Trotter's product formula for projections

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December 12, 1997

1 Introduction

Let A be the generator of a contractive C_0 – semigroup $\left(e^{tA}\right)_{t\geq 0}$ on a Banach space E, and let $B \in \mathcal{L}(E)$ be a dissipative operator. Then $A+B$ generates a C_0 – semigroup which is given by Trotter's formula

$$e^{t(A+B)} = \lim_{n\to\infty} \left(e^{\frac{t}{n}A}e^{\frac{t}{n}B}\right)^n \tag{1}$$

where the limit is taken in the strong operator topology (see e.g. Goldstein [G, Theorem 8.12.] or Chernoff [C] for a comprehensive treatment of product formulas).

Now let $P \in \mathcal{L}(E)$ be a contractive projection. Then we may consider P as a constant degenerate semigroup on E. It is interesting to know under which circumstances the limit in (1) exists if we replace e^{tB} by P. More precisely, we ask for conditions which imply that

$$S(t) := \lim_{n\to\infty} \left(e^{\frac{t}{n}A}P\right)^n \tag{2}$$

converges strongly for all $t \geq 0$ and that S is strongly continuous. In that case $(S(t))_{t\geq 0}$ is a *degenerate semigroup*; i.e.,

$$S : [0,\infty) \to \mathcal{L}(E)$$

is strongly continuous and

$$S(s+t) = S(s)\,S(t)$$

for all $s, t \geq 0$. This implies that $S(0)$ is a projection, its image $E_0 := S(0)\,E$ is invariant under $S(t)$; and the restriction of S to E_0 is a C_0 – semigroup, whereas $S(t)$ equals 0 on $(I - S(0))\,E$. Moreover, $E_0 \subset PE$. Thus, in a certain sense, E_0 is the largest closed subspace of PE on which $\left(e^{tA}\right)_{t\geq 0}$ induces a C_0 – semigroup. Of course, in the trivial case where e^{tA} and P commute, $S(t)$ is just the restriction of e^{tA} to PE.

In the following we describe two classes of examples where the Trotter's product formula holds for projections; we give a counterexample and formulate two conjectures.

2 Positive semigroups

Let (X, Σ, μ) be a σ – finite measure space and let $\left(e^{tA}\right)_{t\geq 0}$ be a positive C_0 – semigroup on $E = L^p(X)$ where $1 \leq p < \infty$. Let $\Omega \subset X$ be measurable. Then $Pf = \mathbf{1}_\Omega f$ defines a projection on E, where $\mathbf{1}_\Omega$ denotes the characteristic function of Ω. In this case, Trotter's formula is valid. In order to formulate this precisely, it is convenient to consider $L^p(\Omega)$ as a subspace of $L^p(X)$ (extending functions by 0). The following result [AB, Theorem 5.3.] holds.

Theorem 1 *Let $f \in E$, $t \geq 0$. Then*

$$S(t) = \lim_{n\to\infty} \left(e^{\frac{t}{n}A}\mathbf{1}_\Omega\right)^n f = \lim_{n\to\infty} \left(\mathbf{1}_\Omega e^{\frac{t}{n}A}\right)^n f \tag{3}$$

exists and $(S(t))_{t\geq 0}$ defines a degenerate semigroup of positive operators. The projection $S(0)$ is of the form $S(0) f = \mathbf{1}_Y f$ where $Y \subset \Omega$ is measurable.

By [AB, Theorem 5.1.], the semigroup $(S(t))_{t\geq 0}$ in the theorem can also be described by the following maximality property: Let $T = (T(t))_{t\geq 0}$ be a positive degenerate semigroup on $L^p(X)$ which leaves $L^p(\Omega)$ invariant and which is dominated by $\left(e^{tA}\right)_{t\geq 0}$; i.e.

$$T(t) f \leq e^{tA} f$$

for all $t \geq 0$ and $0 \leq f \in L^p(\Omega)$. Then

$$T(t) f \leq S(t) f$$

for all $f \geq 0$ and $t \geq 0$. Of course, it can happen that $Y = \emptyset$ (see [AB, Example 5.4.]).

Example 2 (The Dirichlet Laplacian) *Let $X = \mathbb{R}^N$ and let $A = \Delta$ be the Laplacian on $L^2(\mathbb{R}^N)$ so that $\left(e^{t\Delta}\right)_{t\geq 0}$ is the Gaussian semigroup. Let Ω be an open set in $\mathbb{R}^N$. Then $Y = \Omega$. Moreover, we can describe S precisely.*

a) *Assume that Ω is bounded with Lipschitz boundary. Then $S(t)|_{L^2(\Omega)} = e^{t\Delta_\Omega}$ where Δ_Ω is the Dirichlet Laplacian on $L^2(\Omega)$; i.e.*

$$\begin{aligned} D(\Delta_\Omega) &= \left\{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\right\} \\ \Delta_\Omega f &= \Delta f. \end{aligned}$$

b) *For general open sets, one has $S(t)|_{L^2(\Omega)} = e^{t\tilde{\Delta}_\Omega}$ where $\tilde{\Delta}_\Omega$ denotes the pseudo - Dirichlet Laplacian; i.e.*

$$\begin{aligned} D\left(\tilde{\Delta}_\Omega\right) &= \left\{f|_\Omega : \Delta f \in H^1\left(\mathbb{R}^N\right), f(x) = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega\right\} \\ \tilde{\Delta}_\Omega f &= \Delta f. \end{aligned}$$

We refer to [AB, Section 7] for more details.

3 A counterexample

Trotter's product formula does no longer hold for positive semigroups on L^p and rank-one projections. Here is an example.

Example 3 *Let $E = L^p([0,1])$ and define the semigroup $T(.)$ by*

$$T(t)f(s) = \begin{cases} f(t+s) & \text{if } t+s \leq 1 \\ 0 & \text{if } t+s > 1. \end{cases}$$

Define the set A by

$$A = \bigcup_{k=0}^{\infty} \left[1 - 2^{-2k}, 1 - 2^{-2k-1}\right]$$

and define the one dimensional projection P by

$$Pf = \frac{3}{2} \int_0^1 f(s)\, ds\, \mathbf{1}_A.$$

Then $(T(t))_{t \geq 0}$ is a strongly continuous contraction semigroup and P is a projection which is contractive if $p = 1$. However, the sequence

$$\left(\left(T\left(\frac{t}{n}\right) P \right)^n f \right)_{n \in \mathbb{N}}$$

does not converge for any $t > 0$ and any f satisfying $\int_0^1 f(s)\, ds \neq 0$.

Proof. We only prove the case $t = 1$. A similar computation can be carried out for other positive values of t.

An easy computation yields

$$\left(T\left(\frac{1}{n}\right) P \right)^n f(s) = \frac{3}{2} \int_0^1 f(s)\, ds \left(\frac{3}{2} \int_0^{1-\frac{1}{n}} \mathbf{1}_A(s)\, ds \right)^{n-1} \mathbf{1}_A\left(s + \frac{1}{n}\right).$$

For $m \in \mathbb{N}$ and $n = 2^{2m+1}$ we have $\frac{3}{2} \int_0^{1-\frac{t}{n}} \mathbf{1}_A(s) = 1 - 2^{-2m-2}$ and therefore

$$\lim_{m \to \infty} \left(\frac{3}{2} \int_0^{1-2^{-2m-1}} \mathbf{1}_A(s)\, ds \right)^{2^{2m+1}-1} = \lim_{m \to \infty} \left(1 - 2^{-2m-2}\right)^{2^{2m+1}-1} = e^{-\frac{1}{2}}.$$

On the other hand, for $n = 2^{2m+2}$ we have $\frac{3}{2} \int_0^{1-\frac{t}{n}} \mathbf{1}_A(s) = 1 - 2^{-2m-2}$ and therefore

$$\lim_{m \to \infty} \left(\frac{3}{2} \int_0^{1-2^{-2m-2}} \mathbf{1}_A(s)\, ds \right)^{2^{2m+2}-1} = \lim_{m \to \infty} \left(1 - 2^{-2m-2}\right)^{2^{2m+2}-1} = e^{-1}.$$

So the sequence does not converge. ∎

4 Holomorphic contraction semigroups on Hilbert spaces

On Hilbert spaces, the Trotter's formula holds for orthogonal projections in an important special case.

Theorem 4 *Let $-A$ be the generator of a holomorphic semigroup $\left(e^{-zA}\right)_{z\in\Sigma(\theta)}$ on an Hilbert space H, where $\theta \in \left(0, \frac{\pi}{2}\right]$ and $\Sigma(\theta) = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\}$. We assume that*

$$\left\|e^{-zA}\right\| \leq 1$$

for all $z \in \Sigma(\theta)$. Let P be an orthogonal projection. Then

$$S(t)\,x = \lim_{n\to\infty} \left(e^{-\frac{t}{n}A}P\right)^n x \tag{4}$$

exists for all $x \in H$, $t \geq 0$ and defines a degenerate semigroup $(S(t))_{t\geq 0}$.

Note that (4) remains true if we replace A by $A + w$ for $w \in \mathbb{R}$.

Theorem 4 is a consequence of Trotter's formula for non-densly defined sesquilinear forms proved by Kato [Ka2, Theorem] for positive symmetric forms and extended by Simon to closed forms (see [Ka2, Addendum]). We explain in more detail.

Let H be a Hilbert space and let

$$a : D(a) \times D(a) \to \mathbb{C}$$

be sesquilinear where $D(a)$, the domain of a, is a subspace of H. We assume furthermore that a is *semibounded*, i.e. that there exists $\lambda \in \mathbb{R}$ such that

$$\|u\|_a^2 := \operatorname{Re} a(u,u) + \lambda (u \mid u)_H > 0$$

for all $u \in D(a)$, $u \neq 0$. Moreover, we assume that a is closed; i.e., that $(D(a), \|.\|_a)$ is complete.

Let $K = \overline{D(a)}$ be the closure of $D(a)$ in H. Denote by A the operator on K associated with a i.e.

$$\begin{aligned} D(A) &= \{u \in D(a) : \exists v \in K \text{ such that } a(u,\varphi) = (v \mid \varphi)_H \\ &\qquad \text{for all } \varphi \in D(a)\} \\ Au &= v. \end{aligned}$$

Then $-A$ generates a C_0 – semigroup $\left(e^{-tA}\right)_{t\geq 0}$ on K.

Now denote by P the orthogonal Projection onto K and define the operator e^{-ta} on H by

$$e^{-ta}x = \begin{cases} e^{-tA}x & \text{for } x \in K \\ 0 & \text{for } x \notin K. \end{cases}$$

Then $(e^{-ta})_{t\geq 0}$ is a degenerate semigroup on H. We call it the (degenerate) semigroup generated by a on H.

Now let b be a second semibounded, closed sesquilinear form on H. Define the sesquilinear form $a+b$ on H by

$$\begin{aligned} D(a+b) &= D(a)\cap D(b) \\ (a+b)(u,v) &= a(u,v)+b(u,v). \end{aligned}$$

Then it is obvious that $a+b$ is semibounded and closed again. Now the following product formula holds (see [Ka2, Theorem and Addendum]).

Theorem 5 *Let $x \in H$. Then*

$$e^{-t(a+b)}x = \lim_{n\to\infty}\left(e^{-\frac{t}{n}a}e^{-\frac{t}{n}b}\right)^n x \tag{5}$$

for all $t > 0$.

Now let $Q \in \mathcal{L}(H)$ be an orthogonal projection. Define the form b by

$$\begin{aligned} D(b) &= QH \\ b(u,v) &= 0 \end{aligned}$$

for all $u, v \in QH$. Then $e^{-tb} = Q$ for all $t \geq 0$. Thus (5) becomes

$$e^{-t(a+b)}x = \lim_{n\to\infty}\left(e^{-\frac{t}{n}a}Q\right)^n x \tag{6}$$

for all $x \in H$.

In order to establish the relation with Theorem 4, we remark the following. Let T be a holomorphic C_0 – semigroup on a Hilbert space H defined on the sector $\Sigma(\theta)$, $\theta \in \left(0, \frac{\pi}{2}\right]$, such that

$$\|T(z)\| \leq 1$$

for all $z \in \Sigma(\theta)$. Let A be the generator of T. Then there exists a closed semibounded densely defined sesquilinear form a such that A is associated with a (see [Ka1, IV.2.1, Theorem 2.7.]). Now Theorem 4 follows from (6).

5 Two conjectures

The general problem is to find conditions under which the product formula for projections (2) converges strongly to a degenerate semigroup. In view of the above results we formulate two conjectures.

Conjecture 6 *Let $T = (T(t))_{t\geq 0}$ be a contractive C_0 – semigroup on a Hilbert space H, and let P be an orthogonal projection. Then*

$$S(t) := \lim_{n\to\infty}\left(T\left(\frac{t}{n}\right)P\right)^n$$

converges strongly for all $t \geq 0$ to a degenerate semigroup.

Conjecture 7 *Let* $T = (T(t))_{t \geq 0}$ *be a positive contractive* C_0 *– semigroup on a space* $L^p(\Omega)$, $1 < p < \infty$, *and let* P *be a positive contractive projection. Then the same assertion as in Conjecture 6 is true.*

References

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