definition that $s(\Delta + \lambda m) \leqslant \frac{1}{2}(s(\Delta - 2\lambda m^-) + s(\Delta + 2\lambda m^+))$. Hence $\frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0_+} s(\Delta + \lambda m) \leqslant \frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0_+} s(\Delta - \lambda m^-) + \frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0_+} s(\Delta + \lambda m^+) < 0$. Now the claim follows by convexity since $\lim_{\lambda\to\infty} s(\Delta + \lambda m) = \infty$.

From the proof it is apparent that the theorem remains true if we replace (a) and (b) by the weaker condition

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0_+} s(\Delta + \lambda m^+) + \frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0_+} s(\Delta - \lambda m^-) < 0. \tag{3.1}$$

In (2.2) we gave an exact formula for the first term: $\lim_{t\to\infty} \frac{1}{t}\|m^+ * \psi_t\|_\infty$. In [7, Remark 4.7] an upper estimate for the second is given.

Theorem 3.2 implies that $(\mathrm{e}^{t(\Delta_p+\lambda m)})_{t\geqslant 0}$ is exponentially stable for $\lambda < \lambda_1$, and unbounded for $\lambda > \lambda_1$. Similar results have been obtained formerly ([11, Theorem 4.2], [15, Theorem 7.7]); however, our condition (b) is much more general than those given in these papers.

Next we establish existence of a principal eigenvalue.

THEOREM 3.3. *Let* $m \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *such that* $0 \neq m^+$. *Assume that*

(a) $s(\Delta - m^-) < 0$;
(b) $m^+ \in L^p + L^\infty_0$ *where* $\infty > p > \frac{N}{2}$ *if* $N \geqslant 2, p \geqslant 1$ *if* $N = 1$.

Then there exists a unique $\lambda_1 > 0$ *such that* $s(\Delta + \lambda_1 m) = 0$. *Moreover, there exists a unique* $0 \leqslant u \in D(\Delta_2 + \lambda_1 m)$ *such that* $\|u\|_{L^2} = 1$ *and* $(\Delta_2 + \lambda_1 m)u = 0$. *One has* $u(x) > 0$ *a.e. and* $u \in D(\Delta_p + \lambda_1 m), (\Delta_p + \lambda_1 m)u = 0$ *for all* $1 \leqslant p \leqslant \infty$. *Finally, 0 is a pole of* $\mu \mapsto R(\mu, \Delta_p + \lambda_1 m)$ *of order one with residue* $P = u \otimes u$.

Proof. In view of Example 2.8, it follows from Corollary 2.10 that $\frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0_+} s(\Delta + \lambda m^+) = 0$. Thus the first assertion follows from Theorem 3.2. Moreover, m^+ defines a compact mapping from $D(\Delta_1)$ (with the graph norm) to L^1. Since $D(\Delta_1 - \lambda_1 m^-) = D(\Delta_1) \cap D(m^-) \hookrightarrow D(\Delta_1)$, it follows that $m^+ R(\mu, \Delta_1 - \lambda_1 m^-)$ is compact. Hence $R(\mu, \Delta_1 + \lambda_1 m) - R(\mu, \Delta_1 - \lambda_1 m^-) = \lambda_1 R(\mu, \Delta_1 + \lambda_1 m) m^+ R(\mu, \Delta_1 - \lambda_1 m^-)$ is compact. It follows that 0 is a pole of the resolvent of $\Delta_1 + \lambda_1 m$. Since $(\mathrm{e}^{t(\Delta_1 + \lambda_1 m)})_{t \geqslant 0}$ is positive and irreducible, the pole is of order 1 and the residue is a strictly positive rank 1 projection P (see [25, C-III. Prop. 3.5, p. 310]).

We show that 0 is also a pole of order 1 in $L^p (1 < p \leqslant \infty)$. Note that the spectrum is independent of p and the resolvents are consistent (see [3]). Let $\varepsilon > 0$ such that $\mu \in \rho(\Delta + \lambda_1 m)$ whenever $0 < |\mu| \leqslant \varepsilon$. Then $\int_{|z|=\varepsilon} \frac{R(z, \Delta_1 + \lambda_1 m)}{z^n} \, \mathrm{d}z = 0$ for $n = 1, 2, \ldots$. It follows that $\int_{|z|=\varepsilon} \frac{R(z, \Delta_p + \lambda_1 m)}{z^n} \, \mathrm{d}z = 0$ for $n = 1, 2 \ldots$ and all $p \in [1, \infty]$. Thus 0 is a pole of order 1 in L^p. Similarly one sees that the residues P_p in L^p are consistent. Since $\Delta_2 + \lambda m$ is self-adjoint it follows that $P_2 = u \otimes u$ with $u(x) > 0$ a.e., $\|u\|_{L^2} = 1$ □

REMARK 3.4. By the argument used in the proof one sees the following. Let A_p be operators on L^p with $\rho(A_p)$ connected and independent of $p \in [1, \infty]$. Assume that the resolvents $R(\lambda, A_p)$ are consistent for one (equivalently all) $\lambda \in \rho(A_p)$. Then $\sigma_{\text{ess}}(A_p)$ is independent of $p \in [1, \infty)$. Here $\sigma_{\text{ess}}(A_p) = \mathbb{C} \backslash \rho_{\text{ess}}(A_p)$ where $\rho_{\text{ess}}(A_p)$ consists of all points λ in $\mathbb{C}$ such that $\lambda \in \rho(A_p)$ or λ is a pole of the resolvent with finite dimensional residue. Concerning the assumption of consistency see [3].

As a consequence of Theorem 3.3 one has

$$\|e^{t(\Delta_p + \lambda_1 m)} - u \otimes u\|_{\mathcal{L}(L^p)} \leqslant M \ e^{-\varepsilon t} \qquad (t \geqslant 0)$$

for some $\varepsilon > 0$ (cf. [8, Theorem 1.2]). Thus $e^{t(\Delta_p + \lambda_1 m)} f \to (\int fu)u \ (t \to \infty)$ in $L^p(\mathbb{R}^N)$ $(1 \leqslant p < \infty)$. This means that the solutions of the diffusion equation with excitation m^+ and absorption m^- converge to an equilibrium. With the help of the parameter $\lambda_1 > 0$ one has adjusted the excitation-absorption term m such that it is in equilibrium with the diffusion.

REMARK 3.5. (continuity of the principal eigenvector).
If $e^{t(\Delta_\infty + \lambda_1 m)}$ leaves $C_0(\mathbb{R}^N)$ invariant and is strongly continuous, then the spectrum in $C_0(\mathbb{R}^N)$ is the same as in L_p (cf. [23]) and one has $u \in C_0(\mathbb{R}^N)$. This is

the case, e.g. if $m \in L^\infty(\mathbb{R}^N)$ by a recent result of Ouhabaz *et al.* [27].

REMARK 3.6. (uniqueness of the principal eigenvalue). In the situation of Theorem 3.2, λ_1 is the unique $\lambda > 0$ such that the problem

$$P(\lambda, p) \begin{cases} u \in D(\Delta_p + \lambda m), \quad u \geqslant 0, \quad u \neq 0, \\ \Delta_p u + \lambda m u = 0 \end{cases}$$

has a solution for some $p \in [1, \infty)$. We give the short argument for completeness: Let $\lambda > 0$ such that $P(\lambda, p)$ has a solution u. Then $s(\Delta_p + \lambda m) \geqslant 0$. If $s = s(\Delta_p + \lambda m) > 0$, then, since $s \notin \sigma_{\text{ess}}(\Delta_p + \lambda m)$, there exists $0 < \varphi \in L^{p'}$ such that $(\Delta_p + \lambda m)'\varphi = s\varphi$. Then φ is strictly positive by irreducibility. Hence $0 = \langle (\Delta_p + \lambda m)u, \varphi \rangle = s\langle u, \varphi \rangle$. Since $\langle u, \varphi \rangle > 0$, this implies $s = 0$. □

The idea of using relative compactness of perturbation in order to establish a principal eigenvalue is standard (see, e.g. [31]). In this context, using $s(\Delta - m^-) < 0$, it was first used by Brown et al. (see [11, Theorem 4.2], where m^+ is supposed to be of compact support). Similar results in the non-autonomous case are [15, Theorems 7.7 and 7.8]. Principal eigenvalues are obtained by Brown and Tertikas [13, Theorem 4.5] and Daners [16, Theorem 1.3] under more general conditions, but they may no longer belong to L^p.

Next we consider the case when $s(\Delta - m^-) = 0$.

PROPOSITION 3.7. *Let* $m \in L^1_{\text{loc}}(\mathbb{R}^N)$ *where* $N \geqslant 3$ *and assume that*

(a) $m^+ \in L^{N/2+\varepsilon} \cap L^{N/2-\varepsilon}$ *for some* $\varepsilon > 0$ *and*
(b) $s(\Delta - m^-) = 0$.

Let $\lambda_0 := \sup\{\lambda \geqslant 0 \colon s(\Delta + \lambda m^+) = 0\}$. *Then* $\lambda_0 > 0$ *and* 0 *is not an eigenvalue of* $\Delta_p + \lambda m$ *in* $L^p(\mathbb{R}^N)$ *for any* $\lambda \in [0, \lambda_0)$ *and any* $p \in [1, \infty)$.

We use the following special case of [4, Theorem 1.3].

PROPOSITION 3.8. *Let* S, T *be* C_0*-semigroups on a space* $L^p (1 < p < \infty)$ *with generators* A *and* B*, respectively, such that* $0 \leqslant S(t) \leqslant T(t)$ *and* $s(A) = s(B) = 0$. *Assume that* T *is bounded and* S *is irreducible. If* 0 *is an eigenvalue of* A*, then* $A = B$.

Proof of Proposition 3.7. We know from Remark 2.11 (a) or (b) that $\lambda_0 > 0$. Assume that $\lambda_1 \in (0, \lambda_0)$ is an eigenvalue of $\Delta_p + \lambda_1 m$ where $1 < p < \infty$. By Simon's theorem (Remark 2.11 b) one has $\sup_{t \geqslant 0} \|e^{t(\Delta_p + \lambda_1 m^+)}\|_{\mathcal{L}(L^p)} < \infty$. It follows from Proposition 3.8 that $\Delta_p + \lambda_1 m^+ = \Delta_p + \lambda_1 m$; i.e. $m^- = 0$. Let

$\lambda_1 < \lambda_2 < \lambda_0$. Then it follows in a similar way that $\Delta_p + \lambda_1 m = \Delta_p + \lambda_2 m$, hence $m = 0$, a contradiction. Since by [40, 6.3], $e^{t(\Delta_1+\lambda_1 m)}L^1 \subset L^2$, it follows that 0 is not an eigenvalue of $\Delta_1 + \lambda_1 m$ either. □

REMARK 3.9. The situation considered in Proposition 3.7 is different for $p = \infty$. In fact, assume that $N \geqslant 3$ and $m \in L^{N/2-\varepsilon} \cap L^{N/2+\varepsilon}$ for some $\varepsilon > 0$ so that the assumptions of Proposition 3.7 are satisfied. Let $0 < \lambda < \lambda_0$. Then there exists a strictly positive $\psi \in L^\infty(\mathbb{R}^N)$ such that

$$(\Delta_\infty + \lambda m)\psi = \psi. \tag{3.2}$$

Moreover, $\ker(\Delta_\infty + \lambda m) = \mathbb{R}\psi$.

In fact, it has been shown by Simon [34, Theorem 3.4] that there exists $\varphi \in L^\infty \neq 0$ such that $T(t)'\varphi = \varphi$ $(t \geqslant 0)$ where $T(t) = e^{t(\Delta_1+\lambda m)}$, and also that $\dim\ker(\Delta_\infty + \lambda m) = 1$. We show that a strictly positive eigenvector exists. Define $\psi \in L^1(\mathbb{R}^N)' = L^\infty(\mathbb{R}^N)$ by $\langle f, \psi\rangle := \underset{t\to\infty}{\mathrm{LIM}}\,\langle f, T(t)'|\varphi|\rangle$ where LIM denotes a Banach limit on $L^\infty(0,\infty)$. Since $|\varphi| \leqslant T'(t)|\varphi|$ one has $\langle f, \psi\rangle \geqslant \langle f, |\varphi|\rangle$ for $f \geqslant 0$ so that $\psi \neq 0$. Since $\langle f, T(s)'\psi\rangle = \underset{t\to\infty}{\mathrm{LIM}}\,\langle f, T(t+s)'|\varphi|\rangle = \langle f, \psi\rangle$ it follows that $T(s)'\psi = \psi$ $(s \geqslant 0)$ which is (3.2). □

Next we come back to the situation considered in Theorem 3.2. We know that $s(\Delta + \lambda_1 m) = 0$ and $s(\Delta + \lambda m_1) < 0$ for $\lambda \in (0, \lambda_1)$. Since the spectral function is continuous, it has a minumum on $[0, \lambda_1]$. We investigate when this minimum is strict.

THEOREM 3.10. *Let $m \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $0 \neq m^+$. Assume in addition that*

(a) *$\lambda \mapsto s(\Delta - \lambda m^-)$ is strictly decreasing and*
(b) *$m^+ \in L^q(\mathbb{R}^N) + L^\infty_0(\mathbb{R}^N)$ where $\infty > q > \frac{N}{2}$ (if $N \geqslant 2$) and $q \geqslant 1$ if $N = 1$.*

Then there exists a unique $\lambda_0 > 0$ such that

$$s(\Delta + \lambda_0 m) = \min_{\lambda>0} s(\Delta + \lambda m).$$

REMARK. Condition (a) is discussed in detail in [7]. For example, it is shown that if $N = 1$ and $m^- \in L^\infty(\mathbb{R})$, then (a) holds. Condition (b) implies that $m^+ \in K_N$.

Proof. Assume that $\min_{\lambda>0} s(\Delta + \lambda m)$ is not strict. Then there exist $0 < \delta < \lambda_0$, $c < 0$ such that $c = s(\Delta + \lambda m)$ for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Since $s(\Delta - \lambda m^-)$ is strictly decreasing by hypothesis, it follows that $s(\Delta - \lambda m^-) < s(\Delta + \lambda m)$ for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Since $\sigma_{\mathrm{ess}}(\Delta + \lambda m) = \sigma_{\mathrm{ess}}(\Delta - \lambda m^-)$ (cf. Remark 3.4), it follows that c is an eigenvalue of $\Delta_2+\lambda_0 m$. Thus there exists $u \in D(\Delta_2+\lambda_0 m)$, $\|u\|_2 = 1$, such that $\Delta u + \lambda_0 mu = cu$. Consequently, $-|\nabla u|^2 + \lambda_0 \int mu^2 = c$. Since $c = s(\Delta + \lambda m)$ it follows that $c + (\lambda - \lambda_0) \int mu^2 = -\int |\nabla u|^2 + \lambda \int mu^2 \leqslant c$ if $|\lambda - \lambda_0| < \delta$. This implies $\int mu^2 = 0$. Hence $-\int |\nabla u|^2 + \lambda \int mu^2 = c = s(\Delta + \lambda m)$ if $|\lambda - \lambda_0| < \delta$. This implies that $\Delta u + \lambda mu = cu$ if $|\lambda - \lambda_0| < \delta$ (since, if B is a form positive operator, and $u \in D(B)$ such that $(Bu \mid u) = 0$, then $Bu = 0$). This implies that $mu = 0$. Hence $\Delta_2 u = cu$, which is a contradiction. □

Finally, we show that it can happen that the minimum of $s(\Delta + \lambda m)$ is negative but not strict.

THEOREM 3.11. *There exists $m \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ such that $m^+ \neq 0$, m^+ has compact support and*

$$-1 = \min_{\lambda>0} s(\Delta + \lambda m) = s(\Delta + \mu m) \qquad (\mu \in [1,2]);$$

Moreover, $s(\Delta - \lambda m^-) = -1$ for all $\lambda \geqslant 1$.

Proof. (a) It suffices to show that there exists a non-empty open set E, $0 \leqslant V \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\varepsilon > 0$ such that $s(\Delta-\lambda V) \geqslant -1$ $(\lambda \geqslant 0)$, $s(\Delta-V+\varepsilon 1_E) \leqslant -1$, $V = 0$ on E. In fact, this implies that $s(\lambda) := s(\Delta + \lambda(\frac{\varepsilon}{2}1_E - V)) \geqslant s(\Delta - \lambda V) \geqslant -1$ for all $\lambda \geqslant 0$ and $s(1) \leqslant s(\Delta - V + \varepsilon 1_E) \leqslant -1$, $s(2) = s(\Delta + \varepsilon 1_E - 2V) \leqslant s(\Delta + \varepsilon 1_E - V) \leqslant -1$, so that $s(\lambda) = -1$ for $\lambda \in [1,2]$ by convexity. Then $m = \frac{\varepsilon}{2}1_E - V$ fulfills the requirements.
(b) Let $\Omega_n = \{x \in \mathbb{R}^N : a_n < |x| < b_n\}$ where $0 < a_n < b_n < a_{n+1}$, $\lim_{n\to\infty} b_n = \infty$, such that $s(\Delta_{\Omega_n}) < s(\Delta_{\Omega_{n+1}})$, $\lim_{n\to\infty} s(\Delta_{\Omega_n}) = -1$. We construct $0 \leqslant V_n \in L^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$, such that

$$V_n = 0 \text{ on } \bigcup_{j=1}^n \Omega_j;$$

$$V_{n+1} = V_n \text{ for } |x| \leqslant b_n;$$

$$s(\Delta - V_n + \varepsilon 1_E) \leqslant -1 \quad (n \in \mathbb{N}),$$

where $E = \Omega_1$. Then, letting $V(x) = V_n(x)$ for $|x| \leqslant b_n$ we obtain $V \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ which satisfies the requirements of (a). In fact, $s(\Delta - \lambda V) \geqslant \sup_{n\in\mathbb{N}} s(\Delta_{\Omega_n}) = -1$ since $V = 0$ on Ω_n for all $n \in \mathbb{N}$. Since for every compact set $K \subset \mathbb{R}^N$ there exists $n \in \mathbb{N}$ such that $V_{|K} = V_{n|K}$, it follows from the variational formula that

$$s(\Delta - V + \varepsilon 1_E) \leqslant \sup_{n\in\mathbb{N}} s(\Delta - V_n + \varepsilon 1_E) \leqslant -1.$$

We construct the potentials V_n. By [7, Proposition 4.1], $\lim_{b\to\infty} s(\Delta - b1_{E^c}) = s(\Delta_E) < -1$. Let $b > 0, \varepsilon > 0$ such that $s(\Delta - b1_{E^c}) + \varepsilon < -1$, and let $V_1 = b1_{E^c}$. Assume that $V_1, \ldots, V_n$ are constructed. For $K \in \mathbb{N}$ let

$$U_k(x) = \begin{cases} V_n(x) & \text{if } |x| \leqslant b_n, \\ k & \text{if } b_n < |x| < a_{n+1}, \\ 0 & \text{if } a_{n+1} \leqslant |x| \leqslant b_{n+1}, \\ k & \text{if } |x| > b_{n+1}. \end{cases}$$

We show that there exists $k \in \mathbb{N}$ such that $s(\Delta - U_k + \varepsilon 1_E) < -1$ and choose $V_{n+1} := U_k$. Assume on the contrary that $-1 \leqslant s_\infty := \inf_k s_k$ where $s_k = s(\Delta - U_k + \varepsilon 1_E)$. Since s_k is in the approximate point spectrum there exists $u_k \in H^1(\mathbb{R}^N)$ such that $\|u_k\|_{L^2} = 1$ and

$$\Delta u_k - U_k u_k + \varepsilon 1_E u_k - s_k u_k = v_k \to 0 \tag{3.3}$$

$(k \to \infty)$. In particular, $-\int |\nabla u_k|^2 - \int U_k u_k^2 + \varepsilon \int_E u_k^2 - s_k \to 0$ $(k \to \infty)$. Hence (u_k) is bounded in $H^1(\mathbb{R}^N)$ and we can assume that $u_k \to u$ $(k \to \infty)$ weakly in $H^1(\mathbb{R}^N)$. Since the embedding of $H^1(B(0,R))$ in $L^2(B(0,R))$ is compact, it follows that $u_k \to u$ strongly in $L^2(B(0,R))$ for all $R > 0$. Since $U_k \equiv k$ on $F = \{x\colon b_n < |x| < a_{n+1}$ or $|x| > b_{n+1}\}$, it follows that $u = 0$ in F. In particular, $u_{|\Omega_{n+1}} \in H_0^1(\Omega_{n+1})$. Moreover, $u_k \to u$ in $L^2(\mathbb{R}^N)$. Passing to the limit for $k \to \infty$ in (3.3) shows that $\Delta u - s_\infty u = 0$ in $\mathcal{D}(\Omega_{n+1})'$. Since $s(\Delta_{\Omega_{n+1}}) < -1$ and $s_\infty \geqslant -1$, it follows that $u_{|\Omega_{n+1}} = 0$ a.e. Thus $u \in H_0^1(B(0,b_n))$. It follows from (3.3) that $\Delta u - V_n u + \varepsilon 1_E u - s_\infty u = 0$ in $\mathcal{D}(\mathbb{R}^N)'$. Since $s_\infty \geqslant -1 > s(\Delta - V_n + \varepsilon 1_E)$ it follows that $u = 0$. This is a contradiction, since $\|u_k\|_{L^2} = 1$ $(k \in \mathbb{N})$ and $u_k \to u$ in $L^2(\mathbb{R}^N)$. □

4. More General Potentials on $L^2(\mathbb{R}^N)$

In some cases one can define the semigroup $(\mathrm{e}^{t(\Delta_2+m)})_{t\geqslant 0}$ on $L^2(\mathbb{R}^N)$ by form-methods, but it no longer has extensions to all $L^p(\mathbb{R}^N)$ $(1 \leqslant p < \infty)$. Let $m \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. At first we consider the positive part of m and, in contrast to the approach in Sections 1–3, we define the spectral function by the variational formula

$$s(\Delta + \lambda m^+) = \sup\left\{-\int |\nabla u|^2 + \lambda \int m^+ u^2 \colon u \in \mathcal{D}_1\right\}, \tag{4.1}$$

$(\lambda > 0)$ and let $\lambda_\infty(m^+) = \sup\{\lambda \geqslant 0\colon s(\Delta + \lambda m^+) < \infty\}$.

PROPOSITION 4.1. *One has* $\lambda_\infty(m^+) > 0$ *if and only if* $H^1(\mathbb{R}^N) \subset Q(m^+)$. *In that case, for* $0 \leqslant \lambda < \lambda_\infty(m^+)$, $a_\lambda^+(u,v) = \int \nabla u \nabla u - \lambda \int m^+ uv$, $Q(a_\lambda^+) = H^1(\mathbb{R}^N)$ *defines a closed, symmetric, lower bounded form. Moreover, one has* $\lambda_\infty(m^+) = \infty$ *if and only if* m^+ *has form bound* 0 *with respect to* $-\Delta_2$.

Proof. (a) If $\lambda > 0$ such that $c := s(\Delta + \lambda m^+) < \infty$, then

$$\lambda \int m^+ u^2 \leqslant c\|u\|_{L^2}^2 + \int |\nabla u|^2 \leqslant (c+1)\|u\|_{H^1}^2$$

$(u \in \mathcal{D}(\mathbb{R}^N))$. Let $u \in H^1(\mathbb{R}^N)$. There exist $u_n \in \mathcal{D}(\mathbb{R}^N)$ such that $u_n \to u$ in $H^1(\mathbb{R}^N)$ and a.e. It follows from Fatou's lemma that $\lambda \int m^+|u|^2 \leqslant \liminf_{n\to\infty} \lambda \int m^+ u_n^2 \leqslant (c+1)\|u\|_{H^1}^2$. Hence $H^1 \subset Q(m^+)$. Conversely, if $H^1 \subset Q(m^+)$, it follows from the closed graph theorem that there exists a constant $c > 0$ such that $\int m^+ u^2 \leqslant c\|u\|_{H^1}^2 = c(\int |\nabla u|^2 + \int u^2)$ $(u \in H^1)$. Hence $s(\Delta + \lambda m^+) \leqslant 1$ whenever $0 \leqslant \lambda \leqslant \frac{1}{c}$.

(b) Assume that $\lambda_\infty = \lambda_\infty(m^+) > 0$ and let $0 < \lambda < \lambda_\infty(m^+)$. Choose $\alpha > 0$ such that $\lambda(1+\alpha) < \lambda_\infty$, let $s := s(\Delta + \lambda(1+\alpha)m^+)$. Then $-\int |\nabla u|^2 + \lambda(1+\alpha)\int m^+u^2 \leqslant s\int u^2$ $(u \in H^1)$. Hence

$$(1+\alpha)\int |\nabla u|^2 - \lambda(1+\alpha)\int m^+u^2 + (s+\alpha)\int u^2 \geqslant \alpha\|u\|_{H^1}^2.$$

and $a_\lambda^+(u,u) + \frac{s+\alpha}{1+\alpha}\int u^2 \geqslant \frac{\alpha}{1+\alpha}\|u\|_{H^1}^2$. This implies that a_λ^+ is closed and lower bounded.

(c) If $s(\lambda) := s(\Delta + \lambda m^+) < \infty$ for all $\lambda > 0$, then $-\int |\nabla u|^2 + \lambda \int m^+ u^2 \leqslant s(\lambda)\|u\|_2^2$ and so $\int m^+u^2 \leqslant \frac{1}{\lambda}\int |\nabla u|^2 + \frac{s(\lambda)}{\lambda}\|u\|_{L^2}^2$ $(u \in H^1)$. Thus m^+ has form bound 0 with respect to $-\Delta_2$.

Conversely, assume that m^+ has form bound 0 with respect to $-\Delta_2$; i.e. for all $\varepsilon > 0$ there exists $\beta > 0$ such that

$$\int m^+u^2 \leqslant \varepsilon \int |\nabla u|^2 + \beta \int u^2 \qquad (u \in H^1).$$

Then $-\int |\nabla u|^2 + \lambda \int m^+u^2 \leqslant (\lambda - \frac{1}{\varepsilon})\int m^+u^2 + \frac{\beta}{\varepsilon} \leqslant \frac{\beta}{\varepsilon}$ for all $u \in \mathcal{D}_1$ whenever $\lambda \leqslant \frac{1}{\varepsilon}$. □

Assume that $\lambda_\infty(m^+) > 0$. Let $0 < \lambda < \lambda_\infty(m^+)$. Then $a_\lambda(u,v) = a_\lambda^+(u,v) + \lambda\int m^- uv$, $Q(a_\lambda) = H^1(\mathbb{R}^N) \cap Q(m^-)$ defines a closed, lower bounded form. We define $-(\Delta_2 + \lambda m)$ on $L^2(\mathbb{R}^N)$ as the operator associated with the form a_λ. Thus $\Delta_2 + \lambda m$ is self-adjoint and generates a C_0-semigroup $(\mathrm{e}^{t(\Delta_2+\lambda m)})_{t\geqslant 0}$ on $L^2(\mathbb{R}^N)$. It follows from [24, Lemma 4.6, p. 349] that $\mathcal{D}(\mathbb{R}^N)$ is a form core of $\Delta_2 + \lambda m$. Thus

$$s(\Delta_2 + \lambda m) = \sup\left\{-\int |\nabla u|^2 + \int \lambda m u^2 \colon u \in \mathcal{D}_1\right\} \tag{4.2}$$

is the spectral bound of $\Delta_2 + \lambda m$ if $0 \leqslant \lambda < \lambda_\infty(m^+)$.

EXAMPLE 4.2. Let $N \geqslant 3, 0 \leqslant m \in L^{N/2}(\mathbb{R}^N)$. Then $\lambda_\infty(m) = \infty$. In fact, since $H^1(\mathbb{R}^N) \subset L^{2N/(N-2)}$, it follows that $L^{N/2} \hookrightarrow \mathcal{L}(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$. Since the injection $H^1(B(0,R)) \hookrightarrow L^2(B(0,R))$ is compact for all $R > 0$, it follows that $m \in K(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$ (the compact operators) whenever $m \in \mathcal{D}(\mathbb{R}^N)$. The test functions being dense in $L^{N/2}$, it follows that $L^{N/2} \subset K(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$. By [31, Problem 39, p. 369] this implies that each $m \in L^{N/2}$ is relatively form compact with respect to $-\Delta_2$ and has relative form bound 0.

EXAMPLE 4.3. Let $N \geqslant 3, m(x) = |x|^{-2}$. By Hardy's inequality (see [17] or [18, XVIII (7.47), p. 754]) there exists $\lambda_0 > 0$ such that

$$\lambda_0 \int \frac{|u|^2}{|x|^2} \leqslant \int |\nabla u|^2 \qquad (u \in \mathcal{D}(\mathbb{R}^N)). \tag{4.4}$$

Let us assume that $\lambda_0 > 0$ is optimal (e.g. $\lambda_0 = 2.25$ if $N = 5$, see [30, p. 172]). It follows from the definition that $s(\lambda) \leqslant 0$ for $\lambda \in [0, \lambda_0))$, hence by Theorem 3.1, $s(\lambda) = 0$ for $\lambda \in [0, \lambda_0]$. We show that $s(\lambda) = \infty$ for $\lambda > \lambda_0$, i.e. $\lambda_0 = \lambda_\infty(m)$. In fact, let $\lambda > 0$ such that $s := s(\Delta + \lambda m) < \infty$. We show that $s(\Delta + \lambda m) \leqslant 0$. One has $-\int |\nabla u|^2 + \lambda \int \frac{u^2}{x^2} \leqslant s \int u^2$ $(u \in \mathcal{D}(\mathbb{R}^N))$. Replacing u by $u_\alpha(x) = u(\alpha x)$, this yields,

$$-\alpha^2 \int |\nabla u|^2 + \lambda \alpha^2 \int \frac{u^2}{x^2} \leqslant s \int u^2 \qquad (u \in \mathcal{D}(\mathbb{R}^N))$$

for all $\alpha > 0$. It follows that

$$-\int |\nabla u|^2 + \lambda \int \frac{u^2}{x^2} \leqslant 0 \qquad (u \in \mathcal{D}(\mathbb{R}^N)). \qquad \square$$

Next we prove the analogous result of Theorem 3.3 on $L^2(\mathbb{R}^N)$ if $m^+ \in L^{N/2}$. Note that by Example 2.12 (c) one has $m^+ \notin \widehat{K}_N$, in general.

THEOREM 4.4. *Let* $N \geqslant 3$, $m \in L^1_{\text{loc}}(\mathbb{R}^N)$ *such that* $0 \neq m^+ \in L^{N/2}$. *If* $s(\Delta - m^-) < 0$, *then the conclusions of Theorem 3.3 hold for* $p = 2$.

Proof. It follows from Example 4.2 that $\sigma_{\text{ess}}(\Delta + \lambda m^+) = \sigma_{\text{ess}}(\Delta)$ for $\lambda > 0$. Thus, if $s(\Delta + \lambda m^+) > 0$, then $s(\Delta + \lambda m^+)$ is an eigenvalue. It follows from the Cwickel–Lieb–Rosenbljum-bound ([31, p. 101]) that $s(\Delta + \lambda m^+) = 0$ for $\lambda \in [0, \lambda_0]$ for some $\lambda_0 > 0$. Now the proof of Theorem 3.2 and Theorem 3.3 can be used for this case.

The following example shows that if $\lambda_\infty(m^+) < \infty$ it may happen that $s(\Delta + \lambda m) < 0$ for all $0 < \lambda < \lambda_\infty(m^+)$.

EXAMPLE 4.5. Let $N \geqslant 3$, $q_1(x) = |x|^{-2}, q_2 \in L^\infty(\mathbb{R}^N), m = q_1 - q_2$. Denote by $\lambda_0 > 0$ the best constant in Hardy's inequality (4.4). Then the following holds:

(a) One has $s(\Delta_2 + \lambda m) < \infty$ if and only if $\lambda \leqslant \lambda_0$.
(b) If $s(\Delta_2 - q_2) = 0$, then for all $\lambda \in [0, \lambda_0)$, $s(\Delta_2 + \lambda m) = 0$ and 0 is not an eigenvalue of $\Delta_2 + \lambda m$.
(c) Assume that $N \geqslant 5$. If $s(\Delta_2 - q_2) < 0$, then $s(\Delta_2 + \lambda m) < 0$ for all $\lambda \in [0, \lambda_0]$.

Proof. (a) Let $\lambda > 0$ such that $s(\Delta_2 + \lambda m) < \infty$. Then $s(\Delta_2 + \lambda q_1) \leqslant s(\Delta_2 + \lambda m) + \lambda \|q_2\|_\infty < \infty$. Hence $\lambda \leqslant \lambda_0$ (see Example 4.3).
(b) is shown as Proposition 3.7.
(c) Assume that $0 < \lambda \leqslant \lambda_0$ such that $s(\Delta_2 + \lambda m) = 0$. Then $s(\Delta_2 + \lambda m) = 0 > s(\Delta_2 - \lambda q_2)$. Hence by [31, Example 9, p. 119], $\sigma_{\text{ess}}(\Delta_2 + \lambda m) \subset (-\infty, s(\Delta_2 - \lambda q_2)]$. Thus 0 is an eigenvalue of $\Delta_2 + \lambda m$. But $s(\Delta_2 + \lambda q_1) = 0$. It follows from Proposition 3.8 that $\Delta_2 + \lambda q_1 = \Delta_2 + \lambda m$, a contradiction. □

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