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DISCRETE SPECTRUM AND ALMOST PERIODICITY

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Abstract. The purpose of this note is to show that solutions of first and second order Cauchy problems with almost periodic inhomogeneity are almost periodic on the real line whenever the spectrum of the underlying operator is discrete.

1. Introduction

Almost periodicity plays an important role in the theory of differential equations in Banach spaces; see, e.g., the monographs of Levitan-Zhikov [16] and, in the framework of one-parameter semigroups, van Neerven [19] as well as articles by Bart and Goldberg [4], Basit [7], Batty and Chill [9], Batty, van Neerven and Räbiger [10], Chill [12], Ruess and Vũ [21], Vesentini [22], Vũ [24] and Arendt and Batty [1, 2].

A central result is Loomis' theorem, which says that a bounded uniformly continuous function of the real line $\mathbb{R}$ into a Banach space X with countable spectrum is almost periodic provided that $c_0 \not\subseteq X$. Loomis proved the scalar version [17], which was later generalised to vector-valued functions (see, e.g., [16, p. 92] or [1]). The result is false if $c_0 \subseteq X$. We will see, though, that merely the accumulation points in the spectrum are responsible for the failure of Loomis' theorem on c_0. In fact, in Section 3 we show that a bounded uniformly continuous function of $\mathbb{R}$ into a Banach space X with discrete spectrum is almost periodic, without any further conditions on the Banach space X. This result is due to A. G. Baskakov. We give a simple proof establishing some further properties (see Theorem 3.4) needed in the sequel.

The main results of the paper are contained in Section 4, where first and second order Cauchy problems on $\mathbb{R}$ are considered. It is shown that bounded

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uniformly continuous solutions of the first order inhomogeneous Cauchy problem on the real line

$$\text{(CP)} \qquad u'(t) = Au(t) + f(t) \qquad (t \in \mathbb{R})$$

with almost periodic inhomogeneity f are almost periodic provided that $\sigma(A) \cap i\mathbb{R}$ is discrete and $-i\sigma(A)$ contains no accumulation points of the spectrum of f. Similar results hold for solutions of the second order Cauchy problem on the real line. We show in particular that a bounded C_0-group or cosine function is almost periodic whenever its generator has discrete spectrum.

The spectral results for functions on $\mathbb{R}$ can be generalized to vector-valued functions on a locally compact Abelian group G. This is done in the last part of the paper.

2. Some Basic Facts about Spectral Theory

Let $u \in BUC(\mathbb{R}, X)$, the space of all bounded uniformly continuous functions on $\mathbb{R}$ with values in a Banach space X. We assume throughout that $X \neq \{0\}$. Recall that the spectrum of such a function can be defined in several ways (see for example [1]).

Denote by $\mathcal{F}$ the *Fourier transform* of a function $f \in L^1(\mathbb{R})$ which is given by

$$(\mathcal{F}f)(s) := \int_{-\infty}^{+\infty} e^{-ist} f(t)dt$$

for all $s \in \mathbb{R}$. The *Beurling spectrum* of $u \in BUC(\mathbb{R}, X)$ is defined by

$$\begin{aligned} \mathrm{sp}_B(u) \quad := \quad & \{\xi \in \mathbb{R} \,|\, \forall \epsilon > 0 \, \exists f \in L^1(\mathbb{R}) \text{ such that} \\ & \mathrm{supp}(\mathcal{F}f) \subseteq (\xi - \epsilon, \xi + \epsilon) \text{ and } f * u \neq 0\}. \end{aligned}$$

The *Carleman transform* of a function $u \in BUC(\mathbb{R}, X)$ is defined as the holomorphic function $\hat{u}$ on $\mathbb{C} \setminus i\mathbb{R}$ given by

$$\hat{u}(\lambda) \quad := \quad \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} u(t)dt & \text{if } \mathrm{Re}\lambda > 0, \\ \displaystyle -\int_{-\infty}^0 e^{-\lambda t} u(t)dt & \text{if } \mathrm{Re}\lambda < 0. \end{cases}$$

A point $\eta \in \mathbb{R}$ is called a *regular point* if the Carleman transform has a holomorphic extension to a neighbourhood of $i\eta$. Now the *Carleman spectrum* of $u \in BUC(\mathbb{R}, X)$ is defined by

$$\mathrm{sp}_C(u) := \{\xi \in \mathbb{R} \,|\, \xi \text{ is not regular}\},$$

which coincides with the Beurling spectrum $\mathrm{sp}_B(u)$ [20, Prop. 0.5]. Hence we can denote the spectrum of u simply by $\mathrm{sp}(u) := \mathrm{sp}_B(u) = \mathrm{sp}_C(u)$.

Next we recall some facts about spectral theory of bounded C_0-groups. If A is a closed linear operator on a Banach space X, then we denote by $\sigma(A)$ the spectrum, and by $\rho(A)$ the resolvent set of A. For $\lambda \in \rho(A)$, let $R(\lambda, A) = (\lambda - A)^{-1}$. Now assume that A generates a bounded C_0-group $\mathcal{U} = (U(t))_{t\in\mathbb{R}}$.

Recall that the *Arveson spectrum* of $\mathcal{U}$ is defined by

$$\begin{aligned}\mathrm{Sp}(\mathcal{U}) \quad := \quad & \{\xi \in \mathbb{R} \,|\, \forall \epsilon > 0 \,\exists f \in L^1(\mathbb{R}) \text{ such that} \\ & \mathrm{supp}(\bar{\mathcal{F}} f) \subseteq (\xi - \epsilon, \xi + \epsilon) \text{ and } f(\mathcal{U}) \neq 0\},\end{aligned}$$

where for $f \in L^1(\mathbb{R})$ the operator $f(\mathcal{U}) \in \mathcal{L}(X)$ is given by

$$f(\mathcal{U})x := \int_{-\infty}^{+\infty} f(t)U(t)x dt \quad (x \in X)$$

and let $(\bar{\mathcal{F}} f)(s) := \int_{-\infty}^{+\infty} e^{ist} f(t)dt$ for all $s \in \mathbb{R}$. It is known that (see [13, Theorem 8.19])

$$i\mathrm{Sp}(\mathcal{U}) = \sigma(A). \tag{1}$$

For an element $x \in X$, the *Arveson spectrum of x with respect to $\mathcal{U}$* is defined by

$$\begin{aligned}\mathrm{sp}^{\mathcal{U}}(x) \quad := \quad & \{\xi \in \mathbb{R} \,|\, \forall \epsilon > 0 \,\exists f \in L^1(\mathbb{R}) \text{ such that} \\ & \mathrm{supp}(\bar{\mathcal{F}} f) \subseteq (\xi - \epsilon, \xi + \epsilon) \text{ and } f(\mathcal{U})x \neq 0\}.\end{aligned}$$

From the definitions it follows that this coincides with the Arveson spectrum of the group $\mathcal{U}_x$ given by $U_x(t) := U(t)|_{X_x}$, where $X_x = \overline{\mathrm{span}}\{U(t)x | t \in \mathbb{R}\}$. Hence by (1) we have

$$i\mathrm{sp}^{\mathcal{U}}(x) = i\mathrm{Sp}(\mathcal{U}_x) = \sigma(A_x), \tag{2}$$

where A_x is the generator of $\mathcal{U}_x$.

Now consider a special C_0-group, the shift group $\mathcal{S} = (S(t))_{t\in\mathbb{R}}$ on $BUC(\mathbb{R}, X)$ defined by $(S(t)u)(s) := u_t(s) = u(s + t)$. Denote by B its generator; then the domain of B consists of all $u \in BUC(\mathbb{R}, X) \cap C^1(\mathbb{R}, X)$ such that $u' \in BUC(\mathbb{R}, X)$ and $Bu = u'$.

Let $BUC_u := \overline{\mathrm{span}}\{S(t)u | t \in \mathbb{R}\}$ and denote by $\mathcal{S}_u = (S_u(t))_{t\in\mathbb{R}}$ the shift group on BUC_u with generator B_u. Then B_u is the part of B in BUC_u and by [1, (2.4)] and (2) we have

$$(3) \qquad i\mathrm{sp}(u) = \sigma(B_u) = i\mathrm{Sp}(\mathcal{S}_u).$$

In the following, let $AP(\mathbb{R},X)$ be the space of all *almost periodic* functions on $\mathbb{R}$ with values in the Banach space X. For the definition and various characterisations we refer to [16]. In particular, it is known that

$$AP(\mathbb{R},X) = \overline{\mathrm{span}}\{e_\eta \otimes x \,|\, \eta \in \mathbb{R}, x \in X\},$$

where $(e_\eta \otimes x)(s) = e^{i\eta s}x$, $s \in \mathbb{R}$.

For $u \in BUC(\mathbb{R},X)$, define the *reduced spectrum of u with respect to* $AP(\mathbb{R},X)$ by (see [1])

$$\begin{aligned} \mathrm{sp}_{AP}(u) \;\; := \;\; & \{\xi \in \mathbb{R} \,|\, \forall \epsilon > 0 \, \exists f \in L^1(\mathbb{R}) \text{ such that} \\ & \mathrm{supp}(\mathcal{F}f) \subseteq (\xi - \epsilon, \xi + \epsilon) \text{ and } f * u \notin AP(\mathbb{R},X)\}. \end{aligned}$$

As before we consider the shift group $\mathcal{S}$ on $BUC(\mathbb{R},X)$. Since $\mathcal{S}$ leaves $AP(\mathbb{R},X)$ invariant, we can define the quotient group $\bar{\mathcal{S}} = (\bar{S}(t))_{t\in\mathbb{R}}$ on $Y := BUC(\mathbb{R},X)/AP(\mathbb{R},X)$ by

$$\bar{S}(t)\bar{u} = \overline{(S(t)u)}$$

for all $t \in \mathbb{R}$ and $u \in BUC(\mathbb{R},X)$, where $^-\colon BUC(\mathbb{R},X) \to Y$ denotes the quotient mapping. The generator of $\bar{\mathcal{S}}$ is denoted by $\bar{B}$.

Again we let $Y_{\bar{u}}$ be the closed linear span of the orbit $\{\bar{S}(t)\bar{u} | t \in \mathbb{R}\}$ in Y and we denote by $\bar{\mathcal{S}}_{\bar{u}} = (\bar{S}_u(t))_{t\in\mathbb{R}}$ the restricted group on $Y_{\bar{u}}$ with generator $\bar{B}_{\bar{u}}$.

Proposition 2.1. *For $u \in BUC(\mathbb{R},X)$ we have with the above notations*

$$i\mathrm{sp}_{AP}(u) = \sigma(\bar{B}_{\bar{u}}).$$

Proof. Let $f \in L^1(\mathbb{R})$ and $u \in BUC(\mathbb{R},X)$. Then

$$\begin{aligned} (f * u)(t) &= \int_{-\infty}^{+\infty} f(s)u(t-s)ds \\ &= \int_{-\infty}^{+\infty} f(-s)(S(s)u)(t)ds \;=\; (f_-(\mathcal{S})u)(t) \end{aligned}$$

for all $t \in \mathbb{R}$, where $f_-(s) := f(-s)$. Hence $f * u \in AP(\mathbb{R},X)$ if and only if $f_-(\mathcal{S})u \in AP(\mathbb{R},X)$, which is equivalent to $f_-(\bar{\mathcal{S}}_{\bar{u}}) = 0$ in $\mathcal{L}(Y_{\bar{u}})$. Since $\bar{\mathcal{F}}f_- = \mathcal{F}f$, we have $\mathrm{sp}_{AP}(u) = \mathrm{Sp}(\bar{\mathcal{S}}_{\bar{u}})$ and the claim follows from (1). ∎

Since the spectrum of the generator of a bounded group on a Banach space different from $\{0\}$ is never empty (see [18, A-III 7.6] or [19, Lemma 2.4.3]), we obtain as a consequence (see also [7, Prop. 2.5]) the following.

Corollary 2.2. *Let $u \in BUC(\mathbb{R}, X)$. Then $u \in AP(\mathbb{R}, X)$ if and only if $\mathrm{sp}_{AP}(u) = \emptyset$.*

A slight generalization of the previous considerations is the following. Let $\mathcal{G}$ be a closed, translation-invariant subspace of $BUC(\mathbb{R}, X)$. In the same way as before we can define the reduced spectrum of $u \in BUC(\mathbb{R}, X)$ with respect to $\mathcal{G}$ by

$$\begin{aligned} \mathrm{sp}_{\mathcal{G}}(u) \;:=\; & \{\xi \in \mathbb{R} \,|\, \forall \epsilon > 0 \,\exists f \in L^1(\mathbb{R}) \text{ such that} \\ & \operatorname{supp}(\mathcal{F}f) \subseteq (\xi - \epsilon, \xi + \epsilon) \text{ and } f * u \notin \mathcal{G}\}. \end{aligned}$$

Note that $\mathrm{sp}_{\mathcal{G}}(u) \subseteq \mathrm{sp}_{AP}(u)$ if $AP(\mathbb{R}, X) \subseteq \mathcal{G}$. Since $\mathcal{G}$ is closed and translation-invariant, we can consider the quotient group $\tilde{\mathcal{S}}$ on $\tilde{Y} := BUC(\mathbb{R}, X)/\mathcal{G}$ defined by $\tilde{S}(t)\tilde{u} := (S(t)u)\tilde{}$, where $\tilde{}: BUC(\mathbb{R}, X) \to \tilde{Y}$ denotes the quotient mapping.

Then we obtain by the same proof as in Proposition 2.1 that

$$i\,\mathrm{sp}_{\mathcal{G}}(u) = \sigma(\tilde{B}_{\tilde{u}}), \tag{4}$$

where $\tilde{B}_{\tilde{u}}$ is the generator of the group $\tilde{\mathcal{S}}_{\tilde{u}} = (\tilde{S}_{\tilde{u}}(t))_{t \in \mathbb{R}}$ on $\tilde{Y}_{\tilde{u}} = \overline{\mathrm{span}}\{\tilde{S}(t)\tilde{u} | t \in \mathbb{R}\}$. And as in Corollary 2.2, we have for functions $u \in BUC(\mathbb{R}, X)$ that $u \in \mathcal{G}$ if and only if $\mathrm{sp}_{\mathcal{G}}(u) = \emptyset$.

The most interesting classes of closed, translation-invariant subspaces of $BUC(\mathbb{R}, X)$ are the following:

(i) the space $AP(\mathbb{R}, X)$;

(ii) the space $W(\mathbb{R}, X)$ of all *weakly almost periodic functions in the sense of Eberlein*:

$$\begin{aligned} W(\mathbb{R}, X) \;:=\; & \{u \in BUC(\mathbb{R}, X) \,|\, \{S(t)u | t \in \mathbb{R}\} \text{ is relatively weakly} \\ & \text{compact in } BUC(\mathbb{R}, X)\}; \end{aligned}$$

(iii) the space $WAP(\mathbb{R}, X)$ of all *weakly almost periodic functions*:

$$WAP(\mathbb{R}, X) \;:=\; \{u \in BUC(\mathbb{R}, X) \,|\, x' \circ u \in AP(\mathbb{R}) \text{ for all } x' \in X'\};$$

(iv) the space $E(\mathbb{R}, X)$ of all *uniformly ergodic functions*:

$$E(\mathbb{R}, X) := \{u \in BUC(\mathbb{R}, X) \,|\, \lim_{\alpha \downarrow 0} \alpha \int_0^\infty e^{-\alpha t} S(t)u \, dt \text{ exists in } BUC(\mathbb{R}, X)\};$$

(v) the space $TE(\mathbb{R}, X)$ of all *totally uniformly ergodic functions*:

$$TE(\mathbb{R}, X) := \{u \in BUC(\mathbb{R}, X) \,|\, e^{i\eta \cdot} u \in E(\mathbb{R}, X) \ \forall \eta \in \mathbb{R}\};$$

(vi) the space $AAP(\mathbb{R}, X)$ of all *asymptotically almost periodic functions*:

$$AAP(\mathbb{R}, X) := C_0(\mathbb{R}, X) \oplus AP(\mathbb{R}, X).$$

Remark that the space $AP(\mathbb{R}, X)$ is included in all these spaces.

3. Discrete Spectrum

In this section we will show that a function $u \in BUC(\mathbb{R}, X)$ is almost periodic provided that the spectrum $\operatorname{sp}(u)$ of u is *discrete*, i.e., $\operatorname{sp}(u)$ consists only of isolated points.

An important result in the spectral theory of almost periodic functions is Loomis' well-known theorem [17], which Loomis proved in the scalar case, whereas the vector-valued version (see for example [16, p. 92] and [1, Theorem 3.2]) is a consequence of Kadets' theorem ([14] or [16, p. 86]).

Theorem 3.1 (Loomis). *Let $u \in BUC(\mathbb{R}, X)$ and assume that $\operatorname{sp}(u)$ is countable. Then $u \in AP(\mathbb{R}, X)$ provided that $c_0 \not\subseteq X$.*

The following example (compare with [16, p. 81]) shows that Theorem 3.1 fails on c, the space of all convergent sequences. Note that c is isomorphic to c_0.

Example 3.2. Let $u \in BUC(\mathbb{R}, c)$ given by

$$u(t) := (e^{\frac{i}{k}t})_{k \in \mathbb{N}}.$$

Then $u \notin AP(\mathbb{R}, c)$, but $\operatorname{sp}(u) = \{\frac{1}{k} | k \in \mathbb{N}\} \cup \{0\}$ is countable.

Lemma 3.3. *Let $u \in BUC(\mathbb{R}, X)$ and assume that $\eta \in \mathbb{R}$ is an isolated point of $\operatorname{sp}(u)$. Then $u = u_0 + u_1$, where $u_0 = e_\eta \otimes x$ for some $x \in X$ and $\eta \notin \operatorname{sp}(u_1)$.*

Proof. Since η is an isolated point of $\operatorname{sp}(u)$, there exists a function $\psi \in L^1(\mathbb{R})$ with $\mathcal{F}\psi = 1$ in a neighbourhood of η and $\mathcal{F}\psi = 0$ in a neighbourhood of $\operatorname{sp}(u) \setminus \{\eta\}$. Then we can write

$$u = (u * \psi) + (u - u * \psi) =: u_0 + u_1.$$

It follows from the definition of the Beurling spectrum (see also [6, 4.1.4]) that $\mathrm{sp}(u_0) \subseteq \mathrm{sp}(u) \cap \mathrm{supp}(\mathcal{F}\psi) = \{\eta\}$ and $\mathrm{sp}(u_1) \subseteq \mathrm{sp}(u) \cap \mathrm{supp}(1 - \mathcal{F}\psi) = \mathrm{sp}(u) \setminus \{\eta\}$. Hence $u_0 = e_\eta \otimes x$ for some $x \in X$ and $\eta \notin \mathrm{sp}(u_1)$. ∎

With the same notations as in Section 2, we have the following

Theorem 3.4. *Let $u \in BUC(\mathbb{R}, X)$ and assume that $\eta \in \mathrm{sp}_{AP}(u)$. Then η is an accumulation point of* $\mathrm{sp}(u)$.

Proof. Assume that η is an isolated point of $\mathrm{sp}(u)$. Then by Lemma 3.3 we have

$$u = u_0 + u_1,$$

where $u_0 = e_\eta \otimes x \in AP(\mathbb{R}, X)$ and $\eta \notin \mathrm{sp}(u_1)$. Hence the closed linear spans $Y_{\bar{u}}$ and $Y_{\bar{u}_1}$ of the orbit of the elements $\overline{u}$, respectively $\overline{u_1}$, in the quotient space $Y = BUC(\mathbb{R}, X)/AP(\mathbb{R}, X)$ coincide. It follows that

$$\sigma(\bar{B}_{\bar{u}}) = \sigma(\bar{B}_{\bar{u}_1}).$$

Since $\sigma(\bar{B}_{\bar{u}_1}) \subseteq \sigma(B_{u_1})$ and $i\mathrm{sp}(u_1) = \sigma(B_{u_1})$, we obtain from Lemma 3.3 that $i\eta \notin \sigma(\bar{B}_{\bar{u}}) = i\mathrm{sp}_{AP}(u)$ (by Proposition 2.1), which is a contradiction. ∎

From this we deduce immediately the following result [8, Theorem 8, p. 78].

Corollary 3.5. *Let $u \in BUC(\mathbb{R}, X)$ and assume that* $\mathrm{sp}(u)$ *is discrete. Then $u \in AP(\mathbb{R}, X)$.*

Proof. Since $\mathrm{sp}(u)$ does not have accumulation points, it follows from Theorem 3.4 that $\mathrm{sp}_{AP}(u) = \emptyset$. Hence $u \in AP(\mathbb{R}, X)$ by Corollary 2.2. ∎

Remark 3.6. The proof of Corollary 3.5 we present here is designed for the further development in Section 4. However, a more direct argument is possible: Let $u \in BUC(\mathbb{R}, X)$. Take an approximate unit $\rho_n \in L^1(\mathbb{R})$ $(n \in \mathbb{N})$ such that $\mathcal{F}\rho_n$ has compact support and $\rho_n * u \to u$ $(n \to \infty)$ in $BUC(\mathbb{R}, X)$. Then $\mathrm{sp}(\rho_n * u) \subseteq \mathrm{supp}(\mathcal{F}\rho_n) \cap \mathrm{sp}(u)$. Thus if $\mathrm{sp}(u)$ is discrete, $\rho_n * u$ has finite spectrum. This implies that $\rho_n * u$ is a trigonometric polynomial, and hence $u \in AP(\mathbb{R}, X)$.

4. Cauchy Problems

Throughout this section let A be a closed, linear operator on a Banach space X. In the following we consider first and second order Cauchy problems.

First we consider the *first order inhomogeneous Cauchy problem* on the real line:

$$\text{(CP)} \qquad u'(t) = Au(t) + f(t) \quad (t \in \mathbb{R}),$$

where $f : \mathbb{R} \to X$ is continuous.

By a *mild solution* of (CP) we understand a continuous function $u : \mathbb{R} \to X$ such that $\int_0^t u(s)ds \in D(A)$ and

$$u(t) - u(0) = A \int_0^t u(s)ds + \int_0^t f(s)ds$$

for all $t \in \mathbb{R}$.

For $f \in BUC(\mathbb{R}, X)$, denote the accumulation points of $\operatorname{sp}(f)$ by $\operatorname{sp}'(f)$, i.e.,

$$\operatorname{sp}'(f) := \{\eta \in \operatorname{sp}(f) \mid \exists (\eta_n)_{n\in\mathbb{N}} \subseteq \operatorname{sp}(f) \setminus \{\eta\} \text{ such that } \lim_n \eta_n = \eta\}.$$

For the proof of the next theorem we first use the Laplace-transform argument of [1, Theorem 4.3]. But here an additional argument is needed since we do not assume that $c_0 \not\subseteq X$.

Theorem 4.1. *Let $u \in BUC(\mathbb{R}, X)$ be a mild solution of* (CP) *and assume that $\sigma(A) \cap i\mathbb{R}$ is discrete as a subset of $i\mathbb{R}$. Let $f \in AP(\mathbb{R}, X)$ with $i\operatorname{sp}'(f) \cap \sigma(A) = \emptyset$. Then $u \in AP(\mathbb{R}, X)$.*

Proof. Since u is a solution, we obtain from (CP) by taking Laplace transforms

$$\hat{u}(\lambda) = R(\lambda, A)u(0) + R(\lambda, A)\hat{f}(\lambda) \tag{5}$$

for $\operatorname{Re}\lambda \neq 0$ and $\lambda \in \rho(A)$, where $\hat{u}$ is the Carleman transform defined in Section 2. It follows that

$$i\operatorname{sp}(u) \subseteq (\sigma(A) \cap i\mathbb{R}) \cup i\operatorname{sp}(f). \tag{6}$$

By the proof of [1, Theorem 4.3], we obtain

$$i\operatorname{sp}_{AP}(u) \subseteq \sigma(A) \cap i\mathbb{R}.$$

By Theorem 3.4, we deduce from this

$$i\operatorname{sp}_{AP}(u) \subseteq (\sigma(A) \cap i\mathbb{R}) \cap i\operatorname{sp}'(u). \tag{7}$$

On the other hand, since by hypothesis $\sigma(A) \cap i\mathbb{R}$ is discrete, assertion (6) implies that

$$\operatorname{sp}'(u) \subseteq \operatorname{sp}'(f).$$

Hence by (7), $i\mathrm{sp}_{AP}(u) \subseteq \sigma(A) \cap i\mathrm{sp}'(f) = \emptyset$. This implies by Corollary 2.2 that $u \in AP(\mathbb{R}, X)$. ■

Note that in Theorem 4.1 we merely need that $\sigma(A) \cap i\mathbb{R}$ is discrete in $i\mathbb{R}$, but $\sigma(A) \cap i\mathbb{R}$ is allowed to contain limit points of $\sigma(A)$.

From [1, Proposition 4.2] (see also [4, 2. Theorem]), we obtain the following corollary.

Corollary 4.2. *Let A be the generator of a bounded C_0-group on a Banach space X and assume that $\sigma(A)$ is discrete. Then*

$$X = X_{AP} := \overline{\mathrm{span}}\{x \in D(A) \,|\, \exists \eta \in \mathbb{R} \text{ such that } Ax = i\eta x\}$$

and the C_0-group is almost periodic.

Theorem 4.1 is in some sense the best possible result. This is shown by the following example where $A = 0$.

Example 4.3. Define $f \in AP(\mathbb{R}, c)$ by $f(t) := (\frac{i}{k} e^{\frac{i}{k}t})_{k\in\mathbb{N}}$. Note that $\mathrm{sp}(f) = \{\frac{1}{k} | k \in \mathbb{N}\} \cup \{0\}$; thus $\mathrm{sp}'(f) = \{0\}$ has nonempty intersection with the spectrum of the operator $A = 0$. The function $u \in BUC(\mathbb{R}, c)$ given by $u(t) = (e^{\frac{i}{k}t})_{k\in\mathbb{N}}$ is a solution of

$$u'(t) = Au(t) + f(t) \qquad (t \in \mathbb{R}),$$

(see also Example 3.2). But $u \notin AP(\mathbb{R}, c)$.

Let us now consider closed, translation-invariant subspaces $\mathcal{G}$ of $BUC(\mathbb{R}, X)$ where $AP(\mathbb{R}, X)$ is included in $\mathcal{G}$. For example, the weakly almost periodic functions, the uniformly ergodic functions and the totally uniformly ergodic functions belong to this class (see Section 2).

By a slight modification of the proof, we obtain the following generalization of Theorem 4.1.

Theorem 4.4. *Let $\mathcal{G} \subseteq BUC(\mathbb{R}, X)$ be a closed, translation-invariant subspace of $BUC(\mathbb{R}, X)$ containing $AP(\mathbb{R}, X)$, and suppose $f \in \mathcal{G}$. Assume that $\sigma(A) \cap i\mathbb{R}$ is discrete and that $i\mathrm{sp}'(f) \cap \sigma(A) = \emptyset$. Let $u \in BUC(\mathbb{R}, X)$ be a mild solution of* (CP). *Then $u \in \mathcal{G}$.*

Proof. It follows from (6) that $i\mathrm{sp}(u) \subseteq (\sigma(A) \cap i\mathbb{R}) \cup i\mathrm{sp}(f)$. And hence, since $(\sigma(A) \cap i\mathbb{R})$ is discrete, we obtain

$$\mathrm{sp}'(u) \subseteq \mathrm{sp}'(f). \tag{8}$$

Now we consider the quotient space $\tilde{Y} := BUC(\mathbb{R}, X)/\mathcal{G}$ with the induced shift group $\tilde{\mathcal{S}}$ and generator $\tilde{B}$. Then we obtain as in the proof of [1, Theorem 4.3] that $R(\lambda, \tilde{B})\tilde{u} = (R(\lambda, A) \circ u)\tilde{}$. It follows (by (4)) that

$$i\mathrm{sp}_{\mathcal{G}}(u) = \sigma(\tilde{B}_{\tilde{u}}) \subseteq \sigma(A) \cap i\mathbb{R}. \tag{9}$$

Since $AP(\mathbb{R}, X) \subseteq \mathcal{G}$, we have that $\mathrm{sp}_{\mathcal{G}}(u) \subseteq \mathrm{sp}_{AP}(u)$ and $\mathrm{sp}_{AP}(u) \subseteq \mathrm{sp}'(u)$ by Theorem 3.4. So by (8) and (9) we conclude that $\mathrm{sp}_{\mathcal{G}}(u) \subseteq (\sigma(A) \cap i\mathbb{R}) \cap \mathrm{sp}'(f) = \emptyset$. It follows that $u \in \mathcal{G}$. ∎

Remark 4.5. It is interesting to compare Theorem 4.1 with the following stronger result [1, Theorem 4.3] which holds if $c_0 \not\subseteq X$: Assume that $\sigma(A) \cap i\mathbb{R}$ is countable. Let $u \in BUC(\mathbb{R}, X)$ be a solution of (CP). If $f \in AP(\mathbb{R}, X)$, then $u \in AP(\mathbb{R}, X)$. In contrast to Theorem 4.4, this result does not extend to more general spaces than $AP(\mathbb{R}, X)$. We give an example in the scalar case. Let $\mathcal{G} = TE(\mathbb{R})$. Then there exists $f \in \mathcal{G}$ such that $u(t) = \int_0^t f(s)ds$ is bounded, but $u \notin \mathcal{G}$. Thus, if we choose $A = 0$, then $u \in BUC(\mathbb{R})$ is a solution of (CP), but $u \notin \mathcal{G}$. Such f can be defined by

$$f(t) \quad := \quad \begin{cases} \frac{1}{2\sqrt{t}} \cos \sqrt{t} & \text{if } t \geq \frac{\pi^2}{4}, \\ 0 & \text{if } t < \frac{\pi^2}{4}. \end{cases}$$

Then

$$u(t) \quad := \quad \begin{cases} \sin \sqrt{t} - \sin \frac{\pi}{2} & \text{if } t \geq \frac{\pi^2}{4}, \\ 0 & \text{if } t < \frac{\pi^2}{4}. \end{cases}$$

It has been shown in [10, Example 4.2] that this function is not totally uniformly ergodic.

Finally we consider the *second order inhomogeneous Cauchy problem* on the real line:

$$(\mathrm{CP}_2) \qquad \begin{cases} u''(t) = Au(t) + f(t) & t \in \mathbb{R}, \\ u(0) = x,\, u'(0) = y & x, y \in X, \end{cases}$$

where f is a continuous function on $\mathbb{R}$ with values in X.

Recall that a continuous function $u : \mathbb{R} \to X$ is called a *mild solution* of (CP_2) if $\int_0^t (t-s)u(s)ds \in D(A)$ and

$$u(t) = x + ty + A\int_0^t (t-s)u(s)ds + \int_0^t (t-s)f(s)ds \quad (t \in \mathbb{R}).$$

Theorem 4.6. *Let $u \in BUC(\mathbb{R}, X)$ be a mild solution of* (CP$_2$), *where $\sigma(A) \cap (-\infty, 0]$ is discrete. Assume that $f \in AP(\mathbb{R}, X)$ and that $\{\eta \in \mathbb{R} | -\eta^2 \in \sigma(A)\} \cap \mathrm{sp}'(f) = \emptyset$. Then $u \in AP(\mathbb{R}, X)$.*

Proof. Taking Laplace transform we obtain from (CP$_2$) that $\hat{u}(\lambda) = \lambda R(\lambda^2, A)x + R(\lambda^2, A)y + R(\lambda^2, A)\hat{f}(\lambda)$ for $\mathrm{Re}\,\lambda \neq 0$ and $\lambda^2 \in \rho(A)$. Hence

$$\mathrm{sp}(u) \subseteq \{\eta \in \mathbb{R} \,|\, -\eta^2 \in \sigma(A)\} \cup \mathrm{sp}(f). \tag{10}$$

As in the proof of [1, Theorem 4.5], we obtain that

$$\mathrm{sp}_{AP}(u) \subseteq \{\mu \in \mathbb{R} \,|\, -\mu^2 \in \sigma(A)\}. \tag{11}$$

Now suppose that $\eta \in \mathrm{sp}_{AP}(u)$. Then by (11), $\eta \in \{\mu \in \mathbb{R} \,|\, -\mu^2 \in \sigma(A)\}$ and by Theorem 3.4, η is an accumulation point of $\mathrm{sp}(u)$. Since $\sigma(A) \cap (-\infty, 0]$ is discrete, it follows from (10) that $\eta \in \mathrm{sp}'(f)$. Hence $\eta \in \{\mu \in \mathbb{R} \,|\, -\mu^2 \in \sigma(A)\} \cap \mathrm{sp}'(f)$, which is a contradiction. We conclude that $\mathrm{sp}_{AP}(u) = \emptyset$ and so $u \in AP(\mathbb{R}, X)$ by Corollary 2.2. ■

With the help of [1, Proposition 4.8], we obtain the following corollary from Theorem 4.6.

Corollary 4.7. *Let A be the generator of a bounded cosine function on a Banach space X. Assume that $\sigma(A)$ is discrete. Then the eigenvectors of A are total in X, i.e.,*

$$X = \overline{\mathrm{span}}\{x \in D(A) \,|\, \exists \eta \in \mathbb{R} \text{ such that } Ax = -\eta^2 x\}$$

and the cosine function is almost periodic.

For closed, translation-invariant subspaces of $BUC(\mathbb{R}, X)$, we obtain the following generalization of Theorem 4.6. We omit the proof, which is similar.

Theorem 4.8. *Let $\mathcal{G} \subseteq BUC(\mathbb{R}, X)$ be a closed, translation-invariant subspace of $BUC(\mathbb{R}, X)$ containing $AP(\mathbb{R}, X)$, and suppose $f \in \mathcal{G}$. Assume that $\sigma(A) \cap (-\infty, 0]$ is discrete and that $\{\eta \in \mathbb{R} | -\eta^2 \in \sigma(A)\} \cap \mathrm{sp}'(f) = \emptyset$. Let $u \in BUC(\mathbb{R}, X)$ be a mild solution of (CP_2). Then $u \in \mathcal{G}$.*

5. Almost Periodic Functions on Groups

In this section we consider the space $BUC(G, X)$ of all bounded uniformly continuous functions defined on a locally compact Abelian group G with values in a Banach space X. Denote by $\hat{G}$ the dual group of G.

For $u \in BUC(G,X)$, we define the *spectrum* of u as follows

$$\mathrm{sp}(u) := \mathrm{hull}\, I_u = \{\chi \in \hat{G} \,|\, (\mathcal{F}f)(\chi) = 0 \ \forall f \in I_u\},$$

where $I_u = \{f \in L^1(G) | f * u = 0\}$ and $(\mathcal{F}f)(\chi) = \int_G \overline{\chi(s)} f(s) ds$ is the Fourier transform of $f \in L^1(G)$. Here $\overline{\chi(s)}$ denotes complex conjugation in $\mathbb{C}$. For $G = \mathbb{R}$, this definition of the spectrum coincides with the one given in Section 2.

Now consider the shift representation $S : G \longrightarrow \mathcal{L}(BUC(G,X))$ which is defined by

$$(S(t)u)(s) := u(s+t)$$

for all $t, s \in G$.

Recall that the *Arveson spectrum* of a strongly continuous group representation is defined as follows (see [3]):

$$\mathrm{Sp}(S) := \{\chi \in \hat{G} \,|\, (\bar{\mathcal{F}}f)(\chi) = 0 \ \forall f \in I_S\},$$

where $(\bar{\mathcal{F}}f)(\chi) := \int_G \chi(s) f(s) ds$. Here $I_S = \{f \in L^1(G) | f(S) = 0\}$, where $f(S) \in \mathcal{L}(BUC(G,X))$ is defined by $f(S)u := \int_G f(s)S(s)u\, ds$ for all $u \in BUC(G,X)$.

Define the translation-invariant space $BUC_u := \overline{\mathrm{span}}\{S(t)u | t \in G\}$ generated by $u \in BUC(G,X)$ and denote by $S_u : G \longrightarrow \mathcal{L}(BUC_u)$ the induced shift representation on BUC_u. Since the ideal I_u is translation-invariant, one has that

$$\mathrm{sp}(u) = \mathrm{Sp}(S_u).$$

To be complete, we include a short proof of Loomis' theorem which is analogous to the one given in [1] for $G = \mathbb{R}$. However, here in the general case, we cannot use resolvents. We need the following generalization of Kadets' theorem ([14] or [16, p. 86]) for groups, which is due to Basit (see [5]).

Theorem 5.1 (Basit)**.** *Let $f \in BUC(G,X)$ with $S(t)f - f \in AP(G,X)$ for all $t \in G$ and assume that $c_0 \not\subseteq X$. Then $f \in AP(G,X)$.*

Here $AP(G,X)$ denotes the set of all almost periodic functions on G with values in the Banach space X, where a function $u \in BUC(G,X)$ is said to be *almost periodic* if the set $\{S(t)u | t \in G\}$ is relatively compact in $BUC(G,X)$ (see for example [15]). This definition is equivalent to (see [25, §35])

$$AP(G,X) = \overline{\mathrm{span}}\{\chi \otimes x \,|\, \chi \in \hat{G}, x \in X\},$$

where $(\chi \otimes x)(s) := \chi(s)x$ for all $s \in G$ and the closure is taken in $BUC(G,X)$.

Finally, recall that a closed subset M of a topological group is called *scattered* if every closed subset contains an isolated point. In the case $G = \mathbb{R}$ this is equivalent to the countability of M.

Theorem 5.2 (Loomis)**.** *Let $u \in BUC(G,X)$ and assume that $sp(u)$ is scattered. If $c_0 \not\subseteq X$, then $u \in AP(G,X)$.*

Proof. Assume that $u \notin AP(G,X)$. Then the closed linear span $Y_{\bar{u}}$ of the orbit $\{\bar{S}(t)\bar{u} | t \in G\}$ in the quotient space $Y = BUC(G,X)/AP(G,X)$ is nontrivial, where$^-$ denotes the quotient mapping and $\bar{S}$ the induced representation on Y, as before. It follows that the induced representation $\bar{S}_{\bar{u}} : G \longrightarrow \mathcal{L}(Y_{\bar{u}})$ is nontrivial. Hence its spectrum $\mathrm{Sp}(\bar{S}_{\bar{u}})$ is nonempty (see [3, Theorem 2.5]) and scattered as a closed subset of $\mathrm{Sp}(S_u) = \mathrm{sp}(u)$. Therefore there exists an isolated point $\chi \in \mathrm{Sp}(\bar{S}_{\bar{u}})$ which belongs to the point spectrum $\mathrm{Sp}_p(\bar{S}_{\bar{u}})$ ([3] or [11, Proposition 4.1]), i.e., there exists $g \in BUC_u \setminus AP(G,X)$ with

$$\bar{S}_{\bar{u}}(t)(\bar{g}) = \chi(t)(\bar{g})$$

for all $t \in G$. This means that $S(t)g - \chi(t)g \in AP(G,X)$ for all $t \in G$. Now define $h \in BUC(G,X)$ by $h(s) := \overline{\chi(s)}g(s)$, where $\overline{\chi(s)}$ denotes complex conjugation in $\mathbb{C}$. Then we have

$$\begin{aligned} ((S(t)h) - h)(s) &= h(s+t) - h(s) = \overline{\chi(s+t)}g(s+t) - \overline{\chi(s)}g(s) \\ &= \overline{\chi(s+t)}\,(g(s+t) - \chi(t)g(s)) \\ &= \overline{\chi(s+t)}\,(S(t)g - \chi(t)g)(s) \end{aligned}$$

for all $t, s \in G$. Hence $S(t)h - h \in AP(G,X)$ for all $t \in G$. From Basit's theorem, it follows that $h \in AP(G,X)$ and so $g \in AP(G,X)$, which is a contradiction. ∎

From the proof we see that it is actually enough that $\mathrm{Sp}(\bar{S}_{\bar{u}})$ is scattered. It is easy to see that in the case $G = \mathbb{R}$ one has $\mathrm{sp}_{AP}(u) = \mathrm{Sp}(\bar{S}_{\bar{u}})$.

In the following we omit the condition $c_0 \not\subseteq X$ from Loomis' theorem. Again merely the accumulation points of the spectrum of a function $u \in BUC(G,X)$ are responsible for the failure of Theorem 5.2 on c_0.

Since the Banach algebra $L^1(G)$ is regular, the proofs of Lemma 3.3 and Proposition 3.4 can be carried over to the general case. We omit the details.

Proposition 5.3. *Let $u \in BUC(G,X)$. Then the following hold:*

(i) *If $\chi \in \mathrm{sp}(u)$ is an isolated point of $\mathrm{sp}(u)$, then $u = u_0 + u_1$ where $u_0 = \chi \otimes x$ for some $x \in X$ and $\chi \notin \mathrm{sp}(u_1)$.*

(ii) *If* $\chi \in \mathrm{Sp}(\bar{S}_{\bar{u}})$, *then* χ *is an accumulation point of* $\mathrm{sp}(u)$.

As a consequence of Proposition 5.3, we obtain the following spectral condition for vector-valued almost periodic functions on a locally compact Abelian group G. The proof is similar to that of Corollary 3.5.

Corollary 5.4. *Let* $u \in BUC(G, X)$ *and assume that* $\mathrm{sp}(u)$ *is discrete. Then* $u \in AP(G, X)$.

Following the same lines as for the case $G = \mathbb{R}$, we finally consider a strongly continuous representation U of a locally compact Abelian group in a Banach space X.

Theorem 5.5. *Assume that one of the following two conditions is satisfied*:

(a) $c_0 \not\subseteq X$ *and* $\mathrm{Sp}(U)$ *is scattered or*

(b) $\mathrm{Sp}(U)$ *is discrete.*

Then U *is almost periodic*, *i.e.*, $U(.)x \in AP(G, X)$ *for all* $x \in X$.

Proof. This follows from Theorem 5.2, respectively Corollary 5.4, since $\mathrm{sp}(U(.)x) \subseteq \mathrm{Sp}(U)$ for $x \in X$ as is easily seen from the definitions of the spectra. ∎

Finally we remark that as for the case $G = \mathbb{R}$, the representation U is almost periodic if and only if $X = \overline{\mathrm{span}}\{x \in X \,|\, \exists \chi \in \hat{G} \text{ such that } U(t)x = \chi(t)x \;\; \forall t \in G\}$.

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References

1. W. Arendt and C. J. K. Batty, Almost periodic solutions of first and second order Cauchy problems, *J. Differential Equations* **137** (1997), 363-383.
2. W. Arendt and C. J. K. Batty, Asymptotically almost periodic solutions of inhomogeneous Cauchy problems on the half line, *Proc. London Math. Soc.*, to appear.

3. W. Arveson, The harmonic analysis of automorphism groups, *Proc. Sympos. Pure Math.* **38** (1982), 199-269.

4. H. Bart and S. Goldberg, Characterizations of almost periodic strongly continuous groups and semigroups, *Math. Ann.* **236** (1978), 105-116.

5. B. Basit, Generalisation of two theorems of M. I. Kadets concerning the indefinite integral of abstract almost periodic functions, *Math. Notes* **9** (1971), 181-186.

6. B. Basit, Some problems concerning different types of vector valued almost periodic functions, *Dissertationes Math.* **338** (1995).

7. B. Basit, Harmonic analysis and asymptotic behaviour of solutions of the abstract Cauchy Problem, *Semigroup Forum* **54** (1997), 58-74.

8. A. G. Baskakov, Harmonic analysis of cosine and exponential operator-valued functions, *Math. Sbornik* **52** (1985), 63-90.

9. C. J. K. Batty and R. Chill, Bounded solutions of inhomogeneous Cauchy problems, *Forum Math.*, to appear.

10. C. J. K. Batty, J. M. A. M. van Neerven and F. Räbiger, Tauberian theorems and stability of solutions of the Cauchy problem, *Trans. Amer. Math. Soc.*, to appear.

11. C. J. K. Batty and Vũ Quôc Phóng, Stability of strongly continuous representations of Abelian semigroups, *Math. Z.* **209** (1992), 75-88.

12. R. Chill, Tauberian theorems for vector-valued Fourier and Laplace transforms, *Studia Math.*, to appear.

13. E. B. Davies, *One-parameter Semigroups*, Academic Press, London, 1980.

14. M. I. Kadets, On the integration of almost periodic functions with values in a Banach space, *Funktsional. Anal. i Priozhen.* **3** (1969), 71-74.

15. Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover Pub., 1968.

16. B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Cambridge, 1982.

17. L. H. Loomis, The spectral characterisation of a class of almost periodic functions, *Ann. Math.* **22** (1960), 362-368.

18. R. Nagel, ed., *One-parameter Semigroups of Positive Operators*, Lecture Notes in Math. 1184, Springer, Berlin, 1984.

19. J. M. A. M. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators*, Birkhäuser, Basel, 1996.

20. J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, 1993.

21. W. M. Ruess and Vũ Quôc Phóng, Asymptotically almost periodic solutions of evolution equations in Banach spaces, *J. Differential Equations* **122** (1993), 282-301.

22. E. Vesentini, Spectral properties of weakly asymptotically almost periodic semigroups, *Adv. Math.* **128** (1997), 217-241.

23. Vũ Quôc Phóng, On the spectrum, complete trajectories, and asymptotic stability of linear semi-dynamical systems, *J. Differential Equations* **105** (1993), 30-45.

24. Vũ Quôc Phóng, Almost periodic and strongly stable semigroups of operators, in: *Linear Operators*, Banach Center Publications, Vol. 38., Polish Academy of Sciences, Warsawa, 1997.

25. A. Weil, *L'intégration dans les Groupes Topologiques et ses Applications*, Publications de l'Institut de mathématiques de l'Université de Strasbourg, Hermann, Paris, 1965.

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