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Wiener Regularity and Heat Semigroups on Spaces of Continuous Functions

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Dedicated to Professor Herbert Amann on the occasion of his sixtieth birthday

Abstract. Let $\Omega \subset \mathbb{R}^N$ be open. It is shown that the Dirichlet Laplacian generates a (holomorphic) C_0-semigroup on $C_0(\Omega)$ if and only if Ω is regular in the sense of Wiener. The same result remains true for elliptic operators in divergence form.

0. Introduction

Let Ω be an open subset of $\mathbb{R}^N$. By $C_0(\Omega)$ we denote the space of all continuous scalar valued functions f on $\bar{\Omega}$ which are 0 on the boundary $\partial\Omega$ of Ω and 0 at infinity; i.e., $C_0(\Omega) = \{f : \Omega \to \mathbb{C}$ continuous: $\lim_{x\to z} f(x) = 0$ for all $z \in \partial\Omega$ and $\lim_{\substack{|x|\to\infty \\ x\in\Omega}} f(x) = 0\}$. Then $C_0(\Omega)$ is a Banach space for the supremum norm

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)|\,,$$

and a Banach lattice for pointwise ordering. We consider the Laplacian Δ_0 on $C_0(\Omega)$ with maximal distributional domain; i.e.,

$$D(\Delta_0) = \{f \in C_0(\Omega) : \Delta f \in C_0(\Omega)\} \quad \Delta_0 f = \Delta f\,.$$

It is a closed, dissipative operator. Our aim is to characterize those open sets Ω for which Δ_0 is the generator of a C_0-semigroup on $C_0(\Omega)$.

We say that Ω is **regular (in the sense of Wiener)** if at each point $z \in \partial\Omega$ there exists a **barrier** (see Definition 3.1). If Ω is bounded then, by classical potential theory (see e.g. [DL], [GT]), Ω is regular if and only if the Dirichlet problem

$$D(\varphi) \begin{cases} u \in C(\bar{\Omega}),\ u_{|\partial\Omega} = \varphi \\ \Delta u = 0 \quad \text{in} \quad \mathcal{D}(\Omega)' \end{cases}$$

has a solution for all $\varphi \in C(\partial\Omega)$. The purpose of this paper is to establish the following result:

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Theorem 0.1. *Let $\Omega \subset \mathbb{R}^N$ be open. The following assertions are equivalent:*
(i) *Ω is regular;*
(ii) *the resolvent set of Δ_0 is non-empty;*
(iii) *Δ_0 generates a holomorphic C_0-semigroup on $C_0(\Omega)$.*

The proof of Theorem 0.1 is based on classical potential theory and the methods are well known. Still it seems that the result is nowhere given explicitely in the literature. On the other hand, it is certainly of importance. For example, treating the semilinear heat equation and associated dynamical systems, the realization of the Laplacian on the space $C_0(\Omega)$ is precisely what is needed, (cf. Cazanave-Haraux [CH], where, even though merely the case of Lipschitz boundary is treated, only vague references and indications for the proof are given; see [CH], Lemma 2.6.5 and Remark 2.6, in particular). In this paper we give complete proofs and also describe in detail properties of the Laplacian we use.

The paper is organized as follows. Section 1 contains prerequisites concerning the heat semigroups on $L^p(\Omega)$. Then Theorem 0.1 is proved for bounded Ω in Section 3 by establishing the equivalence of *(ii)* and *(iii)* to well-posedness of the Dirichlet problem. In Section 4 we prove the result for arbitrary open sets establishing equivalence to regularity in the sense that there is a barrier at each point. These two sections are completely elementary and self-contained. In Section 4 we prove the corresponding result for elliptic operators. Here we use the fundamental result of De Giorgi and Nash on (Hölder-) continuity of weak solutions and further regularity results by Stampacchia. We also show several spectral and regularity results for the elliptic operators on $C_0(\Omega)$. In particular, we prove holomorphy of the semigroup on $C_0(\Omega)$ using that the adjoint semigroup is holomorphic on $L^1(\Omega)$ (see [AB], [AE], [Ou2]). For previous results exploiting this duality see Amann [Am] and Amann and Escher [AmE].

1. The Dirichlet Laplacian on $\mathbf{L}^p(\Omega)$

In this section we put together some known results on the Laplacian with Dirichlet boundary conditions. Let $\Omega \subset \mathbb{R}^N$ be an open set. By $\mathcal{D}(\Omega)$ we denote the space of all test functions and by $\mathcal{D}(\Omega)'$ the space of all distributions. The first Sobolev space is denoted by

$$H^1(\Omega) := \{u \in L^2(\Omega) : D_j u \in L^2(\Omega),\ j = 1 \dots N\}$$

where $D_j = \frac{\partial}{\partial x_j}$. Moreover, we let

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}.$$

Recall that for real-valued $u \in H^1(\Omega)$ one has $u^+, u^-, (u-1)^+ \in H^1(\Omega)$ and

$$D_j u^+ = 1_{[u>0]} D_j u,\ D_j (u-1)^+ = 1_{[u>1]} D_j u\,. \tag{1.1}$$

Moreover, $u^+, (u-1)^+ \in H_0^1(\Omega)$ if $u \in H_0^1(\Omega)$. Note also that the mappings $u \mapsto u^+$ and $u \mapsto (u-1)^+$ at continuous from $H^1(\Omega)$ into $H^1(\Omega)$. See [GT], Chapter 7, for these simple facts. By Δ_2^Ω we denote the **Laplacian with Dirichlet boundary conditions** on $L^2(\Omega)$, also called the **Dirichlet Laplacian** for short; i.e.

$$D(\Delta_2^\Omega) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\},\ \Delta_2^\Omega u = \Delta u\,.$$

This is the operator associated with the form

$$a(u,v) = \int_\Omega \nabla u \nabla u$$

on $H_0^1(\Omega)$; i.e., for $u \in H_0^1(\Omega)$, $\quad v \in L^2(\Omega)$ we have

$$u \in D(\Delta_2^\Omega), \Delta_2^\Omega u = v \ \Leftrightarrow\ -\int \nabla u \nabla \varphi = \int v\varphi \quad \forall \varphi \in H_0^1(\Omega)\,.$$

The operator Δ_2^Ω is self-adjoint and generates a positive contraction C_0-semigroup $(e^{t\Delta_2^\Omega})_{t\geq 0}$ on $L^2(\Omega)$. Moreover, $0 \in \varrho(\Delta_2^\Omega)$ (the resolvent set of Δ_2^Ω) if Ω is bounded. If $\Omega = \mathbb{R}^N$ we write $\Delta_2 = \Delta_2^{\mathbb{R}^N}$. Then $(e^{t\Delta_2})_{t\geq 0}$ is the *Gaussian semigroup* which we denote by G_2; i.e.,

$$(G_2(t)f)(x) = (4\pi t)^{-N/2} \int_{\mathbb{R}^N} f(y) e^{-(x-y)^2/4t} dy$$

for all $f \in L^2(\mathbb{R}^N)$.

We denote by $\varrho(A)$ the resolvent set of an operator A, and by $R(\lambda, A) = (\lambda - A)^{-1}$ its resolvent ($\lambda \in \varrho(A)$). The following domination property is well known (cf. [Da], Theorem 2.1.6).

For convenience of the reader we give a simple direct proof.

Proposition 1.1. *Let Ω_1, Ω_2 be open subsets of $\mathbb{R}^N$ such that $\Omega_1 \subset \Omega_2$. Then*

$$\begin{aligned} 0 &\leq e^{t\Delta_2^{\Omega_1}} f &&\leq e^{t\Delta_2^{\Omega_2}} f\ , \\ 0 &\leq R(\lambda, \Delta_2^{\Omega_1}) f &&\leq R(\lambda, \Delta_2^{\Omega_2}) f \end{aligned}$$

for $f \in L^2(\Omega)_+,\ \lambda > 0,\ t \geq 0$.

Here we identify $L^2(\Omega_1)$ with a subspace of $L^2(\Omega_2)$ extending functions by 0. We let $L^2(\Omega)_+ = \{f \in L^2(\Omega) : f(x) \geq 0 \ \text{ a.e.}\}, H^1(\Omega)_+ = L^2(\Omega)_+ \cap H^1(\Omega)$, $H_0^1(\Omega) = H_0^1(\Omega) \cap L^2(\Omega)_+$, $\mathcal{D}(\Omega)_+ = \{\varphi \in \mathcal{D}(\Omega) : \varphi(x) \geq 0 \quad \text{for all} \quad x \in \Omega\}$. For $w_1, w_2 \in D(\Omega)'$ we define

$$w_1 \leq w_2 \ :\Leftrightarrow\ \langle w_1, \varphi\rangle \ \leq\ \langle w_2, \varphi\rangle \quad \text{for all} \quad \varphi \in \mathcal{D}(\Omega)_+\,.$$

Note that $\mathcal{D}(\Omega)_+$ is dense in $H_0^1(\Omega)_+$. Moreover, we recall: If $u \in H^1(\Omega)$ has compact support, then $u \in H_0^1(\Omega)$ (see [B], IX.5 p. 171).

The proof of Proposition 1.1 is based on the following.

Lemma 1.2. *Let $\Omega \subset \mathbb{R}^N$ be open, $\lambda > 0$. Let $u \in H_0^1(\Omega)$, $v \in H^1(\Omega)$ such that $v \geq 0$ and*

$$\lambda u - \Delta u \leq \lambda v - \Delta v \quad in \quad \mathcal{D}(\Omega)'\,.$$

Then $u(x) \leq v(x)$ a.e.. Taking in particular $v = 0$, we obtain for $u \in H_0^1(\Omega)$

$$\lambda u - \Delta u \leq 0 \ \Rightarrow u \ \leq 0\,. \tag{1.2}$$

Proof. By hypothesis we have

$$\lambda \int (u-v)\varphi + \int \nabla(u-v)\nabla\varphi \leq 0 \tag{1.3}$$

for all $\varphi \in \mathcal{D}(\Omega)_+$. It follows by density that (1.3) remains true for all $\varphi \in H_0^1(\Omega)_+$. Note that $(u-v)^+ \in H_0^1(\Omega)$. In fact, let $u_n \in \mathcal{D}(\Omega)$, $u_n \to u$ in $H^1(\Omega)$. Then $(u_n - v)^+ \in H^1(\Omega)$ has compact support and thus $(u_n - v)^+ \in H_0^1(\Omega)$. Since $(u_n - v)^+ \to (u-v)^+ \quad (n \to \infty)$ in $H^1(\Omega)$, the claim follows. Now (1.3) for $\varphi = (u-v)^+$ yields

$$\lambda \int ((u-v)^+)^2 + \int |\nabla(u-v)^+|^2 \leq 0\,.$$

Hence $(u-v)^+ = 0$. □

Proof of Proposition 1.1. a) Let $\lambda > 0$, $f \in L^2(\Omega_1)_+$, $u = R(\lambda, \Delta_2^{\Omega_1})f$, $v = R(\lambda, \Delta_2^{\Omega_2})f$. We show that $0 \leq u \leq v$. In fact, $u \in H_0^1(\Omega_1)$, $\lambda u - \Delta u = f$ in $D(\Omega_1)'$, $v \in H_0^1(\Omega_2)$, $\lambda v - \Delta v = f$ in $\mathcal{D}(\Omega_2)'$. It follows from (1.2) that $v(x) \geq 0$ and $u(x) \geq 0$ a.e.. Since $\lambda u - \Delta u = \lambda v - \Delta v$ in $\mathcal{D}(\Omega_2)'$, it follows from Lemma 1.2 that $u \leq v$ a.e..
b) Let $0 \leq f \in L^2(\Omega_1)$. Then it follows from a) that $(I - t\Delta_2^{\Omega_1})^{-1} f \leq (I - t\Delta_2^{\Omega_2})^{-1} f$ for all $t > 0$. Hence $e^{t\Delta_2^{\Omega_1}} f = \lim_{n\to\infty} (I - \frac{t}{n}\Delta_2^{\Omega_1})^{-n} f \leq \lim_{n\to\infty} (I - \frac{t}{n}\Delta_2^{\Omega_2})^{-n} f = e^{t\Delta_2^{\Omega_2}} f$. □

It follows from Proposition 1.1 that

$$0 \leq e^{t\Delta_2^{\Omega}} f \leq G(t)f \quad (t \geq 0)\,, \tag{1.4}$$

$$0 \leq R(\lambda, \Delta_2^{\Omega})f \leq R(\lambda, \Delta_2)f \quad (\lambda > 0) \tag{1.5}$$

for all $f \in L^2(\Omega)_+$.

Since the Gaussian semigroup is a contractive C_0-semigroup on $L^p(\mathbb{R}^N)$, it follows that there exist positive contraction C_0-semigroups $(e^{t\Delta_p^{\Omega}})_{t\geq 0}$ on $L^p(\Omega)$ such that

$$e^{t\Delta_p^{\Omega}} f = e^{t\Delta_p^{\Omega}} f \quad (t \geq 0)$$

for all $1 \leq p,\ q < \infty,\ f \in L^p(\Omega) \cap L^q(\Omega)$. Moreover,

$$0 \leq R(\lambda, \Delta_p^{\Omega})f \leq R(\lambda, \Delta_p)f \tag{1.6}$$

for all $f \in L^p(\Omega)_+$, $\lambda > 0$, $p \in [1, \infty)$. Finally, defining Δ_∞^{Ω} as the adjoint of Δ_1^{Ω}, we have consistency in the sense that

$$R(\lambda, \Delta_p^{\Omega})f = R(\lambda, \Delta_q^{\Omega})f \quad (f \in L^p(\Omega) \cap L^q(\Omega))$$

for $\lambda > 0$, $1 \leq p, q \leq \infty$. Moreover, for each $p \in [1, \infty]$, one has $\Delta_p^{\Omega} f = \Delta f$ in $\mathcal{D}(\Omega)'$. It has been proved by Ouhabaz [Ou1] that Gaussian estimates (and in particular (1.4)) imply that the semigroup generated by Δ_1 is holomorphic of angle

$\frac{\pi}{2}$ (see also [AE]). Denoting by Δ_∞ the adjoint of Δ_1 we obtain in particular that $\sigma(\Delta_\infty) \subset (-\infty, 0]$ and a bound

$$\|\lambda R(\lambda, \Delta_\infty)\| \leq M_\theta \tag{1.7}$$

for all $\lambda \in \Sigma(\theta) = \{re^{i\alpha},\ r > 0,\ |\alpha| < \theta\}$ whenever $\theta \in [0, \pi)$. This estimate will be used later.

Next we establish some spectral properties. Let E, F be Banach spaces such that $F \hookrightarrow E$ (i.e. F is continuously injected into E). Let A be an operator on E. The **part** B of A in F is defined by

$$D(B) = \{x \in D(A) \cap F : Ax \in F\} \quad Bx = Ax\,.$$

This notation is motivated by the following observation. Let $\lambda \in \varrho(A)$. Then

$$\lambda \in \varrho(B) \ \Leftrightarrow\ R(\lambda, A)F \subset F\,, \tag{1.8}$$

and in this case, $R(\lambda, B) = R(\lambda, A)_{|F}$. A proof of the following easy result can be found in [ANS].

Proposition 1.3. *Assume that $\varrho(A) \neq \emptyset$. If there exists $k \in \mathbb{N}$ such that $D(A^k) \subset F$, then $\sigma(A) = \sigma(B)$.*

Now by (1.6) we have

$$0 \leq R(1, \Delta_p^\Omega)^k \leq R(1, \Delta_p)^k \tag{1.9}$$

for $1 \leq p < \infty,\ k \in \mathbb{N}$. Since $D(\Delta_2^k) = H^{2k}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ for $k > \frac{N}{4}$ we conclude that

$$D((\Delta_p^\Omega)^k) \subset L^\infty(\Omega) \quad \text{for} \quad k > \frac{N}{4},\ 2 \leq p \leq \infty\,. \tag{1.10}$$

If Ω has bounded Lebesgue measure, it follows from Proposition 1.3 that $\sigma(\Delta_p^\Omega) = \sigma(\Delta_\infty^\Omega)$ for $p \in [2, \infty]$. By duality and selfadjointness we conclude

Proposition 1.4. *Let $\Omega \subset \mathbb{R}^N$ be open and of finite measure. Then*

$$\sigma(\Delta_p^\Omega) = \sigma(\Delta_2^\Omega) \quad (1 \leq p \leq \infty)\,. \tag{1.11}$$

Moreover, $R(\lambda, \Delta_p^\Omega) = R(\lambda, \Delta_q^\Omega)_{|L^q}$ for all $\lambda \in \sigma(\Delta_p^\Omega),\ \infty \geq p \geq q \geq 1$.

If Ω is arbitrary, then more elaborate arguments are needed, but (1.11) still remains true (see [Ar]). We also note that

$$R(1, \Delta_p)L^p(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$$

for $p > \frac{N}{2}$. Hence

$$D(\Delta_p^\Omega) \subset L^\infty(\Omega) \tag{1.12}$$

for $\frac{N}{2} < p \leq \infty$. Finally we recall the following result by de Pagter [dP].

Proposition 1.5. *Let A be the generator of a C_0-semigroup on a Banach space E. Assume that $R(\lambda, A)^k$ is compact for some $k \in \mathbb{N},\ \lambda \in \varrho(A)$. Then A has compact resolvent.*

2. Bounded domains

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open, nonempty set and $C_0(\Omega) = \{u \in C(\bar{\Omega}) : u_{|\partial\Omega} = 0\}$. By Δ_0^Ω we denote the Laplacian on $C_0(\Omega)$ with maximal domain; i.e.,

$$D(\Delta_0^\Omega) = \{u \in C_0(\Omega) : \Delta u \in C_0(\Omega)\} \quad \Delta_0^\Omega u = \Delta u\,.$$

Here, for $u \in L^1_{\text{loc}}(\Omega)$ we denote by $\Delta u \in \mathcal{D}(\Omega)'$ the distributional Laplacian of u. Note that $D(\Delta_0^\Omega) \not\subset C^2(\Omega)$ as is well known (cf. [DL], II. § 3 Remark 5). We recall the following easy local properties of the Laplacian (see e.g., [DL], II. § 3 Prop. 6 p. 336).

Lemma 2.1. *Let $u \in \mathcal{D}(\Omega)'$.*

a) *If $\Delta u \in L^p_{\text{loc}}(\Omega)$, $p > \frac{N}{2}$, then $u \in C(\Omega)$;*
b) *if $\Delta u \in L^p_{\text{loc}}(\Omega)$, $p > N$, then $u \in C^1(\Omega)$.*

In particular,

$$D(\Delta_0^\Omega) \subset C^1(\Omega)\,. \tag{2.1}$$

Next we show that Δ_0^Ω is the part of Δ_2^Ω in $C_0(\Omega)$.

Lemma 2.2. *a) Let $u \in C_0(\Omega)$ such that $\Delta u \in L^p(\Omega)$ where $p > N$. Then $u \in H_0^1(\Omega)$.*
b) In particular, the operator Δ_0^Ω is the part of Δ_2^Ω in $C_0(\Omega)$.

Proof. Let $u \in C_0(\Omega)$ such that $\Delta u = f \in L^p(\Omega)$ where $p > N$. We can assume that u is real-valued. Let $\varepsilon > 0$. Then $(u-\varepsilon)^+$ has compact support. Let $\omega \subset \Omega$ be open such that $\bar{\omega} \subset \Omega$ and $\operatorname{supp}(u-\varepsilon)^+ \subset \omega$. Since $u \in C^1(\Omega)$ (by (2.1)), we have $(u-\varepsilon)^+ \in H_0^1(\omega)$. By hypothesis, we have

$$\int \nabla u \nabla \varphi = \int f\varphi \quad \text{for all} \quad \varphi \in \mathcal{D}(\Omega)\,,$$

hence also for all $\varphi \in H_0^1(\omega)$ by density. Taking $\varphi = (u-\varepsilon)^+$ we obtain

$$\begin{aligned}\int (\nabla(u-\varepsilon)^+)^2 &= \int \nabla u \nabla (u-\varepsilon)^+ = \int f(u-\varepsilon)^+ \\ &\leq \|f\|_p \|u\|_q, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1\,.\end{aligned}$$

Thus $\{(u-\varepsilon)^+ : 0 < \varepsilon \leq 1\}$ is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is reflexive, we find a sequence $\varepsilon_n \downarrow 0 \quad (n \to \infty)$ such that $(u-\varepsilon_n)^+$ converges weakly in $H_0^1(\Omega)$, to $v \in H_0^1(\Omega)$, say. Then $(u-\varepsilon_n)^+$ converges weakly to v in $L^2(\Omega) \quad (n \to \infty)$. Since $(u-\varepsilon_n)^+ \to u^+$ in $L^2(\Omega)$, it follows that $u^+ = v \in H_0^1(\Omega)$. In the same way one sees that $u^- \in H_0^1(\Omega)$. □

We say that the **Dirichlet problem is well-posed** in Ω if for every $\varphi \in C(\partial\Omega)$ there exists a solution u of $(D(\varphi))$ (see Introduction). It is known that the solution is unique and in $C^\infty(\Omega)$. Moreover, the Dirichlet problem is well-posed whenever $D(\varphi)$ has a solution for all φ in a dense subspace of $C(\partial\Omega)$.

Theorem 2.3. *Let Ω be a bounded open set in $\mathbb{R}^N$ such the Dirichlet problem is well-posed. Assume that Ω is regular. Then Δ_0^Ω generates a holomorphic C_0-semigroup $T_0 = (T_0(t)_{t\geq 0}$ of angle $\frac{\pi}{2}$ on $C_0(\Omega)$.*

Moreover, $T_0(t)$ is compact for all $t > 0$ and $\sigma(\Delta_0^\Omega) = \sigma(\Delta_2^\Omega)$.

Proof. a) We show that $R(0, \Delta_2^\Omega)L^\infty(\Omega) \subset C_0(\Omega)$. Let $f \in L^\infty(\Omega)$. Let $v = E_N * f \in C(\mathbb{R}^N)$, where E_N is the Newtonian potential. Then $\Delta v = f$ in $\mathcal{D}(\Omega)'$. Let $\varphi = v_{|\partial\Omega}$. By hypothesis there exists $w \in C(\bar{\Omega})$ such that $w_{|\partial\Omega} = \varphi$ and $\Delta w = 0$ in $\mathcal{D}(\Omega)'$. Thus $u = w - v \in C_0(\Omega)$ and $-\Delta u = f$ in $\mathcal{D}(\Omega)'$. It follows from Lemma 2.2 that $u \in H_0^1(\Omega)$. Thus $R(0, \Delta_2^\Omega)f = u \in C_0(\Omega)$.
b) It follows from (1.8) that $0 \in \varrho(\Delta_0^\Omega)$. Moreover, $\varrho(\Delta_0^\Omega) = \varrho(\Delta_\infty^\Omega)$ by Proposition 1.3. By (1.8) again, we have $R(\lambda, \Delta_0^\Omega) = R(\lambda, \Delta_\infty^\Omega)_{|C_0(\Omega)}$ for all $\lambda \in \varrho(\Delta_\infty^\Omega)$. Now it follows from (1.7) that Δ_0^Ω generates a bounded holomorphic C_0-semigroup. Note that $D(\Delta_0^\Omega)$ is dense in $C_0(\Omega)$ since $\mathcal{D}(\Omega) \subset D(\Delta_0^\Omega)$.
c) We show that Δ_0 has compact resolvent. By (1.10) we have $R(0, \Delta_2)^k L^2(\Omega) \subset L^\infty(\Omega)$ where $k > \frac{N}{4}$. Thus by a) $R(0, \Delta_2)^{k+1} L^2(\Omega) \subset C_0(\Omega)$. Note that $R(0, \Delta_2) \in \mathcal{L}(L^2(\Omega))$ is compact. Thus we can write $R(0, \Delta_0)^{k+2} = R(0, \Delta_2)^{k+1} \circ R(0, \Delta_2) \circ j$ where $j : C_0(\Omega) \to L^2(\Omega)$ is the canonial embedding. It follows that $R(0, \Delta_0)^{k+2}$ is compact. Consequently, $R(0, \Delta_0)$ is compact by Proposition 1.5.
d) Finally, from Proposition 1.3 one sees that $\sigma(\Delta_0^\Omega) = \sigma(\Delta_2^\Omega)$. □

Let $C^b(\Omega)$ be the space of all bounded continuous scalar-valued functions on Ω with supremum norm. Denote by Δ_b the part of Δ_2 in $C^b(\Omega)$.
Since $R(0, \Delta_2)C^b(\Omega) \subset C^b(\Omega)$ is follows from Proposition 1.3 and 1.4 that

$$\sigma(\Delta_b) = \sigma(\Delta_2) = \sigma(\Delta_p) \tag{2.2}$$

for $1 \leq p \leq \infty$. Moreover, $R(\lambda, \Delta_b) = R(\lambda, \Delta_2)_{|C^b(\Omega)}$ for all $\lambda \in \varrho(\Delta_2)$.

We now consider necessary conditions for well-posedness of the Dirichlet problem.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^N$ be bounded and open. The following are equivalent:*

(i) *the Dirichlet problem is well-posed;*
(ii) $\varrho(\Delta_0^\Omega) \neq \emptyset$;
(iii) $D(\Delta_p^\Omega) \subset C_0(\Omega)$ *for all* $p \in (\frac{N}{2}, \infty]$;
(iv) *there exist* $\frac{N}{2} < p \leq \infty$, $\lambda \in \varrho(\Delta_p^\Omega)$ *and* $f \in L^p(\Omega)$ *such that* $f(x) > 0$ *a.e. and* $R(\lambda, \Delta_p^\Omega)f \in C_0(\Omega)$.

Proof. *(i)* $\Rightarrow$ *(ii)* follows from Theorem 2.3.
(ii) $\Rightarrow$ *(iii)* Since $\varrho(\Delta_b^\Omega)$ has non-empty interior, it follows from the hypothesis that $U := \varrho(\Delta_0^\Omega) \cap \varrho(\Delta_b^\Omega) \neq \emptyset$. The set U is open and relatively closed in $\varrho(\Delta_b^\Omega)$. In fact, let $\lambda \in \varrho(\Delta_b^\Omega)$ such that $\lambda = \lim_{n\to\infty} \lambda_n \in U$. Then $(R(\lambda_n, \Delta_b^\Omega))_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}(C_b(\Omega))$, thus also $(R(\lambda_n, \Delta_0^\Omega))_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}(C_0(\Omega))$. This implies that $\lambda \in \varrho(\Delta_0^\Omega)$. Since $\varrho(\Delta_b^\Omega) = \varrho(\Delta_2^\Omega)$ is connected, it follows that $\varrho(\Delta_b^\Omega) \subset \varrho(\Delta_0^\Omega)$. In particular, $0 \in \varrho(\Delta_0^\Omega)$. It follows from (1.8) that $R(0, \Delta_b^\Omega)C_0(\Omega) \subset C_0(\Omega)$. Let

$\frac{N}{2} < p \le \infty$. Then $R(0, \Delta_p^\Omega)$ is a bounded operator from $L^p(\Omega)$ into $L^\infty(\Omega)$ (by (1.12)). Since $R(0, \Delta_p^\Omega) C_0(\Omega) \subset C_0(\Omega)$ and since $C_0(\Omega)$ is closed in $L^\infty(\Omega)$, it follows that $R(0, \Delta_p^\Omega) L^p(\Omega) \subset C_0(\Omega)$.
$(iii) \Rightarrow (i)$ We show that the Dirichlet problem has a solution for all $\varphi \in C(\partial\Omega)$. At first, assume that $\varphi = w_{|\partial\Omega}$ for some $w \in C^2(\bar{\Omega})$. Let $v = -R(0, \Delta_p^\Omega)\Delta w$. Then $v \in C_0(\Omega)$ and $\Delta v = \Delta w$ in $\mathcal{D}(\Omega)'$. Thus $u = w - v \in C(\bar{\Omega})$, $u_{|\partial\Omega} = \varphi$ and $\Delta u = 0$ in $\mathcal{D}(\Omega)'$; i.e. u is a solution of $D(\varphi)$. Since the set $\{w_{|\partial\Omega} : w \in C^2(\bar{\Omega})\}$ is dense in $C(\partial\Omega)$, it follows that $D(\varphi)$ has a (unique) solution for all $\varphi \in C(\partial\Omega)$.
$(iii) \Rightarrow (iv)$ is obvious.
$(iv) \Rightarrow (ii)$ We can assume that $N/2 < p < \infty$ (since $L^\infty(\Omega) \subset L^p(\Omega)$). Note that by (1.12) and Lemma 2.1, $R(\lambda, \Delta_p^\Omega)$ is a bounded operator from $L^p(\Omega)$ into $C^b(\Omega)$. Let $F = \{g \in L^p(\Omega) : \exists\, c \ge 0,\ |g| \le cf\}$. Then the hypothesis implies that $R(\lambda, \Delta_p^\Omega) F \subset C_0(\Omega)$. Since F is dense in $L^p(\Omega)$, it follows that $R(\lambda, \Delta_p^\Omega) L^p(\Omega) \subset C_0(\Omega)$. In particular, $R(\lambda, \Delta_p^\Omega) C_0(\Omega) \subset C_0(\Omega)$. Now it follows from (1.8) that $\lambda \in \varrho(\Delta_0)$. Since Δ_0 is the part of Δ_p in $C_0(\Omega)$. □

Corollary 2.5. *Let Ω be a bounded, open, non-empty, connected subset of $\mathbb{R}^N$. Then Ω is regular if and only if the eigenfunction u_1 associated with the first eigenvalue of Δ_2^Ω is in $C_0(\Omega)$.*

We conclude this section by a remark concerning the realization of Dirichlet boundary conditions in $L^2(\Omega)$. There is another choice, namely to replace $H_0^1(\Omega)$ by

$$\tilde{H}_0^1(\Omega) := \{u \in L^2(\Omega),\ \tilde{u} \in H^1(\mathbb{R}^N)\} \quad \text{where} \quad \tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \notin \Omega\,. \end{cases}$$

The operator associated with the form

$$a(u, v) = \int_\Omega \nabla u \nabla v$$

on $\tilde{H}_0^1(\Omega)$ is called the **pseudo-Dirichlet Laplacian** in [AB] and is denoted by $\tilde{\Delta}_\Omega$. Thus $D(\tilde{\Delta}_\Omega) = \{u \in \tilde{H}_0^1(\Omega) : \exists\, v \in L^2(\Omega)$ such that $\int_\Omega \nabla u \nabla \varphi = \int v\varphi \quad \forall \varphi \in \tilde{H}_0^1(\Omega)\}$ and $\tilde{\Delta}_\Omega u = v$. Equivalently, $D(\tilde{\Delta}_\Omega) = \{u \in \tilde{H}_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$, $\tilde{\Delta}_\Omega u = \Delta u$. For example, if $N = 1$, $\Omega = (0,1) \cup (1,2)$, then $\tilde{\Delta}_\Omega \ne \Delta_2^\Omega$, but if Ω has Lipschitz boundary then $\tilde{\Delta}_\Omega = \Delta_2^\Omega$.

Now assume that the semigroup $(e^{t\tilde{\Delta}_\Omega})_{t \ge 0}$ generated by $\tilde{\Delta}_\Omega$ on $L^2(\Omega)$ leaves invariant $C_0(\Omega)$. Hence $R(1, \tilde{\Delta}_\Omega) C_0(\Omega) \subset C_0(\Omega)$. But then Lemma 2.2 implies that $R(1, \tilde{\Delta}_\Omega) C_0(\Omega) \subset H_0^1(\Omega)$. Thus $R(1, \tilde{\Delta}_\Omega) f = R(1, \Delta_2^\Omega) f$ for $f \in C_0(\Omega)$. It follows that $\tilde{\Delta}_\Omega = \Delta_2^\Omega$; i.e. $H_0^1(\Omega) = \tilde{H}_0^1(\Omega)$. Concluding we have the following result.

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Assume that*

$$e^{t\tilde{\Delta}_\Omega} C_0(\Omega) \subset C_0(\Omega) \quad (t > 0)\,.$$

Then Ω is regular and $\tilde{\Delta}_\Omega = \Delta_2^\Omega$.

3. Unbounded open sets

Whereas in the preceding paragraph, for bounded open sets, we established an equivalence with well-posedness of the Dirichlet problem, in this section it will be more convenient to consider barriers. Let $\Omega \subset \mathbb{R}^N$ be open.

Definition 3.1. *a) Let $z \in \partial\Omega$. A* **barrier** *is a function $w \in C(\overline{\Omega \cap B})$ such that*

$$\Delta w \leq 0 \quad in \quad \mathcal{D}(\Omega \cap B)' \quad and \quad w(z) = 0,\ w(x) > 0 \quad for \quad x \in (\Omega \cap B) \setminus \{z\}$$

where $B = B(z, r)$ is a ball centered at z. We call w an $\mathbf{H^1}$**−barrier** *if in addition $w \in H^1(\Omega \cap B)$.*
b) We say that Ω is **regular**, *if at each boundary point $z \in \partial\Omega$ there exists a barrier.*

It is well known that a bounded open set Ω is regular if and only if the Dirichlet problem is well-posed (see [GT], § 2.8 or [DL], II). For unbounded Ω the situation is more complicated since the behaviour at infinity has to be taken into account (see [DL], II § 4). Still, the existence of a barrier at each boundary point is the right regularity property in order that Δ_0^Ω be a generator. This is the assertion of Theorem 0.1 which will be proved below. We first show that Δ_0^Ω is dispersive. This is not new (cf. [LP1], [LP2]). In the bounded case, it follows immediately from our arguments since Δ_0^Ω is the part of Δ_∞^Ω in $C_0(\Omega)$. We include a proof in the unbounded case for convenience of the reader. Recall that an operator A defined on the real space $C_0(\Omega)$ is called **dispersive** if for every $u \in D(A)$ such that $u^+ \neq 0$ there exists $x_0 \in \Omega$ such that

$$u^+(x_0) = \|u^+\|_\infty \quad \text{and} \quad (Au)(x_0) \leq 0\,. \tag{3.1}$$

Dispersiveness on $C_0(\Omega)$ implies dissipativity (as is easy to see). In particular, $(\lambda - A)$ is injective for $\lambda > 0$ whenever A is dispersive. This will be used in Theorem 3.7 below. More generally, a densely defined operator A generates a positive contractive C_0-semigroup if and only if A is dispersive and $I - A$ is surjective (see [N], C-II.Theorem 1.2).

Proposition 3.2. *The operator Δ_0 is closed and dispersive.*

Proof. It is obvious that Δ_0 is closed. Let $u \in D(\Delta_0)$, such that $u^+ \neq 0$. Let $(\varrho_n)_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R}^N)$ be an approximate unit; i.e. $\varrho_n \geq 0$, $\int \varrho_n(x)dx = 1$, $\operatorname{supp} \varrho_n \subset B(0, \frac{1}{n})$. Let $u_n = \varrho_n * u$. Then $u_n \in C_0(\mathbb{R}^N)$ and $\lim_{n\to\infty} u_n = u$ in $C_0(\Omega)$. Let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|u_n - u\|_\infty \leq \frac{\delta}{4}$, $\delta = \|u^+\|_\infty$. The set $K = \{x \in \Omega : u(x) \geq \frac{\delta}{2}\}$ is compact. Let $x_n \in \Omega$ such that $u_n(x_n) = \max_{y\in\Omega} u_n(y)$. Then $u_n(x_n) \geq \frac{3}{4}\delta$ and so $u(x_n) \geq \frac{\delta}{2}$ for $n \geq n_0$. Hence $x_n \in K \quad (n \geq n_0)$. Since K is compact, we can assume that $x_0 = \lim_{n\to\infty} x_n$ exists. Then $u(x_0) = \|u^+\|_\infty$. Since $u_n \in C^\infty(\Omega)$ we have $(\varrho_n * \Delta u)(x_n) = \Delta u_n(x_n) \leq 0$. Since $\varrho_n * \Delta u$ converges to u uniformly, it follows that $(\Delta u)(x_0) \leq 0$. $\square$

Remark 3.3. *More generally the following* **maximum principle** *holds, which clearly implies dispersiveness: If $u \in C(\Omega)$ has a local maximum at $x \in \Omega$ and $\Delta u \in C(\Omega)$, then $(\Delta u)(x) \leq 0$.*

We deduce from Proposition 3.2 that the operator Δ_0^Ω is the only possible realization of the Laplacian in $C_0(\Omega)$ which might generate a semigroup.

Corollary 3.4. *Let $A \subset \Delta_0^\Omega$ be the generator of a C_0-semigroup on $C_0(\Omega)$. Then $A = \Delta_0^\Omega$.*

Proof. There exists $\lambda > 0$ such that $(\lambda - A)$ is surjective. Let $u \in D(\Delta_0^\Omega)$. There exists $v \in D(A)$ such that $(\lambda - A)v = (\lambda - \Delta_0^\Omega)u$. Since $Av = \Delta_0^\Omega v$ it follows that $u = v \in D(A)$. □

For the proof of Theorem 0.1 in the unbounded case we need some further preparation.

Lemma 3.5. *Let Ω_1, Ω_2 be regular, open subsets of $\mathbb{R}^N$. Then $\Omega_1 \cap \Omega_2$ is regular.*

Proof. Let $\Omega = \Omega_1 \cap \Omega_2$. Then $\partial\Omega \subset \partial\Omega_1 \cup \partial\Omega_2$. Let $z \in \partial\Omega$. Suppose that $z \in \partial\Omega_1$. Then there exists a barrier $w \in C(\overline{\Omega_1 \cap B})$ where $B = B(z,r)$. Then $w_{|\overline{\Omega \cap B}}$ is clearly a barrier on $\overline{\Omega \cap B}$ at z. □

We deduce from Lemma 3.5 the following

Lemma 3.6. *Let Ω be a regular open subset of $\mathbb{R}^N$. Then for each $z \in \partial\Omega$ there exists an H^1-barrier at z.*

Proof. It follows from Lemma 3.5 that $\Omega \cap B$ is regular, where $B = B(z,r)$, $z \in \partial\Omega$, $r > 0$. Thus the Dirichlet problem is well-posed on $\Omega \cap B$. Let $\varphi(x) = |x - z|^2$. Then there exists $v \in C(\overline{\Omega \cap B})$ such that $v(x) = \varphi(x)$ for all $x \in \partial(\Omega \cap B)$ and $\Delta v = 0$ in $\mathcal{D}(\Omega \cap B)'$. It follows from the maximum principle that $v(x) > 0$ for all $x \in B \cap \Omega$. We show that $v \in H^1(\Omega \cap B)$. Note that $\varphi \in C^2(\overline{\Omega \cap B})$ and $\Delta\varphi = \text{const}$. Let $u = v - \varphi$. Then $u \in C_0(\Omega \cap B)$ and $\Delta u = -\Delta\varphi$ in $\mathcal{D}(\Omega \cap B)'$. It follows from Lemma 2.2 that $u \in H_0^1(\Omega \cap B)$. Thus $v = u + \varphi \in H^1(\Omega \cap B)$. □

We recall the following simple fact which is easy to proof (cf. [B], IX).

Lemma 3.7. *Let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then there exist $u_n \in \mathcal{D}(\Omega)$ such that $\|u_n\|_\infty \leq \|u\|_\infty \quad (n \in \mathbb{N})$ and $\lim_{n\to\infty} u_n = u$ in $H_0^1(\Omega)$.*

Moreover, we recall that for a bounded open set $\Omega_1 \subset \mathbb{R}^N$ one has

$$C_0(\Omega_1) \cap H^1(\Omega_1) \subset H_0^1(\Omega_1) \tag{3.2}$$

(see [B], Remark 20, p. 172 or [Da], Theorem 1.5.7).

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^N$ be a regular, open set. Then Δ_0^Ω generates a holomorphic C_0-semigroup on $C_0(\Omega)$. Moreover, Δ_0^Ω is the part of Δ_∞^Ω in $C_0(\Omega)$ and $\sigma(\Delta_p^\Omega) = \sigma(\Delta_0^\Omega)$ for $1 \leq p \leq \infty$.*

Proof. a) Let $f \in C_0(\Omega) \cap L^2(\Omega)$, $u = R(1, \Delta_2^\Omega)f$, i.e. $u \in H_0^1(\Omega)$, $u - \Delta u = f$ in $\mathcal{D}(\Omega)'$. We show that $u \in C_0(\Omega)$. It follows from local regularity that u is continuous and bounded.

1. Let $z \in \partial\Omega$. We show that $\lim\limits_{x \to z} u(x) = 0$. Let $\varepsilon > 0$. Choose $w \in C^1(\mathbb{R}^N)$ such that $\Delta w = u - f$ in $\mathcal{D}(\Omega)'$ and $w(z) = \varepsilon$ (one can take $w = E_N * (u - f) + \text{const}$). Let $v \in C(\overline{B \cap \Omega}) \cap H^1(\Omega \cap B)$ be an H^1-barrier where $B = B(z, r)$ is so small that $w \geq 0$ on $\bar{B}$. Multiplying v with a positive constant if necessary, we can assume that $v(x) > \|u\|_\infty$ on $\partial B \cap \partial(\Omega \cap B)$. Then $u - v - w \in H^1(\Omega \cap B)$ and $(u - v - w)^+ \in H_0^1(\Omega \cap B)$. To see the last point, choose $u_n \in \mathcal{D}(\Omega)$ such that $\|u_n\|_\infty \leq \|u\|_\infty$ and $u_n \to u$ in $H^1(\Omega)$ (by Lemma 3.7). Then $(u_n - v - w)^+ \to (u - v - w)^+$ in $H^1(\Omega \cap B)$. Note that $(u_n - v - w)^+ \in C(\overline{\Omega \cap B})$ vanishes on $\partial(\Omega \cap B) = (\partial(\Omega \cap B) \cap \partial\Omega) \cup (\partial(\Omega \cap B) \cap \partial B)$. Thus $(u_n - v - w)^+ \in H_0^1(\Omega \cap B)$ by (3.2). Thus $(u - v - w)^+ \in H_0^1(\Omega \cap B)$ and the claim is proved.

Now $\Delta(u - v - w) = -\Delta v \geq 0$ in $\mathcal{D}(\Omega \cap B)'$. Since $u - v - w \in H^1(\Omega)$, it follows that $\int \nabla(u - v - w)\nabla\varphi \leq 0$ for all $\varphi \in \mathcal{D}(\Omega \cap B)_+$ and hence for all $\varphi \in H_0^1(\Omega \cap B)_+$. Taking $\varphi = (u - v - w)^+$, we deduce that $\int |\nabla(u - v - w)^+|^2 \leq 0$. Hence $(u - v - w)^+ = 0$; i.e., $u \leq w + v$. Thus $\overline{\lim\limits_{x \to z}}\, u(x) \leq \overline{\lim\limits_{x \to z}}\,(v(x) + w(x)) = \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\overline{\lim\limits_{x \to z}}\, u(x) \leq 0$. Replacing u by $-u$ we obtain $\lim\limits_{\substack{x \to z \\ x \in \partial\Omega}} u(x) = 0$.

2. Since $|u| \leq R(1, \Delta_2)|f|$ (by (1.5)) and $R(1, \Delta_2)C_0(\mathbb{R}^N) \subset C_0(\mathbb{R}^N)$, it follows that $\lim\limits_{|x| \to \infty} |u(x)| = 0$. Thus a) is proved.

b) Recall that $R(1, \Delta_2^\Omega)$ and $R(1, \Delta_\infty^\Omega)$ are consistent. It follows from a) and by density that $R(1, \Delta_\infty^\Omega)C_0(\Omega) \subset C_0(\Omega)$. We show that $1 \in \varrho(\Delta_0^\Omega)$ and $R(1, \Delta_0^\Omega) = R(1, \Delta_\infty^\Omega)_{|C_0(\Omega)}$. Let $f \in C_0(\Omega)$. Then $u = R(1, \Delta_\infty^\Omega)f \in C_0(\Omega)$ and $u - \Delta u = f$ in $\mathcal{D}(\Omega)'$. (In fact, for $\varphi \in \mathcal{D}(\Omega)$ we have $\langle u - \Delta u,\ \varphi\rangle = \langle u,\ \varphi - \Delta\varphi\rangle = \langle R(1, \Delta_\infty^\Omega)f,\ \varphi - \Delta\varphi\rangle = \langle f,\ R(1, \Delta_1^\Omega)(\varphi - \Delta\varphi)\rangle = \langle f, \varphi\rangle$). Thus $u \in D(\Delta_0)$ and $u - \Delta_0 u = f$. Conversely, let $u \in \mathcal{D}(\Delta_0^\Omega)$ and $f = u - \Delta u$. Let $v = R(1, \Delta_\infty^\Omega)f$. Then $w = u - v \in \mathcal{D}(\Delta_0)$ and $w - \Delta_0 w = 0$. Since Δ_0 is dissipative, it follows that $w = 0$. We have shown that $1 \in \varrho(\Delta_0^\Omega)$ and $R(1, \Delta_0^\Omega) = R(1, \Delta_\infty^\Omega)_{|C_0(\Omega)}$. Thus Δ_0^Ω is a generator and Δ_0^Ω is the part of Δ_∞^Ω in $C_0(\Omega)$.

c) It is clear from analyticity of the resolvent, that the set $U = \{\lambda \in \varrho(\Delta_\infty^\Omega) : R(\lambda, \Delta_\infty^\Omega)C_0(\Omega) \subset C_0(\Omega)\}$ is open and closed in $\varrho(\Delta_\infty^\Omega)$. Since $\varrho(\Delta_\infty^\Omega)$ is connected it follows that $U = \varrho(\Delta_\infty^\Omega)$; i.e. $\varrho(\Delta_\infty^\Omega) \subset \varrho(\Delta_0^\Omega)$. We have shown that $\sigma(\Delta_0^\Omega) \subset \sigma(\Delta_\infty^\Omega) = \sigma(\Delta_2^\Omega)$. In order to show the converse denote by B the adjoint of Δ_0^Ω on $C_0(\Omega)'$. Let $\lambda > 0$, $f \in L^1(\Omega) \cap L^2(\Omega)$. Then for $\varphi \in \mathcal{D}(\Omega)$,
$\langle \varphi,\ R(\lambda, B)f\rangle = \langle R(\lambda, \Delta_0^\Omega)\varphi,\ f\rangle = \langle \varphi,\ R(\lambda, \Delta_2^\Omega)f\rangle = \langle \varphi,\ R(\lambda, \Delta_1^\Omega)f\rangle$. Hence $R(\lambda, B)L^1(\Omega) \subset L^1(\Omega)$ and $R(\lambda, B)_{|L^1(\Omega)} = R(\lambda, \Delta_1^\Omega)$. As before, it follows that $\varrho(\Delta_0^\Omega) = \varrho(B) \subset \varrho(\Delta_1^\Omega) = \varrho(\Delta_2^\Omega)$. Thus $\sigma(\Delta_2^\Omega) \subset \sigma(\Delta_0^\Omega)$. □

Of course, if Ω is regular, the semigroup $(e^{t\Delta_0^\Omega})_{t\geq 0}$ generated by Δ_0^Ω is consistent with $(e^{t\Delta_p^\Omega})_{t\geq 0}$ i.e.

$$e^{t\Delta_0^\Omega} f = e^{t\Delta_p^\Omega} f \quad (f \in C_0(\Omega) \cap L^p(\Omega),\ t \geq 0,\ 1 \leq p \leq \infty). \tag{3.3}$$

This follows from consistency of the resolvents.

Next we show that Δ_0^Ω has compact resolvent whenever Ω is regular and has finite Lebesgue measure. This follows from the following more general result.

Proposition 3.9. *Let Ω be a regular open subset of $\mathbb{R}^N$ with finite Lebesgue measure. Then $e^{t\Delta_0^\Omega}$ is a compact operator on $C_0(\Omega)$ for all $t > 0$.*

Proof. Let $t > 0$. Then $e^{t\Delta_2^\Omega}$ is a bounded operator from $L^2(\Omega)$ into $L^\infty(\Omega)$. We have seen that $e^{t\Delta_2^\Omega} C_0(\Omega) \subset C_0(\Omega)$. Hence by density $e^{t\Delta_2^\Omega} L^2(\Omega) \subset C_0(\Omega)$. Denote by $j : C_0(\Omega) \to L^2(\Omega)$ the canonical injection. Then $e^{t\Delta_0^\Omega}$ can be written as

$$e^{\frac{t}{2}\Delta_2^\Omega} \circ e^{\frac{t}{2}\Delta_2^\Omega} \circ j : C_0(\Omega) \to L^2(\Omega) \to L^2(\Omega) \to C_0(\Omega).$$

Since $e^{\frac{t}{2}\Delta_2^\Omega}$ is a compact operator on $L^2(\Omega)$, the claim follows. □

In order to prove the converse of Theorem 3.1 we establish a spectral characterization of Δ_0^Ω being a generator.

Proposition 3.10. *Let $\Omega \subset \mathbb{R}^N$ be an open set. The following are equivalent:*

(i) *Δ_0^Ω is a generator;*
(ii) *$\varrho(\Delta_0^\Omega) \neq \emptyset$;*
(iii) *$R(\lambda, \Delta_2^\Omega)(L^2(\Omega) \cap C_0(\Omega)) \subset C_0(\Omega)$.*

Proof. *(i) ⇒ (ii)* This is clear.
(ii) ⇒ (iii) If $\varrho(\Delta_0^\Omega) \neq \emptyset$, since $\sigma(\Delta_2^\Omega) \subset \mathbb{R}$, there exists $\lambda \in \varrho(\Delta_0^\Omega) \cap \varrho(\Delta_2^\Omega)$. Let B be the adjoint of Δ_0^Ω. We show that $R(\lambda, B)L^1(\Omega) \subset L^1(\Omega)$ and $R(\lambda, B)_{|L^1} = R(\lambda, \Delta_1^\Omega)$. Let $g \in L^1(\Omega) \cap L^2(\Omega)$, $u = R(\lambda, B)g$. Let $\varphi \in \mathcal{D}(\Omega)$. $v = R(\lambda, \Delta_0^\Omega)\varphi$. Then $v \in C_0(\Omega)$ and $\lambda v - \Delta v = \varphi$ in $\mathcal{D}(\Omega)'$. As in the proof of Lemma 2.2 one sees that $v \in H_0^1(\Omega)$. Thus $v = R(\lambda, \Delta_2^\Omega)\varphi$. Consequently, $\langle \varphi, R(\lambda, B)g\rangle = \langle R(\lambda, \Delta_0^\Omega)\varphi, g\rangle = \langle R(\lambda, \Delta_2^\Omega)\varphi, g\rangle = \langle \varphi, R(\lambda, \Delta_2^\Omega)g\rangle = \langle \varphi, R(\lambda, \Delta_1^\Omega)g\rangle$. We have shown that $R(\lambda, \Delta_1^\Omega)g = R(\lambda, B)g$ for all $g \in L^1(\Omega) \cap L^2(\Omega)$, hence for all $g \in L^1(\Omega)$. As in the proof of Theorem 3.8 we deduce that $\varrho(\Delta_1^\Omega) \subset \varrho(B) = \varrho(\Delta_0^\Omega)$ and $R(\lambda, B)_{|L^1(\Omega)} = R(\lambda, \Delta_1^\Omega)$ for all $\lambda \in \varrho(\Delta_1^\Omega)$. Let $f \in C_0(\Omega) \cap L^2(\Omega)$. Then for $g \in L^1(\Omega) \cap L^2(\Omega)$, $\langle R(\lambda, \Delta_0^\Omega)f, g\rangle = \langle f, R(\lambda, B)g\rangle = \langle f, R(\lambda, \Delta_2^\Omega)g\rangle = \langle R(\lambda, \Delta_2^\Omega)f, g\rangle$. Hence $R(\lambda, \Delta_0^\Omega)f = R(\lambda, \Delta_2^\Omega)f$. Thus *(iii)* is proved.
(iii) ⇒ (i) Let $f \in L^2(\Omega) \cap C_0(\Omega)$. Then by hypothesis $u = R(\lambda, \Delta_2^\Omega)f \in C_0(\Omega)$. Thus $u \in D(\Delta_0^\Omega)$ and $u - \Delta_0 u = f$. We have shown that $\lambda - \Delta_0^\Omega$ has dense image. Since Δ_0^Ω is closed and dissipative, it follows that Δ_0^Ω is a generator. □

Corollary 3.11. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open. If $\Delta_0^{\Omega_1}$ and $\Delta_0^{\Omega_2}$ are generators, then $\Delta_0^{\Omega_1 \cap \Omega_2}$ is a generator.*

Proof. Let $0 \le f \in C_0(\Omega) \cap L^2(\Omega)$, $\Omega = \Omega_1 \cap \Omega_2$. Since by Proposition 1.1 $R(\lambda, \Delta_2^{\Omega})f \le R(\lambda, \Delta_2^{\Omega_1})f$ and $R(\lambda, \Delta_2^{\Omega})f \le R(\lambda, \Delta_2^{\Omega_2})f$ and $\partial\Omega \subset \partial\Omega_1 \cup \partial\Omega_2$, it follows that $R(\lambda, \Delta_2^{\Omega})f \in C_0(\Omega)$. □

Corollary 3.12. *Assume that $\Omega \subset \mathbb{R}^N$ is open. If Δ_0^{Ω} is a generator, then at each $z \in \partial\Omega$ there exists an H^1-barrier. In particular, Ω is regular.*

Proof. Let $z \in \partial\Omega$, $B = B(z, r)$, $r > 0$. By Corollary 3.11, the operator $\Delta_0^{\Omega\cap B}$ is a generator on $C_0(B \cap \Omega)$. By Theorem 2.4 this implies that the Dirichlet problem on $B \cap \Omega$ is well-posed. Taking $\varphi(x) = |x - z|^2$ on the boundary, the solution of the Dirichlet problem on $B \cap \Omega$ gives an H^1-barrier. □

Concluding, we mention that the proofs given in Section 2 and 3 are almost self-contained. In particular, for a bounded open set Ω in $\mathbb{R}^N$, we gave a complete proof of the equivalence of the following three assertions

(a) the Dirichlet problem on Ω is well-posed;
(b) at each $z \in \partial\Omega$ there exists an H^1-barrier;
(c) Δ_0^{Ω} is a generator.

For the fact that condition (b) can be replaces by the more general condition that Ω is regular we refer to classical potential theory. Concerning holomorphy of the semigroup generated by Δ_0 the following should be added: The case where Ω is bounded and of class C^∞ is due to Stewart [Stu]. Lumer and Paquet [LP1] (see also [LP2]) proved by a beautiful dissipativity argument that the semigroup $(e^{t\Delta_0})_{t\ge 0}$ is holomorphic whenever Δ_0 is a generator. The duality argument we give here has the advantage to give the optimal angle. It was first used by Ouhabaz [Ou2].

4. Elliptic Operators on $\mathbf{C_0}(\Omega)$

Whereas Section 2 and 3 were self-contained, using only elementary results of potential theory, the following investigation of parabolic equations on $C_0(\Omega)$ depends on results by Stampacchia [Sta] among others who studied boundary behavior for solutions of elliptic equations. We consider merely bounded open sets generalizing the approach of Section 2.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. We introduce elliptic operators using the notation of Gilbarg-Trudinger [GT], Chapter 8. Let $a_{ij} \in L^\infty(\Omega)$ $(i, j = 1, \dots, N)$ be real functions such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2$$

for all $\xi \in \mathbb{R}^N$, x-a.e., where $\alpha > 0$, and let $d, b_j, c_j \in L^\infty(\Omega)$ be real coefficients, $j = 1, \dots, N$. We consider the elliptic operator L, formally given by

$$Lu = \sum_{i=1}^{N} D_i\Big(\sum_{j=1}^{N} a_{ij} D_j u + b_i u\Big) + \sum_{i=1}^{N} c_i D_i u + du\,.$$

Defining the form

$$a(u,v) = \int_\Omega \{\sum_{i,j=1}^N a_{ij} D_j u D_i v + \sum_{i=1}^N (b_i u D_i v - c_i D_i u v) - duv\} dx \tag{4.1}$$

for $u \in H^1_{\text{loc}}(\Omega),\ v \in \mathcal{D}(\Omega)$, we can realize L as an operator. $L : H^1_{\text{loc}}(\Omega) \to \mathcal{D}(\Omega)'$ given by

$$\langle Lu,\ v\rangle = -a(u,v) \quad (u \in H^1_{\text{loc}}(\Omega),\ v \in \mathcal{D}(\Omega))\,. \tag{4.2}$$

We observe furthermore that

$$Lu \in (H^1_0(\Omega))' \quad \text{whenever} \quad u \in H^1(\Omega)\,. \tag{4.3}$$

We define the operator A_0 on $C_0(\Omega)$ as the part of L in $C_0(\Omega)$; i.e.

$$D(A_0) = \{u \in C_0(\Omega) \cap H^1_{\text{loc}}(\Omega) : Lu \in C_0(\Omega)\} \quad A_0 u = Lu\,.$$

In the following we will assume throughout that

$$\sum_{j=1}^N D_j b_j + d \le 0 \quad \text{in} \quad \mathcal{D}(\Omega)'\,. \tag{4.4}$$

Then the following is the main result of this section.

Theorem 4.1. *Assuming (4.4), the following are equivalent:*

(i) Ω *is regular;*
(ii) $\varrho(A_0) \neq \emptyset$;
(iii) A_0 *generates a positive, contractive C_0-semigroup on $C_0(\Omega)$.*

By A_2 we denote **the realization of L in $L^2(\Omega)$ with Dirichlet boundary conditions**; i.e.

$$D(A_2) = \{u \in H^1_0(\Omega) : Lu \in L^2(\Omega)\} \quad A_2 u = Lu\,.$$

Note that $-A_2$ is associated with the form a on the form domain $H^1_0(\Omega)$; i.e., for $u, v \in L^2(\Omega)$ one has

$$u \in D(A_2),\ A_2 u = v \ \Leftrightarrow\ u \in H^1_0(\Omega) \text{ and } a(u,\varphi) = -\int_\Omega v\varphi \text{ for all } \varphi \in H^1_0(\Omega)\,.$$

The form a is **elliptic**; i.e. for some $\beta > 0,\ w \ge 0$ one has

$$a(u,u) + w(u|u)_{L^2} \ge \beta \|u\|^2_{H^1} \tag{4.5}$$

for all $u \in H^1_0(\Omega)$. Thus A_2 generates a holomorphic C_0-semigroup T_2 on $L^2(\Omega)$. It follows from the first Beurling-Deny criterion that $T_2(t) \ge 0 \quad (t \ge 0)$ (see [Ou1]). In virtue of (4.4), by the second Beurling-Deny criterion one has for $f \in L^2(\Omega) \cap L^\infty(\Omega)$,

$$\|\lambda R(\lambda, A_2) f\|_\infty \le \|f\|_\infty \quad (\lambda > 0) \tag{4.6}$$

or equivalently,

$$\|T_2(t)f\|_\infty \le \|f\|_\infty \quad (t > 0) \tag{4.7}$$

(see [Ou1]). Actually (4.6) (and (4.7)) are equivalent to (4.4) (see [ABBO]). It follows from (4.6) that there are operators A_p on $L^p(\Omega)$, $2 \le p \le \infty$, such that $(0,\infty) \subset \varrho(A_p)$ and $R(\lambda, A_p) = R(\lambda, A_2)_{|L^p(\Omega)}$. Moreover, for $2 < p < \infty$, by (4.7) the restriction of T_2 to L^p is a C_0-semigroup T_p on $L^p(\Omega)$ whose generator is A_p.

Lemma 4.2. *Let $u \in H^1_{\mathrm{loc}}(\Omega) \cap C_0(\Omega)$ such that $Lu \in L^2(\Omega)$. Then $u \in H^1_0(\Omega)$.*

Proof. Let $v = Lu$. Then

$$a(u, \varphi) = -\int v\varphi \quad \text{for all} \quad \varphi \in \mathcal{D}(\Omega)\,.$$

Let $\varepsilon > 0$. Then $(u - \varepsilon)^+ \in H^1_0(\Omega)$. Let ω be open such that $\bar{\omega} \subset \Omega$. Since $u \in H^1(\omega)$, $(u-\varepsilon)^+ \in H^1_0(\Omega)$, we have

$$a(u, (u-\varepsilon)^+) = -\int_\Omega v(u-\varepsilon)^+ dx \le \|v\|_{L^2}\|u\|_{L^2}\,.$$

Since $D_j(u-\varepsilon)^+ = 1_{\{u>\varepsilon\}} D_j u$ we have

$$\begin{aligned}
&a(u, (u-\varepsilon)^+) = \\
&a((u-\varepsilon)^+,\ (u-\varepsilon)^+) + \varepsilon \int_\Omega \sum_{i=1}^N b_i D_i (u-\varepsilon)^+ dx - \varepsilon \int_\Omega d(u-\varepsilon)^+ dx \\
&\ge a((u-\varepsilon)^+,\ (u-\varepsilon)^+) \qquad\qquad\qquad\qquad\qquad\qquad \text{(by (4.4))} \\
&\ge \beta\|(u-\varepsilon)^+\|^2_{H^1} - \omega\|(u-\varepsilon)^+\|^2_{L^2}\,.
\end{aligned}$$

Thus $\beta\|(u-\varepsilon)^+\|^2_{H^1} \le a(u,\ (u-\varepsilon)^+) + \omega\|u\|^2_{L^2} \le \|v\|_{L^2}\|u\|_{L^2} + \omega\|u\|^2_{L^2}$. As in the proof of Lemma 2.2 we deduce from this that $u^+ \in H^1_0(\Omega)$. Similarly, $u^- \in H^1_0(\Omega)$. □

Corollary 4.3. *The operator A_0 is the part of A_2 in $C_0(\Omega)$. In particular, A_0 is closed and dissipative.*

Proof. It follows from Lemma 4.2 that A_0 is the part of A_2 in $C_0(\Omega)$. Since A_2 is closed, A_0 is closed as well. In order to show dissipativity, let $\lambda > 0$, $u \in D(A_0)$, $\lambda u - A_0 u = f$. Since A_0 is the part of A_2 in $C_0(\Omega)$, it follows that $u = R(\lambda, A_2)f$. Now by (4.6), $\|\lambda u\|_\infty \le \|f\|_\infty$. □

Theorem 4.4. *Assume that Ω is regular, and that (4.4) is satisfied. Then A_0 generates a positive contractive C_0-semigroup T_0 on $C_0(\Omega)$. Moreover, $T_0(t) = T_2(t)_{|C_0(\Omega)}$ $(t \ge 0)$.*

Proof. 1. We extend the coefficients to $\mathbb{R}^N$ by setting

$$a_{ij}(x) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

for $x \in \mathbb{R}^N \setminus \Omega$ and $b_i(x) = c_i(x) = d(x) = 0$ for $x \in \mathbb{R}^N \setminus \Omega$. Denote by $\tilde{L} : H^1_{\text{loc}}(\mathbb{R}^N) \to \mathcal{D}(\mathbb{R}^N)'$ the corresponding elliptic operator and by $\tilde{A}_2$ its realization in $L^2(\mathbb{R}^N)$. Let $\lambda > \max\{0, \omega(\tilde{A}_2)\}$, where $\omega(\tilde{A}_2)$ is the type of $\tilde{A}_2$. We show that $\lambda - A_0$ is surjective, which proves our claim in view of Corollary 4.3. Let $g \in C_0(\Omega)$, $v = R(\lambda, \tilde{A}_2)g$. Then $v \in H^1(\mathbb{R}^N)$ and $v - Lv = g$ in $\mathcal{D}(\Omega)'$. By the famous result of De Giorgi and Nash [GT], Theorem 8.22, the function v is continuous on $\mathbb{R}^N$. Let $\varphi = v_{|\partial\Omega}$. By [GT], Theorem 8.31, (in the case where Ω is connected, see [Sta], Section 10 for the general case), there exists $w \in H^1_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$ such that $w - Lw = 0$ in $\mathcal{D}(\Omega)'$ and $\omega_{|\partial\Omega} = \varphi$. Now let $u = v - w$. Then $u \in C_0(\Omega) \cap H^1_{\text{loc}}(\Omega)$ and $u - Lu = g$ in $\mathcal{D}(\Omega)'$. Thus $u \in D(A_0)$ and $u - A_0 u = g$. We have shown that A_0 generates a contractive C_0-semigroup. Since A_0 is the part of A_2, it follows that $R(\lambda, A_2)_{|C_0(\Omega)} = R(\lambda, A_0) \quad (\lambda > 0)$ which implies the last claim in the proposition.

We have shown that $(I - A_0)$ is surjective. Since A_0 is dissipative, it follows that $(0, \infty) \subset \varrho(A_0)$ and $\|\lambda R(\lambda, A_0)\| \leq 1 \quad (\lambda > 0)$. Moreover, A_0 being the part of A_2 in $C_0(\Omega)$, we have $R(\lambda, A_0) = R(\lambda, A_2)_{|C_0(\Omega)}$ for $\lambda > 0$.

2. We show that $D(A_0)$ is dense in $C_0(\Omega)$. For that we have to show that $R(\lambda, A_0)' \in \mathcal{L}(C_0(\Omega)')$ is injective. Let $\mu \in C_0(\Omega)'$ such that $R(\lambda, A_0)'\mu = 0$. Take $\mu_n \in L^1(\Omega) \cap L^2(\Omega)$ such that $\sup\limits_{n \in \mathbb{N}} \|\mu_n\| < \infty$ and $\lim\limits_{n \to \infty} \mu_n = \mu$ for the topology $\sigma(C_0(\Omega)', \ C_0(\Omega))$. Let $v_n = R(\lambda, A_0)'\mu_n$. Let $1 < q < \frac{N}{N-1}$. By a result of Stampacchia [Sta], Théorème 4.4, $R(\lambda, A_0)'$ is a bounded operator from $L^1(\Omega)$ into $W^{1,q}_0(\Omega)$. By reflexivity, we can assume that $(v_n)_{n \in \mathbb{N}}$ converges weakly to v in $W^{1,q}_0(\Omega)$ (choosing a subsequence otherwise). Since $w^* - \lim\limits_{n \to \infty} v_n = R(\lambda, A_0)'\mu = 0$, we have $v = 0$. Since $R(\lambda, A_0) = R(\lambda, A_2)_{|C_0(\Omega)}$ one has $v_n = R(\lambda, A_2)'\mu_n$.

Hence for $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned} &\int \varphi d\mu = \lim_{n \to \infty} \int \mu_n \varphi = \\ &\lim_{n \to \infty} ((I - A_2)' v_n | \varphi)_{L^2} = \\ &\lim_{n \to \infty} \{\int v_n \varphi + a(\varphi, v_n)\} = 0\,. \end{aligned}$$

It follows that $\mu = 0$.

3. We have shown that A_0 generates a contractive C_0-semigroup T_0 on $C_0(\Omega)$. Since $R(\lambda, A_0) = R(\lambda, A_2)_{|C_0(\Omega)} \quad (\lambda > 0)$, it follows that

$$T_0(t)f = \lim_{n \to \infty} (I - \frac{t}{n} A_0)^{-n} f = \lim_{n \to \infty} (I - \frac{t}{n} A_2)^{-n} f = T_2(t)f \quad (f \in C_0(\Omega))\,.$$

□

We introduce the following notation. Let $S \in \mathcal{L}(L^p(\Omega))$. Then

$$\|S\|_{\mathcal{L}(L^q, L^r)} = \sup\{\|Sf\|_r : f \in L^q \cap L^p, \ \|f\|_q \leq 1\}$$

where $1 \le p,\ q,\ r \le \infty$. We recall the following result on ultracontractivity (see [V], Théorème 2 or [C], Lemma 1).

Theorem 4.5. *Let T be a C_0-semigroup on $L^2(\Omega)$ such that*

$$\|T(t)f\|_\infty \le M_\infty \|f\|_\infty \quad (f \in L^2 \cap L^\infty,\ 0 < t \le 1) \tag{4.8}$$

and

$$\|T(t)f\|_q \le M_q t^{-\alpha} \|f\|_2 \quad (f \in L^2 \cap L^q,\ 0 < t \le 1) \tag{4.9}$$

where $2 < q < \infty,\ \alpha > 0,\ M_q \ge 0,\ M_\infty \ge 0$. Then there exists a constant $c > 0$ such that

$$\|T(t)f\|_\infty \le c t^{-\beta} \|f\|_2 \quad (0 < t \le 1) \tag{4.10}$$

where $\beta = \frac{\alpha}{1-\frac{2}{q}}$.

Corollary 4.6. *Assume that Ω is a bounded regular open set. Then the operators $T_0(t) \quad (t > 0)$ are compact. Moreover,*

$$\sigma(A_0) = \sigma(A_2)\,.$$

Proof. Since T_2 is holomorphic, one has $\sup\limits_{0<t\le 1} \|tA_2T_2(t)\|_{\mathcal{L}(L^2)} < \infty$.
Since $H_0^1(\Omega) \hookrightarrow L^{2N/N-2}$, it follows that

$$\sup_{0<t\le 1} \|tT_2(t)\|_{\mathcal{L}(L^2, L^{2N/N-2})} < \infty\,.$$

Now it follows from Theorem 4.5 that

$$\|T_2(t)\|_{\mathcal{L}(L^2, L^\infty)} \le c t^{-N/2} \quad (0 < t \le 1)\,. \tag{4.11}$$

Since the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it follows as in the proof of Proposition 3.8 that $T_0(t)$ is compact $(t > 0)$. In particular, A_0 has compact resolvent. Now it follows from [Ar], Proposition 2.6 that $\sigma(A_0) = \sigma(A_2)$. □

Corollary 4.7. *Assume that $b_i, c_i \in W^{1,\infty}(\Omega)$ $i = 1, \dots, N$. If Ω is a bounded, regular, open set, then A_0 generates a holomorphic C_0-semigroup on $C_0(\Omega)$.*

Proof. It follows from [Ou1] that there exists a C_0-semigroup T_1 on $L^1(\Omega)$ such that $T_1(t)_{|_{L^2(\Omega)}} = T_2(t) \quad (t \ge 0)$. Denote the generator of T_1 by A_1. It follows from [AE] that T_1 is holomorphic. By duality

$$\|\lambda R(\lambda, A_\infty)\| \le M \quad (\operatorname{Re} \lambda > \omega)$$

for some $\omega, M \ge 0$. This implies that $\|\lambda R(\lambda, A_0)\| \le M \quad (\operatorname{Re} \lambda > \omega)$, so A_0 generates a holomorphic C_0-semigroup. □

Remark 4.8. In Corollary 4.7 the hypothesis (4.4) is not needed a priori: it is satisfied if we replace L by $L-\omega$ for a suitable ω.

Lemma 4.9. *Let $\infty > p > N$. Then*

$$D(A_p) \subset L^\infty(\Omega)\,.$$

Moreover,

$$D(A_2^k) \subset L^\infty(\Omega)$$

for $k > \max\{\frac{N}{2},\ 1\}$.

Proof. By (4.11) we have

$$\|T_2(t)\|_{\mathcal{L}(L^2,L^\infty)} \le ct^{-N/2} \quad (0 < t \le 1)\,. \tag{4.12}$$

By interpolation we deduce from this that

$$\|T_p(t)\|_{\mathcal{L}(L^p,L^\infty)} \le c_p t^{-N/p} \quad (0 < t \le 1) \tag{4.13}$$

for some $c_p \ge 0$, $2 \le p < \infty$. Let $p > N$. Let $M \ge 0$, $\omega \ge 0$ such that $\|T_p(t)\|_{\mathcal{L}(L^p)} \le Me^{\omega t}$. Then for $\lambda > \omega$,

$$\begin{aligned}
&\|R(\lambda, A_p)f\|_\infty \le \textstyle\int_0^1 e^{-\lambda t}\|T(t)f\|_\infty\, dt + \int_0^\infty e^{-\lambda(t+1)}\|T(1)T(t)f\|_\infty\, dt\\
&\le c_p \textstyle\int_0^1 t^{-N/p}\, dt\ \|f\|_p + c_p\int_0^\infty e^{-\lambda(t+1)}\|T(t)f\|_p\, dt\\
&\le c_p\{\textstyle\int_0^1 t^{-N/p}\, dt + \int_0^\infty e^{-\lambda(t+1)}Me^{\omega t}\, dt\}\|f\|_p\\
&\le \text{const.}\ \cdot \|f\|_p\,.
\end{aligned}$$

Similarly, for λ sufficiently large, since

$$R(\lambda, A_2)^k = (k-1)! \int_0^\infty e^{-\lambda t} t^{k-1} T_2(t)dt$$

one obtains by (4.13) that $\|R(\lambda, A_2)^k\|_{\mathcal{L}(L^2,L^\infty)} < \infty$ if $k > \frac{N}{2}$. □

In order to prove the remaining implication $(ii) \Rightarrow (i)$ of Theorem 4.1 we consider the Dirichlet problem with respect to $L - I$. For this, fix $\tilde{\Omega}$ a large ball containing $\bar{\Omega}$. Let $h \in C(\partial\Omega)$ be the trace of a function $w \in H^1(\tilde{\Omega}) \cap C(\tilde{\Omega})$; i.e. $h = w_{|\partial\Omega}$. Then there exists a unique function $u \in H^1(\Omega) \cap C^b(\Omega)$ such that

$$Lu - u = 0 \quad \text{in} \quad D(\Omega)' \quad \text{and} \quad u - w \in H_0^1(\Omega) \tag{4.14}$$

(see [Sta], Théorème 10.1 and [GT], 8.22). We interprete (4.14) as a weak form of $u_{|\partial\Omega} = h$. Then by the maximum principle (see [Sta], § 10 or [GT], 8.1),

$$\|u\|_{L^\infty(\Omega)} \le \|h\|_{C(\partial\Omega)}\,. \tag{4.15}$$

We set $u = Bh$. Since by the Stone-Weierstraß theorem traces of functions $w \in C(\tilde{\Omega}) \cap H^1(\tilde{\Omega})$ are dense in $C(\partial\Omega)$, we find a unique linear extension

$$B : C(\partial\Omega) \to H^1(\Omega) \cap C^b(\Omega)$$

such that

$$\|Bh\|_{L^\infty(\Omega)} \le \|h\|_{C(\partial\Omega)}\,. \tag{4.16}$$

Recall that $L : H^1(\Omega) \to \mathcal{D}(\Omega)'$ is a continuous linear mapping. Let $h \in C(\partial\Omega)$, then $u = Bh$ is the solution of the Dirichlet problem

$$\begin{cases} u \in H^1(\Omega) \\ Lu - u = 0 \quad \text{in} \quad \mathcal{D}(\Omega)' \\ u_{|\partial\Omega} = h \end{cases} \tag{4.17}$$

where the last identity has to be understood in the sense of the construction.

It turns out that Ω is regular (which means by our definition regular with respect to the Laplacian) if and only if Ω is regular with respect to L. More precisely, the following remarkable result is due to Stampacchia [Sta], § 10 and Littmann, Stampacchia, Weinberger [LSW].

Theorem 4.10. *The following are equivalent:*

(i) Ω *is regular;*
(ii) $\lim\limits_{\substack{x \to z \\ x \in \Omega}} (Bh)(x) = h(z)$ *and all* $z \in \partial\Omega$ *for all* $h \in C(\partial\Omega)$.

Recall that here L is a fixed elliptic operator satisfying (4.4).

On the basis of this result we now deduce the following.

Proposition 4.11. *Assume (4.4). If* $\varrho(A_0) \neq \emptyset$, *then* Ω *is regular.*

Proof. 1. It follows from Lemma 4.8 and Proposition 1.3 that $\sigma(A_2) = \sigma(A_\infty)$. Since by hypothesis $\varrho(A_0) \neq \emptyset$ and since $\sigma(A_\infty) \subset \mathbb{R}$, it follows that $\varrho(A_0) \cap \varrho(A_\infty) \neq \emptyset$. As in the proof of Theorem 4.4 we conclude that $\lambda \in \varrho(A_0)$. Let $p > N$. Then by Lemma 4.8, $R(\lambda, A_p)L^p(\Omega) \subset L^\infty(\Omega)$. Since $R(\lambda, A_p)C_0(\Omega) = R(\lambda, A_0)C_0(\Omega) \subset C_0(\Omega)$, it follows by density that $R(\lambda, A_p)L^p(\Omega) \subset C_0(\Omega)$.
2. Let $\tilde{\Omega}$ be a ball containing $\bar{\Omega}$. Note that $\tilde{\Omega}$ is regular. Consider the elliptic operator $\tilde{L} : H^1_{\text{loc}}(\tilde{\Omega}) \to \mathcal{D}(\tilde{\Omega})'$ one obtains by extending the coefficients as in the proof of Theorem 4.4 and let A_0 be its realization in $C_0(\tilde{\Omega})$. Then $1 \in \varrho(\tilde{A}_0)$ by Theorem 4.4. Since $\tilde{A}_0$ has dense domain in $C_0(\tilde{\Omega})$, and since each function in $C(\partial\Omega)$ has an extension to a function in $C_0(\tilde{\Omega})$, the space

$$F = \{h \in C(\partial\Omega) : \exists\, g \in C_0(\tilde{\Omega}),\ h = (R(\lambda, \tilde{A}_0)g)_{|\partial\Omega}\}$$

is dense in $C(\partial\Omega)$. Let $h \in F$, $h = w_{|\partial\Omega}$, $w = R(1, \tilde{A}_0)g$, $g \in C_0(\tilde{\Omega})$. Then $g_{|\Omega} \in L^p(\Omega)$. Let $v = R(\lambda, A_p)(g_{|\Omega})$. Then $v \in C_0(\Omega)$ by 1. Moreover, we have

$$\begin{aligned} v - Lv &= g \quad \text{in} \quad \mathcal{D}(\Omega)' \quad \text{and} \\ w - Lw &= g \quad \text{in} \quad \mathcal{D}(\Omega)' . \end{aligned}$$

Let $u = (w - v)_{|\Omega}$. Then $u \in H^1(\Omega) \cap C(\bar{\Omega})$ and $u - Lu = 0$ in $\mathcal{D}(\Omega)'$ and $u_{|\partial\Omega} = h$. Thus *(ii)* of Theorem 4.9 is satisfied for all $h \in F$ and hence for all $h \in C(\partial\Omega)$ by density. $\square$

The proof of Theorem 4.1 is complete now.

Concluding Remark

For elliptic operators we restricted ourselves to bounded open sets. In order to carry over the strategy used for the Laplacian in Section 3 further arguments are needed. One can actually show that L_0 is dissipative under condition (4.4). However, one obtains barriers with respect to L merely in H^1_{loc} which presents an additional difficulty.

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