J. OPERATOR THEORY
38(1997), 87–130

GAUSSIAN ESTIMATES FOR SECOND ORDER ELLIPTIC OPERATORS WITH BOUNDARY CONDITIONS

W. ARENDT and A.F.M. TER ELST

Communicated by Norberto Salinas

ABSTRACT. We prove Gaussian estimates for the kernel of the semigroup generated by a second order operator A in divergence form with real, not necessarily symmetric, second order coefficients on an open subset Ω of $\mathbb{R}^d$ satisfying various boundary conditions. Moreover, we show that $A + \omega I$ has a bounded H_∞-functional calculus and has bounded imaginary powers if ω is large enough.

KEYWORDS: *Gaussian bounds, elliptic operators, boundary conditions, non-symmetric operators, functional calculus, bounded imaginary powers.*

AMS SUBJECT CLASSIFICATION: 35J25, 35J15, 47A60.

1. INTRODUCTION

A large literature has recently arisen on Gaussian estimates for kernels of semigroups generated by elliptic operators, including several books, see Davies ([16]), Robinson ([37]) and Varopoulos–Saloff-Coste–Coulhon ([40]). The starting point was a paper of Aronson ([8]) for real non-symmetric elliptic operators on $\mathbb{R}^d$ with measurable coefficients, which used Moser's parabolic Harnack inequality ([29]). New impetus to the subject came from Davies ([15]), who introduced a perturbation method together with logarithmic Sobolev inequalities to deduce Gaussian upper bounds with optimal constants for symmetric pure second order operators with L_∞-coefficients and Dirichlet boundary conditions or, if the region has the extension property, with Neumann boundary condition. (See [16] for a coherent description.)

In this paper, we consider second order elliptic operators of the form

$$\mathcal{A}u = -\sum_{i,j=1}^{d} D_j\, a_{ij}\, D_i u + \sum_{i=1}^{d} b_i\, D_i u - \sum_{i=1}^{d} D_i(c_i\, u) + c_0 u$$

with real, not necessarily symmetric coefficients $a_{ij} \in L_\infty(\Omega)$ satisfying a uniform ellipticity condition, and lower order coefficients $b_i, c_i \in W^{1,\infty}(\Omega)$ and $c_0 \in L_\infty(\Omega)$ real or complex. We study realizations A of $\mathcal{A}$ in $L_2(\Omega)$ obtained by quadratic form methods. They correspond to various boundary conditions, for example, Dirichlet, Neumann, mixed, or Robin boundary conditions. It turns out that the non-symmetry of the leading coefficients interacts with the boundary conditions and it is not possible to symmetrize the second order part. Our main results show that, in each of these cases, A generates a semigroup $S = (\mathrm{e}^{-tA})_{t>0}$ given by a kernel $(K_t)_{t>0}$ which satisfies a Gaussian estimate

$$|K_t(x;y)| \leqslant ct^{-d/2}\mathrm{e}^{-b|x-y|^2t^{-1}}\mathrm{e}^{\omega t} \quad (x,y)\text{-a.e.}$$

for all $t > 0$. We establish this by two different methods.

The first method (Section 3) works for Dirichlet boundary conditions and once differentiable second order coefficients. The proof is very short and elementary and relies on the Beurling–Deny criterion for forms in a non-symmetric version recently given by Ouhabaz ([31], [32]). Besides its simplicity, one advantage of the method is that complex lower order coefficients are allowed. This approach is, however, restricted to Dirichlet boundary conditions.

The second method (Section 4) is based on an iteration process of Fabes–Stroock ([22]), which is also used in Robinson ([37]) for second order real symmetric operators on Lie groups with constant coefficients. The advantage of this more elaborate method is that we no longer need to assume the once differentiability of the second order coefficients. Moreover, it works for all boundary conditions considered here. On the other hand, the lower order coefficients have to be real.

Gaussian estimates have various interesting consequences. In Section 5 we show that for each of the considered boundary conditions one obtains a holomorphic semigroup on all the L_p-spaces with $1 \leqslant p \leqslant \infty$ with the same sector as in $L_2(\Omega)$. Moreover, using recent results of Duong–Robinson ([20]) we show that, for all boundary conditions considered here, the operator $A + \omega I$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_p(\Omega)$ for each $p \in \langle 1,\infty\rangle$ and large ω, where $\nu > 0$ is such that $\Sigma(\nu)$ contains the numerical range of the matrix $(a_{ij}(x))$ for a.e. $x \in \Omega$. In particular, the fractional powers $(A + \omega I)^{is}$ are bounded, which is of interest in view of the regularity theorem of Dore–Venni ([18]) (see also [36], [35]). In this context it is interesting to determine the range of ω for which this is true. If ω_0 is such that $\|S_z\|_{2\to 2} \leqslant e^{\omega_0|z|}$ for all $z \in \mathbb{C}$ with $|\arg z| \leqslant \pi/2 - \nu$, then one obviously requires $\omega \geqslant \omega_0$. We prove that the imaginary powers are bounded for any $\omega > \omega_0$.

2. PRELIMINARIES

In this section we fix some notations and give some basic results on semigroups and Sobolev spaces as they are needed throughout this paper.

Let $\Omega \subset \mathbf{R}^d$ be an open set and let $1 \leqslant p_1 < p_2 \leqslant \infty$. A family of operators $T^{(p)} \in \mathcal{L}(L_p(\Omega))$, $p_1 \leqslant p \leqslant p_2$, is called *consistent* if

$$T^{(p)}\varphi = T^{(q)}\varphi$$

for all $p, q \in [p_1, p_2]$ and $\varphi \in L_p(\Omega) \cap L_q(\Omega)$. Similarly we refer to a *consistent family of semigroups* $(S_t^{(p)})_{t>0}$ on $L_p(\Omega)$, $p_1 \leqslant p \leqslant p_2$, if for every fixed $t > 0$ the family $S_t^{(p)}$, $p_1 \leqslant p \leqslant p_2$, is consistent. We shall briefly say that S is consistent on L_p, $p_1 \leqslant p \leqslant p_2$ and drop the suffix p in $S^{(p)}$.

Let $1 \leqslant p_1 \leqslant 2 \leqslant p_2 \leqslant \infty$. Let S be a C_0-semigroup on $L_2(\Omega)$. We say that S *interpolates on* $L_p(\Omega)$, $p_1 \leqslant p \leqslant p_2$, if there exists a consistent family of semigroups $(S_t^{(p)})_{t>0}$ on L_p, $p_1 \leqslant p \leqslant p_2$, such that $S^{(p)}$ is strongly continuous if $p \in [p_1, p_2]$, $p \neq \infty$, and in the case $p_2 = \infty$, $S^{(\infty)}$ is weakly* continuous, and, moreover, $S_t = S_t^{(2)}$ for all $t > 0$. In that case, there exist $M \geqslant 1$ and $\omega \in \mathbf{R}$ such that

$$\|S_t^{(p)}\|_{p\to p} \leqslant M e^{\omega t}$$

uniformly for all $p \in [p_1, p_2]$ and $t > 0$. In order to show that a given semigroup S on L_2 interpolates, frequently the strong continuity in the endpoints p_1, p_2 is not a trivial problem. In the following lemma we give some sufficient conditions.

LEMMA 2.1. *Let S be a C_0-semigroup on $L_2(\Omega)$ satisfying $S_t(L_1 \cap L_2) \subset L_1$ for all $t > 0$ and*

$$\|S_t\varphi\|_1 \leqslant M\|\varphi\|_1 \qquad (2.1)$$

uniformly for all $t \in \langle 0, 1]$ and all $\varphi \in L_1 \cap L_2$. (We use $\|\varphi\|_p$ to denote the norm of φ in $L_p(\Omega)$.) Then S interpolates on $L_p(\Omega)$, $1 \leqslant p \leqslant 2$, if one of the following conditions is satisfied:

(i) $M = 1$.

(ii) Ω *has finite measure.*

(iii) $S_t \geqslant 0$ *for all* $t > 0$.

(iv) *There exists* $\omega \in \mathbf{R}$ *such that* $\|S_t\varphi\|_1 \leqslant e^{\omega t}\|\varphi\|_1$ *for all* $\varphi \in L_1 \cap L_2$ *and* $t > 0$.

(v) *There exist* $c > 0$, *open* $\Omega' \subset \mathbf{R}^d$ *with* $\Omega \subset \Omega'$ *and an interpolating semigroup* T *on* $L_p(\Omega')$, $1 \leqslant p \leqslant 2$, *such that* $|S_t\varphi| \leqslant cT_t|\varphi|$ *for all* $t \in \langle 0, 1]$ *and* $\varphi \in L_1(\Omega) \cap L_2(\Omega)$.

Proof. It is clear that one obtains consistent semigroups $(S_t^{(p)})_{t>0}$ on L_p, $1 \leqslant p \leqslant 2$ and it follows from the interpolation inequality ([12], p. 57) that $S^{(p)}$ is strongly continuous for $p > 1$. The strong continuity of $S^{(1)}$ demands further arguments and is proved in Voigt ([41]) (see also Davies, pp. 22–23 in [16]) if one of the first four above conditions is satisfied.

The sufficiency of condition (v) can be proved as follows: Let $p \in [1, 2\rangle$ and $\varphi \in L_p(\Omega) \cap L_2(\Omega)$. We identify a function on Ω with the function on Ω' by extending it by 0 on $\Omega' \setminus \Omega$. Moreover, let $t_1, t_2, \ldots \in \langle 0, 1]$ and suppose that $\lim t_n = 0$. Then $\lim S_{t_n}\varphi = \varphi$ in $L_2(\Omega)$, so there exists a subsequence such that $\lim\limits_{k\to\infty} S_{t_{n_k}}\varphi = \varphi$ a.e. Since $\lim\limits_{k\to\infty} T_{t_{n_k}}|\varphi| = |\varphi|$ in $L_p(\Omega')$, there exist a subsubsequence (which we can assume to be the subsequence) and a $\psi \in L_p(\Omega')$ such that $T_{t_{n_k}}|\varphi| \leqslant \psi$ a.e. for all $k \in \mathbf{N}$. Then $|S_{t_{n_k}}\varphi| \leqslant cT_{t_{n_k}}|\varphi| \leqslant c\psi$ a.e. for all $k \in \mathbf{N}$. Therefore, $\lim\limits_{k\to\infty} S_{t_{n_k}}\varphi = \varphi$ in $L_p(\Omega)$ by an application of the Lebesgue dominated convergence theorem, and S is continuous on $L_p(\Omega)$. ∎

Similarly, if $S_t(L_2 \cap L_\infty) \subset L_\infty$ and

$$\|S_t\varphi\|_\infty \leqslant M\|\varphi\|_\infty$$

uniformly for all $t \in \langle 0, 1]$ and $\varphi \in L_2 \cap L_\infty$, then the semigroup interpolates on L_p, $2 \leqslant p \leqslant \infty$, if one of the conditions (i)–(v) of Lemma 2.1 is satisfied (with L_1 replaced by L_∞). Note that, in that case, S^* satisfies (2.1) and one can define $S^{(\infty)}$ by $S_t^{(\infty)} = (S_t^{*(1)})^*$.

An operator T on L_p is called *positive*, notation $T \geqslant 0$, if $T\varphi \geqslant 0$ a.e. for all $\varphi \in L_p$ with $\varphi \geqslant 0$ a.e. We call T L_∞*-contractive* if $\|T\varphi\|_\infty \leqslant \|\varphi\|_\infty$ for all $\varphi \in L_p \cap L_\infty$. Thus, if S is a C_0-semigroup on $L_2(\Omega)$ and S_t and S_t^* are L_∞-contractive for all $t > 0$, then S interpolates on $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$. Finally, a semigroup S on L_2 is called *quasi-contractive* on L_∞ if there exists an $\omega \in \mathbf{R}$ such that $\|S_t\varphi\|_\infty \leqslant e^{\omega t}\|\varphi\|_\infty$ for all $\varphi \in L_2 \cap L_\infty$ and $t > 0$.

Next we give some results on Sobolev spaces. As before, Ω denotes an open set in $\mathbf{R}^d$. For $p \in [1, \infty]$ let $W^{1,p}(\Omega) = \{u \in L_p(\Omega) : D_i u \in L_p(\Omega)$ for all $i \in$

$\{1,\ldots,d\}\}$. Here $D_i u = \partial u/\partial x_i$ is the distributional derivative in $\mathcal{D}'(\Omega)$. If $p=2$, then the space $H^1(\Omega) = W^{1,2}(\Omega)$ is a Hilbert space for the norm

$$\|u\|^2_{H^1(\Omega)} = \sum_{i=1}^{d} \|D_i u\|_2^2 + \|u\|_2^2.$$

Here and in Section 4 we consider real spaces. In Sections 3 and 5 the spaces are complex and the notation and field will be clear from the context.

The following results follow from p. 152 in [23].

LEMMA 2.2. *Let $u \in H^1(\Omega)$. Then $u^+ = u \vee 0 \in H^1(\Omega)$ and*

$$D_i u^+ = 1_{[u>0]} D_i u \quad \text{a.e.}$$

for all $i \in \{1,\ldots,d\}$. As a consequence, $u^- = (-u)^+ \in H^1(\Omega)$ and $|u| = u^+ + u^- \in H^1(\Omega)$ and

$$D_i|u| = (\operatorname{sgn} u)\, D_i u \quad \text{a.e.,} \tag{2.2}$$

where

$$(\operatorname{sgn} u)(x) = \begin{cases} 1 & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) = 0, \\ -1 & \text{if } u(x) < 0. \end{cases}$$

Moreover, one has

$$D_i u = 0 \text{ a.e. on the set } \{x : u(x) = 0\} \tag{2.3}$$

for all $i \in \{1,\ldots,d\}$.

We note some further consequences. Set $L_2(\Omega)_+ = \{u \in L_2(\Omega) : u \geqslant 0 \text{ a.e.}\}$ and $H^1(\Omega)_+ = H^1(\Omega) \cap L_2(\Omega)_+$.

LEMMA 2.3. (i) *If $v \in H^1(\Omega)$, then the mappings $u \mapsto u \wedge v$ and $u \mapsto u \vee v$, and in particular $u \mapsto u^+$, $u \mapsto u^-$ and $u \mapsto |u|$ from $H^1(\Omega)$ into $H^1(\Omega)$ are continuous.*

(ii) *If $u \in H^1(\Omega)$, then $\|\,|u|\,\|_{H^1(\Omega)} = \|u\|_{H^1(\Omega)}$.*

(iii) *If $0 \leqslant u \in H^1(\Omega)$, then $u \wedge 1 \in H^1(\Omega)$ and the mapping $u \mapsto u \wedge 1$ is continuous on $H^1(\Omega)_+$.*

(iv) *If $u \in H^1_0(\Omega)$, then $u^+, u^-, |u|, |u| \wedge 1 \in H^1_0(\Omega)$.*

Proof. Since $u \vee v = u + (v-u)^+$ and $u \wedge v = -((-u) \vee (-v))$, it suffices to show that $u \mapsto u^+$ is continuous. Let $u, u_1, u_2, \ldots \in H^1(\Omega)$ and suppose that $\lim u_n = u$ in $H^1(\Omega)$. It suffices to show that every subsequence of (u_n^+)

has a subsubsequence which converges to u^+. Therefore, we can assume that $\lim u_n = u$ a.e., $\lim D_i u_n = D_i u$ a.e. and, moreover, $|u_n| \leqslant f$ and $|D_i u_n| \leqslant f$ for some $f \in L_2(\Omega)$, uniformly for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$. Then $\lim D_i u_n^+ = \lim 1_{[u_n>0]} D_i u_n = 1_{[u>0]} D_i u = D_i u^+$ a.e. in virtue of (2.3). Now Statement (i) follows from the Lebesgue dominated convergence theorem.

Statement (ii) follows from (2.2) and (2.3).

It follows from p. 152 in [23] that $u \wedge 1 = u + (1-u)^+ \in H^1_{\mathrm{loc}}(\Omega)$ and

$$D_i(u \wedge 1) = D_i(u + (1-u)^+) = 1_{[u<1]} D_i u \in L_2(\Omega).$$

Therefore, $u \wedge 1 \in H^1(\Omega)$ whenever $0 \leqslant u \in H^1(\Omega)$. It follows from (2.2) that $D_i u = 0$ a.e. on $[u = 1]$. So the proof of continuity is as in Statement (i).

Next we prove Statement (iv). Let $u \in H^1_0(\Omega)$ and $u_1, u_2, \ldots \in C_c^\infty(\Omega)$ be such that $\lim u_n = u$ in $H^1(\Omega)$. Let $e_1, e_2, \ldots \in C_c^\infty(\mathbb{R}^d)$ be a regularizing sequence. Fix $n \in \mathbb{N}$. Then $e_m * u_n^+ \in C_c^\infty(\Omega)$ for m sufficiently large and $\lim_m e_m * u_n^+ = u_n^+$ in $H^1(\Omega)$. Hence $u_n^+ \in H^1_0(\Omega)$ and $u^+ = \lim_n u_n^+ \in H^1_0(\Omega)$. The proof for $|u| \wedge 1$ is similar. ∎

REMARK 2.4. (i) The assertions of Lemma 2.3 remain valid if $H^1(\Omega)$ is replaced by $W^{1,p}(\Omega)$ with $p \in [1, \infty\rangle$.

(ii) It should be noted that $H^1(\Omega)$ is not a Banach lattice. In fact, the intervals $[0, u] = \{v \in H^1(\Omega) : 0 \leqslant v \leqslant u\}$ are not norm bounded, in general.

(iii) If $H^1(\Omega)$ is the complex space, then one has

$$D_i |u| = \mathrm{Re}(\overline{\mathrm{sgn}\, u}\, D_i u)$$

for all $u \in H^1(\Omega)$ (cf. B-II, Lemma 2.4 and C-II.2, p. 251 in [30]). In particular, one has

$$\| \, |u| \, \|_{H^1(\Omega)} \leqslant \|u\|_{H^1(\Omega)}. \tag{2.4}$$

In general, however, the inequality in (2.4) is strict. An example is $\Omega = \langle 0, 1 \rangle$ and $u(x) = e^{ix}$. Then $\| \, |u| \, \|_{H^1(\Omega)} = 1$ but $\|u\|_{H^1(\Omega)} = \sqrt{2}$.

Let V be a closed subspace of $H^1(\Omega)$. We say that V has the *L_1-H^1-extension property* if there exists a continuous linear operator $\mathfrak{E} : V \to H^1(\mathbb{R}^d)$, called *extension operator*, with the following two properties:

(i) $(\mathfrak{E}\varphi)|_\Omega = \varphi$ for all $\varphi \in V$,

(ii) there exists a constant $c > 0$ such that $\|\mathfrak{E}\varphi\|_{L_1(\mathbb{R}^d)} \leqslant c \|\varphi\|_{L_1(\Omega)}$ for all $\varphi \in V \cap L_1(\Omega)$.

EXAMPLE 2.5. Let $\Omega \subset \mathbb{R}^d$ be open and set $V = H_0^1(\Omega)$. Then the extension operator which extends functions on Ω by 0 on $\mathbb{R}^d \setminus \Omega$ obviously satisfies properties (i) and (ii).

EXAMPLE 2.6. Let $\Omega \subset \mathbb{R}^d$ be open and suppose the boundary $\partial\Omega$ of Ω is *minimally smooth* in the sense of Stein ([38]), i.e., there exist an $\varepsilon > 0$, an integer N, an $M > 0$ and a possibly infinite sequence of open sets U_n such that:

(i) For all $x \in \partial\Omega$ there exists an $n \in \mathbb{N}$ such that $\{y \in \mathbb{R}^d : |y - x| < \varepsilon\} \subset U_n$.

(ii) No point of $\mathbb{R}^d$ is contained in more than N of the U_n's.

(iii) For all $n \in \mathbb{N}$ there exists an isometry $T : \mathbb{R}^d \to \mathbb{R}^d$ and a function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $|\varphi(x) - \varphi(y)| \leqslant M|x - y|$ for all $x, y \in \mathbb{R}^{d-1}$ and $U_n \cap \Omega = U_n \cap (TZ)$, where $Z = \{(x,t) \in \mathbb{R}^{d-1} \times \mathbb{R} : \varphi(x) < t\}$.

Then it follows from Theorem VI.5 in [38] that $H^1(\Omega)$ has the L_1-H^1-extension property.

Note that Ω need not be bounded.

The reason why we consider spaces with the L_1-H^1-extension property is that certain properties of $H^1(\mathbb{R}^d)$ are inherited by V. We will use the following inequality of Nash.

LEMMA 2.7. *Let V be a subspace of $H^1(\Omega)$ which has the L_1-H^1-extension property. Then there exists a $c_N > 0$ such that*

$$\|\varphi\|_2^{2+4/d} \leqslant c_N \|\varphi\|_V^2 \|\varphi\|_1^{4/d} \tag{2.5}$$

for all $\varphi \in V \cap L_1(\Omega)$.

Proof. There exists a constant $c_N > 0$ such that

$$\|\varphi\|_{L_2(\mathbb{R}^d)}^{2+4/d} \leqslant c_N \|\varphi\|_{H^1(\mathbb{R}^d)}^2 \|\varphi\|_{L_1(\mathbb{R}^d)}^{4/d}$$

for all $\varphi \in H^1(\mathbb{R}^d)$. (See p. 169 in [37] for a short proof.) Let $\mathfrak{E}$ be the extension operator and $\varphi \in V \cap L_1(\Omega)$. Then

$$\|\varphi\|_{L_2(\Omega)}^{2+4/d} \leqslant \|\mathfrak{E}\varphi\|_{L_2(\mathbb{R}^d)}^{2+4/d} \leqslant c_N \|\mathfrak{E}\varphi\|_{H^1(\mathbb{R}^d)}^2 \|\mathfrak{E}\varphi\|_{L_1(\mathbb{R}^d)}^{4/d} \leqslant c'_N \|\varphi\|_V^2 \|\varphi\|_1^{4/d},$$

where $c'_N = c_N \|\mathfrak{E}\|_{V \to H^1(\mathbb{R}^d)}^2 \|\mathfrak{E}\|_{L_1(\Omega) \to L_1(\mathbb{R}^d)}^{4/d}$. ∎

REMARK 2.8. Note that the Nash inequality does not hold in $H^1(\Omega)$ for general Ω.

We frequently use the following proposition on semigroups associated with continuous coercive forms.

PROPOSITION 2.9. *Let $V, \mathcal{H}$ be Hilbert spaces, V dense and continuously embedded in $\mathcal{H}$ and $a : V \times V \to \mathbf{C}$ a continuous sesquilinear form. Suppose the form a is* **coercive**, *i.e., there exist $\omega \in \mathbf{R}$ and $\mu > 0$ such that*

$$\operatorname{Re} a(u,u) + \omega \|u\|_{\mathcal{H}}^2 \geqslant \mu \|u\|_V^2$$

for all $u \in V$. Define the operator A **associated with the form** *a by*

$$D(A) = \{u \in V : \exists_{v \in \mathcal{H}} \forall_{\varphi \in V} [a(u,\varphi) = (v,\varphi)_{\mathcal{H}}]\}$$

and $Au = v$ for all $u \in D(A)$ if $a(u,\varphi) = (v,\varphi)_{\mathcal{H}}$ for all $\varphi \in V$. Then A generates a holomorphic semigroup $S = (\mathrm{e}^{-tA})_{t>0}$ on $\mathcal{H}$.

Proof. See Chapter XVII, p. 450 in [14], or Theorem 3.6.1 in [39]. ∎

In the last part of this preliminary section we put together some basic properties of traces. For that we assume that Ω is a bounded open subset of $\mathbf{R}^d$ with Lipschitz boundary $\Gamma = \partial\Omega$.

There exists a unique linear bounded operator $B : H^1(\Omega) \to L_2(\Gamma)$ such that $Bu = u|_\Gamma$ for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$. Here Γ is considered as a measure space with the surface measure. The operator B is called the *trace operator* and Bu the *trace* of u. (See Adams ([1]) or Alt ([2], p. 168) for trace properties.) The operator B is a lattice homomorphism, i.e.,

$$B(u \vee v) = (Bu) \vee (Bv), \quad B(u \wedge v) = (Bu) \wedge (Bv) \tag{2.6}$$

and in particular

$$B(u^+) = (Bu)^+, \quad B(u \wedge 1) = (Bu) \wedge 1 \tag{2.7}$$

for all $u, v \in H^1(\Omega)$. In fact (2.6) and (2.7) are trivially valid for $u|_\Omega$ with $u \in C_c^\infty(\mathbf{R}^d)$. Since the lattice operations are continuous in $H^1(\Omega)$ and $L_2(\Gamma)$, the claim follows by taking limits. Note that $H_0^1(\Omega) = \{u \in H^1(\Omega) : Bu = 0\}$.

3. DIRICHLET BOUNDARY CONDITIONS

Given an elliptic operator arising from a form with Dirichlet boundary conditions, we show in this section that the corresponding semigroup has a kernel which satisfies Gaussian bounds, provided the second order coefficients are once differentiable. Since we do not assume that the lower order coefficients are real, all spaces are complex in this section. The method we use here consists in proving uniform L_∞-estimates for the semigroup perturbed by the Davies' method. This is done via a criterion of quasi L_∞-contractivity for non-symmetric forms due to Ouhabaz. Then the Gaussian estimates follow easily from the Nash inequality. The main theorem of this section is the following.

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^d$ open, let $a_{ij} \in W^{1,\infty}(\Omega)$ be real functions for all $i,j \in \{1,\dots,d\}$ and let $b_i, c_i \in W^{1,\infty}(\Omega)$ (complex) for all $i \in \{1,\dots,d\}$. Let $c_0 \in L_\infty(\Omega)$. Consider the form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ defined by*

$$a(u,v) = \sum_{i,j=1}^d \int_\Omega a_{ij}\, D_i u\, \overline{D_j v} + \sum_{i=1}^d \int_\Omega b_i\, D_i u\, \overline{v} + \sum_{i=1}^d \int_\Omega c_i\, u\, \overline{D_i v} + \int_\Omega c_0\, u\, \overline{v}.$$

Suppose there exists a $\mu > 0$ such that

$$\sum_{i,j=1}^d a_{ij}(x)\, \xi_i\, \xi_j \geqslant \mu |\xi|^2$$

for all $\xi \in \mathbb{R}^d$, for a.e. *$x \in \Omega$. Let A be the operator associated with the continuous coercive form a and $S = (\mathrm{e}^{-tA})_{t>0}$ the semigroup generated by A (see Proposition 2.9). Then S interpolates on L_p, $1 \leqslant p \leqslant \infty$, and there exists $b, c > 0$, $\omega \in \mathbb{R}$ and $K_t \in L_\infty(\Omega \times \Omega)$ such that*

$$|K_t(x;y)| \leqslant c t^{-d/2} \mathrm{e}^{-b|x-y|^2 t^{-1}} \mathrm{e}^{\omega t} \quad (x,y)\text{-a.e.}$$

and

$$(S_t\varphi)(x) = \int_\Omega K_t(x;y)\, \varphi(y)\, \mathrm{d}y \quad x\text{-a.e.}$$

for all $t > 0$ and $\varphi \in L_2(\Omega)$.

The proof relies on the Davies perturbation method to obtain Gaussian upper bounds. In order to be complete we describe briefly this method. For $K \in L_\infty(\Omega \times \Omega)$ define the integral operator $T_K \in \mathcal{L}(L_1(\Omega), L_\infty(\Omega))$ by

$$(T_K\varphi)(x) = \int_\Omega K(x;y)\, \varphi(y)\, \mathrm{d}y. \tag{3.1}$$

Then it is well known that $K \mapsto T_K$ is an isometric isomorphism from $L_\infty(\Omega \times \Omega)$ onto $\mathcal{L}(L_1(\Omega), L_\infty(\Omega))$. (See, e.g., [7], Theorem 1.3 for a short proof.) In particular, if $T \in \mathcal{L}(L_2(\Omega))$ is such that

$$\|T\|_{1\to\infty} = \sup\{\|T\varphi\|_\infty : \varphi \in L_1 \cap L_2\} < \infty$$

then there exists a $K \in L_\infty(\Omega \times \Omega)$ such that (3.1) holds x-a.e. for all $\varphi \in L_1 \cap L_2$.

Next, let

$$W = \{\psi \in C_b^\infty(\mathbf{R}^d) : \psi \text{ is real and } \|D_i\psi\|_\infty \leqslant 1,\ \|D_iD_j\psi\|_\infty \leqslant 1 \text{ for all } i,j \in \{1,\ldots d\}\}.$$

Then clearly $d(x;y) = \sup\{\psi(x) - \psi(y) : \psi \in W\}$ defines a distance on $\mathbf{R}^d$. This distance is equivalent to the Euclidean metric.

LEMMA 3.2. *There exists an $\alpha > 0$ such that*

$$\alpha|x-y| \leqslant d(x;y) \leqslant \alpha^{-1}|x-y| \tag{3.2}$$

for all $x, y \in \mathbf{R}^d$.

Proof. See [37], pp. 200–202. ∎

Now let S be a semigroup on $L_2(\Omega)$, where Ω is an open subset of $\mathbf{R}^d$. For $\rho \in \mathbf{R}$ and $\psi \in W$ we define the perturbed semigroup S^ρ on L_2 by $S_t^\rho = U_\rho S_t U_\rho^{-1}$, where $(U_\rho\varphi)(x) = \mathrm{e}^{-\rho\psi(x)}\varphi(x)$. Here we deliberately omit the dependence of S^ρ and U_ρ on ψ in our notation.

Gaussian upper estimates for the kernel of S can be obtained from ultracontractivity of S^ρ, uniformly in ρ and ψ. The following useful device is due to Davies ([15]) (see also [16]). We include a proof for the convenience of the reader, since only variations of the criterion are explicitly given in the literature, cf. Chapter III, p. 189 ff. and the proof of Proposition IV.2.2 in [37], or Section 3.2 in [16]. Moreover, the technical measure theoretical problem is usually left to the reader or circumvented by approximation of the operator by operators with smooth coefficients. In our situation the last method is impossible.

PROPOSITION 3.3. *Let S be a semigroup on $L_2(\Omega)$ and $c, \omega_1 \in \mathbf{R}$. Then the following are equivalent:*

(i) *There exists a constant $\omega_2 > 0$ such that*

$$\|S_t^\rho\|_{1\to\infty} \leqslant ct^{-d/2}\mathrm{e}^{\omega_1 t + \omega_2\rho^2 t} \tag{3.3}$$

uniformly for all $\rho \in \mathbf{R}$, $t > 0$ and $\psi \in W$.

(ii) *There exists a constant $b > 0$ such that the operators S_t have a kernel $K_t \in L_\infty(\Omega \times \Omega)$ which verifies*

$$|K_t(x;y)| \leqslant ct^{-d/2}\mathrm{e}^{-b|x-y|^2t^{-1}}\mathrm{e}^{\omega_1 t} \quad (x,y)\text{-a.e.} \tag{3.4}$$

for all $t > 0$.

Moreover, if one of the two conditions is valid then S interpolates on $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$ and there exists a constant $c_1 > 0$, depending only on the constants b and c in (3.4) *such that $\|S_t\|_{p\to p} \leqslant c_1\mathrm{e}^{\omega_1 t}$ uniformly for all $t > 0$ and $p \in [1,\infty]$.*

Proof. (i) $\Rightarrow$ (ii). Taking $\rho = 0$ we see that S_t has a kernel $K_t \in L_\infty(\Omega \times \Omega)$. Then for each ρ and t the operator S_t^ρ has a kernel K_t^ρ, given by

$$K_t^\rho(x;y) = \mathrm{e}^{-\rho(\psi(x)-\psi(y))}K_t(x;y) \quad (x,y)\text{-a.e.}$$

Then (3.3) implies that for all $t > 0$, $\rho \in \mathbf{R}$ and $\psi \in W$ one has

$$|K_t(x;y)| \leqslant ct^{-d/2}\mathrm{e}^{\omega_1 t + \omega_2\rho^2 t}\mathrm{e}^{\rho(\psi(x)-\psi(y))} \quad (x,y)\text{-a.e.}$$

Replacing ρ by $-\rho$ one deduces that

$$|K_t(x;y)| \leqslant ct^{-d/2}\mathrm{e}^{\omega_1 t + \omega_2\rho^2 t}\mathrm{e}^{-\rho|\psi(x)-\psi(y)|} \quad (x,y)\text{-a.e.}$$

Next, Lemma 3.4 below implies that

$$|K_t(x;y)| \leqslant ct^{-d/2}\mathrm{e}^{\omega_1 t + \omega_2\rho^2 t}\mathrm{e}^{-\rho d(x;y)} \quad (x,y)\text{-a.e.}$$

for each $t > 0$ and $\rho \in \mathbf{R}$. For fixed $t > 0$ and $x, y \in \Omega$ the minimum over ρ of the right hand side is attained in $\rho = (2\omega_2 t)^{-1}d(x;y)$. Thus, applying Lemma 3.4 again we obtain

$$|K_t(x;y)| \leqslant ct^{-d/2}\mathrm{e}^{-(4\omega_2 t)^{-1}d(x;y)^2}\mathrm{e}^{\omega_1 t} \quad (x,y)\text{-a.e.}$$

Now (3.4) follows from Lemma 3.2 with $b = (4\omega_2)^{-1}\alpha^2$.

(ii) $\Rightarrow$ (i). Let α be as in Lemma 3.2. Then

$$\begin{aligned}\|S_t^\rho\|_{1\to\infty} &= \operatorname*{ess\,sup}_{x\in\Omega}\operatorname*{ess\,sup}_{y\in\Omega}|K_t^\rho(x;y)| \leqslant \operatorname*{ess\,sup}_{x,y\in\Omega}|K_t(x;y)|\mathrm{e}^{|\rho|\,|\psi(x)-\psi(y)|}\\ &\leqslant \sup_{x,y\in\Omega} ct^{-d/2}\mathrm{e}^{-b|x-y|^2t^{-1}+\alpha^{-1}|\rho|\,|x-y|}\mathrm{e}^{\omega_1 t} \leqslant ct^{-d/2}\mathrm{e}^{\omega_2\rho^2 t}\mathrm{e}^{\omega_1 t}\end{aligned}$$

with $\omega_2 = (4\alpha^2 b)^{-1}$.

Finally, suppose (ii) is valid. Let T be the semigroup on $L_2(\mathbf{R}^d)$ generated by the operator $-\sum_{i=1}^{d}\partial^2/\partial x_i^2$. Then T interpolates on $L_p(\mathbf{R}^d)$, $1 \leqslant p \leqslant \infty$ and T has the Gaussian kernel K^Δ. Then

$$|K_t(x;y)| \leqslant c(\pi b^{-1})^{d/2}\mathrm{e}^{\omega_1 t}K^\Delta_{(4b)^{-1}t}(x;y) \quad \text{a.e.-}(x,y) \in \Omega \times \Omega$$

for all $t > 0$. Therefore, $|S_t\varphi| \leqslant c(\pi b^{-1})^{d/2}\mathrm{e}^{\omega_1 t}T_{(4b)^{-1}t}|\varphi|$ for all $\varphi \in L_1(\Omega)\cap L_2(\Omega)$ and $t > 0$. So by Lemma 2.1 (v) it follows that S interpolates on $L_p(\Omega)$, $1 \leqslant p \leqslant 2$. By duality, S interpolates on $L_p(\Omega)$, $2 \leqslant p \leqslant \infty$. Moreover, $\|S_t\|_{p\to p} \leqslant c(\pi b^{-1})^{d/2}\mathrm{e}^{\omega_1 t}$ for all $t > 0$ and $p \in [1,\infty]$. ∎

In the previous proposition we needed the following result on infima, which can be stated in a more general context.

LEMMA 3.4. *Let Y be a σ-compact topological space and let $F \subset C(Y)$. Let $f_0 \in C(Y)$ and assume that $f_0(x) = \inf_{f \in F} f(x)$ for all $x \in Y$. Then there exist $f_1, f_2, \ldots \in F$ such that $f_0(x) = \inf_{n \in \mathbf{N}} f_n(x)$ for all $x \in Y$. In particular, if (Y, Σ, μ) is a measure space and $h : Y \to \mathbf{R}$ is a measurable function such that $h \leqslant f$ μ-a.e. for all $f \in F$ then $h \leqslant f_0$ μ-a.e.*

Proof. First we can assume that Y is compact. Secondly, replacing F by $F - f_0$ we can (and do) assume that $f_0 = 0$. Let $m \in \mathbf{N}$. For all $x \in Y$ there exists an $f_{x,m} \in F$ such that $f_{x,m}(x) < m^{-1}$ and hence $f_{x,m} < m^{-1}$ on an open neighbourhood $U_{x,m}$ of x. By compactness we find $x_{m,1}, \ldots, x_{m,n_m} \in Y$ such that $Y = \bigcup_{j=1}^{n_m} U_{x_{m,j},m}$. Then $\inf_j f_{x_{m,j},m}(x) < m^{-1}$ for all $x \in Y$. Now the set $F_0 = \{x_{m,j} : m \in \mathbf{N},\ j \in \{1, \ldots, n_m\}\}$ is countable and $\inf_{f \in F_0} f(x) = 0$ for all $x \in Y$. ∎

In view of Proposition 3.3, we have to show (3.3) in order to prove Theorem 3.1. This will be done in two steps. At first we show L_∞-contractivity with help of the following criterion.

PROPOSITION 3.5. *Denote by $S = (\mathrm{e}^{-tA})_{t>0}$ the semigroup on $L_2(\Omega)$ generated by the operator A of Theorem 3.1. Assume that*

$$\text{(3.5)}\quad \operatorname{Re}\Big(\sum_{i,j=1}^{d} a_{ij}\, D_i u\, \overline{D_j u} + \sum_{i=1}^{d} (b_i - c_i)\, D_i u\, \overline{u} + \Big(c_0 + \sum_{i=1}^{d} D_i c_i\Big) |u|^2 \Big) \geqslant 0 \quad \text{a.e.}$$

for all $u \in H_0^1(\Omega)$. Then S is L_∞-contractive. In particular, S interpolates on L_p, $2 \leqslant p \leqslant \infty$.

Proof. Using integration by parts we obtain

$$a(u,v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij}\, D_i u\, \overline{D_j v} + \sum_{i=1}^{d} \int_\Omega (b_i - c_i)\, D_i u\, \overline{v} + \int_\Omega \Big(c_0 + \sum_{i=1}^{d} D_i c_i\Big)\, u\, \overline{v}$$

for all $u, v \in H_0^1(\Omega)$. Moreover, $(1 \wedge |u|) \operatorname{sgn} u \in H_0^1(\Omega)$ for all $u \in H_0^1(\Omega)$. Therefore, the L_∞-contractivity follows from Theorem 4.2 (3) in [32]. The last statement follows from Lemma 2.1 (i). ∎

LEMMA 3.6. *Let $\psi \in W$ be fixed and $\rho \in \mathbb{R}$. Denote by $S = (e^{-tA})_{t>0}$ the semigroup on $L_2(\Omega)$ generated by the operator A of Theorem 3.1. Then the generator A^ρ of the perturbed semigroup S^ρ is associated with the form a^ρ on $H_0^1(\Omega) \times H_0^1(\Omega)$ given by*

$$a^\rho(u,v) = \sum_{i,j=1}^d \int_\Omega a_{ij}\, D_i u\, \overline{D_j v} + \sum_{i=1}^d \int_\Omega b_i^\rho\, D_i u\, \overline{v} + \sum_{i=1}^d \int_\Omega c_i^\rho\, u\, \overline{D_i v} + \int_\Omega c_0^\rho\, u\, \overline{v},$$

where

$$b_i^\rho = b_i - \rho \sum_{j=1}^d a_{ij}\, \psi_j,$$

$$c_i^\rho = c_i + \rho \sum_{k=1}^d a_{ki}\, \psi_k,$$

$$c_0^\rho = c_0 - \rho^2 \sum_{i,j=1}^d a_{ij}\, \psi_i\, \psi_j + \rho \sum_{i=1}^d b_i\, \psi_i - \rho \sum_{i=1}^d c_i\, \psi_i$$

and $\psi_i = D_i \psi$ for all $i \in \{1, \dots, d\}$.

Proof. Note that $A^\rho = U_\rho A U_\rho^{-1}$. Furthermore, one has $e^{\rho\psi} H_0^1(\Omega) = H_0^1(\Omega)$ and $U_\rho D_i U_\rho^{-1} = D_i + \rho\psi_i$. Therefore,

$$\begin{aligned} a(U_\rho^{-1}u, U_\rho v) &= \sum_{i,j=1}^d \int_\Omega a_{ij}\,(D_i + \rho\psi_i)u\,\overline{(D_j - \rho\psi_j)v} \\ &\quad + \sum_{i=1}^d \int_\Omega b_i\,(D_i + \rho\psi_i)u\,\overline{v} + \sum_{i=1}^d \int_\Omega c_i\, u\,\overline{(D_i - \rho\psi_i)v} + \int_\Omega c_0\, u\,\overline{v} \\ &= a^\rho(u,v) \end{aligned}$$

for all $u, v \in H_0^1(\Omega)$. This proves the lemma. ∎

The second statement in the following lemma shows again the well-known fact that the form a is coercive, which we have used already. For the sequel, however, we need a uniform coercivity estimate for the form a^ρ.

LEMMA 3.7. *Denote by $S = (e^{-tA})_{t>0}$ the semigroup on $L_2(\Omega)$ generated by the operator A of Theorem 3.1.*

(i) *There exists an $\omega > 0$ such that*

$$\|S_t^\rho \varphi\|_\infty \leqslant e^{\omega(1+\rho^2)t} \|\varphi\|_\infty$$

uniformly for all $\rho \in \mathbf{R}$, $\psi \in W$, $t > 0$ *and* $\varphi \in L_2 \cap L_\infty$. *The constant* ω *depends only on* μ, $\|a_{ij}\|_{W^{1,\infty}}$, $\|b_i\|_\infty$, $\|c_i\|_{W^{1,\infty}}$ *and* $\|c_0\|_\infty$.

(ii) *There exists an* $\omega > 0$ *such that*

$$\operatorname{Re} a^\rho(u,u) + \omega(1+\rho^2)\|u\|_2^2 \geqslant 2^{-1}\mu\|u\|^2_{H_0^1(\Omega)}$$

uniformly for all $\rho \in \mathbf{R}$, $\psi \in W$, $t > 0$ *and* $u \in H_0^1(\Omega)$. *The constant* ω *depends only on* μ, $\|a_{ij}\|_\infty$, $\|b_i\|_\infty$, $\|c_i\|_\infty$ *and* $\|c_0\|_\infty$.

Proof. We show that there exists an $\omega \in \mathbf{R}$ such that

$$\begin{aligned}
&\operatorname{Re}\Bigg(\sum_{i,j=1}^d a_{ij}\, D_i u\, \overline{D_j u} + \sum_{i=1}^d (b_i^\rho - c_i^\rho)\, D_i u\, \overline{u} \\
&\qquad + \Big(c_0^\rho + \sum_{i=1}^d D_i c_i^\rho\Big)\, |u|^2 + \omega(1+\rho^2)|u|^2\Bigg) \qquad (3.6)\\
&\qquad \geqslant 2^{-1}\mu \sum_{i=1}^d |D_i u|^2 \quad \text{a.e.}
\end{aligned}$$

for all $u \in H_0^1(\Omega)$, $\rho \in \mathbf{R}$ and $\psi \in W$. Here b_i^ρ, c_i^ρ and c_0^ρ are as in Lemma 3.6. Let

$$M = 1 + \max\{\|a_{ij}\|_{W^{1,\infty}}, \|b_i\|_\infty, \|c_i\|_{W^{1,\infty}}, \|c_0\|_\infty\}.$$

The first term in (3.6) can be estimated by

$$\operatorname{Re} \sum_{i,j=1}^d a_{ij}\, D_i u\, \overline{D_j u} \geqslant \mu \sum_{i=1}^d |D_i u|_2^2 \quad \text{a.e.},$$

for all $u \in H_0^1(\Omega)$. The second term can be majorated in the following manner,

$$\begin{aligned}
&\left|\operatorname{Re}\sum_{i=1}^d (b_i^\rho - c_i^\rho)\,(D_i u)\,\overline{u}\right| \\
&\quad\leqslant \sum_{i=1}^d |b_i - c_i|\,|D_i u|\,|u| + |\rho| \left|\sum_{i=1}^d\sum_{k=1}^d (a_{ki} + a_{ik})\,\psi_k\,(D_i u)\,\overline{u}\right| \\
&\quad\leqslant 2M \sum_{i=1}^d |D_i u|\,|u| + 2dM|\rho| \sum_{i=1}^d |D_i u|\,|u| \\
&\quad= 2M(d+|\rho|) \sum_{i=1}^d |D_i u|\,|u| \\
&\quad\leqslant 2M(d+|\rho|)\varepsilon \sum_{i=1}^d |D_i u|^2 + (2\varepsilon)^{-1} dM(d+|\rho|)|u|^2 \\
&\quad\leqslant 2^{-1}\mu \sum_{i=1}^d |D_i u|^2 + 4d^3M^2\mu^{-1}(1+\rho^2)|u|^2 \quad \text{a.e.},
\end{aligned}$$

where we have chosen $\varepsilon = (4M(d+|\rho|))^{-1}\mu$ and used the inequality $xy \leqslant \delta x^2 + (4\delta)^{-1}y^2$. Finally, we majorate the coefficient in the third term in the following manner,

$$\begin{aligned}\Big|\mathrm{Re}\Big(c_0^\rho + \sum_{i=1}^d D_i c_i^\rho\Big)\Big| &\leqslant M + d^2 M\rho^2 + dM|\rho| + dM|\rho| \\ &\quad + \Big|\sum_{i=1}^d \Big(D_i c_i + \rho\sum_{k=1}^d ((D_i a_{ki})\,\psi_k + a_{ki}\,D_i\psi_k)\Big)\Big| \\ &\leqslant 2d^2M(1+\rho^2) + M(d + 2d^2|\rho|).\end{aligned}$$

Here we have used the differentability of the second order coefficients. Note that in case $\rho = 0$ these terms vanish. Hence, for all $\rho \in \mathbb{R}$

$$\Big|\mathrm{Re}\Big(c_0^\rho + \sum_{i=1}^d D_i c_i^\rho\Big)\Big| \leqslant 4d^2M(1+\rho^2) \quad \text{a.e.,}$$

for all $\rho \in \mathbb{R}$. Therefore, (3.6) holds if $\omega = 4d^3M^2\mu^{-1} + 4d^2M$. Now Statement (i) follows from Proposition 3.5.

Similarly one can estimate

$$\begin{aligned}\mathrm{Re}\Big(\sum_{i,j=1}^d a_{ij}\,D_i u\,\overline{D_j u} + \sum_{i=1}^d b_i^\rho\,D_i u\,\overline{u} + \sum_{i=1}^d c_i^\rho\,u\,\overline{D_i u} + c_0^\rho|u|^2 + \omega(1+\rho^2)|u|^2\Big) \\ \geqslant 2^{-1}\mu\sum_{i=1}^d |D_i u|^2 \quad \text{a.e.}\end{aligned}$$

if $\omega = 4d^3M_0^2\mu^{-1} + 2d^2M_0$ and

$$M_0 = 1 + \max\{\|a_{ij}\|_\infty, \|b_i\|_\infty, \|c_i\|_\infty, \|c_0\|_\infty\}.$$

Integrating this inequality one obtains

$$\mathrm{Re}\,a^\rho(u,u) + \omega(1+\rho^2)\|u\|_2^2 \geqslant 2^{-1}\mu\sum_{i=1}^d \|D_i u\|_2^2$$

for all $u \in H_0^1(\Omega)$. Hence

$$\mathrm{Re}\,a^\rho(u,u) + (\omega + 2^{-1}\mu)(1+\rho^2)\|u\|_2^2 \geqslant 2^{-1}\mu\|u\|^2_{H_0^1(\Omega)}.$$

Replacing ω by $\omega + 2^{-1}\mu$ proves Statement (ii). ∎

We now know that the perturbed semigroup is quasi-contractive on L_∞ and hence by duality one has a bound on $\mathcal{L}(L_1)$. Next we convert the L_2-ellipticity estimate and the $\mathcal{L}(L_1)$-bound in a $\mathcal{L}(L_1, L_2)$-bound for S (cf. Step 2 of the proof of Proposition III.4.2 in [37] or Theorem 2.4.6 in [16]). For our purposes, it is important to obtain independent constants.

PROPOSITION 3.8. *Let a be a continuous form with domain $D(a) = V$, with V a Hilbert space which is continuous embedded in $L_2(X)$, where (X, Σ, m) is a σ-finite measure space. Assume there exists a constant $\mu > 0$ such that $\operatorname{Re} a(\varphi, \varphi) \geqslant \mu \|\varphi\|_V^2$ for all $\varphi \in V$. Let S be the semigroup on L_2 generated by the operator associated with the form a. Suppose that S interpolates on L_p, $1 \leqslant p \leqslant 2$. Assume there exists a $c_1 > 0$ such that $\|S_t\|_{1\to 1} \leqslant c_1$ for all $t > 0$. Further, let c_N and $n > 0$, and suppose that the Nash inequality*

$$\|\varphi\|_2^{2+4/n} \leqslant c_N \|\varphi\|_V^2 \|\varphi\|_1^{4/n}$$

is valid for all $\varphi \in L_1 \cap V$. Then there exists a constant $c > 0$, depending continuously on μ, c_1, c_N and n and which is otherwise independent of a, such that

$$\|S_t\|_{1\to 2} \leqslant c t^{-n/4}$$

uniformly for all $t > 0$.

Proof. Let $\varphi \in L_1(\Omega) \cap L_2(\Omega)$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|S_t\varphi\|_2^2 &= -2 \operatorname{Re} a(S_t\varphi, S_t\varphi) \leqslant -2\mu \|S_t\varphi\|_V^2 \leqslant -\frac{2\mu}{c_N} \frac{\|S_t\varphi\|_2^{2+4/n}}{\|S_t\varphi\|_1^{4/n}} \\ &\leqslant -\frac{2\mu}{c_N\, c_1^{4/n}} \frac{(\|S_t\varphi\|_2^2)^{1+2/n}}{\|\varphi\|_1^{4/n}}.\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|S_t\varphi\|_2^2)^{-2/n} = -\frac{2}{n}(\|S_t\varphi\|_2^2)^{-1-2/n}\frac{\mathrm{d}}{\mathrm{d}t}\|S_t\varphi\|_2^2 \geqslant \frac{4\mu}{n c_N\, c_1^{4/n}} \|\varphi\|_1^{-4/n}$$

and by integration

$$\|S_t\varphi\|_2^{-4/n} = (\|S_t\varphi\|_2^2)^{-2/n} \geqslant t \frac{4\mu}{n c_N\, c_1^{4/n}} \|\varphi\|_1^{-4/n}.$$

Now the theorem follows if one takes $c = (4\mu)^{-n/4}(n c_N)^{n/4} c_1$. ∎

We continue the proof of Theorem 3.1.

COROLLARY 3.9. *Denote by* $S = (e^{-tA})_{t>0}$ *the semigroup on* $L_2(\Omega)$ *generated by the operator* A *of Theorem* 3.1. *Then there exist* $c, \omega > 0$ *such that*

$$\|S_t^\rho\|_{1\to\infty} \leqslant ct^{-d/2}e^{\omega(1+\rho^2)t}$$

uniformly for all $t > 0$, $\rho \in \mathbf{R}$ *and* $\psi \in W$.

Proof. Since the form-adjoint of a is of the same form as the form a it follows from Lemma 3.7 that there exist $\mu, \omega > 0$ such that $\operatorname{Re} a^\rho(\varphi,\varphi) + \omega(1+\rho^2)\|\varphi\|_2^2 \geqslant \mu\|\varphi\|_{H^1}^2$ and $\|S_t^\rho e^{-\omega(1+\rho^2)t}\|_{1\to1} \leqslant 1$ uniformly for all $\rho \in \mathbf{R}$, $\psi \in W$ and $t > 0$. Here a^ρ is as in Lemma 3.6. Moreover, by Example 2.5 and Lemma 2.7 there exists a $c_N > 0$ such that the Nash inequality

$$\|\varphi\|_2^{2+4/d} \leqslant c_N \|\varphi\|_{H^1}^2 \, \|\varphi\|_1^{4/d}$$

is valid for all $\varphi \in L_1(\Omega) \cap H_0^1(\Omega)$. Then by Proposition 3.8 there exists a $c > 0$ such that $\|S_t^\rho e^{-\omega(1+\rho^2)t}\|_{1\to2} \leqslant ct^{-d/4}$ uniformly for all $\rho \in \mathbf{R}$, $\psi \in W$ and $t > 0$. So

$$\|S_t^\rho\|_{1\to2} \leqslant ct^{-d/4}e^{\omega(1+\rho^2)t}. \tag{3.7}$$

But by duality it then follows that

$$\|S_t^\rho\|_{2\to\infty} \leqslant ct^{-d/4}e^{\omega(1+\rho^2)t},$$

possibly by enlarging c and ω. Then

$$\|S_t^\rho\|_{1\to\infty} \leqslant \|S_{t/2}^\rho\|_{1\to2}\|S_{t/2}^\rho\|_{2\to\infty} \leqslant 2^{d/2}c^2t^{-d/2}e^{\omega(1+\rho^2)t}$$

uniformly for all $t > 0$, $\rho \in \mathbf{R}$ and $\psi \in W$. ∎

Now Theorem 3.1 has been proved completely by an application of Proposition 3.3. ∎

REMARK 3.10. (i) A version of Theorem 3.1 with somewhat complementary assumptions has been obtained by [21] for $\Omega = \mathbf{R}^d$: if $a_{ij} \in W^{2,\infty}(\mathbf{R}^d)$ are complex coefficients and satisfy

$$\operatorname{Re} \sum_{i,j=1}^{d} a_{ij}(x)\,\xi_i\,\xi_j \geqslant \mu|\xi|^2$$

for all $\xi \in \mathbf{R}^d$, for a.e. $x \in \mathbf{R}^d$, with $\mu > 0$, and $b_i, c_i, c_0 \in L_\infty$ then the assertions in Theorem 3.1 are valid. Moreover, extensions for higher order operators have been presented. In [9], Gaussian bounds have been proved for second order elliptic operators in divergence form with uniformly continuous second order coefficients in case $\Omega = \mathbf{R}^d$.

(ii) If the coefficients a_{ij} in Theorem 3.1 are real and symmetric and $b_i = c_i = 0$, then one can deduce Theorem 3.1 for Ω from the corresponding theorem for $\mathbf{R}^d$ since the semigroup on $L_p(\Omega)$ is dominated by the corresponding semigroup on $L_p(\mathbf{R}^d)$ (see Examples 4.9 and 5.6 and Theorem 6.2 in [6]).

4. GENERAL BOUNDARY CONDITIONS

In this section we consider second order operators in divergence form with real, L_∞, non-symmetric second order coefficients. Moreover, we drop the assumption that the operator satisfies Dirichlet boundary conditions. Since here all coefficients are supposed to be real we will only work over the real field in this section. So all spaces are real spaces. In general there are no Gaussian bounds for an elliptic operator defined on an open subset $\Omega \subset \mathbf{R}^d$ with Neumann boundary conditions, even if the operator has constant coefficients. An example is the Laplacian Δ on $\Omega = \bigcup_{n=1}^{\infty} (2^{-(n+1)}, 2^{-n}) \subset [0,1] \subset \mathbf{R}$. Then $1_{(2^{n+1}, 2^{-n})}$ is an eigenvector of Δ with eigenvalue 0 for all $n \in \mathbf{N}$. Therefore, S_t has an eigenvalue with infinite multiplicity and S_t is not compact for any $t > 0$. But the existence of a kernel for S_t with Gaussian bounds on the pre-compact set $\Omega \times \Omega \subset [0,1] \times [0,1]$ implies that S_t is a Hilbert–Schmidt operator and therefore compact. There are also examples of bounded connected domains Ω where S_t is not compact on $L_2(\Omega)$, see Hempel–Seco–Simon ([25]) for a systematic study of spectral properties of these kind of operators. Thus, in order to establish Gaussian estimates for the kernel, one needs some kind of regularity of Ω or of the domain on which the sectorial form is defined. We will assume throughout that V has the L_1-H^1-extension property (see Section 2). This is true for all open sets $\Omega \subset \mathbf{R}^d$ if $V = H_0^1(\Omega)$ but one demands some regularity of the boundary of Ω if $V \neq H_0^1(\Omega)$.

Now let A be the (formal) elliptic operator

$$Au = -\sum_{i,j=1}^{d} D_j\, a_{ij}\, D_i u + \sum_{i=1}^{d} b_i\, D_i u - \sum_{i=1}^{d} D_i(c_i\, u) + c_0 u \tag{4.1}$$

with real coefficients. For the coefficients we suppose that $a_{ij} \in L_\infty(\Omega)$ $(i,j \in \{1,\ldots,d\})$, $b_i, c_i \in W^{1,\infty}(\Omega)$ $(i \in \{1,\ldots,d\})$ and $c_0 \in L_\infty(\Omega)$ are real valued functions such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\, \xi_i\, \xi_j \geqslant \mu |\xi|^2 \tag{4.2}$$

for all $\xi \in \mathbf{R}^d$, for a.e. $x \in \Omega$, where $\mu > 0$ is a fixed constant. We emphasize that the coefficients a_{ij} need not be symmetric. We consider realizations of $\mathcal{A}$ in $L_2(\Omega)$ with various boundary conditions. They will be defined by a form domain V satisfying the following hypotheses:

(4.3) V is a closed subspace of $H^1(\Omega)$,

(4.4) $H_0^1(\Omega) \subset V$,

(4.5) V has the L_1-H^1-extension property,

(4.6) $v \in V$ implies $|v|, |v| \wedge 1 \in V$,

(4.7) $v \in V,\ u \in H^1(\Omega),\ |u| \leqslant v$ implies $u \in V$.

Assumption (4.7) means that V is an *ideal* in $H^1(\Omega)$. Furthermore, we assume that the first order coefficients satisfy

(4.8) $i \in \{1, \ldots, d\}$ and $v \in V$ implies $b_i v, c_i v \in H_0^1(\Omega)$.

Now we consider the form $a : V \times V \to \mathbf{R}$ given by

$$(4.9)\quad a(u, v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij}\, D_i u\, D_j v + \sum_{i=1}^{d} \int_\Omega b_i\, D_i u\, v + \sum_{i=1}^{d} \int_\Omega c_i\, u\, D_i v + \int_\Omega c_0\, u\, v.$$

Then a is clearly continuous and coercive, i.e., there exists an $\omega \in \mathbf{R}$ such that

$$a(u, u) + \omega \|u\|_2^2 \geqslant 2^{-1} \mu \|u\|_V^2$$

for all $u \in V$. Let A be the operator on $L_2(\Omega)$ associated with the form a on V. It follows from Proposition 2.9 that the complexification of the operator A associated with the complexified form (still denoted by a) generates a holomorphic semigroup S on $L_2(\Omega)$. Recall that we assume throughout this section that the spaces are real.

If $V = H_0^1(\Omega)$ we say that A is the realization of $\mathcal{A}$ in $L_2(\Omega)$ with *Dirichlet boundary conditions*. In that case (4.8) is satisfied whenever $b_i, c_i \in W^{1,\infty}(\Omega)$.

If $V = H^1(\Omega)$ and $\partial\Omega$ is minimally smooth we say that A is the realization of $\mathcal{A}$ in $L_2(\Omega)$ with *Neumann boundary conditions*. In that case (4.8) is satisfied whenever $b_i, c_i \in W_0^{1,\infty}(\Omega)$. If Ω is bounded, then $b_i, c_i \in H_0^1(\Omega)$ is a necessarily condition for (4.8), since $1 \in V$.

EXAMPLE 4.1. If $a_{ij} = \delta_{ij}$, $V = H^1(\Omega)$ with Ω regular, then one obtains the Neumann–Laplacian with Neumann boundary conditions (cf. Example 4.8).

EXAMPLE 4.2. In general the boundary conditions depend on the coefficients. As an example we consider a concrete non-symmetric case. Let $\Omega = \{re^{i\varphi} : r \in [0,1\rangle,\ \theta \in \mathbf{R}\}$ be the open disk in $\mathbf{R}^2$ and let $V = H^1(\Omega)$. Consider the pure second order operator with constant coefficients $(a_{ij}) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Then one can easily see by Green's formula that $Au = -\Delta u$ for all $u \in D(A)$, and for $u \in C^2(\mathbf{R}^2)$ one has

$$u \in D(A) \Leftrightarrow u_r = u_\varphi \text{ on } \partial\Omega.$$

Similarly, if we choose $\Omega = \langle 0,1\rangle \times \langle 0,1\rangle$ and the same matrix for the coefficients then

$$u \in D(A) \Leftrightarrow \begin{cases} u_x = -u_y & \text{on } \langle 0,1\rangle \times \{0\} \cup \langle 0,1\rangle \times \{1\} \\ u_x = u_y & \text{on } \{0\} \times \langle 0,1\rangle \cup \{0\} \times \langle 0,1\rangle \end{cases}$$

for all $u \in C^2(\mathbf{R}^2)$.

Finally one may consider *mixed boundary conditions* in the following way.

EXAMPLE 4.3. *Suppose Ω has a minimally smooth boundary, let $\Gamma_1 \subset \partial\Omega$ be a closed set and*

$$V = \overline{\{u|_\Omega : u \in C_c^\infty(\mathbf{R}^d \setminus \Gamma_1)\}}^{H^1(\Omega)}.$$

Let $\Gamma_2 \subset \partial\Omega$ be closed such that

$$\Gamma_1 \cup \Gamma_2 = \partial\Omega \quad \text{and} \quad b_i, c_i \in \overline{\{\varphi|_\Omega : \varphi \in C_c^\infty(\mathbf{R}^d \setminus \Gamma_2)\}}^{W^{1,\infty}(\Omega)}.$$

Then (4.3)–(4.8) *is satisfied.*

Proof. The domain V clearly satisfies (4.3) and (4.4). It follows from Example 2.6 that V has the L_1-H^1-extension property (4.5). Let $u \in V$. Then there exist $u_1, u_2, \ldots \in C_c^\infty(\mathbf{R}^d \setminus \Gamma_1)$ such that $\lim u_n = u$ in $H^1(\Omega)$. Then $\lim u_n^+ = u^+$ in $H^1(\Omega)$. Let $e_1, e_2, \ldots \in C_c^\infty(\mathbf{R}^d)$ be a regularizing sequence. Fix $n \in \mathbf{N}$. Then for sufficiently large m one has $e_m * u_n^+ \in C_c^\infty(\mathbf{R}^d \setminus \Gamma_1)$ and $\lim_m e_m * u_n^+ = u_n^+$ in $H^1(\mathbf{R}^d)$. Therefore, $u^+ \in V$. It follows that $|u| = u^+ \vee u^- \in V$. Using the regularizing sequence again one proves in a similar way that $u \wedge 1 \in V$ whenever $0 \leqslant u \in V$. This proves the condition (4.6).

Next we prove the ideal condition (4.7). Let $v \in V$, $u \in H^1(\Omega)$ and suppose that $|u| \leqslant v$. Let $\mathfrak{E}$ be the extension operator with respect to $H^1(\Omega)$. There exist $v_1, v_2, \ldots \in C_c^\infty(\mathbf{R}^d \setminus \Gamma_1)$, $u_1, u_2 \in \cdots \in C_c^\infty(\mathbf{R}^d)$ and $f \in L_2(\Omega)$ such that

$\lim v_n|_\Omega = v$ in $H^1(\Omega)$, $\lim u_n = \mathfrak{E}u$ in $H^1(\mathbb{R}^d)$, $\lim v_n|_\Omega = v$ a.e., $\lim D_i v_n|_\Omega = D_i v$ a.e., $\lim u_n = \mathfrak{E}u$ a.e., $\lim D_i u_n = D_i \mathfrak{E}u$ a.e., and, $|v_n| \leqslant f$ a.e., $|D_i v_n| \leqslant f$ a.e., $|u_n|_\Omega| \leqslant f$ a.e. and $|(D_i u_n)|_\Omega| \leqslant f$ a.e. on Ω for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$. Then $\lim u_n^+|_\Omega = (\mathfrak{E}u)^+|_\Omega = u^+$ in $H^1(\Omega)$ and $\lim(u_n^+ \wedge v_n)|_\Omega = u^+ \wedge v = u^+$ in $H^1(\Omega)$. For all $n \in \mathbb{N}$ one has $e_m * (u_n^+ \wedge v_n) \in C_c^\infty(\mathbb{R}^d \setminus \Gamma_1)$ for large m and $\lim_m e_m * (u_n^+ \wedge v_n) = u_n^+ \wedge v_n$ in $H^1(\mathbb{R}^d)$. So $u^+ \in V$. Similarly $u^- \in V$ and therefore $u = u^+ - u^- \in V$.

Finally, let $b \in \overline{\{\varphi|_\Omega : \varphi \in C_c^\infty(\mathbb{R}^d \setminus \Gamma_2)\}}^{W^{1,\infty}(\Omega)}$ and $u \in V$. We show that $bu \in H_0^1(\Omega)$. There exists $b_1, b_2, \ldots \in C_c^\infty(\mathbb{R}^d \setminus \Gamma_2)$ and $u_1, u_2, \ldots \in C_c^\infty(\mathbb{R}^d \setminus \Gamma_1)$ such that $\lim b_n|_\Omega = b$ in $W^{1,\infty}(\Omega)$ and $\lim u_n|_\Omega = u$ in $H^1(\Omega)$. Then $(b_n u_n)|_\Omega \in C_c^\infty(\Omega)$ and $\lim(b_n u_n)|_\Omega = bu$ in $H^1(\Omega)$. This shows the condition (4.8). ∎

THEOREM 4.4. *Let V satisfy (4.3)–(4.7). Let A be the operator associated with the form a given by (4.9) with domain V and real coefficients $a_{ij} \in L_\infty(\Omega)$, $b_i, c_i \in W^{1,\infty}(\Omega)$ and $c_0 \in L_\infty(\Omega)$ satisfying the ellipticity condition (4.2) and the condition (4.8). Then A generates a positive semigroup $(\mathrm{e}^{-tA})_{t>0}$ which interpolates on $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$, and which is given by a kernel K for which $K_t \in L_\infty(\Omega \times \Omega)$ for all $t > 0$ satisfying*

$$0 \leqslant K_t(x;y) \leqslant ct^{-d/2}\mathrm{e}^{-b|x-y|^2t^{-1}}\mathrm{e}^{\omega t} \quad (x,y)\text{-a.e.}$$

for some constants $b, c > 0$ and $\omega \in \mathbb{R}$, uniformly for all $t > 0$.

In the proof of Theorem 4.4 we will again use Davies' perturbation method and prove ultracontractivity of S^ρ uniformly for all real $\psi \in C_b^\infty(\mathbb{R}^d)$ with $\|D_i\psi\|_1 \leqslant 1$. In case of Neumann boundary conditions the method of Section 3 is, however, not applicable since $(S_t^\rho \mathrm{e}^{-\omega t})_{t>0}$ is not L_∞-contractive for any $\omega \in \mathbb{R}$, in general, even for the Laplacian as the following example shows.

EXAMPLE 4.5. Let $\Omega = (0,1) \subset \mathbb{R}$, $V = H^1(\Omega)$ and $a(u,v) = \int_0^1 u'v'$. Let $\rho = 1$ and $\psi \in C_b^\infty(\mathbb{R})$ be such that $\psi(x) = x$ for all $x \in [-1,2]$. Then S^ρ is associated with the form

$$a^\rho(u,v) = \int_0^1 (u'+u)(v'-v).$$

Let $\omega \in \mathbb{R}$ and suppose that $\|S_t^\rho \mathrm{e}^{-\omega t}\|_{\infty\to\infty} \leqslant 1$ for all $t > 0$. Then $\mathrm{e}^{-\omega t} S_t^\rho 1 \leqslant 1$ for all $t > 0$ in $L_\infty(\Omega)$. Denote by A^ρ the generator of S^ρ. Since $1 \in D(A^\rho)$ it follows that

$$(A^\rho + \omega I)1 = \lim_{t\downarrow 0} \frac{(I - \mathrm{e}^{-\omega t} S_t^\rho)1}{t} \geqslant 0.$$

Hence by density of $D(A)_+$ in $H^1(\Omega)_+$ one deduces that

$$a^\rho(1,u)+\omega(1,u)_{L_2}\geqslant 0$$

for all $u\in H^1(\Omega)_+$. Next for $n\in\mathbb{N}$ set $u_n(x)=(1-x)^n$. Then $u_n\in H^1(\Omega)_+$ and

$$\begin{aligned}0\leqslant a^\rho(1,u_n)+\omega(1,u_n)_{L_2}&=\int_0^1(u_n'-u_n)+\omega\int_0^1 u_n\\&=u_n(1)-u_n(0)+(\omega-1)\int_0^1 u_n=-1+\frac{\omega-1}{n+1}.\end{aligned}$$

This gives a contradiction if one chooses n sufficiently large.

This example has been considered before by Ouhabaz ([32], Remark 4.3 (b)) in a different context.

The method of proving ultracontractivity we use in this section is based on the following proposition (cf. Chapter IV, pp. 262–264 in [37]). Again, it is important for us to obtain constants which do not depend explicitly on the coefficients of the operator.

PROPOSITION 4.6. *Let S be a real continuous semigroup on $L_2(X)$ whose complexification is a holomorphic semigroup, where (X,Σ,m) is a σ-finite measure space. Assume that S is consistent on $L_p(X)$, $2\leqslant p\leqslant\infty$. Let $c_1,\mu>0$ and V be a Hilbert space which is continuously embedded in L_2. Suppose that $(S_t\varphi)^p\in V$, $t\mapsto\|S_t\varphi\|_{2p}^{2p}$ is differentiable and*

$$\frac{d}{dt}\|S_t\varphi\|_{2p}^{2p}\leqslant-\mu\|(S_t\varphi)^p\|_V^2+c_1p^2\|(S_t\varphi)^p\|_2^2$$

for all $t>0$, all real $\varphi\in L_2\cap L_\infty$ and $p\in 2\mathbb{N}$. Let $c_N,n>0$ and suppose that the Nash inequality

$$\|\varphi\|_2^{2+4/n}\leqslant c_N\|\varphi\|_V^2\|\varphi\|_1^{4/n}$$

is valid for all $\varphi\in L_1\cap V$. Moreover, let $M\geqslant 1$ and $\omega\geqslant 0$ be such that

$$\|S_t\|_{2\to 2}\leqslant Me^{\omega t}$$

for all $t>0$. Then there exists a $c_2>0$, depending only on c_N and n, such that

$$\|S_t\|_{2\to\infty}\leqslant c_2M\mu^{-n/4}t^{-n/4}e^{\omega t}e^{tc_1/2}$$

for all $t>0$.

The proof follows from the estimates in Chapter IV, pp. 262–264 of [37]. In order to make this paper more self-contained we include the proof in the Appendix.

Proof of Theorem 4.4. It follows from Lemma 3.7 (ii) and Proposition 2.9 that the complexification of the operator A associated with the complexified form a generates a holomorphic semigroup $S = (e^{-tA})_{t>0}$ on $L_2(\Omega)$. Note that the proof of Lemma 3.7 (ii) is valid for $a_{ij} \in L_\infty(\Omega)$ and $u \in H^1(\Omega)$. Recall that we assume throughout this section that the spaces are real.

First we show that S is positive. Let $\varphi \in V$. Since $D_i\varphi^+ = 1_{[\varphi>0]}D_i\varphi$ and $D_i\varphi^- = -1_{[\varphi<0]}D_i\varphi$ one has $a(\varphi^+, \varphi^-) = 0$. It then follows from Theorem 2.4 in [32] (which is also valid in case of real spaces) that S is positive.

Secondly we show that there exists a constant $\omega \in \mathbb{R}$ such that

$$\|S_t\varphi\|_\infty \leqslant e^{\omega t}\|\varphi\|_\infty \tag{4.10}$$

for all $\varphi \in L_2(\Omega) \cap L_\infty(\Omega)$ and $t > 0$. Since the proof is very similar to a proof in Section 3 we discuss the critical steps. We wish to apply the proof of Lemma 3.7 (i) in case $\rho = 0$. In that case we do not need the differentiability of the second order coefficients. Secondly, we used integration by parts in the proof of Proposition 3.5. But by assumption (4.8) one has $(c_iu) \in H_0^1(\Omega)$ for all $u \in V$ and $i \in \{1, \ldots, d\}$. Hence $\int c_iu\, D_iv = -\int D_i(c_iu)\, v = -\int (D_ic_i)\, u\, v - \int c_i\, (D_iu)\, v$ for all $u, v \in V$. Thirdly, one needs to verify that Theorem 4.2 (3) (or Theorem 2.7) in [32] is also valid for real spaces and that $(1 \wedge |u|)\operatorname{sgn} u \in V$ for all $u \in V$. But $(1 \wedge |u|)\operatorname{sgn} u = u - (u-1)^+ + (-u-1)^+ \in V$ for all $u \in V$. Therefore, the semigroup S is quasi-contractive on L_∞.

Thirdly, replacing A by A^*, S by S^*, $a(u,v)$ by $a^*(u,v) = \overline{a(v,u)}$ one obtains by duality the L_1-bound

$$\|S_t\varphi\|_1 \leqslant e^{\omega t}\|\varphi\|_1 \tag{4.11}$$

for some $\omega > 0$, uniformly for all $t > 0$ and $\varphi \in L_1 \cap L_2$. It follows from (4.10), (4.11) and Lemma 2.1 (i) that S interpolates on $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$.

Fourthly, let $\psi \in W$ (see Section 3), $\rho \in \mathbb{R}$ and define $U_\rho\varphi = e^{-\rho\psi}\varphi$ as before. We show that $U_\rho\varphi \in V$ for all $\varphi \in V$ and $\rho \in \mathbb{R}$. Obviously $e^{-\rho\psi}\varphi \in H^1(\Omega)$ because $\varphi \in V \subset H^1(\Omega)$. Since $|e^{-\rho\psi}\varphi| \leqslant c|\varphi|$ it follows from the ideal assumption (4.7) that $U_\rho\varphi = e^{-\rho\psi}\varphi \in V$. Now define the form $a^\rho : V \times V \to \mathbb{R}$ by

$$a^\rho(u,v) = \sum_{i,j=1}^d \int_\Omega a_{ij}\,(D_i + \rho\psi_i)u\,(D_j - \rho\psi_j)v$$
$$+ \sum_{i=1}^d \int_\Omega b_i\,(D_i + \rho\psi_i)u\,v + \sum_{i=1}^d \int_\Omega c_i\,u\,(D_i - \rho\psi_i)v + \int_\Omega c_0\,u\,v$$

and let A^ρ be the operator associated with the form a^ρ. Then $a^\rho(u,v) = a(U_\rho^{-1}u, U_\rho v)$ for all $u, v \in V$, so $A^\rho = U_\rho A U_\rho^{-1}$. Hence $S_t^\rho = U_\rho S_t U_\rho^{-1}$ for all $t > 0$, where S^ρ is the holomorphic semigroup generated by A^ρ. It then follows as in the proof of Lemma 3.7 (ii) that there exists an $\omega > 0$ such that $a^\rho(\varphi,\varphi) + \omega(1+\rho^2)\|\varphi\|_2^2 \geqslant 0$ for all $\varphi \in V$. Note that the second order coefficients a_{ij} need not be differentiable in Lemma 3.7 (ii). Hence

$$\|S_t^\rho\|_{2\to 2} \leqslant e^{\omega(1+\rho^2)t}$$

for all $t > 0$.

Fifthly one has $(S_t^\rho \varphi)^p \in V$ whenever $t > 0$, $\varphi \in L_2(\Omega) \cap L_\infty(\Omega)$ and $p \in 2\mathbf{N}$. In fact, let $f = S_t^\rho \varphi$. Then $f \in V \subset H^1(\Omega)$ and therefore $f \in H^1(\Omega) \cap L_\infty(\Omega)$ by the second step. By the product formula (7.18) of [23] it follows that $f^p \in H^1(\Omega)$. But $|f^p| \leqslant \|f\|_\infty^{p-1}|f| = c|f|$. Therefore, it follows again from the ideal assumption (4.7) that $f^p \in V$.

Sixthly, let $\varphi \in L_2 \cap L_\infty$ and $p \in 2\mathbf{N}$. We show that $t \mapsto \|S_t^\rho \varphi\|_{2p}^{2p}$ is differentiable on $\langle 0, \infty\rangle$ and that

$$\frac{d}{dt}\|S_t^\rho \varphi\|_{2p}^{2p} = -2p(A^\rho \varphi_t, \varphi_t^{2p-1}) = -2p\, a^\rho(\varphi_t, \varphi_t^{2p-1}), \tag{4.12}$$

where we set $\varphi_t = S_t^\rho \varphi$. Note that $\varphi_t' = \frac{d}{dt}\varphi_t = -A^\rho \varphi_t$ exists in $L_2(\Omega)$ since S^ρ is holomorphic. Moreover, $\varphi_t \in L_2 \cap L_\infty$. Let $t > 0$. Then

$$\begin{aligned}
&\Big|h^{-1}(\|\varphi_{t+h}\|_{2p}^{2p} - \|\varphi_t\|_{2p}^{2p}) - 2p\int \varphi_t^{2p-1}\varphi_t'\Big| \\
&\quad = \Big|\int h^{-1}(\varphi_{t+h}^{2p} - \varphi_t^{2p}) - 2p\int \varphi_t^{2p-1}\varphi_t'\Big| \\
&\quad = \Big|\int h^{-1}(\varphi_{t+h} - \varphi_t)(\varphi_{t+h}^{2p-1} + \varphi_{t+h}^{2p-2}\varphi_t + \cdots + \varphi_{t+h}\varphi_t^{2p-2} + \varphi_t^{2p-1}) \\
&\qquad\qquad - 2p\int \varphi_t^{2p-1}\varphi_t'\Big| \\
&\quad = \Big|\int \Big(h^{-1}(\varphi_{t+h} - \varphi_t) - \varphi_t'\Big)(\varphi_{t+h}^{2p-1} + \varphi_{t+h}^{2p-2}\varphi_t + \cdots + \varphi_{t+h}\varphi_t^{2p-2} + \varphi_t^{2p-1}) \\
&\qquad\qquad + \int \varphi_t'\Big((\varphi_{t+h}^{2p-1} - \varphi_t^{2p-1}) + (\varphi_{t+h}^{2p-2}\varphi_t - \varphi_t^{2p-1}) + \cdots \\
&\qquad\qquad\qquad + (\varphi_{t+h}\varphi_t^{2p-2} - \varphi_t^{2p-1}) + (\varphi_t^{2p-1} - \varphi_t^{2p-1})\Big)\Big| \\
&\quad \leqslant \Big\|h^{-1}(\varphi_{t+h} - \varphi_t) - \varphi_t'\Big\|_2 \Big\|\varphi_{t+h}^{2p-1} + \varphi_{t+h}^{2p-2}\varphi_t + \cdots + \varphi_{t+h}\varphi_t^{2p-2} + \varphi_t^{2p-1}\Big\|_2 \\
&\qquad\qquad + \Big|\int \varphi_t'(\varphi_{t+h} - \varphi_t)g_{t,h}\Big| \\
&\quad \leqslant c\|h^{-1}(\varphi_{t+h} - \varphi_t) - \varphi_t'\|_2 + \|g_{t,h}\|_\infty \|\varphi_t'\|_2 \|\varphi_{t+h} - \varphi_t\|_2,
\end{aligned}$$

which tends to 0 if h tends to 0. Here $g_{t,h}$ is an element of $L_\infty(\Omega)$ which is uniformly bounded for small h by the estimates (4.10). (Note that t, ρ and ψ are fixed.)

Seventhly, we show that there exists a constant $c > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|S_t^\rho \varphi\|_{2p}^{2p} \leqslant -2^{-1}\mu \sum_{i=1}^{d} \|D_i\varphi_t\|_2^2 + c(1+\rho^2)p^2\|\varphi_t^p\|_2^2$$

uniformly for all $t > 0$, $\rho \in \mathbf{R}$, $\psi \in W$, $\varphi \in V \cap L_\infty(\Omega)$ and $p \in 2\mathbf{N}$. By (4.12) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|S_t^\rho \varphi\|_{2p}^{2p} &= -2p \sum_{i,j=1}^{d} (a_{ij}(D_i + \rho\psi_i)\varphi_t, (D_j - \rho\psi_j)\varphi_t^{2p-1}) \\
&\quad - 2p \sum_{i=1}^{d} (b_i(D_i + \rho\psi_i)\varphi_t, \varphi_t^{2p-1}) \\
&\quad - 2p \sum_{i=1}^{d} (c_i\varphi_t, (D_i - \rho\psi_i)\varphi_t^{2p-1}) - 2p \int c_0 \varphi_t^{2p} \\
&= -2p \sum_{i,j=1}^{d} (a_{ij} D_i\varphi_t, D_j\varphi_t^{2p-1}) + \tau_2 + \tau_3 + \tau_4,
\end{aligned}$$

where τ_2 is the sum of terms of the form $p\rho(k_i D_i\varphi_t, \varphi_t^{2p-1})$, τ_3 is the sum of terms of the form $p\rho(k_i'\varphi_t, D_i\varphi_t^{2p-1})$, and τ_4 is a term of the form $p((k_0 + k_0'\rho + k_0''\rho^2)\varphi_t, \varphi_t^{2p-1})$, with $k_0, k_0', k_0'', k_i, k_i' \in L_\infty(\Omega)$ functions of which the L_∞-norm is bounded uniformly in $\psi \in W$, and is independent of ρ, p, φ and t. We estimate the first term.

$$\begin{aligned}
-2p \sum_{i,j=1}^{d} (a_{ij} D_i\varphi_t, D_j\varphi_t^{2p-1}) &= -2p(2p-1) \sum_{i,j=1}^{d} (a_{ij} D_i\varphi_t, \varphi_t^{2p-2} D_j\varphi_t) \\
&= -2p(2p-1) \sum_{i,j=1}^{d} (a_{ij}\varphi_t^{p-1} D_i\varphi_t, \varphi_t^{p-1} D_j\varphi_t) \\
&= -2p^{-1}(2p-1) \sum_{i,j=1}^{d} (a_{ij} D_i\varphi_t^p, D_j\varphi_t^p) \\
&\leqslant -2p^{-1}(2p-1)\mu \sum_{i=1}^{d} \|D_i\varphi_t^p\|_2^2 \\
&\leqslant -2\mu \sum_{i=1}^{d} \|D_i\varphi_t^p\|_2^2.
\end{aligned}$$

The second term can be estimated by

$$|\tau_2| \leqslant \left|p\rho \sum_{i=1}^{d} (k_i D_i \varphi_t, \varphi_t^{2p-1})\right| = |\rho| \left|\sum_{i=1}^{d} (k_i \, D_i \varphi_t^p, \varphi_t^p)\right|$$
$$\leqslant c_2 |\rho| \sum_{i=1}^{d} \|D_i \varphi_t^p\|_2 \, \|\varphi_t^p\|_2 \leqslant \varepsilon \sum_{i=1}^{d} \|D_i \varphi_t^p\|_2^2 + (4\varepsilon)^{-1} c_2^2 d \rho^2 \|\varphi_t^p\|_2^2$$

for all $\varepsilon > 0$. The third term can be estimated by

$$|\tau_3| = \left|p\rho \sum_{i=1}^{d} (k_i' \varphi_t, D_i \varphi_t^{2p-1})\right| = \left|p(2p-1)\rho \sum_{i=1}^{d} (k_i' \varphi_t, \varphi_t^{2p-2} D_i \varphi_t)\right|$$
$$= \left|(2p-1)\rho \sum_{i=1}^{d} (k_i' \varphi_t^p, D_i \varphi_t^p)\right| \leqslant c_3 p |\rho| \sum_{i=1}^{d} \|D_i \varphi_t^p\|_2 \, \|\varphi_t^p\|_2$$
$$\leqslant \varepsilon \sum_{i=1}^{d} \|D_i \varphi_t^p\|_2^2 + (4\varepsilon)^{-1} c_3^2 d p^2 \rho^2 \|\varphi_t^p\|_2^2.$$

The fourth term is trivial:

$$|p((k_0 + k_0' \rho + k_0'' \rho^2)\varphi_t, \varphi_t^{2p-1})| \leqslant c_4 p (1 + \rho^2) \|\varphi_t^p\|_2^2.$$

The constants c_2, c_3 and c_4 are independent of ρ, p, $\psi \in W$, φ and t. Choosing ε appropriate one obtains that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|S_t^\rho \varphi\|_{2p}^{2p} \leqslant -\mu \sum_{i=1}^{d} \|D_i \varphi_t^p\|_2^2 + c' p^2 (1 + \rho^2) \|\varphi_t^p\|_2^2$$
$$\leqslant -\mu \|\varphi_t^p\|_V^2 + (c' + \mu) p^2 (1 + \rho^2) \|\varphi_t^p\|_2^2$$

for some constant $c' > 0$, independent of ρ, p, $\psi \in W$, φ and t.

Recall that one has the estimate $\|S_t^\rho\|_{2\to 2} \leqslant \mathrm{e}^{\omega'(1+\rho^2)t}$ for some $\omega' > 0$, uniformly for all $t > 0$, $\rho \in \mathbf{R}$ and $\psi \in W$. Now one can apply Proposition 4.6 and deduce that

$$\|S_t^\rho\|_{2\to\infty} \leqslant c t^{-d/4} \mathrm{e}^{\omega'(1+\rho^2)t} \mathrm{e}^{2^{-1}(c'+\mu)(1+\rho^2)t} = c t^{-d/4} \mathrm{e}^{\omega(1+\rho^2)t} \tag{4.13}$$

for a constant $c > 0$, independent of ρ, ψ and t and $\omega = \omega' + (c' + \mu)/2$. Since the adjoint of S^ρ is of the same form we obtain by duality

$$\|S_t^\rho\|_{1\to 2} \leqslant c t^{-d/4} \mathrm{e}^{\omega(1+\rho^2)t}$$

possibly by enlarging c and ω. Hence

$$\|S_t^\rho\|_{1\to\infty} \leqslant 2^{d/2} c^2 t^{-d/2} \mathrm{e}^{\omega(1+\rho^2)t}$$

for all $t > 0$ and $\rho \in \mathbf{R}$. Now the theorem follows from Proposition 3.3. ∎

REMARK 4.7. (i) One would expect to obtain the results of Theorem 4.4 also for coefficients $b_i, c_i \in L_\infty$. The main point in the above argument is to prove that S operates consistently on L_1 and L_∞. This could be proved if the D_i are small perturbations of A. However, this is not true in general. In fact, even the domain of the Dirichlet Laplacian on $L_p(\Omega)$ is not contained in $W^{1,p}(\Omega)$ for p sufficiently large, if Ω is not regular, in general (see [24]). This also shows that in general there are no Gaussian type bounds for the derivatives of the kernel if the domain is not regular (even if the coefficients are constant).

(ii) In general the theorem is false if all coefficients are complex. A counter-example on a subset of $\mathbb{R}^d$ has been presented by Maz'ya–Nazarov–Plamenevskii ([28]) and on $\mathbb{R}^d$ by Auscher–Tchamitchian ([11]) in case $d \geqslant 5$. Semigroups generated by complex operators on $\mathbb{R}^1$ and $\mathbb{R}^2$ have Gaussian kernel bounds by Auscher–McIntosh–Tchamitchian ([10]).

Finally we consider the realization of $\mathcal{A}$ (see (4.1)) with *Robin boundary conditions.* For this we assume that Ω is a bounded open set in $\mathbb{R}^d$ with Lipschitz boundary $\Gamma = \partial\Omega$ and we let $\beta \in L_\infty(\Gamma)$ be a positive function. We still assume the conditions (4.3)–(4.7) on the form domain V and the condition (4.8) on the coefficients. By a we continue to denote the form (4.9) defined on V. Let $b : V \times V \to \mathbb{R}$ be defined by

$$b(u,v) = \int_\Gamma \beta(x)\,(Bu)(x)\,(Bv)(x)\,\mathrm{d}\gamma(x),$$

where $B : H^1(\Omega) \to L_2(\Gamma)$ denotes the trace operator (see Section 2). Then b is a continuous bilinear form on V. Set

$$q = a + b.$$

Then q is a continuous bilinear form on V which is coercive. Let A be the operator associated with the form q. We call A the realization of $\mathcal{A}$ with *Robin boundary conditions.* Note that Robin boundary conditions coincide with Dirichlet boundary conditions if $V = H_0^1(\Omega)$ and with Neumann boundary conditions if $V = H^1(\Omega)$ and $\beta = 0$.

EXAMPLE 4.8. Let $a_{ij} = \delta_{ij}$, $b_i = c_i = 0$, $c_0 = 0$ and $V = H^1(\Omega)$. Assume that $u \in D(A) \cap C^2(\overline{\Omega})$. Then

$$\frac{\partial u}{\partial n} = -\beta u \text{ on } \Gamma. \tag{4.14}$$

Conversely, if $u \in C^2(\overline{\Omega})$ is such that (4.14) holds then $u \in D(A)$. This follows by applying Green's formula. We call A the *Laplacian with Robin boundary conditions.*

THEOREM 4.9. *Let A be the realization of $\mathcal{A}$ with Robin boundary conditions. Then A generates a semigroup $S = (\mathrm{e}^{-tA})_{t>0}$ on $L_2(\Omega)$ which interpolates on $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$. The semigroup S is positive and is given by a kernel K. Moreover, there exist $b, c > 0$ and $\omega \in \mathbf{R}$ such that*

$$0 \leqslant K_t(x;y) \leqslant ct^{-d/2}\mathrm{e}^{-|x-y|^2t^{-1}}\mathrm{e}^{\omega t} \quad (x,y)\text{-}a.e.$$

uniformly for all $t > 0$.

Proof. First we show that S is positive. Let $u \in V$. By Theorem 2.4 in [32] we have to show that $q(u^+, u^-) \leqslant 0$. Since $a(u^+, u^-) = 0$ (see the proof of Theorem 4.4) and $Bu^+ = (Bu)^+$ and $Bu^- = (Bu)^-$ (by (2.7)), we have

$$b(u^+, u^-) = \int_\Gamma \beta(x)\,(Bu)^+(x)\,(Bu)^-(x)\,\mathrm{d}\gamma(x) = 0.$$

Thus $q(u^+, u^-) \leqslant 0$.

Secondly, it follows from Proposition 2.9 that A generates a semigroup on $L_2(\Omega)$.

Thirdly, we show that S interpolates on $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$. By the properties (2.7) of the trace operator we have $B((|u|-1)^+ \operatorname{sgn} u) = B(u-1)^+ - B(-u-1)^+ = (Bu-1)^+ - (-(Bu)-1)^+$ for all $u \in H^1(\Omega)$. Therefore,

$$b(u, (|u|-1)^+ \operatorname{sgn} u) = \int \beta\,(Bu)\Big((Bu-1)^+ - (-(Bu)-1)^+\Big)\,\mathrm{d}\gamma \geqslant 0.$$

Now one argues as in the proof of Theorem 4.4 and deduces that S generates a quasi contraction semigroup on L_∞ and by duality it interpolates.

Finally, let $S_t^\rho = U_\rho S_t U_\rho^{-1}$ where $\rho \in \mathbf{R}$ and $\psi \in W$. Then the associated form is given by

$$q^\rho(u,v) = q(U_\rho^{-1}u, U_\rho v) = a^\rho(u,v) + b(u,v)$$

since $b(U_\rho^{-1}u, U_\rho v) = b(u,v)$. Then the proof of Theorem 4.4 carries over to the present case. ∎

REMARK 4.10. (i) An alternative proof of Theorem 4.9 using the results of Theorem 4.4 can be given by domination. Denote by $A^{(a)}$ the operator associated by the form a and $S^{(a)} = (e^{-tA^{(a)}})_{t>0}$ the semigroup generated by $A^{(a)}$. Then S and $S^{(a)}$ are positive semigroups and $q(u,v) \geqslant a(u,v)$ for all $u, v \in V_+$. So it follows from Proposition 3.2 and Theorem 3.7 in [33] that S is dominated by $S^{(a)}$, i.e., $|S_t\varphi| \leqslant S_t^{(a)}|\varphi|$ for all $\varphi \in L_2(\Omega)$. Then $K_t \leqslant K_t^{(a)}$ and Gaussian estimates follow.

(ii) Similarly, one could prove Theorem 4.4 first for Neumann boundary conditions (i.e. $V = H^1(\Omega)$) and then deduce the Gaussian estimates for the general V by domination. However, this requires b_i, c_i to be elements of $H_0^1(\Omega)$ which is stronger than our assumption (4.8).

5. APPLICATIONS

In this section we give two kinds of applications of the previous results. They concern the holomorphy of the semigroup in L_p and the bounded H_∞-functional calculus.

If T is a holomorphic semigroup on $L_2(\Omega)$ which interpolates on $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$, then it follows from Stein's interpolation theorem that T is also holomorphic on L_p, $1 < p < \infty$, but it may not be holomorphic on L_1. For elliptic operators with boundary conditions, holomorphy in L_1 has first been proved by Amann ([3]) for regular bounded domains and later for Dirichlet boundary conditions and no regularity assumptions on the domain in [6] and [5]. More recently Ouhabaz ([31] and [34]) used Gaussian estimates and a Phragmen–Lindelöf argument (cf. Theorem 3.4.8 in [16]) to show holomorphy for symmetric operators (see also Lemma 2 in [17]). Here we prove holomorphy on $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$ on a sector where $\|S_z\|_{2\to2} \leqslant e^{\omega|z|}$ by a direct short proof avoiding the Phragmen–Lindelöf theorem (see Theorem 5.3). In order to obtain a possibly larger sector, however, we adapt the Phragmen–Lindelöf argument to the non-symmetric case (see Theorem 5.4).

Adopt the notation and assumptions of Theorems 3.1, 4.4 or 4.9. In case of Theorems 4.4 and 4.9 we complexify the form domain V and the form a. Set

$$\theta_a = \frac{\pi}{2} - \inf\Big\{\theta > 0 : \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\overline{\xi_j} \in \Sigma(\theta) \text{ for all } \xi \in \mathbb{C}^d, \text{ for a.e. } x \in \Omega\Big\}.$$

Note that $\theta_a = \pi/2$ if the a_{ij} are symmetric, i.e., $a_{ij}(x) = a_{ji}(x)$ for a.e. $x \in \Omega$ and all $i, j \in \{1, \ldots, d\}$.

It is a standard exercise to show that the semigroup $S = (e^{-tA})_{t>0}$ generated by the operator A associated with the form is a holomorphic semigroup on L_2, with a holomorphy sector which contains at least $\Sigma(\theta_a)$. In fact one has the following.

LEMMA 5.1. *Adopt the notation and assumptions of Theorems 3.1, 4.4 or 4.9. Then for all $\rho \in \mathbf{R}$ the operator A^ρ generates a holomorphic semigroup S^ρ on $L_2(\Omega)$, holomorphic in the sector $\Sigma(\theta_a)$. Moreover, for all $\theta \in \langle 0, \theta_a \rangle$ there exists an $\omega \in \mathbf{R}$, depending only on θ, μ, $\|a_{ij}\|_\infty$, $\|b_i\|_\infty$, $\|c_i\|_\infty$ and $\|c_0\|_\infty$, such that*

$$\|S_z^\rho\|_{2\to 2} \leqslant e^{\omega(1+\rho^2)|z|}$$

for all $z \in \Sigma(\theta)$, $\rho \in \mathbf{R}$ and $\psi \in W$.

Proof. Let $\theta \in \langle 0, \theta_a \rangle$. There exists $\nu > 0$ such that $\mathrm{Re} \sum_{i,j=1}^{d} e^{i\alpha} a_{ij}(x)\, \xi_i \overline{\xi_j} \geqslant \nu |\xi|^2$ uniformly for all $\alpha \in [-\theta, \theta]$, $\xi \in \mathbf{C}^d$ and a.e. $x \in \Omega$. Then one can argue as in the proof of Lemma 3.7 and deduce that

$$\begin{aligned} \mathrm{Re}\, e^{i\alpha} \Big(\sum_{i,j=1}^{d} a_{ij}\, D_i u\, \overline{D_j u} + \sum_{i=1}^{d} b_i^\rho\, D_i u\, \overline{u} + \sum_{i=1}^{d} c_i^\rho\, u\, \overline{D_i u} + c_0^\rho |u|^2 \Big) + \omega(1+\rho^2)|u|^2 \\ \geqslant 2^{-1} \nu \sum_{i=1}^{d} |D_i u|^2 \quad \text{a.e.} \end{aligned}$$

uniformly for all $\alpha \in [-\theta, \theta]$, $u \in V$, $\rho \in \mathbf{R}$ and $\psi \in W$ if one chooses $\omega = 4d^3 M_0^2 \nu^{-1} + 2d^2 M_0$ and where

$$M_0 = 1 + \max\{\|a_{ij}\|_\infty, \|b_i\|_\infty, \|c_i\|_\infty, \|c_0\|_\infty\},$$

as before. Again integrating this inequality gives

$$\mathrm{Re}(e^{i\alpha} a^\rho(u,u)) + \omega(1+\rho^2)\|u\|_2^2 \geqslant 2^{-1} \mu \sum_{i=1}^{d} \|D_i u\|_2^2.$$

Hence S^ρ is holomorphic on $\Sigma(\theta)$ and

$$\|S_z^\rho\|_{2\to 2} \leqslant e^{\omega(1+\rho^2)|z|}$$

uniformly for all $z \in \Sigma(\theta)$, $\rho \in \mathbf{R}$ and $\psi \in W$. ∎

We next show the remarkable fact that S is even holomorphic on any L_p, $1 \leqslant p \leqslant \infty$, with a holomorphy sector which contains at least $\Sigma(\theta_a)$.

REMARK 5.2. Here a holomorphic semigroup S on L_∞ of angle $\theta \in \langle 0, \pi/2]$ is by definition a holomorphic mapping $S : \Sigma(\theta) \to \mathcal{L}(L_\infty)$ such that $S_{z+z'} = S_z S_{z'}$ for all $z, z' \in \Sigma(\theta)$ and

$$\lim_{\substack{z \to 0 \\ z \in \Sigma(\theta - \varepsilon)}} (S_z \varphi, \psi) = (\varphi, \psi)$$

for all $\varphi \in L_\infty$, $\psi \in L_1$ and $\varepsilon \in \langle 0, \theta \rangle$.

THEOREM 5.3. *Adopt the notation and assumptions of Theorems* 3.1, 4.4 *or* 4.9. *Then the semigroup S generated by the operator A is holomorphic on any L_p, $1 \leqslant p \leqslant \infty$, with a holomorphy sector which contains at least $\Sigma(\theta_a)$. Moreover, S_z has a kernel $K_z \in L_\infty(\Omega \times \Omega)$ for all $z \in \Sigma(\theta_a)$, and for all $\theta \in \langle 0, \theta_a \rangle$ there exist $b, c > 0$ and $\omega > 0$ such that*

$$|K_z(x;y)| \leqslant c(\operatorname{Re} z)^{-d/2} e^{-b|x-y|^2 |z|^{-1}} e^{\omega |z|} \quad (x,y)\text{-}a.e.$$

uniformly for all $z \in \Sigma(\theta)$.

Proof. Let $\theta \in \langle 0, \theta_a \rangle$. Choose $\theta_1 \in \langle \theta, \theta_a \rangle$. There exists a $\delta > 0$ such that $\delta t + is \in \Sigma(\theta_1)$ for all $t + is \in \Sigma(\theta)$. By Lemma 5.1 there exists $\omega_1 > 0$ such that

$$\|S_z^\rho\|_{2 \to 2} \leqslant e^{\omega_1 (1+\rho^2)|z|}$$

uniformly for all $\rho \in \mathbf{R}$, $\psi \in W$ and $z \in \Sigma(\theta_1)$. By (3.7), (4.13) and duality there exist c, $\omega_2 > 0$ such that

$$\|S_t^\rho\|_{1 \to 2} \leqslant c t^{-d/4} e^{\omega_2 (1+\rho^2) t}, \quad \|S_t^\rho\|_{2 \to \infty} \leqslant c t^{-d/4} e^{\omega_2 (1+\rho^2) t}$$

uniformly for all $\rho \in \mathbf{R}$, $\psi \in W$ and $t > 0$. Now let $z = t + is \in \Sigma(\theta)$. Then

$$\begin{aligned} \|S_z^\rho\|_{1 \to \infty} &\leqslant \|S^\rho_{(1-\delta)t/2}\|_{1 \to 2} \|S^\rho_{\delta t + is}\|_{2 \to 2} \|S^\rho_{(1-\delta)t/2}\|_{2 \to \infty} \\ &\leqslant \left(c((1-\delta)t/2)^{-d/4} e^{\omega_2 (1+\rho^2)(1-\delta)t/2} \right)^2 e^{\omega_1 (1+\rho^2)|\delta t + is|} \\ &\leqslant c' t^{-d/2} e^{\omega' (1+\rho^2)|z|} \end{aligned}$$

for some $c', \omega' > 0$, independent of z and uniformly for all $\rho \in \mathbf{R}$ and $\psi \in W$. Now the complex Gaussian bounds follow as in Proposition 3.3.

Moreover, by Proposition 3.3 there also exists a $c_1 > 0$ such that $\|S_{te^{i\alpha}}\|_{p \to p} \leqslant c_1 e^{\omega' t}$, uniformly for all $t > 0$ and $\alpha \in [-\theta, \theta]$. The holomorphy now follows from Kato ([27], Theorem IX.1.23). ∎

The above short proof for the complex Gaussian bounds works well for elliptic differential operators. More generally, any holomorphic semigroup on $L_2(\Omega)$ with real time Gaussian bounds is holomorphic on $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$. This is proved in the next theorem. It was known before for symmetric semigroups (see Ouhabaz, [31] and [34]).

THEOREM 5.4. *Let S be a holomorphic semigroup on $L_2(\Omega)$, where Ω is an open subset of $\mathbf{R}^d$. Suppose S is holomorphic in the sector $\Sigma(\theta_0)$, where $\theta_0 \leqslant \pi/2$ and suppose that S_t $(t > 0)$ has a kernel K_t which satisfies Gaussian bounds*

$$|K_t(x;y)| \leqslant ct^{-d/2}e^{-b|x-y|^2t^{-1}}e^{\omega t} \quad (x,y)\text{-a.e.}$$

for some $b, c > 0$ and $\omega \in \mathbf{R}$, uniformly for all $t > 0$. Then S interpolates on L_p, $1 \leqslant p \leqslant \infty$ and S is a holomorphic semigroup on L_p, $1 \leqslant p \leqslant \infty$, with holomorphy sector $\Sigma(\theta_0)$. Moreover, for all $z \in \Sigma(\theta_0)$ the operator S_z has a kernel $K_z \in L_\infty(\Omega \times \Omega)$ and for all $\theta \in \langle 0, \theta_0 \rangle$ there are $b, c > 0$ and $\omega \in \mathbf{R}$ such that

$$|K_z(x;y)| \leqslant c|z|^{-d/2}e^{-b|x-y|^2|z|^{-1}}e^{\omega|z|} \quad (x,y)\text{-a.e.} \tag{5.1}$$

uniformly for all $z \in \Sigma(\theta)$.

Proof. It follows from Proposition 3.3 that the Gaussian bounds imply that S interpolates on L_p, $1 \leqslant p \leqslant \infty$. Moreover, one has bounds $\|S_t\|_{1\to 2} \leqslant c_1 t^{-d/4}e^{\omega_1 t}$ and $\|S_t\|_{2\to\infty} \leqslant c_2 t^{-d/4}e^{\omega_2 t}$, together with the bounds $\|S_z\|_{2\to 2} \leqslant M_\theta e^{\omega_\theta |z|}$ for all $z \in \Sigma(\theta)$, if $\theta \in \langle 0, \theta_0\rangle$. Then one deduces as in the proof of Theorem 5.3 that $\|S_z\|_{1\to\infty} \leqslant c_3(\operatorname{Re} z)^{-d/2}e^{\omega_3|z|}$ for all $z \in \Sigma(\theta)$. Next, one derives from Theorem 3.1 in [7] that there exists a measurable function $K : \Sigma(\theta) \times \Omega \times \Omega \to \mathbf{C}$ such that $z \mapsto K(z,x,y)$ is analytic from $\Sigma(\theta) \to \mathbf{C}$ for all $(x,y) \in \Omega \times \Omega$ and K_z is the kernel of S_z, where $K_z(x;y) = K(z,x,y)$. By replacing S_z by $e^{-\omega_4 z}S_z$ we may assume that ω_θ, $\omega_3 < 0$. Now one can argue as in Davies ([16], Theorem 3.4.8) to deduce that K_z has the complex Gaussian bounds (5.1) by an application of the Phragmen–Lindelöf theorem. Finally it can be proved as in the proof of Theorem 5.3 that S is a holomorphic semigroup on L_p, holomorphic on a sector which contains $\Sigma(\theta_0)$. ∎

REMARK 5.5. By a similar argument one proves that if S is holomorphic on L_p in a sector $\Sigma(\theta_p)$ then the semigroup on L_2 is holomorphic on a sector which contains $\Sigma(\theta_p)$. Therefore, the maximal holomorphy sector is independent of p, $1 \leqslant p \leqslant \infty$.

Now consider again the semigroup S generated by an elliptic operator under the assumptions of Theorems 3.1, 4.4 or 4.9. We have proved that S is a holomorphic semigroup and has complex Gaussian kernel estimates

$$|K_z(x;y)| \leqslant c|z|^{-d/2}e^{-b|x-y|^2|z|^{-1}}e^{\omega|z|} \quad (x,y)\text{-a.e.} \tag{5.2}$$

uniformly on each closed sector

$$\widetilde{\Sigma}(\theta) = \{z \in \mathbf{C} : z \neq 0,\ |\arg z| \leqslant \theta\}$$

for all $\theta \in [0, \theta_a\rangle$. If the bounds (5.2) are valid, then

$$\|S_z\|_{2\to 2} \leqslant M e^{\omega|z|} \tag{5.3}$$

where M depends on b and c, but with the same ω as in (5.2). For the applications to H_∞-functional calculus given below, it is important to have a good control over ω in (5.2). In general, if (5.3) is valid for some ω then there are no kernel bounds (5.2) with the same ω. An example is minus the Laplace operator $-\Delta$ on a bounded regular open set Ω with Neumann boundary conditions and $\theta = 0$. Then the constant function 1 is in the domain of $-\Delta$ and $-\Delta 1 = 0$. Therefore, $S_t 1 = 1$ on $L_2(\Omega)$. Gaussian kernel bounds with $\omega \leqslant 0$, however, imply that $\lim_{t\to\infty} S_t 1 = 0$, which is impossible.

We have shown in Lemma 5.1 that there are always bounds (5.3) with $M = 1$. We next establish that there are complex kernel bounds with a slightly larger ω than the ω in (5.3) in case $M = 1$.

THEOREM 5.6. *Adopt the notation and assumptions of Theorems* 3.1, 4.4 *or* 4.9. *Let $\theta \in [0, \theta_a\rangle$ and let $\omega_0 \in \mathbf{R}$ be such that*

$$\|S_z\|_{2\to 2} \leqslant e^{\omega_0|z|}$$

for all $z \in \widetilde{\Sigma}(\theta) = \{z \in \mathbf{C} : z \neq 0,\ |\arg z| \leqslant \theta\}$. Then for all $\omega > \omega_0$ there exist $b, c > 0$ such that

$$|K_z(x;y)| \leqslant c|z|^{-d/2}e^{-b|x-y|^2|z|^{-1}}e^{\omega|z|} \quad (x,y)\text{-a.e.}$$

uniformly for all $z \in \widetilde{\Sigma}(\theta)$.

Proof. We have to give a better estimate for Lemma 5.1. There exists $\nu > 0$ such that $\operatorname{Re} \sum_{i,j=1}^{d} e^{i\alpha} a_{ij}(x)\,\xi_i\,\overline{\xi_j} \geqslant \nu|\xi|^2$ uniformly for all $\alpha \in [-\theta, \theta]$, $\xi \in \mathbf{C}^d$ and a.e. for $x \in \Omega$. It follows from the Lumer–Phillips theorem that

$$\operatorname{Re} e^{i\alpha} a(\varphi, \varphi) + \omega_0(\varphi, \varphi) \geqslant 0$$

for all $\varphi \in V$. Let $\omega > \omega_0$ and $\delta \in \langle 0, 1]$. Note that

$$\operatorname{Re} e^{i\alpha} \int_\Gamma \beta(x)\,(B\varphi)(x)\,\overline{(B\varphi)(x)}\,d\gamma(x) \geqslant 0$$

in case of Robin boundary conditions, since $\beta \geqslant 0$. Then

$$\begin{aligned}\operatorname{Re} e^{i\alpha} a^\rho(\varphi,\varphi) + \omega(\varphi,\varphi) &= (1-\delta)\Big(\operatorname{Re} e^{i\alpha} a(\varphi,\varphi) + \omega_0(\varphi,\varphi)\Big) + \delta \operatorname{Re} e^{i\alpha} a(\varphi,\varphi) \\ &\quad + \operatorname{Re} e^{i\alpha} b_\rho(\varphi,\varphi) + (\omega - (1-\delta)\omega_0)\|\varphi\|_2^2 \\ &\geqslant \delta \operatorname{Re} e^{i\alpha} a(\varphi,\varphi) + \operatorname{Re} e^{i\alpha} b_\rho(\varphi,\varphi) + (\omega - (1-\delta)\omega_0)\|\varphi\|_2^2,\end{aligned}$$

where

$$\begin{aligned}b_\rho(\varphi,\varphi) = -\rho \sum_{i,j=1}^d \int e^{i\alpha} a_{ij}\,(D_i\varphi)\,\psi_j\,\overline{\varphi} + \rho \sum_{i,j=1}^d \int e^{i\alpha} a_{ij}\,\psi_i\,\varphi\,\overline{D_j\varphi} \\ - \rho^2 \sum_{i,j=1}^d \int e^{i\alpha} a_{ij}\,\psi_i\,\varphi\,\psi_j\,\overline{\varphi} \\ + \rho \sum_{i=1}^d \int e^{i\alpha} b_i\,\psi_i\,\varphi\,\overline{\varphi} - \rho \sum_{i=1}^d \int e^{i\alpha} c_i\,\varphi\,\psi_i\,\overline{\varphi}.\end{aligned}$$

Now

$$\begin{aligned}\delta \operatorname{Re} e^{i\alpha} a(\varphi,\varphi) &\geqslant \delta\nu \sum_{i=1}^d \|D_i\varphi\|_2^2 - \delta\Big|\sum_{i=1}^d \int b_i\,D_i\varphi\,\overline{\varphi}\Big| \\ &\quad - \delta\Big|\sum_{i=1}^d \int c_i\,\varphi\,\overline{D_i\varphi}\Big| - \delta \int |c_0|\,|\varphi|^2 \\ &\geqslant \delta\nu \sum_{i=1}^d \|D_i\varphi\|_2^2 - 2\delta\eta \sum_{i=1}^d \|D_i\varphi\|_2^2 - \delta(2\eta)^{-1} d M_0^2 \|\varphi\|_2^2 - \delta M_0 \|\varphi\|_2 \\ &\geqslant 2^{-1}\delta\nu \sum_{i=1}^d \|D_i\varphi\|_2^2 - c\delta\|\varphi\|_2^2\end{aligned}$$

for some $c > 0$, independent of δ and an appropriate choice of η. Here M_0 is as in the proof of Lemma 5.1. As in the proof of Lemma 3.7 one proves that there exists a $c' > 0$ such that

$$\begin{aligned}|b_\rho(\varphi,\varphi)| &\leqslant \varepsilon \sum_{i=1}^d \|D_i\varphi\|_2^2 + c'((1+\varepsilon^{-1})\rho^2 + |\rho|)\|\varphi\|_2^2 \\ &\leqslant \varepsilon \sum_{i=1}^d \|D_i\varphi\|_2^2 + c'((1+\varepsilon^{-1})\rho^2 + \delta + (4\delta)^{-1}\rho^2)\|\varphi\|_2^2\end{aligned}$$

for all $\varepsilon > 0$. Combining these estimates one obtains

$$\operatorname{Re} e^{i\alpha} a^\rho(\varphi, \varphi) + \omega(\varphi, \varphi) \geqslant (2^{-1}\delta\nu - \varepsilon) \sum_{i=1}^{d} \|D_i\varphi\|_2^2 + (\omega - (1-\delta)\omega_0 - c\delta - c'\delta)\|\varphi\|_2^2 \\ - (c'(1+\varepsilon^{-1}) + (4\delta)^{-1}c')\rho^2\|\varphi\|_2^2.$$

Since $\lim_{\delta\to 0} \omega - (1-\delta)\omega_0 - c\delta - c'\delta = \omega - \omega_0 > 0$ there exists $\delta > 0$ such that $\omega - (1-\delta)\omega_0 - c\delta - c'\delta > 0$. Next take $\varepsilon = 2^{-1}\delta\nu$. Then

$$\operatorname{Re} e^{i\alpha} a^\rho(\varphi, \varphi) + \omega(\varphi, \varphi) \geqslant -\omega_1\rho^2\|\varphi\|_2^2$$

for some $\omega_1 > 0$, uniformly for all $\alpha \in [-\theta, \theta]$ and $\rho \in \mathbb{R}$. Therefore,

$$\|S_z^\rho\|_{2\to 2} \leqslant e^{\omega|z|} e^{\omega_1\rho^2|z|}$$

uniformly for all $z \in \widetilde{\Sigma}(\theta)$ and $\rho \in \mathbb{R}$.

By Theorem 5.3 there exist $b, c > 0$ and $\omega_2 \in \mathbb{R}$ such

$$|K_z(x;y)| \leqslant c|z|^{-d/2} e^{-b|x-y|^2|z|^{-1}} e^{\omega_2|z|} \quad (x,y)\text{-a.e.}$$

uniformly for all $z \in \widetilde{\Sigma}(\theta)$. Let $\alpha > 0$ be as in Lemma 3.2. Then

$$\begin{aligned}
\|S_z^\rho\|_{2\to\infty}^2 &= \sup_{\|\varphi\|_2 \leqslant 1} \|S_z^\rho\varphi\|_\infty^2 = \sup_{\|\varphi\|_2\leqslant 1} \operatorname*{ess\,sup}_{x\in\Omega} \Big| \int_\Omega K_z^\rho(x;y)\,\varphi(y)\,dy \Big|^2 \\
&= \operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega |K_z^\rho(x;y)|^2\,dy \leqslant \operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega |K_z(x;y) e^{|\rho|\,|\psi(x)-\psi(y)|}|^2\,dy \\
&\leqslant \operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega |K_z(x;y) e^{\alpha^{-1}|\rho|\,|x-y|}|^2\,dy \\
&\leqslant \sup_{x\in\Omega} \int_\Omega \Big(c|z|^{-d/2} e^{-b|x-y|^2|z|^{-1} + \alpha^{-1}|\rho|\,|x-y|} e^{\omega_2|z|} \Big)^2\,dy \\
&\leqslant \int_{\mathbb{R}^d} \Big(c|z|^{-d/2} e^{-b|y|^2|z|^{-1} + \alpha^{-1}|\rho|\,|y|} e^{\omega_2|z|} \Big)^2\,dy \\
&= \Big(c'|z|^{-d/4} e^{\omega_3\rho^2|z|} e^{\omega_2|z|} \Big)^2
\end{aligned}$$

for some $c', \omega_3 > 0$, uniformly for all $z \in \widetilde{\Sigma}(\theta)$ and $\rho \in \mathbb{R}$. So

$$\|S_z^\rho\|_{2\to\infty} \leqslant c'|z|^{-d/4} e^{\omega_3\rho^2|z|} e^{\omega_2|z|}$$

and by duality

$$\|S_z^\rho\|_{1\to 2} \leqslant c'|z|^{-d/4} e^{\omega_3 \rho^2 |z|} e^{\omega_2 |z|}$$

possibly by enlarging c' and ω_2 and ω_3. Then for all $\varepsilon > 0$ one establishes

$$\begin{aligned} \|S_z^\rho\|_{1\to\infty} &\leqslant \|S_{\varepsilon z}^\rho\|_{1\to 2}\|S_{(1-2\varepsilon)z}^\rho\|_{2\to 2}\|S_{\varepsilon z}^\rho\|_{2\to\infty} \\ &\leqslant \left(c'(\varepsilon|z|)^{-d/4} e^{\varepsilon\omega_3\rho^2|z|} e^{\varepsilon\omega_2|z|}\right)^2 e^{(1-2\varepsilon)\omega|z|} e^{(1-2\varepsilon)\omega_1\rho^2|z|} \\ &= (c')^2\varepsilon^{-d/2}|z|^{-d/2} e^{(\omega+\varepsilon(2\omega_2-2\omega))|z|} e^{(2\varepsilon\omega_3+(1-2\varepsilon)\omega_1)\rho^2|z|} \end{aligned}$$

uniformly for all $\rho \in \mathbf{R}$. Since $\omega > \omega_0$ and $\varepsilon > 0$ are arbitrary, the theorem follows by a minimalization over ρ and $\psi \in W$ as in the proof of Proposition 3.3. ∎

Next we show that the operator $A+\omega I$ has a bounded H_∞-functional calculus in L_p, $1 < p < \infty$. Frequently it is easy to establish a bounded H_∞-functional calculus in L_2; for example, m-accretivity is a sufficient condition. Recently, Duong and Robinson ([20]) proved the remarkable fact that this functional calculus can be carried over to L_p, $1 < p < \infty$, whenever a complex Gaussian estimate is valid. (See also [19] for the case where the coefficients are Hölder continuous.) Their result can be applied directly to $\Omega = \mathbf{R}^d$. But it is valid for arbitrary open $\Omega \subset \mathbf{R}^d$. In case Ω satisfies a very mild regularity condition (namely that $\partial\Omega$ is a null set in $\mathbf{R}^d$) this can be seen by a direct sum argument which we will give in the proof below. In the general case one can modify the proof given by Duong and Robinson. We are grateful to X.-T. Duong and J. Prüss for discussions on this point. Concerning the definition and basic facts on H_∞-functional calculus we refer to [20] and the references given there.

THEOREM 5.7. *Let $\Omega \subset \mathbf{R}^d$ be open. Let $S = (e^{-tA})_{t>0}$ be a holomorphic semigroup on $L_2(\Omega)$ with generator A. Suppose that S is holomorphic in the sector $\Sigma(\theta)$, where $\theta \in \langle 0, \pi/2\rangle$. Assume that:*

(a) *A is accretive in $L_2(\Omega)$,*

(b) *S_z is given by a kernel $K_z \in L_\infty(\Omega \times \Omega)$ satisfying*

$$|K_z(x;y)| \leqslant c|z|^{-d/2} e^{-b|x-y|^2|z|^{-1}} \quad (x,y)\text{-a.e.} \tag{5.4}$$

uniformly for all $z \in \Sigma(\theta)$ and some $b, c > 0$.

Then S interpolates on $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$ and A has a bounded $H_\infty(\Sigma(\nu))$-functional calculus for all $\nu > \pi/2-\theta$ in $L_p(\Omega)$ for all $p \in \langle 1, \infty\rangle$. Moreover, $f(A)$ is of weak type $(1,1)$ for each $f \in H_\infty(\Sigma(\nu))$. Here A denotes the generator of S in $L_p(\Omega)$.

REMARK 5.8. (i) Condition (a) implies that:

(a′) A has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_2(\Omega)$ for some $\nu > \pi/2 - \theta$.

Theorem 5.7 remains valid if one replaces (a) by the more general condition (a′).

(ii) A special case of Theorem 5.7 had been obtained by Hieber ([26]) who applied it to a pure second order symmetric elliptic operator on a bounded domain with Lipschitz boundary.

Proof. It follows from (5.4) and Theorem 5.4 that S interpolates in $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$ and that S is holomorphic on the sector $\Sigma(\theta)$ on L_p. Moreover, S is bounded on $\Sigma(\theta)$ in $\mathcal{L}(L_p)$ by Proposition 3.3. If $\Omega = \mathbb{R}^d$, the assertion follows from Theorem 3.1 in [20].

Now assume $\partial\Omega$ is a null set. Then we can reduce the problem to the case where the domain is $\mathbb{R}^d$ in the following way. Let $\Omega_1 = \mathbb{R}^d \setminus \overline{\Omega}$ and let $A_1 = -\sum_{i=1}^{d} \partial^2/\partial x_i^2$ with Dirichlet boundary conditions on $L_2(\Omega_1)$. Since $\partial\Omega$ is a null set one has $L_2(\mathbb{R}^d) = L_2(\Omega) \oplus L_2(\Omega_1)$, where the decomposition is given by $f = f\mathbb{1}_\Omega + f\mathbb{1}_{\Omega_1}$. Let $\widetilde{A} = A \oplus A_1$. Then $\widetilde{A}$ satisfies the hypotheses of the theorem on $L_2(\mathbb{R}^d)$ and consequently, $\widetilde{A}$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_p(\mathbb{R}^d)$ for $p \in \langle 1, \infty\rangle$ whenever $\nu > \pi/2 - \theta$. Then A has the same property.

Similarly the $(1,1)$-estimate follows from Theorem 3.1 in [20]. ∎

In virtue of Theorem 5.4 one obtains a bounded H_∞-functional calculus for $A + \omega I$ for some ω if one has merely real time Gaussian bounds. More precisely, assume that the hypotheses of Theorem 5.4 are satisfied. Denote the generator of S in $L_p(\Omega)$ by A. Then for all $\nu > \pi/2 - \theta$ there exists an $\omega \in \mathbb{R}$ such that the operator $A + \omega I$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_p(\Omega)$, $1 < p < \infty$. Of course, if $A + \omega_0 I$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus then the same is true for $A + \omega I$ for all $\omega > \omega_0$. For the elliptic operators obtained here, Theorem 5.6 allows us to consider the result for small ω.

THEOREM 5.9. *Adopt the notation and assumptions of Theorems* 3.1, 4.4 *or* 4.9. *Let $\nu > \pi/2 - \theta_a$, $\nu < \pi/2$ and $\omega_0 \in \mathbb{R}$ be such that*

$$\|S_z\|_{2\to 2} \leqslant e^{\omega_0 |z|}$$

for all $z \in \widetilde{\Sigma}(\pi/2 - \nu) = \{z \in \mathbb{C} : z \neq 0,\ |\arg z| \leqslant \pi/2 - \nu\}$. Then for all $\omega > \omega_0$ the operator $A + \omega I$ has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_p(\Omega)$ for each $p \in \langle 1, \infty\rangle$. Moreover, $f(A + \omega I)$ is of weak type $(1,1)$ *for each $f \in H_\infty(\Sigma(\nu))$.*

Proof. This is a direct consequence of Theorems 5.6 and 5.7. ∎

Corollary 5.10. *Adopt the notation and assumptions of Theorems* 3.1, 4.4 *or* 4.9. *Let* $\nu > \pi/2 - \theta_a$, $\nu < \pi/2$ *and* $\omega_0 \in \mathbf{R}$ *be such that*

$$\|S_z\|_{2\to 2} \leqslant e^{\omega_0|z|}$$

for all $z \in \widetilde{\Sigma}(\pi/2 - \nu) = \{z \in \mathbf{C} : z \neq 0, \ |\arg z| \leqslant \pi/2 - \nu\}$. *Then for all* $\omega > \omega_0$ *the operator* $A + \omega I$ *has bounded imaginary powers and there exists a* $c > 0$ *such that*

$$\|(A + \omega I)^{is}\|_{p\to p} \leqslant c e^{\nu|s|}$$

uniformly for all $s \in \mathbf{R}$ *and* $p \in \langle 1, \infty\rangle$.

Proof. Apply Theorem 5.9 to the holomorphic function $z \mapsto z^{is}$. ∎

Note that the value of ν in the previous theorem is less than $\pi/2$. This is important in order to apply the Dore–Venni theorem ([18]) and its extensions (see Theorem II.8.4, p. 218 in [35]).

Example 5.11. Suppose the operator A is pure second order (not necessarily symmetric) with L_∞-coefficients and Dirichlet boundary conditions. Moreover, suppose that Ω is contained in a strip

$$\{x \in \mathbf{R}^d : l < x \cdot \xi < r\}$$

for some $l < r$ and $\xi \in \mathbf{R}^d$, $\xi \neq 0$. Then for all $\theta \in \langle 0, \theta_a\rangle$ there exists $\mu' > 0$ such that

$$\mathrm{Re}\, e^{i\alpha} a(\varphi, \varphi) \geqslant \mu' \sum_{i=1}^{d} \|D_i\varphi\|_2^2$$

for all $\alpha \in [-\theta, \theta]$ and $\varphi \in H_0^1(\Omega)$. Therefore, by the Poincaré inequality, one deduces that

$$\mathrm{Re}\, e^{i\alpha} a(\varphi, \varphi) \geqslant 2(r - l)^{-2}\mu' \|\varphi\|_2^2$$

(see p. 920 in [13]). So

$$\|S_z\|_{2\to 2} \leqslant e^{-(r-l)^{-2}\mu'|z|}$$

for all $z \in \widetilde{\Sigma}(\theta)$. As a result, one obtains from Theorems 5.6, 5.9 and Corollary 5.10 that for all $\theta \in \langle 0, \theta_a\rangle$ there exist $b, c > 0$ and a negative $\omega < 0$ such that

$$|K_z(x;y)| \leqslant c|z|^{-d/2} e^{-b|x-y|^2|z|^{-1}} e^{\omega|z|} \quad (x,y)\text{-a.e.}$$

uniformly for all $z \in \widetilde{\Sigma}(\theta)$ and A has a bounded $H_\infty(\Sigma(\nu))$-functional calculus on $L_p(\Omega)$ for all $p \in \langle 1, \infty\rangle$ and $\nu \in \langle \pi/2 - \theta_a, \pi/2\rangle$. In particular, there exists a $c > 0$, depending on ν, such that $\|A^{is}\|_{p\to p} \leqslant c e^{\nu|s|}$ for all $s \in \mathbf{R}$.

The next remark clarifies the nature of the angle θ_a.

REMARK 5.12. Assume that $b_i = c_i = c_0 = 0$ for all $i \in \{1, \ldots, d\}$. Let A be any of the operators considered in Theorems 3.1, 4.4 or 4.9. Then

$$\|S_z\|_{2\to 2} \leqslant 1 \quad \text{for all } z \in \Sigma(\theta_a)$$

by the proof of Lemma 5.1 for $\rho = 0$. If Ω is bounded with minimally smooth boundary, $V = H^1(\Omega)$ and the coefficients a_{ij} are constant, then $\Sigma(\theta_a)$ is the largest sector on which S_z is a contraction. In fact, by the Lumer–Phillips theorem we have to show that θ_a is the smallest angle in $\langle 0, \pi/2\rangle$ such that the numerical range $\theta(A)$ of A is included in $\{z \in \mathbf{C} \setminus \{0\} : |\arg z| \leqslant \pi/2 - \theta_a\} \cup \{0\}$. We will show the following identity

$$\theta(A) = \overline{\mathbf{R}_+\theta(B)} = \{z \in \mathbf{C} \setminus \{0\} : |\arg z| \leqslant \pi/2 - \theta_a\} \cup \{0\}, \tag{5.5}$$

where $B = (a_{ji})$ and $\theta(B)$ is the numerical range of the matrix B. Obviously the second equality is valid by definition of θ_a, the convexity of the numerical range of B and the fact that B is a real matrix. Let $\lambda \in \theta(B)$ and $r \geqslant 0$. Let $\xi \in \mathbf{C}^d$ be such that $|\xi| = 1$ and $\lambda = (B\xi, \xi)$. Let $u \in C_c^\infty(\mathbf{R}^d)$ and $\alpha \in \langle 0, \infty\rangle$ be such that $u(x) = \alpha e^{r\xi_1 x_1 + \cdots + r\xi_d x_d}$ for all $x \in \Omega$ and $\|u|_\Omega\|_2 = 1$. Then $u|_\Omega \in H^1(\Omega)$ and $D_i u = r\xi_i u$ on Ω for all $i \in \{1, \ldots, d\}$. Therefore,

$$a(u,u) = \int_\Omega (B\nabla u, \nabla u) = \int_\Omega r^2 (B\xi, \xi)\, |u|^2 = \lambda r^2$$

and $\overline{\mathbf{R}_+\theta(B)} \subset \theta(A)$. Conversely, if $u \in H^1(\Omega)$ with $\|u\|_2 = 1$ then

$$a(u,u) = \int_\Omega (B\nabla u, \nabla u) = \int_\Omega (Bv, v)\, |\nabla u|^2 \in \{z \in \mathbf{C}\setminus\{0\} : |\arg z| \leqslant \pi/2 - \theta_a\} \cup \{0\}$$

since $(Bv(x), v(x)) \in \{z \in \mathbf{C} \setminus \{0\} : |\arg z| \leqslant \pi/2 - \theta_a\} \cup \{0\}$ for a.e. $x \in \Omega$, where

$$v(x) = \begin{cases} \frac{(\nabla u)(x)}{|(\nabla u)(x)|} & \text{if } (\nabla u)(x) \neq 0, \\ 0 & \text{if } (\nabla u)(x) \neq 0. \end{cases}$$

Now (5.5) follows.

The equality (5.5) even implies that S cannot be holomorphic and quasi-contractive on L_2 on a sector strictly larger than $\Sigma(\theta_a)$.

We conclude by a consequence concerning the spectrum of the different realizations of $\mathcal{A}$ in $L_p(\Omega)$.

THEOREM 5.13. (p-independence of the spectrum) *Adopt the notation and assumptions of Theorems 3.1, 4.4 or 4.9, so A is the realization of the elliptic operator $\mathcal{A}$ in $L_p(\Omega)$ with boundary conditions. Then the component $\rho_\infty(A)$ of the resolvent set of A which contains a left half-plane is independent of p, $1 \leqslant p \leqslant \infty$. Moreover, $(\lambda I + A)^{-1}$ is a kernel operator for all $\lambda \in \rho_\infty(A)$.*

Proof. This follows immediately from Theorem 4.2 in [4] the remark following Corollary 4.3 in [4] and the Gaussian estimates established here. ∎

APPENDIX

Proof of Proposition 4.6. Let $\varphi \in L_2 \cap L_\infty$. Set $\varphi_t = S_t\varphi$ for all $t > 0$. If $\varphi_{t_0} = 0$ for some $t_0 > 0$, then $\varphi_t = 0$ for all $t > t_0$ and by holomorphy of S it follows that $\varphi_t = 0$ for all $t > 0$ and hence $\varphi = 0$. So we may assume that $\varphi_t \neq 0$ for all $t > 0$. Then it follows from the Nash inequality that

$$\begin{aligned}\frac{d}{dt}\|\varphi_t\|_{2p}^{2p} &\leqslant -\frac{\mu}{c_N}\frac{\|\varphi_t^p\|_2^{2+4/n}}{\|\varphi_t^p\|_1^{4/n}} + c_1p^2\|\varphi_t^p\|_2^2 \\ &= -\frac{\mu}{c_N}\frac{\|\varphi_t\|_{2p}^{2p+4p/n}}{\|\varphi_t\|_p^{4p/n}} + c_1p^2\|\varphi_t\|_{2p}^{2p}.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|\varphi_t\|_{2p} \leqslant -\frac{\mu}{2c_Np}\|\varphi_t\|_{2p}^{1+4p/n}\|\varphi_t\|_p^{-4p/n} + 2^{-1}c_1p\|\varphi_t\|_{2p}$$

and

$$\frac{d}{dt}\Big(\|\varphi_t\|_{2p}e^{-2^{-1}c_1pt}\Big)^{-4p/n} \geqslant 2\mu(c_Nn)^{-1}\Big(\|\varphi_t\|_pe^{-2^{-1}c_1pt}\Big)^{-4p/n}. \tag{5.6}$$

Since $\lim_{p\to\infty} p\big(1-(1-p^{-2})^{p-1}\big) = 1$ there exists a $\sigma > 0$ such that

$$p\big(1-(1-p^{-2})^{p-1}\big) \geqslant \sigma$$

for all $p \geqslant 2$. Next define

$$f_2(t) = Me^{\omega t}\|\varphi\|_2$$

and by induction for all $p \in \{2^r : r \in \mathbb{N}\}$ define

$$f_{2p}(t) = (c_3\,\mu)^{-n/(4p)}e^{2^{-1}c_1t/p}p^{n/(2p)}f_p(t),$$

where $c_3 = 2\sigma(c_N n)^{-1}$.

Note that f_p is an increasing function. We shall prove by induction that

$$\|\varphi_t\|_p \leqslant t^{-2^{-1}n(2^{-1}-p^{-1})} f_p(t) \tag{5.7}$$

for all $p \in \{2^r : r \in \mathbb{N}\}$ and $t > 0$.

Clearly (5.7) is valid if $p = 2$. Let $p \in \{2^r : r \in \mathbb{N}\}$ and suppose that (5.7) is valid for all $t > 0$. Then it follows by integration from (5.6) that

$$\begin{aligned}
&\Big(\|\varphi_t\|_{2p} e^{-2^{-1}c_1 pt}\Big)^{-4p/n} \\
&\geqslant 2\mu(c_N n)^{-1} \int_0^t \Big(s^{-2^{-1}n(2^{-1}-p^{-1})} f_p(s) e^{-2^{-1}c_1 ps}\Big)^{-4p/n} \, ds \\
&\geqslant 2\mu(c_N n)^{-1} f_p(t)^{-4p/n} \int_0^t s^{p-2} e^{2c_1 p^2 s/n} \, ds \\
&\geqslant 2\mu(c_N n)^{-1} f_p(t)^{-4p/n} \int_{(1-p^{-2})t}^t s^{p-2} e^{2c_1 p^2 s/n} \, ds \\
&\geqslant 2\mu(c_N n)^{-1} e^{2c_1 p^2 (1-p^{-2})t/n} f_p(t)^{-4p/n} \int_{(1-p^{-2})t}^t s^{p-2} \, ds \\
&= 2\mu(c_N n)^{-1} e^{2c_1 p^2 (1-p^{-2})t/n} f_p(t)^{-4p/n} (p(p-1))^{-1} t^{p-1} p\Big(1 - (1-p^{-2})^{p-1}\Big) \\
&\geqslant 2\mu\,\sigma(c_N n)^{-1} e^{2c_1 p^2(1-p^{-2})t/n} p^{-2} t^{p-1} f_p(t)^{-4p/n}
\end{aligned}$$

for all $t > 0$. Therefore,

$$\|\varphi_t\|_{2p} e^{-2^{-1}c_1 pt} \leqslant (c_3\,\mu)^{-n/(4p)} e^{-2^{-1}c_1 p(1-p^{-2})t} p^{n/(2p)} t^{-2^{-1}n(2^{-1}-(2p)^{-1})} f_p(t)$$

and

$$\begin{aligned}
\|\varphi_t\|_{2p} &\leqslant (c_3\,\mu)^{-n/(4p)} e^{2^{-1}c_1 t/p} p^{n/(2p)} t^{-2^{-1}n(2^{-1}-(2p)^{-1})} f_p(t) \\
&= t^{-2^{-1}n(2^{-1}-(2p)^{-1})} f_{2p}(t).
\end{aligned}$$

It follows from the definition of f_p that

$$\begin{aligned}
f_{2^r}(t) &= M\Big(\prod_{k=1}^{r-1} (c_3\,\mu)^{-2^{-k-2}n} e^{2^{-k-1}c_1 t} 2^{2^{-k-1}nk}\Big) e^{\omega t} \|\varphi\|_2 \\
&\leqslant c_3^{-n/4} M\Big(\prod_{k=1}^{\infty} 2^{2^{-k-1}nk}\Big) \mu^{-n/4} e^{2^{-1}c_1 t} e^{\omega t} \|\varphi\|_2
\end{aligned}$$

for all $r \in \mathbb{N}$. Hence by (5.7),

$$\|S_t\varphi\|_{2^r} \leqslant c_2 M \mu^{-n/4} t^{-n/4} t^{2^{-r-1}n} e^{2^{-1}c_1 t} e^{\omega t} \|\varphi\|_2,$$

where $c_2 = c_3^{-n/4} \prod_{k=1}^{\infty} 2^{2^{-k-1}nk}$. Thus

$$\|S_t\varphi\|_\infty \leqslant \limsup_{r\to\infty} \|S_t\varphi\|_{2^r} \leqslant c_2 M \mu^{-n/4} t^{-n/4} e^{2^{-1}c_1 t} e^{\omega t} \|\varphi\|_2$$

and Proposition 4.6 has been proved. ∎

Acknowledgements. This paper has been written while the second named author was visiting the Department of Mathematics of the University of Besançon. He wishes to thank the first named author for his generous hospitality and the department for financial support.

Note added in proof. Theorem 5.4 has also been obtained by M. Hieber: Gaussian estimates and holomorphy of semigroups on L^p spaces, *J. London Math. Soc. (2)* **54**(1996), 148–160.

REFERENCES

1. R.A. ADAMS, *Sobolev Spaces*, Pure Appl. Math., vol. 65, Academic Press, New York 1975.
2. H.W. ALT, *Lineare Funktionalanalysis*, Springer-Verlag, Berlin 1985.
3. H. AMANN, Dual semigroups and second order linear elliptic boundary value problems, *Isreal J. Math.* **45**(1983), 225–254.
4. W. ARENDT, Gaussian estimates and interpolation of the spectrum in L^p, *Differential Integral Equations* **7**(1994), 1153–1168.
5. W. ARENDT, C.J.K. BATTY, L'holomorphie du semi-groupe engendré par le laplacian Dirichlet sur $L^1(\Omega)$, *C. R. Acad. Sci. Paris, Sér. I Math.* **315**(1992), 31–35.
6. W. ARENDT, C.J.K. BATTY, Absorption semigroups and Dirichlet boundary conditions, *Math. Ann.* **295**(1993), 427–448.
7. W. ARENDT, A.V. BUKHVALOV, Integral representations of resolvents and semigroups, *Forum Math.* **6**(1994), 111–135.
8. D.G. ARONSON, Bounds for the fundamental solution of a parabolic equation, *Bull. Amer. Math. Soc. (N.S.)* **73**(1967), 890–896.
9. P. AUSCHER, Regularity theorems and heat kernel for elliptic operators, *J. London Math. Soc. (2)* **54**(1996), 284–296.
10. P. AUSCHER, A. MCINTOSH, P. TCHAMITCHIAN, Noyau de la chaleur d'opérateurs elliptiques complexes, *Math. Res. Letters* **1**(1994), 37–45.
11. P. AUSCHER, P. TCHAMITCHIAN, Sur un contre-exemple aux estimations gaussiennes pour les opérateurs elliptiques complexes, 1994.

12. H. BREZIS, *Analyse fonctionnelle, Théorie et applications*, Collect. Maîtrise Math. Pures, Masson, Paris 1983.

13. R. DAUTRAY, J.-L. LIONS, *Analyse mathématique et calcul numérique*, vol. 3, Masson, Paris 1987.

14. R. DAUTRAY, J.-L. LIONS, *Analyse mathématique et calcul numérique*, vol. 8, Masson, Paris 1987.

15. E.B. DAVIES, Explicit constants for Gaussian upper bounds on heat kernels, *Amer. J. Math.* **109**(1987), 319–333.

16. E.B. DAVIES, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989.

17. E.B. DAVIES, E.B., L^p spectral independence and L^1 analyticity, *J. London Math. Soc. (2)*, to appear.

18. G. DORE, A. VENNI, On the closedness of the sum of two closed operators, *Math. Z.* **196**(1987), 189–201.

19. X.T. DUONG, H_∞ functional calculus of elliptic partial differential operators in L^p spaces, Ph.D. Dissertation, Macquarie University, Sydney 1990.

20. X.T. DUONG, D.W. ROBINSON, Semigroup kernels, Poisson bounds and holomorphic functional calculus, *J. Funct. Anal.* **142**(1996), 89–129.

21. A.F.M. TER ELST, D.W. ROBINSON, Subcoercive and subelliptic operators on Lie groups: variable coefficients, *Publ. Res. Inst. Math. Sci.* **29** (1993), 745–801.

22. E.B. FABES, D.W. STROOCK, A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash, *Arch. Rational. Mech. Anal.* **96**(1986), 327–338.

23. D. GILBARG, N.S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Second edition, Grundlehren math. Wiss., vol. 224, Springer-Verlag, Berlin 1983.

24. P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*, Pitman Monographs Surveys Pure Appl. Math., vol. 24, Pitman, Boston 1985.

25. R. HEMPEL, L.A. SECO, B. SIMON, The essential spectrum of Neumann Laplacians on some bounded singular domains, *J. Funct. Anal.* **102**(1991), 448–483.

26. M. HIEBER, Heat kernels and bounded H^∞-calculus on L^p-spaces, in *Partial Differential Equations. Models in Physics and Biology*, vol. 82 of Math. Res., Akademie Verlag, Berlin, 1994, pp. 166–173.

27. T. KATO, *Perturbation Theory for Linear Operators*. Second edition, Grundlehren math. Wiss., vol. 132, Springer-Verlag, Berlin 1984.

28. V.G. MAZ'YA, S.A. NAZAROV, B.A. PLAMENEVSKII, Absence of the De Giorgi-type theorems for strongly elliptic equations with complex coeffients, *J. Soviet. Math.* **28**(1985), 726–739.

29. J. MOSER, A Harnack inequality for parabolic differential equations, *Commun. Pure Appl. Math.* **17**(1964), 101–134.

30. R. NAGEL, *One-parameter semigroups of positive operators*, Lecture Notes in Math., vol. 1184, Springer-Verlag, Berlin 1986.

31. E.-M. OUHABAZ, Propriétés d'ordre et de contractivité des semi-groups avec applications aux opérateurs elliptiques, Ph.D. Dissertation, Université de Franche-Comté, Besançon 1992.

32. E.-M. OUHABAZ, L^∞-contractivity of semigroups generated by sectorial forms, *J. London Math. Soc. (2)* **46**(1992), 529–542.

33. E.-M. OUHABAZ, Gaussian estimates and holomorphy of semigroups, *Proc. Amer. Math. Soc.* **123**(1995), 1465–1474.

34. E.-M. OUHABAZ, Invariance of closed convex sets and domination criteria for semigroups, *Potential Anal.*, to appear.
35. J. PRÜSS, *Evolutionary Integral Equations and Applications*, Monographs Math., vol. 87, Birkhäuser, Basel 1993.
36. J. PRÜSS, H. SOHR, Imaginary powers of elliptic second order differential operators in L^p-spaces, *Hiroshima Math. J.* **23**(1993), 161–192.
37. D.W. ROBINSON, *Elliptic Operators and Lie Groups*, Oxford Math. Monographs, Oxford University Press, Oxford 1991.
38. E.M. STEIN, *Singular integrals and differential properties of functions*, Princeton Math. Ser., vol. 30, Princeton University Press, Princeton, New Jersey 1970.
39. H. TANABE, *Equations of Evolution*, Pitman Monographs Surveys Pure Appl. Math., vol. 6, Pitman, London 1979.
40. N.T. VAROPOULOS, L. SALOFF-COSTE, T. COULHON, *Analysis and Geometry on Groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, Cambridge 1992.
41. J. VOIGT, One-parameter semigroups acting simultaniously on different L_p-spaces, *Bull. Soc. Roy. Sci. Liège* **61**(1992), 465–470.

W. ARENDT
Laboratoire de Mathématiques
Université de Franche-Comté
F-25030 Besançon Cedex
FRANCE

Current address:

Abteilung Mathematik V
Universität Ulm
D-89069 Ulm
GERMANY

A.F.M. ter ELST
Laboratoire de Mathématiques
Université de Franche-Comté
F-25030 Besançon Cedex
FRANCE

Home institution:

Dept. of Math. and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
THE NETHERLANDS

Received April 27, 1996.