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On the Infinite Product of C_0-Semigroups

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Given a family $(e^{tA_k})_{t\geqslant 0}$ $(k\in\mathbb{N})$ of commuting contraction semigroups, we investigate when the infinite product $\prod_{k=1}^{\infty} e^{tA_k}$ converges and defines a C_0-semigroup. A particular case is the heat semigroup in infinite dimension introduced by Cannarsa and Da Prato (*J. Funct. Anal.* **118** (1993), 22–42).

1. INTRODUCTION

Recently, parabolic equations in infinite dimensions have received much attention in literature (see, for example, Pa Prato [DP] and Da Prato–Zabczyk [DZ]). In particular, Cannarsa and Da Prato [CD1] showed that the Laplacian (with a certain weight) generates a semigroup on $BUC(H)$, the space of all bounded uniformly continuous functions on a separable Hilbert space H, which is called the *heat semigroup* (see also [CD2]). This semigroup can be expressed as an infinite product,

$$\prod_{k=1}^{\infty} e^{tA_k}, \tag{1.1}$$

of a commuting family of contraction semigroups $(e^{tA_k})_{t\geqslant 0}$, $k\in\mathbb{N}$.

Motivated by this example, in the present paper, we start a systematic study of such infinite products. Already the simple example $E=\mathbb{C}$, $A_k=i$ $(k\in\mathbb{N})$ shows that (1.1) does not always converge. In order to obtain positive results, one has to allow a "change of speed" represented by a

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sequence $\lambda = (\lambda_k)_{k \in \mathbb{N}}$ of positive numbers and replace the semigroups $(e^{tA_k})_{t \geqslant 0}$ by the semigroup $(e^{t\lambda_k A_k})_{t \geqslant 0}$. Our main result in Section 2 shows that, if E is a separable Banach space, there always exists a change of speed such that the product $\prod_{k=1}^{\infty} e^{t\lambda_k A_k}$ converges to a semigroup. On the other hand, if E is not separable, such a sequence λ may not exist.

The heat semigroup on $BUC(H)$ is actually of the form $\prod_{k=1}^{\infty} e^{tB_k^2}$, where B_k generates an isometric group (the kth shift group with change of speed on $BUC(H)$). Convergence of the group product $\prod_{k=1}^{\infty} e^{tB_k}$ is immediate in that case. So the question arises of when this implies convergence of $\prod_{k=1}^{\infty} e^{tB_k^2}$. We give an affirmative answer in Section 3 for a slightly stronger notion of convergence for infinite products (namely l^p-continuity). This allows us to prove the result of Cannarsa and Da Prato by a completely different approach. Moreover, we show that the heat semigroup also exists on $BUC(X)$ for more general spaces X than Hilbert spaces. The change of speed $\lambda \in l^1$ is known to be optimal in the Hilbert space case. We obtain different conditions on λ when X is a weighted l^p-space.

In the last section we investigate a regularity property of the heat semigroup G on $BUC(X)$. Denoting its generator by A, we show that it does not have the Riemann–Lebesgue property, which means that $R(i\lambda, A)$ does not converge to 0 if $|\lambda| \to \infty$. In particular, this shows that G is not eventually norm continuous and in particular not analytic. This extends and gives alternative easy proofs of recent results by Guiotto [G], Desch and Rhandi [DR], and van Neerven and Zabczyk [NZ].

2. THE INFINITE PRODUCT OF SEMIGROUPS

Let E be a Banach space. By a semigroup or group we always understand a C_0-semigroup (C_0-group, respectively). Let B be the generator of a semigroup T. Then we frequently use the notation $T(t) = e^{tB}$ $(t \geqslant 0)$. If $\lambda > 0$, then λB generates the semigroup $T(\lambda \cdot) = (T(\lambda t))_{t \geqslant 0}$. Thus λ corresponds to a change of speed, which will play an important role in the article. In the following, for each $k \in \mathbb{N}$, let there be given a semigroup $T_k = (T_k(t))_{t \geqslant 0}$ on E with generator A_k. We assume that T_k is *contractive*; i.e., $\|T_k(t)\| \leqslant 1$ $(t \geqslant 0)$, for all $k \in \mathbb{N}$. Moreover, we assume that the family $\{T_k : k \in \mathbb{N}\}$ *commutes*, which means by definition that

$$T_k(t)\, T_l(s) = T_l(s)\, T_k(t) \tag{2.1}$$

for all $k, l \in \mathbb{N}$, $s, t \geqslant 0$. Then for all $n \in \mathbb{N}$, $(\prod_{k=1}^{n} T_k(t))_{t \geqslant 0}$ is a semigroup and $\overline{A_1 + \cdots + A_n}$ its generator. Here $\overline{A_1 + \cdots + A_n}$ is the closure of the operator $A_1 + \cdots + A_n$ which is considered with domain $\bigcap_{j=1}^{n} D(A_j)$. This is easy to see and well known (see [N, A-1.3.8]).

DEFINITION 2.1. We say that *the product* $\prod_{k=1}^{\infty} T_k$ *exists*, if for every $x \in X$,

$$\left(\prod_{k=1}^{\infty} T_k(t) \right) x := \lim_{n \to \infty} \prod_{k=1}^{n} T_k(t)\, x \tag{2.2}$$

converges uniformly on compact subsets of $[0, \infty)$. In that case, $\prod_{k=1}^{\infty} T_k(t) \in \mathscr{L}(E)$ for all $t \geqslant 0$ and $(\prod_{k=1}^{\infty} T_k(t))_{t \geqslant 0}$ is a contraction semigroup on E which we denote by $\prod_{k=1}^{\infty} T_k$ and which we call *the product semigroup* of the family of semigroups $\{T_k : k \in \mathbb{N}\}$.

We use the same terminology if each T_k is a C_0-group of contractions. In that case we ask that the convergence in (2.2) be uniform on all compact subsets of $\mathbb{R}$. Then $(\prod_{k=1}^{\infty} T_k(t))_{t \in \mathbb{R}}$ is a C_0-group of isometries which we call the *product group* of the family of C_0-groups $\{T_k : k \in \mathbb{N}\}$.

EXAMPLE 2.2. Let $T = (T(t))_{t \geqslant 0}$ be a semigroup with generator A. Let $\lambda_k \geqslant 0$ $(k \in \mathbb{N})$ and consider the semigroups $T_k(t) = T(\lambda_k t)$. If $\sum_{k=1}^{\infty} \lambda_k < \infty$, then the product $P = \prod_{k=1}^{\infty} T_k$ exists and is given by $P(t) = T(bt)$, where $b = \sum_{k=1}^{\infty} \lambda_k$. If $\sum_{k=1}^{\infty} \lambda_k = \infty$, then two cases are possible:

(a) $\lim_{t \to \infty} T(t) = Q$ exists strongly. Then $\lim_{n \to \infty} \prod_{k=1}^{n} T_k(t) = Q$ strongly but not uniformly on compact intervals unless $T(t) = I$ for all $t \geqslant 0$.

(b) If $T(t)$ does not converge strongly as $t \to \infty$, then $\prod_{k=1}^{n} T_k(t)$ does not converge strongly.

Our aim is to show that in the separable case there always exists a "change of speed" $(\lambda_j)_{j \in \mathbb{N}} \subset (0, \infty)$ such that the product

$$\prod_{j=1}^{\infty} T_j(\lambda_j \cdot)$$

converges. For this, we first show that the intersection of the domains of all generators is dense in E. The proof is based on the most useful but not very well-known abstract version of the Mittag–Leffler theorem (see Esterle [E] for a proof and further applications; see also [Am, V.1.1, AEK]). Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

THEOREM 2.3 (Mittag–Leffler theorem). *Let (M_n, d_n) be complete metric spaces and $\Theta_n : M_{n+1} \to M_n$ continuous mappings with dense image $(n \in \mathbb{N}_0)$. Let $x_0 \in M_0$, $\varepsilon > 0$. Then there exist $y_n \in M_n$ $(n \in \mathbb{N}_0)$ such that*

(a) $d_0(x_0, y_0) < \varepsilon$

(b) $\Theta_n y_{n+1} = y_n$.

Calling a sequence $(y_n)_{n\in\mathbb{N}_0}$ with $y_n\in M_n$ *projective* if $\Theta_n y_{n+1}=y_n$, and calling y_0 the *final point* of such a sequence, the theorem says that the set of all final points of projective sequences is dense in M_0.

PROPOSITION 2.4. *The space* $D=\bigcap_{k\in\mathbb{N}} D(A_k)$ *is dense in* E.

Proof. Let $M_0=E$ with the given norm and, for $n\in\mathbb{N}$, let $M_n=\bigcap_{k=1}^n D(A_k)$ with the norm $\|x\|_n=\sum_{k=1}^n \|A_k x\|+\|x\|$. Since the operators are closed, M_n is a Banach space. Moreover, the injection $\Theta_n\colon M_{n+1}\to M_n$ is continuous. Let $x\in M_n$. Since the operators commute, one has $R(\lambda, A_{n+1})\,x\in M_{n+1}$ and $\lim_{\lambda\to\infty}\lambda R(\lambda, A_{n+1})\,x=x$ in M_n. Thus Θ_n has dense image for all $n\in\mathbb{N}$. Here, every projective sequence is constant and final points are the same as elements of D. Thus the Mittag–Leffler theorem says that D is dense in E. ∎

Remark 2.5. In the same way one sees that the set $\bigcap_{k\in\mathbb{N}}\bigcap_{m\in\mathbb{N}} D(A_k^m)$ is dense in E.

LEMMA 2.6. *Suppose in addition that E is separable. Then there exist $\lambda_j>0$ such that the set*

$$D_1=\left\{x\in D\colon \sum_{j=1}^{\infty}\lambda_j\,\|A_j x\|<\infty\right\}$$

is dense in E.

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be a dense sequence in E. By Proposition 2.4, for $k\in\mathbb{N}$, $n\in\mathbb{N}$ there exists $d_{k,n}\in D$ such that $\|d_{k,n}-x_k\|<1/n$. Thus the countable set $D_0=\{d_{k,n}\colon k, n\in\mathbb{N}\}$ is dense in E. Write $D_0=\{y_m\colon m\in\mathbb{N}\}$. By induction we find sequences $(\lambda_j^m)_{j\in\mathbb{N}}$ in $(0,\infty)$ such that $\lambda_j^{m+1}\leqslant\lambda_j^m$ for all $m, j\in\mathbb{N}$ and

$$\sum_{j=1}^{\infty}\lambda_j^m\,\|A_j y_m\|<\infty \qquad \text{for all} \quad m\in\mathbb{N}.$$

Let $\lambda_j=\lambda_j^j$ $(j\in\mathbb{N})$. Let $m\in\mathbb{N}$. Then $\lambda_j\leqslant\lambda_j^m$ for all $j\geqslant m$. Thus $\sum_{j=1}^{\infty}\lambda_j\,\|A_j y_m\|<\infty$. This shows that $D_0\subset D_1$. Thus D_1 is dense in E. ∎

PROPOSITION 2.7. *Let $\{T_k\colon k\in\mathbb{N}\}$ be a commuting family of contraction semigroups (or groups) with generators A_k $(k\in\mathbb{N})$. Assume that the space*

$$D_1:=\left\{x\in\bigcap_{k\in\mathbb{N}} D(A_k)\colon \sum_{k=1}^{\infty}\|A_k x\|<\infty\right\} \tag{2.3}$$

is dense in E. Then the semigroup (resp. group) product $\prod_{k=1}^{\infty} T_k$ *exists. Define A on* D_1 *by* $Ax = \sum_{k=1}^{\infty} A_k x$. *Then A is closable and* $\bar{A}$ *is the generator of the product semigroup.*

Proof. We consider the semigroup case only. The group case is analogous. Let $x \in D_1$. Let $\tau > 0$. Then for $t \in [0, \tau]$,

$$\begin{aligned}
&\left\| \left(\prod_{j=1}^{n+k} T_j(t) \right) x - \left(\prod_{j=1}^{n} T_j(t) \right) x \right\| \\
&\quad \leqslant \left\| \left(\prod_{j=n+1}^{n+k} T_j(t) \right) x - x \right\| \\
&\quad \leqslant \left\| \left(\prod_{j=n+1}^{n+k} T_j(t) \right) x - \left(\prod_{j=n+1}^{n+k-1} T_j(t) \right) x \right\| \\
&\qquad + \left\| \left(\prod_{j=n+1}^{n+k-1} T_j(t) \right) x - \left(\prod_{j=n+1}^{n+k-2} T_j(t) \right) x \right\| + \cdots + \| T_{n+1}(t) x - x \| \\
&\quad \leqslant \sum_{j=n+1}^{n+k} \| T_j(t) x - x \| = \sum_{j=n+1}^{n+k} \left\| \int_0^t T_j(s) A_j x \, ds \right\| \\
&\quad \leqslant \tau \sum_{j=n+1}^{n+k} \| A_j x \|.
\end{aligned}$$

This shows that $(\prod_{k=1}^{n} T_k(t)\, x)_{n \in \mathbb{N}}$ converges uniformly on $[0, \tau]$ for all $x \in D_1$; and then, since $\bar{D}_1 = E$, also for all $x \in E$. Hence the product $\prod_{k=1}^{\infty} T_k$ exists in the sense of our definition. We denote by P the product semigroup, and let $P_n(t) = \prod_{k=1}^{n} T_j(t)$. Then $\lim_{n \to \infty} P_n(t)\, y = P(t)\, y$ uniformly on $[0, \tau]$ for all $\tau > 0$ and all $y \in E$. Let $x \in D$; then $(d/dt)\, P_n(t)\, x = P_n(t) \sum_{j=1}^{n} A_j x$ (note that $A_j T_k(t)\, x = T_k(t)\, A_j x$ for $t \geqslant 0$, $k, j \in \mathbb{N}$). Hence $P_n(t)\, x = x + \int_0^t P_n(s) \sum_{j=1}^{n} A_j x \, ds$. But $P_n(t) \sum_{j=1}^{n} A_j x = P_n(t)(\sum_{j=1}^{n} A_j x - Ax) + P_n(t)\, Ax$ converges to $P(t)\, Ax$ uniformly on $[0, \tau]$ for all $\tau > 0$. Thus

$$P(t)\, x = x + \int_0^t P(s)\, Ax \, ds.$$

This shows that the generator of P is an extension of A. Let $x \in D_1$. Then $\lim_{n \to \infty} A_j P_n(t)\, x = \lim_{n \to \infty} P_n(t)\, A_j x = P(t)\, A_j x$. Since A_j is closed, this shows that $P(t)\, x \in D(A_j)$ and $A_j P(t)\, x = P(t)\, A_j x$. It follows that $P(t)\, x \in D_1$. We have shown that D_1 is invariant by the semigroup P. Consequently D_1 is a core of the generator (by [N, A-II. Corollary 1.34)] or [Da, Theorem 1.9]. ∎

Remark 2.8 (Trotter–Kato Theorem). In the situation of Proposition 2.7 one is tempted to apply the Trotter–Kato theorem. In fact, the operator A with domain $D(A)=D_1$ is dissipative and densely defined. Moreover, $Ax=\lim_{n\to\infty} B_n x$, where $B_n=\overline{(A_1+\cdots+A_n)}$ is the generator of P_n. However, in order to apply the Trotter–Kato theorem, one needs to know that $(I-A)$ has dense range. It is remarkable that in the special case considered here, the range condition can be omitted.

THEOREM 2.9. *Suppose that E is separable. Let $\{T_j\colon j\in\mathbb{N}\}$ be a commuting family of contraction semigroups or groups. Then there exist $\lambda_j>0$ $(j\in\mathbb{N})$ such that the semigroup (resp. group) product*

$$\prod_{j=1}^{\infty} T_j(\lambda_j\,\cdot)$$

exists.

Proof. It follows from Lemma 2.6 that there exist $\lambda_j>0$ such the space $D_1=\{x\in D\colon \sum_{j=1}^{\infty}\|A_j x\|<\infty\}$ is dense in E. Now the claim follows from Proposition 2.7. ▮

Theorem 2.9 no longer holds if the Banach space E is not separable. We give an example.

EXAMPLE 2.10. Let $I=\{(\alpha_n)_{n\in\mathbb{N}}\colon \alpha_n>0 \text{ for all } n\in\mathbb{N}\}$ be the set of all positive sequences and let $E=l^2(I)$. For $j\in\mathbb{N}$ we define the unitary group T_j on E by

$$T_j(t)\,x=(e^{it\alpha_j}x_\alpha)_{\alpha\in I}$$

$(t\in\mathbb{R},\ x\in E)$. Then clearly, $T_j(t)\,T_k(s)=T_k(s)\,T_j(t)$ for all $k, j\in\mathbb{N}$, $s, t\in\mathbb{R}$. Let $\lambda_j>0$ $(j\in\mathbb{N})$, $t>0$. We show that there exists $x\in E$ such that the product $(\prod_{j=1}^{n} T_j(\lambda_j t))\,x$ does not converge in E (and in fact not even component wise). Let $\alpha\in I$ be given by $\alpha_j=\pi/\lambda_j t$. Let $x\in E$ be given by

$$x_\beta=\begin{cases}0 & \text{if } \beta\neq\alpha\\ 1 & \text{if } \beta=\alpha.\end{cases}$$

Then

$$\left(\prod_{j=1}^{n} T_j(\lambda_j t)\,x\right)_\alpha=\exp\left(it\sum_{j=1}^{n}\lambda_j\alpha_j\right)=e^{it\pi n}=(-1)^n.\quad ▮$$

3. THE HEAT SEMIGROUP IN INFINITE DIMENSION

The motivation of this investigation is as follows. Consider the Laplacian Δ_n on $BUC(\mathbb{R}^n)$ (with maximal distributional domain). Then Δ_n generates the Gaussian semigroup $(e^{t\Delta_n})_{t\geqslant 0}$ on $BUC(\mathbb{R}^n)$. The question arises whether the Laplacian also generates a semigroup if we replace $\mathbb{R}^n$ by an infinite dimensional space X. We will show this in the following if we take as X a weighted l^2-space.

The idea of proof is that the Laplacian is the sum of squares of the generators of the shift groups. It is easy to show that the product of the shift groups converges on suitable spaces. We make this more precise. If X is a metric space we denote by $BUC(X)$ the Banach space of all bounded uniformly continuous scalar-valued functions on X with the supremum norm. In our context, weighted l^p-spaces will be useful examples for X. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $(0, \infty)$ which serves as weight. Let $0<p<\infty$. By l^p_w we denote the space of all sequence $\lambda=(\lambda_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} w_n\lambda_n^p<\infty$. Then l^p_w is a Banach space for the norm $\|\lambda\|_{l^p_w}=(\sum_{n=1}^{\infty} w_n\lambda_n^p)^{1/p}$ if $1\leqslant p<\infty$. If $0<p<1$, then l^p_w is a metric space for the metric $d^p_w(\lambda, \mu)=\sum_{n=1}^{\infty} w_n |\lambda_n-\mu_n|^p$. By l^p_{w+} we denote the cone of all positive sequences in l^p_w. The following example will be useful.

EXAMPLE 3.1. Let $X=l^p_w$, where $0<p<\infty$ and $w=(w_n)_{n\in\mathbb{N}}\in(0, \infty)^{\mathbb{N}}$. For $k\in\mathbb{N}$ we denote by $e_k=(0, ..., 0, 1, ...)$ the kth unit vector. Let $E=BUC(X)$. Then

$$(T_k(t)f)(x)=f(x-te_k)$$

defines an isometric group on E which we call the kth shift group. Let $\lambda=(\lambda_j)_{j\in\mathbb{N}}\in X_+$. For $n\in\mathbb{N}$, let $P_n(t)=\prod_{k=1}^{n} T_k(\lambda_k t)$. Then $(P_n(t)f)(x)=f(x-t\sum_{k=1}^{n}\lambda_k e_k)$. Since $\lim_{n\to\infty}\sum_{k=1}^{n}\lambda_k e_k=\sum_{k=1}^{\infty}\lambda_k e_k$ exists in X, it follows that $P_n(t)f$ converges uniformly on $[-\tau, \tau]$ to $P(t)f$ for every $\tau>0$, where $(P(t)f)(x)=f(x-t\sum_{k=1}^{\infty}\lambda_k e_k)$. Thus the product group

$$P=\prod_{k=1}^{\infty} T_k(\lambda_k\,\cdot)$$

exists (in the sense of our terminology).

We recall a simple result of semigroup theory.

PROPOSITION 3.2. *Let B be the generator of an isometric C_0-group $T=(T(t))_{t\in\mathbb{R}}$ on E. Then B^2 generates a contraction C_0-semigroup S given by*

$$S(t)f=\frac{1}{(4\pi t)^{1/2}}\int_{\mathbb{R}} e^{-s^2/4t}T(s)f\,ds \qquad (f\in E).$$

Moreover, the semigroup S is holomorphic of angle $\pi/2$.

We refer to [N, A-IIII 1.13] of [Da, Theorem 2.31] for the easy proof. The Gaussian semigroup is the prototype of Proposition 3.2. Let $E=BUC(\mathbb{R})$, $(T(t)f)(x)=f(x-t)$. Then $(S(t)f)(x)=(1/(4\pi t)^{1/2})\int_{\mathbb{R}} e^{-s^2/4t}f(x-s)\,ds$. In the context of Example 3.1 we may consider the group T_k with generator B_k. Then $A_k=B_k^2$ generates the semigroup S_k on $BUC(X)$ given by $S_k(t)f(x)=(1/(4\pi t)^{1/2})\int_{\mathbb{R}} e^{-s^2/4t}f(x-se_k)\,ds$. Our aim is to find sequences $(\lambda_k)_{k\in\mathbb{N}}\subset\mathbb{R}_+^{\mathbb{N}}$ such that the product

$$\prod_{k=1}^{\infty} S_k(\lambda_k\cdot)$$

exists. We do not know whether in general, the semigroup product $\prod_{k=1}^{\infty} e^{tB_k^2}$ exists if the group product $\prod_{k=1}^{\infty} e^{tB_k}$ exists (where we suppose that the operators B_k generate commuting isometric groups). For this reason, we introduce a slightly stronger notion of convergence of the infinite semigroup product which enjoys the desired permanence property. It can be expressed as a joint strong continuity of the finite products at the origin. This can be done for any norm on the finite sequences. We restrict ourselves to weighted p-norms $(0<p<\infty)$. In view of the main example, the heat semigroup, it is useful to include also the case $0<p<1$.

DEFINITION 3.3. Let $w_n>0$ $(n\in\mathbb{N})$ and let $0<p<\infty$. A commuting family of contraction semigroups (resp., groups) on E is called *l^p_w-continuous*, if for every $x\in E$ and every $\varepsilon>0$ there exists $\delta>0$ such that

$$\sum_{k=1}^{n}|\lambda_k|^p\,w_k\leqslant\delta \qquad \text{implies} \qquad \left\|\left(\prod_{k=1}^{n}T_k(\lambda_k)\right)x-x\right\|\leqslant\varepsilon \tag{3.1}$$

for all finite sequences $(\lambda_1,...,\lambda_n)$ in $\mathbb{R}_+$ (in $\mathbb{R}$, respectively, in the group case). In the case where $w_n=1$ for all $n\in\mathbb{N}$ we simply speak of l^p-continuity.

It is obvious that the family of the shift groups considered in Example 3.1 is l^p_w-continuous.

If $a=(\alpha_k)_{k\in\mathbb{N}}$ and $b=(\beta_k)_{k\in\mathbb{N}}$ are two real sequences such that $\alpha_k\leqslant\beta_k$ for all $k\in\mathbb{N}$, we denote by $[a,b]$ the set of all sequences $\lambda=(\lambda_k)_{k\in\mathbb{N}}$ such that $\alpha_k\leqslant\lambda_k\leqslant\beta_k$ for all $k\in\mathbb{N}$.

PROPOSITION 3.4. *Let $0<p<\infty$, $w=(w_n)_{n\in\mathbb{N}}\in(0,\infty)^{\mathbb{N}}$. Let $\{T_k: k\in\mathbb{N}\}$ be a l^p_w-continuous, commuting family of contraction semigroups or groups. Let $\mu=(\mu_k)_{k\in\mathbb{N}}\in l^p_{w+}$. Let $x\in X$. Then*

$$\lim_{n\to\infty}\prod_{k=1}^{n}T_k(\lambda_k)\,x$$

converges uniformly for $\lambda\in[0,\mu]$ (and for $\lambda\in[-\mu,\mu]$, in the group case). In particular, for every $\lambda\in l^p_{w+}$ ($\lambda\in l^p_w$, respectively), the semigroup product (resp. group product) $\prod_{k=1}^{\infty}T_k(\lambda_k\,\cdot)$ exists.

Proof. Let $x\in E$, $\varepsilon>0$. Choose $\delta>0$ according to Definition 3.2. There exists $n_o\in\mathbb{N}$ such that $\sum_{k=n_o}^{\infty}\mu_k^p\leqslant\delta$. Let $n\geqslant n_o$, $k\in\mathbb{N}$. Then for $\lambda\in[0,\mu]$ (resp. $\lambda\in[-\mu,\mu]$),

$$\begin{aligned}\left\|\left(\prod_{j=1}^{n+k}T_j(\lambda_j)\right)x-\left(\prod_{j=1}^{n}T_j(\lambda_j)\right)x\right\| &= \left\|\prod_{j=1}^{n}T_j(\lambda_j)\left[\left(\prod_{j=n+1}^{n+k}T_j(\lambda_j)\right)x-x\right]\right\|\\ &\leqslant\left\|\left(\prod_{j=n+1}^{n+k}T_j(\lambda_j)\right)x-x\right\|\leqslant\varepsilon.\end{aligned}$$

This implies the desired assertion. ∎

Now we establish the desired permanence property.

THEOREM 3.5. *Let $0<p<\infty$, $w=(w_n)_{n\in\mathbb{N}}\in(0,\infty)^{\mathbb{N}}$. Let $\{T_k: k\in\mathbb{N}\}$ be an l^p_w-continuous commuting family of isometric groups with generators B_k ($k\in\mathbb{N}$). Let S_k be the semigroup generated by B_k^2. Then $\{S_k: k\in\mathbb{N}\}$ is an $l^{p/2}_w$-continuous commuting family of contraction semigroups. In particular, the product*

$$\prod_{k=1}^{\infty}S_k(\lambda_k\,\cdot)$$

exists for every $(\lambda_k)_{k\in\mathbb{N}}\in l^{p/2}_{w+}$.

Proof. Let $x\in E$, $\varepsilon>0$. There exists $\delta>0$ such that $\|\prod_{k=1}^{n}T_k(t_k)\,x-x\|\leqslant\varepsilon/2$ whenever $t_k\in(0,\infty)$ and $\sum_{k=1}^{n}|t_k|^p\,w_k\leqslant\delta$. Let

$$K_p=\frac{1}{(4\pi)^{1/2}}\int_{\mathbb{R}}|r|^p\,e^{-r^2/4}\,dr.$$

Then

$$\begin{aligned}
&\left\| \left(\prod_{j=1}^{n} S_j(t_j) \right) x - x \right\| \\
&\quad = \left\| \int_{\mathbb{R}^n} \left(\prod_{j=1}^{n} (4\pi t_j)^{-1/2} e^{-s_j^2/4t_j} T_j(s_j) \right) x \, ds_1 \cdots ds_n - x \right\| \\
&\quad = \left\| \int_{\mathbb{R}^n} (4\pi)^{-n/2} e^{-|s|^2/4} \left\{ \left(\prod_{j=1}^{n} T_j(s_j \sqrt{t_j}) \right) x - x \right\} ds_1 \cdots ds_n \right\| \\
&\quad \leqslant \int_{\mathbb{R}^n} (4\pi)^{-n/2} e^{-|s|^2/4} \left\| \left(\prod_{j=1}^{n} T_j(s_j \sqrt{t_j}) \right) x - x \right\| ds_1 \cdots ds_n \\
&\quad = \int_{\sum_{j=1}^n w_j |s_j|^p t_j^{p/2} \leqslant \delta} \cdots + \int_{\sum_{j=1}^n w_j |s_j|^p t_j^{p/2} > \delta} \cdots \\
&\quad \leqslant \frac{\varepsilon}{2} + 2 \|x\| \cdot \frac{1}{\delta} (4\pi)^{-n/2} \int_{\mathbb{R}^n} \sum_{j=1}^{n} w_j |s_j|^p t_{2}^{p/j} e^{-|s|^2/4} \, ds \\
&\quad \leqslant \frac{\varepsilon}{2} + 2 \|x\| \frac{1}{\delta} K_p \cdot \sum_{j=1}^{n} w_j t_j^{p/2} \leqslant \varepsilon, \\
&\qquad \text{if} \quad \sum_{j=1}^{n} w_j t_j^{p/2} \leqslant \frac{\varepsilon}{2} \delta (2 \|x\| K_p)^{-1}. \quad \blacksquare
\end{aligned}$$

Now we can define the heat semigroup in infinite dimension.

THEOREM 3.6. *Let $0 < p < \infty$, $w = (w_n)_{n \in \mathbb{N}} \in (0, \infty)^{\mathbb{N}}$. Let $E = BUC(l_w^p)$. Let $\lambda = (\lambda_j)_{j \in \mathbb{N}} \in l_w^{p/2}$. Define the n-dimensional Gaussian semigroup G_n on E by*

$$(G_n(t) f)(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|s|^2/4t} f\left(x - \sum_{j=1}^{n} \sqrt{s_j} e_j \right) ds$$

for $f \in E$. Then

$$G(t) f = \lim_{n \to \infty} G_n(t) f$$

converges uniformly on $[0, \tau]$ in E for every $\tau > 0$, $f \in E$ and defines a semigroup G on E. We call it the **heat semigroup** *on E (of "speed" λ). In particular, if $p = 2$ and $w \in l^1$, then one may choose $\lambda_j = 1$ for all $j \in \mathbb{N}$.*

Proof. It is clear from the definition that the family of shift-groups $\{T_k : k \in \mathbb{N}\}$ is l_w^p-continuous on E. Thus Theorem 3.7 is a consequence of Proposition 3.4 and Theorem 3.5. $\blacksquare$

In the case where $w \in l^1$ and $p=2$ we may take $\lambda_j = 1$ ($j \in \mathbb{N}$). Then formally, the orbit u: $\mathbb{R}_+ \to BUC(l^2_w)$ given by $u(t) = G(t)f$ is solution of the infinite dimensional heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{j=1}^{\infty} \dfrac{\partial^2}{\partial^2 x_j} u \\ u(0) = f \end{cases} \tag{3.2}$$

where $f \in BUC(l^2_w)$. But one should be aware that in this case we do not diagonalize the equation with respect to an orthonormal basis. Theorem 3.6 is due to Cannarsa and Da Prato [CD2] in the case $p=2$, $w_n = 1$ ($n \in \mathbb{N}$). They give a completely different proof, though. Their argument is based on a non-trivial result on the differential structure of $BUC(l^2)$ (viz., the density of the set of all Fréchet differentiable functions with Lipschitz continuous derivative). It is known [DP, 3.1] that in the case $p=2$, $w_n = 1$ ($n \in \mathbb{N}$) the condition that $\lambda = (\lambda_j)_{j \in \mathbb{N}} \in l^1$ is optimal; this means that, if $G_n(t)$ converges strongly to a semigroup, then $\lambda \in l^1$.

4. FURTHER PROPERTIES OF l^p_w-CONTINUITY

In this section we investigate further the notion of l^p_w-continuity. Let $\{T_j\colon j \in \mathbb{N}\}$ be a commuting family of contraction semigroups. By c_{00+} we denote the cone of all finitely supported sequences in $[0, \infty)$ (i.e., the set of all sequences $\lambda = (\lambda_j)_{j \in \mathbb{N}}$ in $[0, \infty)$ such that $\lambda_j = 0$ for almost all $j \in \mathbb{N}$). Then we can define the mapping

$$P : c_{00} \to \mathscr{L}(E)$$

by

$$P(\lambda)\, x = \prod_{j=1}^{\infty} T_j(\lambda_j)\, x \qquad (\lambda \in c_{00+},\ x \in E).$$

Let $0 < p < \infty$ and let $w = (w_m)_{n \in \mathbb{N}}$ be a sequence in $(0, \infty)$. Then c_{00} is dense in the space l^p_w.

PROPOSITION 4.1. *The family $\{T_j\colon j \in \mathbb{N}\}$ is l^p_w-continuous if and only if P extends to a strongly continuous mapping $\tilde{P}$: $l^p_{w+} \to \mathscr{L}(E)$. In that case one has*

$$\tilde{P}(\lambda + \gamma) = \tilde{P}(\lambda)\, \tilde{P}(\gamma) \tag{4.1}$$

for all $\lambda, \gamma \in l^p_{w+}$.

This is easy to see (cf. Proposition 3.4).

Remark 4.2. If we merely assume that the semigroup product $\prod_{j=1}^{\infty} T_j(\lambda_j \cdot)$ exists for all λ in l^p_{w+}, then we obtain a mapping $\widetilde{P}$: $l^p_{w+} \to \mathscr{L}(E)$ satisfying (4.1) which is merely strongly continuous on lines; i.e., $t \mapsto \widetilde{P}(t\lambda)x$: $\mathbb{R}_+ \to E$ is continuous for all $x \in E$, $\lambda \in l^p_{w+}$.

It is obvious that $\{T_j: j \in \mathbb{N}\}$ is l^p_w-continuous whenever the condition in Definition 3.3 is satisfied for all x in a dense subspace of E. Using this we obtain

PROPOSITION 4.3. *Let $1 \leqslant p < \infty$. Assume that the space*

$$D_q := \left\{ x \in \bigcap_{k=1}^{\infty} D(A_k) : (\|A_k x\|)_{k \in \mathbb{N}} \in l^q_w \right\}$$

is dense in E, where $1/p + 1/q = 1$. Then the family $\{T_j: j \in \mathbb{N}\}$ is l^p_w-continuous.

Proof. Let $x \in D_q$, $\lambda \in c_{00+}$. As in the proof of Proposition 2.7 we have

$$\begin{aligned} \left\| \left(\prod_{j=1}^{n} T_j(\lambda_j) \right) x - x \right\| &\leqslant \prod_{j=1}^{n} \|T_j(\lambda_j)\, x - x\| \\ &= \sum_{j=1}^{n} \left\| \int_0^{\lambda_j} T_j(s)\, A_j x\, ds \right\| \\ &\leqslant \sum_{j=1}^{n} \lambda_j \|A_j x\| \\ &\leqslant \|\lambda\|_{l^p_w} \cdot \|(A_j x)_{j \in \mathbb{N}}\|_{l^q_w}. \end{aligned}$$

This implies the claim. ∎

From Lemma 2.6 we now deduce.

THEOREM 4.4. *Assume that E is separable. Then there exist $\lambda_j > 0$ $(j \in \mathbb{N})$ such that the family $\{T_j(\lambda_j \cdot): j \in \mathbb{N}\}$ is l^p-continuous for all $1 \leqslant p < \infty$.*

Proof. By Lemma 2.6 there exist $\lambda_j > 0$ such that $\sum_{j=1}^{\infty} \|\lambda_j A_j x\| < \infty$ for all $x \in E$. Hence $(\|\lambda_j A_j x\|)_{j \in \mathbb{N}} \in l^q$ for all $1 \leqslant q \leqslant \infty$. Now the claim follows from Proposition 4.3. ∎

The analogous assertions of Propositions 4.1 and 4.3, Theorem 4.2 also hold for groups.

Remark 4.5 (Equicontinuity). Let $0 < p < \infty$. Every l^p-continuous family of commuting contraction semigroups $\{T_j: j \in \mathbb{N}\}$ is equicontinuous (i.e., for all $x \in E$, $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T_j(t)\, x - x\| \leqslant \varepsilon$ for all $t \in [0, \delta]$, $j \in \mathbb{N}$). For example, the family $\{(e^{itk})_{t \geqslant 0}: k \in \mathbb{N}\}$ in $E = \mathbb{C}$ is not equicontinuous and thus not l^p-continuous. However, it is l^1_w-continuous with $w_k = k$ $(k \in \mathbb{N})$.

In the preceding section we proved convergence of the product by using information on the generators A_k (namely density of D_1 in Proposition 2.7). It would be desirable to obtain such information from l^p-continuity. Here is a result of this kind.

PROPOSITION 4.6. *Let $0 < p < \infty$. Let $\{T_k : k \in \mathbb{N}\}$ be an l^p-continuous commuting family of contraction semigroups with generators A_k $(k \in \mathbb{N})$. Let $\lambda \in l^p_+$, such that $\lambda_j > 0$ for all $j \in \mathbb{N}$. Then the set $D^\lambda := \{x \in \bigcap_{k \in \mathbb{N}} D(A_k) : \sup_{k \in \mathbb{N}} \lambda_k \|A_k x\| < \infty\}$ is dense in X.*

Proof. Let $n \in \mathbb{N}$. Let

$$M^n_m x = \int_{(0,1)^m} T_1\left(\frac{\lambda_1 s_1}{n}\right) T_2\left(\frac{\lambda_2 s_2}{n}\right) \cdots T_m\left(\frac{\lambda_m s_m}{n}\right) x \, ds_1 \cdots ds_m.$$

Then $M^n_m \in \mathscr{L}(E)$, $\|M^n_m\| \leqslant 1$. Moreover,

$$\|M^n_{m+k} x - M^n_m x\|$$

$$\leqslant \int_{(0,1)^k} \|T_{m+1}(\lambda_{m+1} \cdot s_{m+1}) \cdots T_{m+k}(\lambda_{m+k} \cdot s_{m+k}) x - x\| \, ds_{m+1} \cdots ds_{m+k}.$$

This shows that $M^n x = \lim_{m \to \infty} M^n_m x$ exists for all $x \in E$. Hence $M^n \in \mathscr{L}(E)$, $\|M^n\| \leqslant 1$. Since $\lim_{n \to \infty} M^n_m x = x$ for all $m \in \mathbb{N}$, it follows that $\lim_{n \to \infty} M^n x = x$ for all $x \in E$. Let $j \in \mathbb{N}$. Then for $x \in E$, $m \geqslant j$, $A_j M^n_m x = 1/\lambda_j \int_{(0,1)^{m-1}} \prod_{i=1;\, t \neq j}^m T_i(\lambda_i s_i) \, ds_1 \cdots ds_{j-1} \, ds_{j+1} \cdots ds_m (T_j(\lambda_j) x - x)$, which converges as $m \to \infty$. Thus $M^n x \in D(A_j)$ and $\lambda_j \|A_j M^n x\| \leqslant 2 \|x\|$. ∎

Finally, we mention that for convergence of the infinite product, instead of contractivity, we may merely assume that finite products are bounded. Then the boundedness condition may be easier to satisfy if we do not change speed of one of the semigroups, T_0, say.

PROPOSITION 4.7. *Let $\{T_k : k \in \mathbb{N}_0\}$ be an l^p-continuous family of commuting C_0-semigroups such that for all $\tau > 0$ there exists $M \geqslant 0$ such that*

$$\left\| \prod_{k=1}^n T_k(t_k) \, T_0(t) \right\| \leqslant M \tag{4.2}$$

whenever $t_k \in [0, \tau]$, $n \in \mathbb{N}$. *Then for all* $x \in E$ *and* $\lambda \in l^p_+$

$$\prod_{k=1}^{n} T_k(\lambda_k t)\, T_0(t)\, x \tag{4.3}$$

converges as $n \to \infty$ *uniformly in* $[0, \tau]$ *for all* $\tau > 0$.

The proof is a slight modification of the proof of Proposition 3.4.

We give an example where (4.3) is valid without l^p-continuity being satisfied. Consider the Laplacian on $L^1(\mathbb{R})$ which generates a holomorphic C_0-semigroup $(e^{z\Delta})_{\operatorname{Re} z>0}$ of angle $\pi/2$. In particular, for all $r>0$, $\theta \in |0, \pi/2)$, one has

$$\sup_{z \in \Sigma(\theta, r)} \|e^{z\Delta}\| < \infty, \tag{4.4}$$

where $\Sigma(\theta, r) = \{\rho e^{i\alpha}: 0 \leqslant \rho \leqslant r, |\alpha| \leqslant \theta\}$. However, for all $\varepsilon > 0$,

$$\sup_{\substack{\operatorname{Re} z>0 \\ |z| \leqslant \varepsilon}} \|e^{z\Delta}\| = \infty \tag{4.5}$$

(see [AEH, Sect. 2]).

Moreover, for $\operatorname{Re} z > 0$, $z \notin \mathbb{R}$ the semigroup $(e^{tz\Delta})_{t \geqslant 0}$ is not contractive in $L^1(\mathbb{R})$ (see [O, Sect. 4, Example 2]). Now let $(z_k)_{k \in \mathbb{N}}$ be a bounded sequence in $\mathbb{C}$ with $\operatorname{Re} z_k > 0$ for all $k \in \mathbb{N}$. Let $T_k(t) = e^{tz_k\Delta}$ on $L^1(\mathbb{R})$. Let $\lambda = (\lambda_k)_{k \in \mathbb{N}} \in l^1_+$. Then for all $f \in L^1(\mathbb{R})$, $t > 0$,

$$P_n(t) f = \prod_{k=1}^{n} T_k(\lambda_k t) f$$

converges as $n \to \infty$ uniformly for $t \in [0, \tau]$. In fact, we can assume that $\lambda_{k_0} > 0$ for some $k_0 \in \mathbb{N}$. Let $\mu_n = \sum_{k=1}^n \lambda_k z_k$. Then $P_n(t)\, f = e^{t\mu_n\Delta} f$.

Since $\operatorname{Re} \mu_n \geqslant \lambda_{k_0} \operatorname{Re} z_{k_0}$, there exist $r>0$, $\theta \in (0, \pi/2)$ such that $t\mu_n \in \Sigma(\theta, r)$ for all $t \in [0, \tau]$, $n \in \mathbb{N}$. Now the claim follows, since $(e^{z\Delta})_{\operatorname{Re} z>0}$ is holomorphic. Because of (4.3) we can choose the z_k such that the semigroups $\{T_k: k \in \mathbb{N}\}$ are not equicontinuous and so not l^p-continuous (by Remark 4.5). This example illustrates the situation described in Remark 4.2. If we take $T_0(t) = e^{t\Delta}$ and choose z_k such that in addition $\sum_{k=1}^{\infty} |z_k| < \infty$, then also condition (4.2) is satisfied.

5. NON-CONTINUITY IN NORM OF THE HEAT SEMIGROUP

Let A be the generator of a contraction semigroup T on a Banach space E. If T is *norm continuous* (i.e., T: $(0, \infty) \to \mathscr{L}(E)$ is continuous with respect to the operator norm on $\mathscr{L}(E)$), then it follows from the Riemann–Lebesgue lemma that

$$\lim_{|\eta| \to \infty} \|R(w+i\eta, A)\| = 0 \tag{5.1}$$

for all $w>0$. (Observe that $R(w+i\eta, A)=\int_0^\infty e^{-i\eta s} e^{-ws} T(s)\, ds$). It follows from the resolvent identity that (5.1) is independent of $w>0$. For simplicity, we call (5.1) the *Riemann–Lebesgue property* in the following. If E is a Hilbert space, then it is known, conservely, that the Riemann–Lebesgue property implies the norm-continuity of the semigroup (see [EE1, EE2]). So far it is not known whether this implication holds on all Banach spaces; it is not even known whether it holds on L^p-spaces. More generally, if T: $[\tau, \infty) \to \mathscr{L}(E)$ is continuous for the operator norm, where $\tau>0$, then

$$\lim_{|\eta| \to \infty} \|R(w+i\eta, A)\, T(t)\| = 0 \tag{5.2}$$

for all $t \geqslant \tau$ (since $R(w+i\eta, A)\, T(t) = \int_0^\infty e^{-i\eta s} e^{-ws} R(t+s)\, ds$).

Our aim is to prove the following.

THEOREM 5.1. *Let $0<p<\infty$ and $w=(w_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$. Let $\lambda=(\lambda_j)_{j\in\mathbb{N}} \in l_w^{p/2}$ such that $\lambda_j>0$ for all $j\in\mathbb{N}$. Denote by G the heat semigroup with change of speed λ on $BUC(l_w^p)$ and by A its generator. Then G fails to have the Rieman–Lebesgue property* (5.1) *and also the more general property* (5.2) *for all $t>0$. In particular, G is not eventually norm-continuous.*

Remark 5.2. For $p=2$, $w_n=1$, it has been proved by Desch and Rhandi [DR] that G is not norm-continuous. It seems of independent interest that the Riemann–Lebesgue property falls; in addition, our argument is very simple.

Because of the analyticity and hence norm continuity of the Gaussian semigroup on $BUC(\mathbb{R}^m)$, it is clear that (5.1) holds for the Laplacian Δ_m on $BUC(\mathbb{R}^m)$. But we show in the following lemma that (5.1) does not hold uniformly in $m\in\mathbb{N}$.

LEMMA 5.3. *There exist $\eta_m \in \mathbb{R}$ and $f_m \in BUC(\mathbb{R}^m)$ such that*

$$\|f_m\|_\infty = 1, \qquad \lim_{m\to\infty} \eta_m = -\infty$$

and

$$\|R(1+i\eta_m, \Delta_m) f_m\|_\infty \geqslant \tfrac{1}{4} \qquad (m\in\mathbb{N}).$$

Proof. For $m \geqslant 2$ let $\theta \in (0, \pi/2)$ such that $\cos\theta = (1/\sqrt{m}) \sin\theta$. Then $\lim_{m\to\infty} \eta_m = -\infty$. Let $z = (e^{i\theta}/\sqrt{m})$ and

$$g(x) = e^{-zx^2/2} \qquad \left(x = (x_1, ..., x_m),\, x^2 = \sum_{j=1}^{m} x_j^2\right).$$

Then $g \in BUC(\mathbb{R}^m)$ and $\|g\|_\infty = 1$. Since g is radial, we have

$$\Delta_m g(r) = g''(r) + \frac{m-1}{r}\, g'(r) = (z^2 r^2 - mz)\, e^{-zr^2/2}, \qquad r = |x|.$$

Thus

$$\begin{aligned}
&\|(1 + i\eta_m - \Delta_m)\, g\|_\infty \\
&= \sup_{x \in \mathbb{R}^m} \left(1 - i\sqrt{m} \cdot \sin\theta + \sqrt{m}\, e^{i\theta} - \frac{e^{i2\theta}}{m}\, x^2\right) e^{-\cos\theta \cdot x^2/2\sqrt{m}} \\
&\leqslant \sup_{x \in \mathbb{R}^m} \left\{ |1 + \sqrt{m}\cos\theta| + \frac{x^2}{m}\, e^{-\cos\theta \cdot x^2/2\sqrt{m}} \right\} \\
&= 2 + \frac{2}{\sqrt{m}\cos\theta} \sup_{x \in \mathbb{R}^m} \frac{\cos\theta \cdot x^2}{2\sqrt{m}}\, e^{-\cos\theta \cdot x^2/2\sqrt{m}} \\
&\leqslant 2 + \frac{2}{\sqrt{m} \cdot \cos\theta} = 4 \qquad \text{since} \quad re^r \leqslant 1 \quad (r \geqslant 0).
\end{aligned}$$

Let $f_m = 1/4(1 + i\eta_m - \Delta_m)\, g$. Then $\|f_m\|_\infty \leqslant 1$ and

$$\|R(1 + i\eta_m - \Delta_m) f_m\|_\infty = \tfrac{1}{4} \|g\|_\infty = \tfrac{1}{4}. \quad \blacksquare$$

Proof of Theorem 5.1. Let $\lambda_j > 0$ such that $(\lambda_j)_{j\in\mathbb{N}} \in l_w^{p/2}$. Let

$$(G_n(t)\, g)(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-s^2/4t} g\left(x - \sum_{j=1}^{n} s_j \sqrt{\lambda_j} e_j\right) ds$$

($g \in BUC(l_w^p)$, $x \in l_w^p$). Then $\lim_{n\to\infty} G_n(t) = G(t)$ strongly. Let $m \in \mathbb{N}$ be fixed. Define

$$J : BUC(\mathbb{R}^m) \to BUC(l_w^p) \qquad \text{by} \quad (Jf)(x) = f(\sqrt{\lambda_1} x_1, ..., \sqrt{\lambda_m} x_m).$$

Then J is a linear isometry and $Je^{t\Delta_m} = G_m(t)\, J$ for all $n \geqslant m$. In fact, for $f \in BUC(\mathbb{R}^m)$, $x \in l^p_w$,

$$\begin{aligned}(G_n(t)\, Jf)(x) &= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-s^2/4t} f\left(\sqrt{\lambda_i}\left(\frac{x_1}{\sqrt{\lambda_1}} - s_1\right), ..., \sqrt{\lambda_m}\left(\frac{x_m}{\sqrt{\lambda_m}} - s_m\right)\right) ds \\ &= (Je^{t\Delta_m} f)(x).\end{aligned}$$

Letting $n \to \infty$ we conclude that $Je^{t\Delta_m} = G(t)\, J$. Taking Laplace transforms we obtain $R(1 + i\lambda_m, A)\, J = JR(1 + i\lambda_m, \Delta_m)$. Let $f_m \in BUC(\mathbb{R}^m)$ such that $\|f_m\|_\infty = 1$, $\|R(1 + i\lambda_m, \Delta_m)\, f_m\| \geqslant 1/4$. Then $\|Jf_m\| = \|f_m\| = 1$ and $\|R(1 + i\lambda_m, A)\, Jf_m\| = \|JR(1 + i\lambda_m, \Delta_m) f_m\| = \|R(1 + i\lambda_m, \Delta_m) f_m\| \geqslant 1/4$. Hence $\|R(1 + i\lambda_m, A)\| \geqslant 1/4$. This shows that (5.1) fails.

In order to show that (5.2) fails as well, we modify the argument. Let $t > 0$ and for $m \in \mathbb{N}$ let $f_m = 1/4(1 + i\eta_m - \Delta_m)\, g$ where $g(x) = e^{-zx^2/2}$ as in Lemma 5.2. Then $\|f_m\|_\infty = 1$. We show that for all $m \in \mathbb{N}$,

$$\|(1 + i\eta_m - \Delta_m)^{-1}\, e^{t\Delta_m} f_m\| \geqslant e^{-(2t^2 + t)}. \tag{5.3}$$

For $w \in \mathbb{C}$, $\operatorname{Re} w > 0$ let $K_z(x) = (4\pi z)^{-m/2}\, e^{-x^2/4z}$. Then $e^{w\Delta_m} h = K_w * h$ ($h \in BUC(\mathbb{R}^m)$) and $K_{w_1} * K_{w_2} = K_{w_1 + w_2}$ ($\operatorname{Re} w_1 > 0$, $\operatorname{Re} w_2 > 0$). Note that $g = (2\pi/z)^{m/z}\, K_{1/2z}$. Hence

$$\begin{aligned}&(1 + i\eta_m - \Delta_m)^{-1}\, e^{t\Delta_m} f_m \\ &\qquad = \tfrac{1}{4} e^{t\Delta_m} g = \tfrac{1}{4}(2\pi/z)^{m/2}\, K_t * K_{1/2z} = \tfrac{1}{4}(2\pi/z)^{m/2}\, K_{t + 1/2z}.\end{aligned}$$

Hence

$$\begin{aligned}&\|(1 + i\eta_m - \Delta_m)^{-1}\, e^{t\Delta_m} f_m\|_\infty \\ &\qquad \geqslant \frac{1}{4}\, |2\pi/z|^{m/2} \cdot \left|4\pi\left(t + \frac{1}{2z}\right)\right|^{-m/2} = \frac{1}{4}\, |2tz + 1|^{-m/2}.\end{aligned}$$

Using that $z = (1/\sqrt{m})\, e^{i\theta}$ and $\cos\theta = 1/\sqrt{m}$ we obtain

$$\|(1 + i\eta_m - \Delta_m)^{-1}\, e^{t\Delta_m} f\|_\infty \geqslant \left(1 + \frac{8t^2 + 4t}{m}\right)^{-m/4} \geqslant e^{-(2t^2 + t)}$$

independent of $m \in \mathbb{N}$. Using that

$$R(1 + i\eta_m, A)\, G(t)\, J = JR(1 + i\eta_m, \Delta_m)\, e^{t\Delta_m}$$

it follows as before that $\|R(1 + i\eta_m, A)\, G(t)\| \geqslant e^{-(2t^2 + t)}$ for all $m \in \mathbb{N}$. ∎

The Gaussian semigroups G_n on $E = BUC(l^p_w)$ are all holomorphic of angle $\pi/2$. But for every sector $\Sigma(\theta) = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leqslant \theta\}$ where $0 < \theta < \pi/2$ and every $r > 0$ one has

$$\sup_{n \in \mathbb{N}} \sup_{\substack{z \in \Sigma(\theta) \\ |z| \leqslant r}} \|G_n(z)\| = \infty. \tag{5.4}$$

Otherwise holomorphy would follow from the following result which is a simple consequence of Vitali's theorem ([HP, Sect. 3.14]; see also [AN] for a short proof).

Theorem 5.4. *Let $\theta \in (0, \pi/2]$ and let T_n be a holomorphic semigroup of angle θ. Assume that there exist $c \geqslant 0$, $r > 0$ such that*

$$\|T_n(z)\| \leqslant c \qquad (z \in \mathbb{C},\ |z| \leqslant r,\ |\arg z| < r) \tag{5.5}$$

for all $n \in \mathbb{N}$. Assume that T is a semigroup, such that

$$\lim_{n \to \infty} T_n(t)\, x = T(t)\, x$$

for all $t \geqslant 0$, $x \in E$. Then T is holomorphic of angle θ.

Proof. It follows from (5.5) that there exist $w \geqslant 0$, $M \geqslant 0$ such that

$$\|e^{-wz} T_n(z)\| \leqslant M$$

for all $z \in \Sigma(\theta)$, $n \in \mathbb{N}$. Now it follows from Vitali's theorem that there exists a holomorphic function S: $\Sigma(\theta) \to \mathscr{L}(E)$ such that

$$\lim_{n \to \infty} e^{-wz} T_n(z)\, x = S(x)\, x$$

for all $x \in E$, $z \in \Sigma(\theta)$. Since $T(t) = e^{wt} S(t)$ it follows that T has a holomorphic extension $(z \mapsto e^{wz} S(z))$ to $\Sigma(\theta)$. The semigroups property follows from analytic continuation, strong continuity in 0 from Vitali's theorem again; see [HP, Theorem 3.14.3]. ∎

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