ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS OF INHOMOGENEOUS CAUCHY PROBLEMS ON THE HALF-LINE

WOLFGANG ARENDT AND CHARLES J. K. BATTY

ABSTRACT

Let u be a bounded, uniformly continuous, mild solution of an inhomogeneous Cauchy problem on $\mathbf{R}_+$: $u'(t) = Au(t) + \phi(t)$ $(t \geqslant 0)$. Suppose that u has uniformly convergent means, $\sigma(A) \cap i\mathbf{R}$ is countable, and ϕ is asymptotically almost periodic. Then u is asymptotically almost periodic. Related results have been obtained by Ruess and Vũ, and by Basit, using different methods. A direct proof is given of a Tauberian theorem of Batty, van Neerven and Räbiger, and applications to Volterra equations are discussed.

1. *Introduction*

Several Tauberian theorems have been obtained which describe the behaviour of a bounded function $f(t)$ for large values of t under assumptions on the behaviour of its Laplace transform $\hat{f}(z)$ near the imaginary axis [**17**, **18**, **1**, **3**, **9**]. These theorems have been closely related to results describing the asymptotic behaviour of C_0-semigroups on Banach spaces, solutions of abstract Cauchy problems and Volterra equations, under certain spectral assumptions involving countability of the purely imaginary part of the spectrum [**1**, **3**, **9**, **12**, **21**, **22**]. Indeed, the proof of the Tauberian theorem given in [**9**, Theorem 4.1] depended on a result for individual orbits of bounded C_0-semigroups obtained in [**8**], which itself depended on the global stability theorem for bounded C_0-semigroups given in [**1**, **21**]. On the other hand, the stability theorems of [**1**], [**21**] and [**8**] are all immediate consequences of this Tauberian theorem, and it was explained in [**9**, Section 5] how the Tauberian theorem may be applied to solutions of some abstract Cauchy problems on the half-line $\mathbf{R}_+$, in particular giving new results for individual orbits of unbounded C_0-semigroups. An account of this theory may be found in [**25**, Sections 5.1, 5.3].

In [**27**], some important results were obtained concerning asymptotic behaviour of solutions of inhomogeneous Cauchy problems on the line, under assumptions on the spectrum of the operator A and the behaviour of the inhomogeneity. The authors reintroduced the condition that a function have convergent means, uniformly over translates, which was previously used in [**19**] in the context of almost periodic functions and indefinite integrals (inhomogeneous Cauchy problems when $A = 0$), and subsequently featured in the Tauberian theorem of [**9**]. The method of [**27**] used some arguments which are very different from those of [**9**] and are apparently confined to solutions defined on $\mathbf{R}$. In [**9**, Section 5], some results about inhomogeneous Cauchy problems were derived from the Tauberian theorem, but they depended on less satisfactory assumptions about the spectrum of the solutions.

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In this paper, we first give a direct proof of the Tauberian theorem from [**9**], by considering a C_0-group on a quotient of $\mathrm{BUC}(\mathbf{R}_+, X)$ induced by translations. This method of proof seems to be the easiest way to obtain the stability result of [**1**, **21**], and it sheds new light on the role of the hypotheses of countability of the unitary spectrum and ergodicity. In Section 3, we show how the same technique may be applied to inhomogeneous Cauchy problems on $\mathbf{R}_+$, thus obtaining the full analogue of the result of [**27**]. (Some partial results on $\mathbf{R}_+$ have been given in [**5**].) In Section 4, we discuss the connection between these results and those of [**3**] for solutions of Volterra equations.

The Tauberian theorem is based on an assumption that the Laplace transform has a holomorphic extension across the imaginary axis except at countably many points. In some situations [**17**, **18**, **3**], it is known that it suffices to have a continuous extension to the imaginary axis, except at countably many points. Although we are unable to prove the general Tauberian theorem under assumptions which cover these refinements, we show in Section 5 that a continuous extension implies a holomorphic extension in the case of orbits of isometric semigroups. As isometric semigroups form one of the main ingredients in the proof of the Tauberian theorem, this is a step towards such an improvement of the theorem.

Closely related results concerning functions on the line, and second-order Cauchy problems, are described in [**2**].

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2. *The Tauberian theorem*

Let X be a complex Banach space, and let $\mathrm{BUC}(\mathbf{R}_+, X)$ be the space of all bounded, uniformly continuous functions $f: \mathbf{R}_+ \to X$, with the norm

$$\|f\| = \sup_{t \in \mathbf{R}_+} \|f(t)\|_X.$$

Let $\mathscr{S} = \{S(t): t \geqslant 0\}$ be the C_0-semigroup of surjective contractions on $\mathrm{BUC}(\mathbf{R}_+, X)$ given by

$$(S(t)f)(s) = f(s+t).$$

Let B be the generator of $\mathscr{S}$. Let $Y_0 = \mathrm{BUC}(\mathbf{R}_+, X)/C_0(\mathbf{R}_+, X)$, and let $\pi_0: \mathrm{BUC}(\mathbf{R}_+, X) \to Y_0$ be the quotient map, so

$$\|\pi_0 f\| = \inf\{\|f-g\| : g \in C_0(\mathbf{R}_+, X)\} = \limsup_{t \to \infty} \|f(t)\|.$$

Now $\mathscr{S}$ induces a C_0-semigroup $\{S_0(t): t \geqslant 0\}$ on Y_0, consisting of surjective isometries, since

$$\|S_0(t)\pi_0 f\| = \|\pi_0 S(t) f\| = \limsup_{s \to \infty} \|(S(t)f)(s)\| = \limsup_{s \to \infty} \|f(s)\| = \|\pi_0 f\|.$$

Thus $\{S_0(t): t \geqslant 0\}$ extends to a C_0-group $\mathscr{S}_0 = \{S_0(t): t \in \mathbf{R}\}$ of isometries on Y_0.

A closed subspace $\mathscr{F}$ of $\mathrm{BUC}(\mathbf{R}_+, X)$ will be said to be *translation-biinvariant* if, for each $t \geqslant 0$, $\mathscr{F} = \{f \in \mathrm{BUC}(\mathbf{R}_+, X): S(t)f \in \mathscr{F}\}$. Clearly, a closed translation-biinvariant subspace contains all functions of compact support, and therefore contains $C_0(\mathbf{R}_+, X)$. Indeed, a closed subspace $\mathscr{F}$ is translation-biinvariant if and only if $\mathscr{F}$ contains $C_0(\mathbf{R}_+, X)$ and $\mathscr{F}_0 := \mathscr{F}/C_0(\mathbf{R}_+, X)$ is invariant under the group $\mathscr{S}_0$. Then $\mathscr{S}_0$ induces a C_0-group $\mathscr{S}_{\mathscr{F}}$ on $Y_{\mathscr{F}} := \mathrm{BUC}(\mathbf{R}_+, X)/\mathscr{F} = Y_0/\mathscr{F}_0$, so $S_{\mathscr{F}}(t)\pi_{\mathscr{F}}f = \pi_{\mathscr{F}}S(t)f$ for all f in $\mathrm{BUC}(\mathbf{R}_+, X)$, where $\pi_{\mathscr{F}}: \mathrm{BUC}(\mathbf{R}_+, X) \to Y_{\mathscr{F}}$ is the quotient map. Let $B_{\mathscr{F}}$ be the generator of $\mathscr{S}_{\mathscr{F}}$.

Examples. 1. The space $C_0(\mathbf{R}, X)$ is translation-biinvariant.

2. Let $\mathrm{AP}(\mathbf{R}_+, X)$ be the closed linear span in $\mathrm{BUC}(\mathbf{R}_+, X)$ of $\{e_\eta \otimes x : \eta \in \mathbf{R}, x \in X\}$, where $(e_\eta \otimes x)(t) = e^{i\eta t}x$. Then $\mathrm{AP}(\mathbf{R}_+, X)$ is the space of (*Maak*) *almost periodic functions* from $\mathbf{R}_+$ to X. Each $S(t)$ maps $\mathrm{AP}(\mathbf{R}_+, X)$ isometrically onto $\mathrm{AP}(\mathbf{R}_+, X)$, but $\mathrm{AP}(\mathbf{R}_+, X)$ is not translation-biinvariant. If L is a closed $\mathscr{S}$-invariant subspace of $\mathrm{AP}(\mathbf{R}_+, X)$, then $S(t)$ maps L onto L.

For g in $C_0(\mathbf{R}_+, X)$ and h in $\mathrm{AP}(\mathbf{R}_+, X)$,

$$\|g+h\| \geqslant \limsup_{t\to\infty} \|S(t)(g+h)\| = \limsup_{t\to\infty} \|S(t)h\| = \|h\|.$$

Thus the space

$$\mathrm{AAP}(\mathbf{R}_+, X) := C_0(\mathbf{R}_+, X) \oplus \mathrm{AP}(\mathbf{R}_+, X),$$

of all *asymptotically almost periodic functions* from $\mathbf{R}_+$ to X, is closed. It follows from the previous paragraph that $\mathrm{AAP}(\mathbf{R}_+, X)$ is translation-biinvariant. Moreover, if L is any closed $\mathscr{S}$-invariant subspace of $\mathrm{AP}(\mathbf{R}_+, X)$, then $C_0(\mathbf{R}_+, X) \oplus L$ is closed and translation-biinvariant.

It is known that $\mathrm{AAP}(\mathbf{R}_+, X)$ is the space of all functions f such that $\{S(t)f : t \geqslant 0\}$ is relatively compact (see [**15**, Theorem 9.3] for this and other characterisations).

3. Let $W(\mathbf{R}_+, X)$ be the space of all functions f in $\mathrm{BUC}(\mathbf{R}_+, X)$ such that $\{S(t)f : t \geqslant 0\}$ is relatively weakly compact in $\mathrm{BUC}(\mathbf{R}_+, X)$. This space is an appropriate weak analogue of the space $\mathrm{AAP}(\mathbf{R}_+, X)$. It originates from the work of Eberlein [**14**], and these functions are variously called *weakly almost periodic*, *Eberlein almost periodic*, *Eberlein weakly almost periodic*, or *weakly asymptotically almost periodic* (*in the sense of Eberlein*). Let $W_0(\mathbf{R}_+, X)$ be the space of all functions f in $W(\mathbf{R}_+, X)$ such that 0 is in the weak closure of $\{S(t)f : t \geqslant 0\}$. Both $W(\mathbf{R}_+, X)$ and $W_0(\mathbf{R}_+, X)$ are translation-biinvariant spaces.

An application of Glicksberg–deLeeuw theory shows that

$$W(\mathbf{R}_+, X) = W_0(\mathbf{R}_+, X) \oplus \mathrm{AP}(\mathbf{R}_+, X). \tag{2.1}$$

Details of this and many other properties of $W(\mathbf{R}_+, X)$ are given in [**28**, **30**]. (See [**16**, **23**] for discussions of some related translation-biinvariant classes of functions.)

4. Any Λ-class in the sense of Basit [**4**, **5**] is translation-biinvariant.

For f in $\mathrm{BUC}(\mathbf{R}_+, X)$, let $\hat{f}$ be the Laplace transform of f, defined on the right half-plane:

$$\hat{f}(\lambda) = \int_0^\infty e^{-t\lambda} f(t)\, dt \quad (\operatorname{Re}\lambda > 0).$$

Let $\mathrm{Sp}_{\mathbf{R}_+}(f)$ be the set of all real ξ such that $i\xi$ is a singular point of $\hat{f}$, that is, $\hat{f}$ does not have a holomorphic extension to a neighbourhood of $i\xi$.

We shall need the following two lemmas, proofs of which are given elsewhere but which we state here for completeness.

LEMMA 2.1 ([**9**, Proposition 3.2]; see also [**25**, Lemma 5.3.3]). *Let* $f \in \mathrm{BUC}(\mathbf{R}_+, X)$ *and* $\xi \in \mathbf{R} \setminus \mathrm{Sp}_{\mathbf{R}_+}(f)$. *Then there is a neighbourhood* V *of* $i\xi$ *in* $\mathbf{C}$ *and a holomorphic function* $h: V \to \mathrm{BUC}(\mathbf{R}_+, X)$ *such that* $h(\lambda) = R(\lambda, B)f$ *whenever* $\lambda \in V$ *and* $\mathrm{Re}\,\lambda > 0$.

LEMMA 2.2 ([**2**, Lemma 2.2]; see also [**8**, Theorem 2.2], [**25**, Lemma 5.1.7]). *Let* A *be the generator of a* C_0*-group* $\mathscr{U}$ *of isometries on a Banach space* Z. *Let* $z \in Z$, $\xi \in \mathbf{R}$, *and suppose that there exist a neighbourhood* V *of* $i\xi$ *in* $\mathbf{C}$ *and a holomorphic function* $h: V \to Z$ *such that* $h(\lambda) = R(\lambda, A)z$ *whenever* $\lambda \in V$ *and* $\mathrm{Re}\,\lambda > 0$. *Then* $i\xi \in \rho(A_z)$, *where* A_z *is the generator of the restriction of* $\mathscr{U}$ *to the closed linear span of* $\{U(t)z : t \in \mathbf{R}\}$ *in* Z.

Let $i\eta$ ($\eta \in \mathbf{R}$) be a point on the imaginary axis. A function f in $\mathrm{BUC}(\mathbf{R}_+, X)$ is said to be *uniformly ergodic* at $i\eta$ if

$$M_\eta(f)(s) := \lim_{\alpha \downarrow 0} \alpha \hat{f_s}(\alpha + i\eta)$$

exists, uniformly for $s \geqslant 0$, where $f_s = S(s)f$. This is equivalent to requiring the existence of the Abel limit

$$M_\eta(f) = \lim_{\alpha \downarrow 0} \alpha \int_0^\infty e^{-\alpha t} e^{-i\eta t} S(t) f \, dt = \lim_{\alpha \downarrow 0} \alpha R(\alpha + i\eta, B) f$$

of $t \mapsto e^{-i\eta t} S(t) f$ in $\mathrm{BUC}(\mathbf{R}_+, X)$. It follows immediately from Lemma 2.1 that f is uniformly ergodic at $i\eta$, and $M_\eta(f) = 0$, whenever $\eta \notin \mathrm{Sp}_{\mathbf{R}_+}(f)$.

Since our functions are uniformly bounded, f is uniformly ergodic at $i\eta$ if and only if the Cesàro limit

$$M_\eta(f) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{-i\eta t} S(t) f \, dt$$

exists in $\mathrm{BUC}(\mathbf{R}_+, X)$. Thus our notion of ergodicity is the same as those appearing in [**4**, **5**, **12**, **19**, **27**].

A simple calculation shows that

$$\alpha \hat{f_s}(\alpha + i\eta) = e^{(\alpha + i\eta)s} \alpha \hat{f}(\alpha + i\eta) - \alpha \int_0^s e^{(\alpha + i\eta)(s-t)} f(t) \, dt,$$

so whenever f is uniformly ergodic at $i\eta$, $M_\eta(f)$ is a function of the form

$$M_\eta(f) = e_\eta \otimes x_{f,\eta}.$$

We say that $x_{f,\eta}$ is the *mean* of f associated with the *frequency* η.

Let E be a subset of $i\mathbf{R}$. We shall say that f is *totally ergodic on* E if f is uniformly ergodic at each point of E. We shall say that f is *totally ergodic* if f is totally ergodic on $i\,\mathrm{Sp}_{\mathbf{R}_+}(f)$, or equivalently, if f is totally ergodic on $i\mathbf{R}$.

Any function f in $W(\mathbf{R}_+, X)$ is totally ergodic, and the means of f determine the almost periodic part of f (in the sense of the decomposition (2.1)) [**30**, p. 426], [**4**, Theorem 2.4.7].

Now we can give the direct proof of (a slightly generalised version of) the Tauberian theorem [**9**, Theorem 4.1].

THEOREM 2.3. *Let* $f: \mathbf{R}_+ \to X$ *be bounded, uniformly continuous and totally ergodic, and suppose that* $\mathrm{Sp}_{\mathbf{R}_+}(f)$ *is countable. Let* $\mathscr{F}$ *be a closed, translation-biinvariant subspace of* $\mathrm{BUC}(\mathbf{R}_+, X)$, *and suppose that* $\mathscr{F}$ *contains* $M_\eta(f)$ *for each* η *in* $\mathrm{Sp}_{\mathbf{R}_+}(f)$. *Then* $f \in \mathscr{F}$.

Proof. Let $B_{f,\mathscr{F}}$ be the generator of the restriction of $\mathscr{S}_{\mathscr{F}}$ to the closed linear span $Z_{f,\mathscr{F}}$ of $\{S_{\mathscr{F}}(t)\,\pi_{\mathscr{F}} f : t \in \mathbf{R}\}$ in $Y_{\mathscr{F}}$. Let $\xi \in \mathbf{R}\backslash \mathrm{Sp}_{\mathbf{R}_+}(f)$, and let V and h be as in Lemma 2.1. Define $\bar{h}: V \to Y_{\mathscr{F}}$ by $\bar{h}(\lambda) = \pi_{\mathscr{F}}(h(\lambda))$. Then $\bar{h}(\lambda) = R(\lambda, B_{\mathscr{F}})\,\pi_{\mathscr{F}} f$ whenever $\operatorname{Re}\lambda > 0$. By Lemma 2.2, $i\xi \in \rho(B_{f,\mathscr{F}})$. Thus $\sigma(B_{f,\mathscr{F}})$ is countable.

Suppose that $f \notin \mathscr{F}$. Then $Z_{f,\mathscr{F}}$ is a non-trivial Banach space, so $\sigma(B_{f,\mathscr{F}})$ is non-empty. Since $\sigma(B_{f,\mathscr{F}})$ is countable and closed in $i\mathbf{R}$, it has an isolated point $i\eta$. Then $i\eta$ is an eigenvalue of $B_{f,\mathscr{F}}$ [**11**, Theorem 8.16], so there is a non-zero z in $Z_{f,\mathscr{F}}$ such that $S_{\mathscr{F}}(t)\,z = e^{i\eta t} z$ for all t. However,

$$\alpha \int_0^\infty e^{-(\alpha+i\eta)\,t} S(t) f\, dt \to M_\eta(f)$$

in $\mathrm{BUC}(\mathbf{R}_+, X)$, as $\alpha \downarrow 0$. Applying $S_{\mathscr{F}}(s)\,\pi_{\mathscr{F}}$,

$$\alpha \int_0^\infty e^{-(\alpha+i\eta)\,t} S_{\mathscr{F}}(t)\, S_{\mathscr{F}}(s)\,\pi_{\mathscr{F}} f\, dt \to S_{\mathscr{F}}(s)\,\pi_{\mathscr{F}}(M_\eta(f)) = 0$$

for all $s \geqslant 0$. Taking linear combinations and limits.

$$z = \alpha \int_0^\infty e^{-(\alpha+i\eta)\,t} S_{\mathscr{F}}(t)\, z\, dt \to 0.$$

This contradiction proves the result.

The following corollary illustrates well the Tauberian character of Theorem 2.3.

COROLLARY 2.4 ([**9**, Theorem 4.1]; see also [**25**, Theorem 5.3.5]). *Let $f: \mathbf{R}_+ \to X$ be bounded, uniformly continuous and totally ergodic, and suppose that $\mathrm{Sp}_{\mathbf{R}_+}(f)$ is countable. Then f is asymptotically almost periodic. If the means $x_{f,\eta}$ are zero for all except finitely many η, then $\|f(t) - \sum_\eta e^{i\eta t} x_{f,\eta}\| \to 0$ as $t \to \infty$.*

Proof. The first statement follows from Theorem 2.3 by taking $\mathscr{F} = \mathrm{AAP}(\mathbf{R}_+, X)$, and the second by applying Theorem 2.3 to $f - \sum_\eta M_\eta(f)$ with $\mathscr{F} = C_0(\mathbf{R}_+, X)$ (see also the remarks following Corollary 2.5).

In particular, Corollary 2.4 shows that a function in $W(\mathbf{R}_+, X)$ with countable spectrum is asymptotically almost periodic. A family of functions in $W(\mathbf{R}_+, X)$ which are not asymptotically almost periodic is given in [**28**, Example 3.7], and it is easy to check that each function in that family has $\mathbf{R}$ as its spectrum.

A remarkable aspect of our approach is that the following well-known stability theorem for semigroups now becomes a direct corollary of the Tauberian theorem (Corollary 2.4).

COROLLARY 2.5 ([**1**, **21**, **22**]; see also [**25**, Theorems 5.1.5, 5.7.10]). *Let $\mathscr{T}$ be a bounded C_0-semigroup on X, with generator A. Suppose that $\sigma(A) \cap i\mathbf{R}$ is countable, and, for each $i\eta$ in $\sigma(A) \cap i\mathbf{R}$,*

$$\operatorname{Ker}(A - i\eta I) + \operatorname{Ran}(A - i\eta I) \quad \text{is dense in } X. \tag{2.2}$$

Then we have the following.

(1) *For each x in X, the map $t \mapsto T(t)\,x$ is asymptotically almost periodic.*

(2) $X = X_0 \oplus X_1$, *where*

$$X_0 = \{x \in X \colon \|T(t)\,x\| \to 0 \text{ as } t \to \infty\},$$
$$X_1 = \overline{\operatorname{span}}\{x \in X \colon \text{there exists } \eta \text{ in } \mathbf{R} \text{ such that } T(t)\,x = e^{i\eta t}x \text{ for all } t\}.$$

(3) *If $\sigma_p(A^*) \cap i\mathbf{R}$ is empty, then $\|T(t)\,x\| \to 0$ as $t \to \infty$, for each x in X.*

Proof. Let $x \in X$ and $f(t) = T(t)\,x$. Then $\hat{f}(\lambda) = R(\lambda, A)\,x$, so $\mathrm{Sp}_{\mathbf{R}_+}(f) \subseteq \sigma(A) \cap i\mathbf{R}$, which is countable. The condition (2.2) implies that the Abel limit

$$y := \lim_{\alpha \downarrow 0} \alpha \int_0^\infty e^{-(\alpha+i\eta)\,t} f(t)\,dt$$

exists (and is 0 under the assumption of (3)) [**11**, Theorem 5.1]. Since

$$\alpha \int_0^\infty e^{-(\alpha+i\eta)\,t}(S(t)f)\,(s)\,dt = T(s)\left(\alpha \int_0^\infty e^{-(\alpha+i\eta)\,t} f(t)\,dt\right),$$

it follows that f is totally ergodic, and $M_\eta(f)\,(s) = T(s)\,y = e^{i\eta s}y$, so $y \in X_1$.

Now (1) and (3) follow immediately from Corollary 2.4. Since $\mathscr{T}$ acts as a bounded group on X_1, (2) follows from Theorem 2.3 on taking

$$\mathscr{F} = C_0(\mathbf{R}_+) \oplus \{T(\cdot)\,z : z \in X_1\}.$$

REMARKS. 1. An equivalent reformulation of Theorem 2.3 is the statement that if f is bounded, uniformly continuous and totally ergodic, and if $\mathrm{Sp}_{\mathbf{R}_+}(f)$ is countable, then

$$f \in C_0(\mathbf{R}_+) \oplus \overline{\operatorname{span}}\{M_\eta(f) : \eta \in \mathbf{R}\} = C_0(\mathbf{R}_+) \oplus \overline{\operatorname{span}}\{M_\eta(f) : \eta \in \mathrm{Sp}_{\mathbf{R}_+}(f)\}.$$

Another reformulation of the conclusion is that

$$\lim_{t \to \infty} \inf \sup_{s \geqslant t} \left\| f(s) - \sum_{j=1}^{n} \alpha_j\, e^{i\eta_j s} x_{f,\eta_j} \right\| = 0,$$

where the infimum is taken over all possible choices of integers n, complex numbers α_j and η_j in $\mathrm{Sp}_{\mathbf{R}_+}(f)$.

2. It is well known (see [**20**, p. 24]) that if $h \in \mathrm{AP}(\mathbf{R}_+, X)$, then $h \in \overline{\operatorname{span}}\{M_\eta(h) : \eta \in \mathbf{R}\}$. It follows directly from this that

$$f \in C_0(\mathbf{R}_+) \oplus \overline{\operatorname{span}}\{M_\eta(f) : \eta \in \mathrm{Sp}_{\mathbf{R}_+}(f)\} \quad \text{whenever } f \in \mathrm{AAP}(\mathbf{R}_+, X). \tag{2.3}$$

Thus the general case of Theorem 2.3 follows from the special case when $\mathscr{F} = \mathrm{AAP}(\mathbf{R}_+, X)$ (Corollary 2.4) and the known fact (2.3). However, the proof of (2.3) is not trivial, while our proof of Theorem 2.3 works just as easily for general $\mathscr{F}$ as for $\mathscr{F} = \mathrm{AAP}(\mathbf{R}_+, X)$.

3. If $\{T(t) : t \geqslant 0\}$ is any bounded C_0-semigroup on X for which (1) of Corollary 2.5 holds, then (2) also holds. This follows easily from (2.3) for $f(t) = T(t)\,x$.

4. There exist bounded C_0-semigroups on reflexive spaces whose non-zero orbits $f_x : t \mapsto T(t)\,x$ are not asymptotically almost periodic [**1**, Example 2.5(a)] (see also [**25**, Example 5.1.6]). Then $f_x \in W(\mathbf{R}_+, X)$ [**29**, Theorem 2.1], and $\mathrm{Sp}_{\mathbf{R}_+}(f_x)$ is uncountable (Corollary 2.5). The example in [**1**] shows that, given any closed uncountable subset E of $\mathbf{R}$, one can arrange that $\mathrm{Sp}_{\mathbf{R}_+}(f_x) \subseteq E$.

3. *Inhomogeneous Cauchy problems*

Let A be a closed operator on X, let $\phi \in \mathrm{BUC}(\mathbf{R}_+, X)$, and consider the abstract inhomogeneous Cauchy problem

$$f'(t) = Af(t) + \phi(t) \quad (t \geqslant 0). \tag{3.1}$$

By definition, a *mild solution* is a continuous function $f: \mathbf{R}_+ \to X$ such that, for each $t \in \mathbf{R}_+$, $\int_0^t f(s)\,ds \in D(A)$ and

$$f(t) = A\left(\int_0^t f(s)\,ds\right) + \int_0^t \phi(s)\,ds + f(0).$$

PROPOSITION 3.1. *Let* $\phi \in \mathrm{BUC}(\mathbf{R}_+, X)$, *and let* f *be a bounded, uniformly continuous, mild solution of* (3.1). *Then we have the following.*

(1) *For* $s \geqslant 0$,

$$\hat{f}_s(\lambda) = R(\lambda, A)(\hat{\phi}_s(\lambda)) + R(\lambda, A)(f(s))$$

whenever $\mathrm{Re}\,\lambda > 0$ *and* $\lambda \in \rho(A)$.

(2) $\mathrm{Sp}_{\mathbf{R}_+}(f) \subseteq \mathrm{Sp}_{\mathbf{R}_+}(\phi) \cup \{\eta \in \mathbf{R} : i\eta \in \sigma(A)\}$.

(3) *If* ϕ *is uniformly ergodic at a point* $i\eta$ *in* $\rho(A) \cap i\mathbf{R}$, *then* f *is uniformly ergodic at* $i\eta$ *and* $x_{f,\eta} = R(i\eta, A)\, x_{\phi,\eta}$.

Proof. (1) Let $g(t) = \int_0^t f_s(r)\,dr$ and $\psi(t) = \int_0^t \phi_s(r)\,dr$. Then

$$f_s(t) = A(g(t)) + \psi(t) + f(s).$$

For $\mathrm{Re}\,\lambda > 0$, the integrals $\int_0^\infty e^{-\lambda t} g(t)\,dt$ and $\int_0^\infty e^{-\lambda t} A(g(t))\,dt$ are both absolutely convergent. Since A is closed, $\lambda^{-1}\hat{f}_s(\lambda) = \hat{g}(\lambda) \in D(A)$ and

$$\lambda^{-1} A(\hat{f}_s(\lambda)) = A(\hat{g}(\lambda)) = \hat{f}_s(\lambda) - \lambda^{-1}\hat{\phi}_s(\lambda) - \lambda^{-1} f(s).$$

Hence

$$(\lambda - A)\hat{f}_s(\lambda) = \hat{\phi}_s(\lambda) + f(s),$$

and (1) follows.

(2) Suppose that $i\eta \in \rho(A)$ and $\eta \notin \mathrm{Sp}_{\mathbf{R}_+}(\phi)$. Then $\hat{\phi}$ has a holomorphic extension g near $i\eta$, and (by (1), with $s = 0$) $\hat{f}$ has a holomorphic extension given by $R(\lambda, A)\, g(\lambda) + R(\lambda, A) f(0)$.

(3) By (1),

$$\alpha \hat{f}_s(\alpha + i\eta) = R(\alpha + i\eta, A)(\alpha \hat{\phi}_s(\alpha + i\eta)) + \alpha R(\alpha + i\eta, A)(f(s)) \to R(i\eta, A)\, M_\eta(\phi)(s)$$

as $\alpha \downarrow 0$, uniformly for $s \geqslant 0$.

THEOREM 3.2. *Suppose that* $\sigma(A) \cap i\mathbf{R}$ *is countable*, $\phi \in \mathrm{BUC}(\mathbf{R}_+, X)$, *and* f *is a bounded, uniformly continuous, mild solution of* (3.1) *which is totally ergodic on* $\sigma(A) \cap i\mathbf{R}$. *Let* $\mathscr{F}$ *be a closed, translation-biinvariant subspace of* $\mathrm{BUC}(\mathbf{R}_+, X)$, *satisfying the following conditions*:

(1) *for all* $i\eta$ *in* $\sigma(A) \cap i\mathbf{R}$, $\mathscr{F}$ *contains* $M_\eta(f)$;

(2) *for all* λ *in* $\rho(A)$, $\mathscr{F}$ *contains the function* $R(\lambda, A) \circ \phi : s \mapsto R(\lambda, A)(\phi(s))$.

Then $f \in \mathscr{F}$.

Proof. By Proposition 3.1 (1),

$$(R(\lambda, B)f)(s) = \int_0^\infty e^{-\lambda t} f(s+t)\,dt = \hat{f}_s(\lambda) = R(\lambda, A)(\hat{\phi}_s(\lambda)) + R(\lambda, A)(f(s))$$

whenever $\operatorname{Re}\lambda > 0$ and $\lambda \in \rho(A)$. But

$$R(\lambda, A)(\hat{\phi}_s(\lambda)) = \left(\int_0^\infty e^{-\lambda t} S(t)(R(\lambda, A)\circ\phi)\,dt\right)(s).$$

Thus $s \mapsto R(\lambda, A)(\hat{\phi}_s(\lambda))$ belongs to $\mathscr{F}$, by (2) and the translation-invariance of $\mathscr{F}$, so

$$R(\lambda, B_{\mathscr{F}})\pi_{\mathscr{F}} f = \pi_{\mathscr{F}} R(\lambda, B) f = \pi_{\mathscr{F}}(R(\lambda, A)\circ f)$$

whenever $\operatorname{Re}\lambda > 0$ and $\lambda \in \rho(A)$. This shows that the map $\lambda \mapsto R(\lambda, B_{\mathscr{F}})\pi_{\mathscr{F}} f$ has a holomorphic extension to a map $g: \rho(A) \to Y_{\mathscr{F}}$, given by $g(\lambda) = \pi_{\mathscr{F}}(R(\lambda, A)\circ f)$. By Lemma 2.2,

$$\sigma(B_{f,\mathscr{F}}) \subseteq \sigma(A) \cap i\mathbf{R},$$

where $B_{f,\mathscr{F}}$ is the generator of the restriction of $\mathscr{S}_{\mathscr{F}}$ to the closed linear span of $\{S_{\mathscr{F}}(t)\pi_{\mathscr{F}} f : t \in \mathbf{R}\}$ in $Y_{\mathscr{F}}$. In particular, $\sigma(B_{f,\mathscr{F}})$ is countable, and $\mathscr{F}$ contains $M_\eta(f)$ whenever $i\eta \in \sigma(B_{f,\mathscr{F}})$. The proof is now completed exactly as in Theorem 2.3.

COROLLARY 3.3. *Suppose that $\sigma(A) \cap i\mathbf{R}$ is countable, and $\phi \in \mathrm{AAP}(\mathbf{R}_+, X)$ (respectively, $\phi \in W(\mathbf{R}_+, X)$). Let f be a bounded, uniformly continuous, mild solution of* (3.1), *which is totally ergodic on $\sigma(A) \cap i\mathbf{R}$. Then $f \in \mathrm{AAP}(\mathbf{R}_+, X)$ (respectively, $f \in W(\mathbf{R}_+, X)$). If the means $x_{f,\eta}$ ($i\eta \in \sigma(A) \cap i\mathbf{R}$) and $x_{\phi,\eta}$ ($i\eta \in \rho(A) \cap i\mathbf{R}$) are all zero, then $f \in C_0(\mathbf{R}_+, X)$ (respectively, $f \in W_0(\mathbf{R}_+, X)$).*

Proof. The first statement is immediate from Theorem 3.2 with $\mathscr{F} = \mathrm{AAP}(\mathbf{R}_+, X)$ or $\mathscr{F} = W(\mathbf{R}_+, X)$, respectively. The second statement follows from Proposition 3.1 and the fact that the almost periodic part of a function in $W(\mathbf{R}_+, X)$ is uniquely determined by its means.

The analogue of Corollary 3.3 for solutions of inhomogeneous Cauchy problems on $\mathbf{R}$ was proved in [**27**, Theorem 4.3], under the additional assumption that A generates a C_0-semigroup, and in [**12**] for solutions of homogeneous problems without that additional assumption. Some results in the direction of Corollary 3.3 were also given in [**5**, Section 3].

There are numerous examples in the literature of closed operators A where $\sigma(A) \cap i\mathbf{R}$ is countable. They include all operators with compact resolvent, many elliptic differential operators and Schrödinger operators (see [**7**, Section 6]), examples from mathematical biology (see [**24**, Section C-IV, Example 2.15], [**31**], [**32**, Proposition 6.2]), and various examples constructed to exhibit the sharpness of Corollary 2.5 and its variants (see [**10**, Section 3], [**6**, Section 4]). Moreover, [**1**, Example 2.5(a)] shows that the countability assumption cannot be improved, even in the case of homogeneous Cauchy problems. There are also various examples [**28**, Examples 4.1, 4.7, 4.10, 4.14] showing that the conclusions of Corollary 3.3 are sharp even when $A = 0$.

On the other hand, in practical examples, it may not be easy to verify that a mild solution f of (3.1) satisfies the assumptions of being bounded and uniformly continuous (and totally ergodic on $\sigma(A) \cap i\mathbf{R}$). While our present results have theoretical interest and may have theoretical applications, they will become more applicable when combined with results showing that classes of solutions of inhomogeneous Cauchy problems are automatically bounded and uniformly continuous (and totally ergodic). Some results of this nature appear in [**5**, Sections 5 and 6], and research in this area is continuing.

There are two special cases when Theorem 3.2 and Corollary 3.3 simplify. One is when $A = 0$. Then a mild solution of the inhomogeneous Cauchy problem is simply the indefinite integral, and Corollary 3.3 becomes the result that if $\phi \in \mathrm{AAP}(\mathbf{R}_+, X)$ (respectively, $\phi \in W(\mathbf{R}_+, X)$), $f(t) = \int_0^t \phi(s)\,ds$ and f is bounded and uniformly ergodic at 0, then $f \in \mathrm{AAP}(\mathbf{R}_+, X)$ (respectively, $f \in W(\mathbf{R}_+, X)$). This case is relatively elementary, and has already been proved in [**4**, Theorem 3.1.1], [**27**, Theorem 2.1]. (The case of almost periodic functions on $\mathbf{R}$ was proved in [**19**].)

The second case is when $\sigma(A) \cap i\mathbf{R}$ is empty (see [**27**, Corollary 4.5]). Then the condition of total ergodicity in Corollary 3.3 becomes vacuous. Moreover, the almost periodic part of f is determined by its means, which are given in terms of the means of ϕ by Proposition 3.1 (3).

We conclude this section by recording the fact (which is part of the folklore of the subject) that when the inhomogeneity ϕ and solution f are both (weakly) asymptotically almost periodic (for example, the situation of Corollary 3.3), their decompositions into an almost periodic part and a singular part produce solutions of corresponding Cauchy problems.

PROPOSITION 3.4. *Let $f \in W(\mathbf{R}_+, X)$, $\phi \in W(\mathbf{R}_+, X)$, and suppose that f is a mild solution of* (3.1). *Let $f = f_0 + f_1$ and $\phi = \phi_0 + \phi_1$, where $f_0 \in W_0(\mathbf{R}_+, X)$, $f_1 \in \mathrm{AP}(\mathbf{R}_+, X)$, $\phi_0 \in W_0(\mathbf{R}_+, X)$ and $\phi_1 \in \mathrm{AP}(\mathbf{R}_+, X)$. Then f_0 and f_1 are mild solutions of*

$$f_0'(t) = Af_0(t) + \phi_0(t) \quad (t \geqslant 0),$$
$$f_1'(t) = Af_1(t) + \phi_1(t) \quad (t \geqslant 0),$$

respectively.

Proof. There is a net $\tau_i \to \infty$ such that $\|S(\tau_i)f_1 - f_1\| \to 0$, $S(\tau_i)f_0 \to 0$ weakly in $\mathrm{BUC}(\mathbf{R}_+, X)$, $\|S(\tau_i)\phi_1 - \phi_1\| \to 0$ and $S(\tau_i)\phi_0 \to 0$ weakly in $\mathrm{BUC}(\mathbf{R}_+, X)$. For each $t \geqslant 0$, $f(t+\tau_i) = (S(\tau_i)f)(t) \to f_1(t)$ weakly in X,

$$\int_0^t f(s+\tau_i)\,ds = \int_0^t (S(\tau_i)f)(s)\,ds \to \int_0^t f_1(s)\,ds$$

weakly in X, and similarly for ϕ. Since f is a mild solution,

$$f(t+\tau_i) = A\left(\int_0^t f(s+\tau_i)\,ds\right) + \int_0^t \phi(s+\tau_i)\,ds + f(\tau_i).$$

Since A is (weakly) closed, it follows on taking the limit that

$$f_1(t) = A\left(\int_0^t f_1(s)\,ds\right) + \int_0^t \phi_1(s)\,ds + f_1(0),$$

so f_1 is a mild solution with inhomogeneity ϕ_1. By linearity, $f_0 = f - f_1$ is a mild solution with inhomogeneity ϕ_0.

4. *Volterra equations*

Consider an inhomogeneous abstract linear Volterra equation:

$$u(t) = \psi(t) + \int_0^t a(t-s)\, Au(s)\, ds \quad (t \geqslant 0). \tag{4.1}$$

Here, $a \in L^1_{\text{loc}}(\mathbf{R}_+)$, and $\psi : \mathbf{R}_+ \to X$ is continuous. A *mild solution* of (4.1) is a continuous function $u : \mathbf{R}_+ \to X$ such that, for each $t \geqslant 0$,

$$(a * u)(t) := \int_0^t a(t-s)\, u(s)\, ds \in D(A),$$

and

$$u(t) = \psi(t) + A((a*u)(t)).$$

Assuming that the Laplace transforms all exist, we obtain

$$\hat{u}(\lambda) = \hat{\psi}(\lambda) + \hat{a}(\lambda)\, A(\hat{u}(\lambda)),$$

so

$$(I - \hat{a}(\lambda)\, A)\, \hat{u}(\lambda) = \hat{\psi}(\lambda). \tag{4.2}$$

Thus knowledge of the singularities of $\hat{\psi}$ and $\hat{a}$, and the spectrum of A, can provide enough information to ensure that $\mathrm{Sp}_{\mathbf{R}_+}(u)$ is countable, and therefore to apply Corollary 2.4 to the solutions.

THEOREM 4.1. *Let u be a bounded, uniformly continuous, mild solution of* (4.1), *where $\int_0^\infty e^{-\omega t}(|a(t)| + \|\psi(t)\|)\, dt < \infty$ for some $\omega > 0$. Suppose that there is an open set U in $\mathbf{C}$ and holomorphic functions $g : U \to \mathbf{C}$ and $h : U \to X$ such that*:

(1) *$i\mathbf{R} \backslash U$ is countable*;

(2) *each connected component of $\{\lambda \in U : \mathrm{Re}\, \lambda > 0\}$ intersects $\{\lambda : \mathrm{Re}\, \lambda > \omega\}$*;

(3) *for λ in U with $\mathrm{Re}\, \lambda > \omega$, $g(\lambda) = \hat{a}(\lambda)$ and $h(\lambda) = \hat{\psi}(\lambda)$*;

(4) *for λ in U, $g(\lambda) \neq 0$ and $g(\lambda)^{-1} \in \rho(A)$*;

(5) *u is totally ergodic on $i\mathbf{R} \backslash U$.*

Then u is asymptotically almost periodic. If the means $x_{u,\eta}$ are zero except for finitely many η, then $\|u(t) - \sum_\eta e^{i\eta t} x_{u,\eta}\| \to 0$ as $t \to \infty$.

Proof. From (4.2) and the assumptions (3) and (4),

$$\hat{u}(\lambda) = (I - g(\lambda)\, A)^{-1}\, h(\lambda) \tag{4.3}$$

whenever $\lambda \in U$ with $\mathrm{Re}\, \lambda > \omega$. The assumption (2) and analytic continuation imply that (4.3) is valid whenever $\lambda \in U$ with $\mathrm{Re}\, \lambda > 0$. Hence $\hat{u}$ has a holomorphic extension near each point of $U \cap i\mathbf{R}$. The assumption (1) implies that $\mathrm{Sp}_{\mathbf{R}_+}(u)$ is countable, and (5) implies that u is totally ergodic, so the result now follows from Corollary 2.4.

In [**3**, Theorem 5.1], the homogeneous case, $\psi(t) \equiv x$, was considered. Then $\hat{\psi}(\lambda) = \lambda^{-1} x$ for $\mathrm{Re}\, \lambda > 0$. It was assumed that the equation is well-posed, so that there is a strongly continuous family $\{S(t) : t \geqslant 0\}$ of bounded linear operators on X, such that $S(t)$ commutes with A and $u(t, x) = S(t)\, x$ is a mild solution of the homogeneous equation with initial value x in X. It was further assumed that $\sup_t \|S(t)\| < \infty$. This implies that $\hat{a}$ has a meromorphic extension to the open right half-plane, and that $\hat{a}(\lambda) \neq 0$ and $\hat{a}(\lambda)^{-1} \in \rho(A)$ whenever $\mathrm{Re}\, \lambda > 0$ [**26**, p. 43]. It was also assumed that $\{S(t)\}$ is strongly Abel ergodic, with limit Q. It is straightforward to verify that the other assumptions of [**3**] are sufficient to ensure that all solutions are bounded, uniformly continuous and totally ergodic, with $M_0(u) \equiv Qx$ and $M_\eta(u) = 0$ for non-zero η. Thus the result of [**3**, Theorem 5.1] that $\|S(t)x - Qx\| \to 0$ as $t \to \infty$ can be obtained from Theorem 4.1, if we assume that $\hat{u}$ has a holomorphic extension to a neighbourhood of each point of $i(\mathbf{R}\backslash E)$, for some closed countable set E.

In [**3**], it was not assumed that $\hat{u}$ has a holomorphic extension, but only that it has a continuous extension to $\mathbf{C}_+ \cup i(\mathbf{R}\backslash E)$. To recover Theorem 5.1 of [**3**] in full from a Tauberian theorem, we would therefore need to prove a version of Theorem 2.3 in which the spectrum of f is taken to be the set of points ξ such that $\hat{f}$ does not have a continuous extension from the right half-plane to an open interval in $i\mathbf{R}$ containing $i\xi$. This could be achieved by establishing versions of Lemmas 2.1 and 2.2 for continuous (as opposed to holomorphic) extensions. We do this in the next section for the former result, but we have been unable to adjust the latter result in this way.

The most appropriate inhomogeneous equations are those when $\psi = a * \phi$ for some function ϕ. Then

$$\hat{u}_s(\lambda) = \lambda^{-1}u(s) + \hat{a}(\lambda)A\hat{u}_s(\lambda) + \hat{a}(\lambda)\hat{\phi}_s(\lambda) + A((b(\lambda)*u)(s)) + (b(\lambda)*\phi)(s),$$

where $u_s = S(s)u$, $\phi_s = S(s)\phi$, $a_t = S(t)a$ and $b(\lambda)(t) = \hat{a}_t(\lambda) - a(t)/\lambda$. To proceed with the argument of Theorem 3.2, one would like to show that there is a holomorphic function G, defined near most points of the imaginary axis, with values in $\mathrm{BUC}(\mathbf{R}_+, X)$, such that

$$G(\lambda)(s) = A(I - \hat{a}(\lambda)A)^{-1}((b(\lambda)*u)(s)) + (I - \hat{a}(\lambda)A)^{-1}((b(\lambda)*\phi)(s))$$

for $\mathrm{Re}\,\lambda > 0$. Except in the special cases when $a(t) = 1$ (first-order Cauchy problems, discussed in Section 3) and when $a(t) = t$ (second-order Cauchy problems, discussed in [**2**]), there do not seem to be natural conditions which ensure this.

5. *Continuous extensions of the resolvent of isometric semigroups*

In this section, we improve Lemma 2.2 (and [**8**, Theorem 2.2], the analogue for semigroups of isometries) by showing that the assumption of a holomorphic extension of $R(\lambda, A)x$, defined on $\mathbf{C}_+ := \{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > 0\}$, to a neighbourhood of $i\xi$ in $\mathbf{C}$ can be weakened to the assumption of a continuous extension to an open subset of $i\mathbf{R}$ containing $i\xi$.

LEMMA 5.1. *Let A be the generator of a C_0-group $\{U(t): t \in \mathbf{R}\}$ of isometries on a Banach space Z, let $x \in Z$, and suppose that*

$$\lim_{\substack{\lambda \to 0 \\ \mathrm{Re}\,\lambda > 0}} R(\lambda, A)x := y$$

exists. Then

$$\lim_{\substack{\lambda \to 0 \\ \mathrm{Re}\,\lambda < 0}} R(\lambda, A)x = y.$$

Proof. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $\|R(\alpha + i\beta, A)x - y\| < \varepsilon$ whenever $0 < \alpha < \delta, |\beta| < \delta$. Now we consider $R(-\alpha + i\beta)x$ for such α and β, by applying the resolvent identity on three sides of a triangle. Let

$$\gamma = \begin{cases} \beta - \alpha & \text{if } \beta \geqslant 0, \\ \beta + \alpha & \text{if } \beta < 0. \end{cases}$$

By the resolvent identity,

$$\|\alpha R(\alpha+i\gamma, A)\, R(\alpha+i\beta, A)\, x\| = \|R(\alpha+i\gamma, A)\, x - R(\alpha+i\beta, A)\, x\| < 2\varepsilon. \tag{5.1}$$

Since $\|\alpha R(-\alpha+i\beta, A)\| \leqslant 1$, it follows that

$$\|\alpha^2 R(-\alpha+i\beta, A)\, R(\alpha+i\gamma, A)\, R(\alpha+i\beta, A)\, x\| < 2\varepsilon.$$

By the resolvent identity,

$$\begin{aligned} &\|\alpha R(-\alpha+i\beta, A)\, R(\alpha+i\beta, A)\, x - \alpha R(\alpha+i\gamma, A)\, R(\alpha+i\beta, A)\, x\| \\ &= \|\alpha(2\alpha \pm i\alpha)\, R(-\alpha+i\beta, A)\, R(\alpha+i\gamma, A)\, R(\alpha+i\beta, A)\, x\| < 2\sqrt{5}\varepsilon. \end{aligned}$$

By (5.1),

$$\|\alpha R(-\alpha+i\beta, A)\, R(\alpha+i\beta, A)\, x\| < 2(1+\sqrt{5})\,\varepsilon.$$

Using the resolvent identity again,

$$\begin{aligned} \|R(-\alpha+i\beta, A)\, x - y\| &\leqslant \|R(\alpha+i\beta, A)\, x - y\| + \|R(-\alpha+i\beta, A)\, x - R(\alpha+i\beta, A)\, x\| \\ &< \varepsilon + \|2\alpha R(-\alpha+i\beta, A)\, R(\alpha+i\beta, A)\, x\| \\ &< (5+4\sqrt{5})\,\varepsilon. \end{aligned}$$

Proposition 5.2. *Let A be the generator of a C_0-group $\mathscr{U}$ of isometries on Z, let $x \in Z$, and let W be an open subset of* $\mathbf{R}$. *Suppose that there is a continuous map $g : \mathbf{C}_+ \cup iW \to Z$ such that $g(\lambda) = R(\lambda, A)\, x$ whenever* $\operatorname{Re}\lambda > 0$. *Then g has an extension to a holomorphic map $h : \mathbf{C} \setminus i(\mathbf{R} \setminus W) \to Z$. Hence $iW \subseteq \rho(A_x)$, where A_x is the generator of the restriction of $\mathscr{U}$ to the closed linear span of $\{U(t)\, x : t \in \mathbf{R}\}$ in Z.*

Proof. Define

$$h(\lambda) = \begin{cases} R(\lambda, A)\, x & (\lambda \in \mathbf{C} \setminus i\mathbf{R}), \\ g(\lambda) & (\lambda \in iW). \end{cases}$$

By the assumption and by Lemma 5.1 applied with A replaced by $A - i\eta$ for η in W, h is continuous. Moreover, h is holomorphic in $\mathbf{C} \setminus i\mathbf{R}$. By a standard application of Morera's Theorem, it follows that h is holomorphic throughout its domain. The final statement follows from Lemma 2.2.

Corollary 5.3. *Let A be the generator of a C_0-semigroup $\mathscr{T}$ of isometries on Z, let $x \in Z$, and let W be an open subset of* $\mathbf{R}$. *Suppose that there is a continuous map $g : \mathbf{C}_+ \cup iW \to Z$ such that $g(\lambda) = R(\lambda, A)\, x$ whenever* $\operatorname{Re}\lambda > 0$. *Then $iW \subseteq \rho(A_x)$, where A_x is the generator of the restriction of $\mathscr{T}$ to the closed linear span of $\{T(t)\, x : t \geqslant 0\}$ in Z.*

Proof. There is an extension of $\mathscr{T}$ to a C_0-group $\mathscr{U}$ of isometries, with generator B, on a Banach space Y containing Z [**13**]. Then $g(\lambda) = R(\lambda, A)\, x = R(\lambda, B)\, x$ whenever $\operatorname{Re}\lambda > 0$. By Proposition 5.2, $iW \subseteq \rho(B_x)$, where B_x is the generator of the restriction of $\mathscr{U}$ to the closed linear span of $\{U(t)\, x : t \in \mathbf{R}\}$. However, $\sigma(A_x) \cap i\mathbf{R}$ consists of approximate eigenvalues of A_x and hence of B_x. Hence $iW \subseteq \rho(A_x)$.

Remark. In the context of the proof of Corollary 5.3, suppose that W is non-empty. It follows from the result of Corollary 5.3 and [**25**, p. 152] that $U(t)\, x \in \overline{\operatorname{span}}\{T(s)\, x : s \geqslant 0\}$, so $A_x = B_x$.

NOTE ADDED IN PROOF. After work on this paper was completed, Ralph Chill ('Tauberian theorems for vector-valued Fourier and Laplace transforms', preprint) succeeded in improving the Tauberian theorem (Theorem 2.3) in the way discussed in the fourth paragraph of the introduction.

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Abteilung Mathematik V
Universität Ulm
89069 Ulm
Germany

St John's College
Oxford OX1 3JP