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Slowly oscillating solutions of Cauchy problems with countable spectrum

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Let u be a bounded slowly oscillating mild solution of an inhomogeneous Cauchy problem, $\dot{u}(t) = Au(t) + f(t)$, on $\mathbb{R}$ or $\mathbb{R}_+$, where A is a closed operator such that $\sigma_{\mathrm{ap}}(A) \cap \mathrm{i}\mathbb{R}$ is countable, and the Carleman or Laplace transform of f has a continuous extension to an open subset of the imaginary axis with countable complement. It is shown that u is (asymptotically) almost periodic if u is totally ergodic (or if X does not contain c_0 in the case of a problem on $\mathbb{R}$). Similar results hold for second-order Cauchy problems and Volterra equations.

1. Introduction

We shall consider mild solutions of inhomogeneous Cauchy problems of the form

$$\left.\begin{aligned} \dot{u}(t) &= Au(t) + f(t), \quad t \in \mathbb{J}, \\ u(0) &= x, \end{aligned}\right\} \tag{1.1}$$

where A is a closed linear operator on a complex Banach space X, and $\mathbb{J}$ is either the line $\mathbb{R}$ or the half-line $\mathbb{R}_+ := [0, \infty)$. We seek conditions on A and f which ensure that a mild solution u is (asymptotically) almost periodic. Letting $\hat{f}$ denote the Carleman or Laplace transform of f and $R(\lambda, A) := (\lambda I - A)^{-1}$ be the resolvent of A, the problem (1.1) may be rewritten as

$$\hat{u}(\lambda) = R(\lambda, A)(x + \hat{f}(\lambda)). \tag{1.2}$$

Thus $\hat{u}$ may have singularities at points of the spectrum $\sigma(A)$ and at singularities of $\hat{f}$, but not elsewhere. When $\mathbb{J} = \mathbb{R}$, X does not contain c_0, u is bounded and uniformly continuous and the Carleman transform $\hat{u}$ has only countably many singularities in $\mathrm{i}\mathbb{R}$, a vector-valued version of a theorem of Loomis [15], [14, p. 92] ensures that u is almost periodic, since the Carleman and Beurling spectra coincide. When $\mathbb{J} = \mathbb{R}$, there is no ambiguity about the notion of singularity, because a continuous extension of $\hat{u}$ from $\mathbb{C} \setminus \mathrm{i}\mathbb{R}$ to an open interval in $\mathrm{i}\mathbb{R}$ is automatically holomorphic.

When $\mathbb{J} = \mathbb{R}_+$, the Laplace transform $\hat{u}$ may have a continuous extension to an interval without having a holomorphic extension. A Tauberian theorem somewhat analogous to Loomis's theorem was proved in [9], with a more direct proof in [2], where it was assumed that $\hat{u}$ has a holomorphic extension except at countably many points of $i\mathbb{R}$. This theorem was improved in [11] by allowing continuous extensions except at countably many points of $i\mathbb{R}$. Using (1.2), this theorem can be applied to some solutions of (1.1) when $\sigma(A) \cap i\mathbb{R}$ is countable and $\hat{f}$ has only countably many singularities.

In this paper, we consider more carefully solutions of (1.1) when $\hat{f}$ has a continuous extension near all except countably many points of the imaginary axis, but f is not necessarily almost periodic. We assume that the approximate point spectrum $\sigma_{\mathrm{ap}}(A)$ of A, rather than the spectrum $\sigma(A)$, contains only countably many points of $i\mathbb{R}$, and we show that $\hat{u}$ then has only countably many singularities. Moreover, we assume that u is slowly oscillating at infinity, rather than uniformly continuous. Under some supplementary conditions, we are then able to deduce that u is (asymptotically) almost periodic (theorems 4.3 and 4.5). Our method also works for second-order Cauchy problems, and for Volterra equations on $\mathbb{R}_+$, and we give the details of the latter case in § 5. Section 3 contains some background material on slowly oscillating functions, weakly almost periodic functions and countable spectra (we are grateful to the referee for several helpful suggestions about this section).

We mention here that other circumstances when solutions of (1.1) are almost periodic are considered in [1, 2, 5–7, 18]. In all these papers, it is assumed that f is (asymptotically) almost periodic, whereas we assume a spectral condition on f, namely that the singularities of $\hat{f}$ are countable.

2. Preliminaries

We shall consider functions from $\mathbb{R}$ or $\mathbb{R}_+ := [0, \infty)$ into a complex Banach space X. To save repetition, we shall use the symbol $\mathbb{J}$ to denote $\mathbb{R}$ or $\mathbb{R}_+$ when statements are valid in both cases. We denote the left and right half-planes in $\mathbb{C}$ by

$$\mathbb{C}_- := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\} \quad \text{and} \quad \mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}.$$

We let $L^\infty(\mathbb{J}, X)$ be the space of all (equivalence classes of) bounded (Bochner) measurable functions from $\mathbb{J}$ to X, and we consider this as a Banach space in the norm

$$\|f\|_\infty := \operatorname*{ess\,sup}_{t \in \mathbb{J}} \|f(t)\|.$$

We let $\mathrm{BUC}(\mathbb{J}, X)$ be the closed subspace of $L^\infty(\mathbb{J}, X)$ consisting of all uniformly continuous functions, and let $\mathrm{AP}(\mathbb{J}, X)$ be the subspace of all almost periodic functions. Thus $\mathrm{AP}(\mathbb{J}, X)$ is the closure in $\mathrm{BUC}(\mathbb{J}, X)$ of the set of all trigonometric polynomials. Moreover, $\mathrm{AP}(\mathbb{J}, X)$ is invariant under the strongly continuous group of translations, so $\rho * f \in \mathrm{AP}(\mathbb{R}, X)$ whenever $\rho \in L^1(\mathbb{R}, X)$ and $f \in \mathrm{AP}(\mathbb{R}, X)$ (see [12, 14] for various characterizations of almost periodic functions). When $X = \mathbb{C}$, we shall denote these spaces by $L^\infty(\mathbb{J})$, etc.

For $f \in L^\infty(\mathbb{R}, X)$, we consider the *Carleman transform* $\hat{f}$ of f, defined by

$$\hat{f}(\lambda) = \begin{cases} \displaystyle\int_0^\infty \mathrm{e}^{-\lambda s} f(s)\,\mathrm{d}s, & \lambda \in \mathbb{C}_+, \\ \displaystyle -\int_0^\infty \mathrm{e}^{\lambda s} f(-s)\,\mathrm{d}s, & \lambda \in \mathbb{C}_-. \end{cases}$$

The *Carleman spectrum* $\operatorname{sp}(f)$ is the closed set of all $\eta \in \mathbb{R}$ such that $\hat{f}$ does not have a holomorphic extension to a neighbourhood of the point $\mathrm{i}\eta$ in the complex plane. A standard argument with contour integrals shows that if $\hat{f}$ has a continuous extension to such a neighbourhood, then the extension is holomorphic. Moreover, $\operatorname{sp}(f)$ coincides with the support of the distributional Fourier transform of f, and with the Beurling spectrum of f [13, § VI.6] [14, p. 87] [16, proposition 0.5, p. 22], and hence with the Arveson spectrum of f with respect to the group of translations on $L^\infty(\mathbb{R}, X)$ given by $(S(t)f)(s) = f(s+t)$ [1, § 2]. The following properties are then standard.

(1) For $\rho \in L^1(\mathbb{R})$, $\operatorname{sp}(\rho * f) \subseteq \operatorname{supp}(\mathcal{F}\rho) \cap \operatorname{sp}(f)$, where $\mathcal{F}\rho$ is the Fourier transform of ρ.

(2) If $\operatorname{sp}(f)$ is empty, then $f(t) = 0$ almost everywhere.

(3) If $\operatorname{sp}(f)$ is bounded, then f is smooth.

(4) If $\operatorname{sp}(f)$ is finite, then f is a trigonometric polynomial.

(5) For $\tau > 0$, $\operatorname{sp}(f) \subseteq (2\pi/\tau)\mathbb{Z}$ if and only if f is τ-periodic, i.e. $f(t+\tau) = f(t)$ a.e.(t).

(6) If $f \in \mathrm{BUC}(\mathbb{R}, X)$ and $\operatorname{sp}(f)$ is discrete, then f is almost periodic.

Parts (5) and (6) can easily be deduced from (1) and (4) (see [16, p. 19], [4,10]).

We also recall Loomis's theorem [15] that if $f : \mathbb{R} \to \mathbb{C}$ is bounded and uniformly continuous and $\operatorname{sp}(f)$ is countable, then f is almost periodic. This theorem is also true for functions taking values in a Banach space X which does not contain c_0 [14, theorem 4, p. 92].

A function $f \in L^\infty(\mathbb{R}, X)$ is said to be *totally ergodic* if the Cesaro limits

$$\lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} \mathrm{e}^{-\mathrm{i}\eta t} f(s+t)\,\mathrm{d}t$$

exist, uniformly for $s \in \mathbb{R}$, for every $\eta \in \mathbb{R}$. This is equivalent to requiring the existence of the Abel limits

$$\lim_{\alpha\to 0} \alpha \hat{f}_s(\alpha + \mathrm{i}\eta)$$

uniformly in s, where $f_s(t) = f(s+t)$. The limits exist uniformly and equal 0 when $\eta \in \mathbb{R} \setminus \operatorname{sp}(f)$. Any almost periodic function f is totally ergodic, and $\operatorname{sp}(f)$ is then the closure of the set of all η where the Cesaro limit is non-zero.

For $f \in L^\infty(\mathbb{R}_+, X)$, we consider the *Laplace transform* $\hat{f}$ of f, defined by

$$\hat{f}(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda s} f(s)\,\mathrm{d}s, \quad \lambda \in \mathbb{C}_+.$$

The *half-line spectrum* $\mathrm{sp}_+(f)$ is defined to be the closed set $\mathbb{R}\setminus U$, where

$$U := \{\eta \in \mathbb{R} : \text{there exists } \varepsilon > 0 \text{ such that } \hat{f} \text{ has a continuous extension to } \mathbb{C}_+ \cup \mathrm{i}(\eta-\varepsilon, \eta+\varepsilon)\}.$$

Note that this spectrum is in general smaller than the one used in [2], where holomorphic extensions were required near $\mathrm{i}U$. For example, if $X = \ell^2$ and

$$f(t) = (n^{-1}\mathrm{e}^{-t/n}),$$

then $\hat{f}$ has a continuous extension to $\mathbb{C}_+ \cup \mathrm{i}\mathbb{R}$, but there is no holomorphic extension to a neighbourhood of 0. Actually, the results in this paper which involve the half-line spectrum (theorems 4.5 and 5.1) remain true for the still smaller notion of spectrum defined to be $\mathbb{R}\setminus V$, where

$$V := \{\eta \in \mathbb{R} : \text{there exists } \varepsilon > 0 \text{ such that } \sup_{0<\alpha<\varepsilon,\, |\beta-\eta|<\varepsilon} \|\hat{f}(\alpha+\mathrm{i}\beta)\| < \infty, \text{ and } \lim_{\alpha\to 0+} \hat{f}(\alpha+\mathrm{i}\beta) \text{ exists whenever } |\beta-\eta|<\varepsilon\}.$$

Let $f \in L^\infty(\mathbb{R}_+, X)$. It is clear that

$$\mathrm{sp}_+(f) \subseteq \bigcap\{\mathrm{sp}(g) : g \in L^\infty(\mathbb{R}, X), g|_{\mathbb{R}_+} = f\}. \tag{2.1}$$

In general, this inclusion is strict. Indeed, it follows from the uniqueness theorems for Laplace transforms and holomorphic functions that, given $f \in L^\infty(\mathbb{R}_+, X)$ and $\eta \in \mathbb{R}$, there is at most one extension g such that $\eta \notin \mathrm{sp}(g)$. Consequently, the right-hand side of (2.1) is non-empty unless $f = 0$ almost everywhere. On the other hand, $\mathrm{sp}_+(f)$ is empty whenever f decays exponentially fast. Indeed, the following proposition shows that the inclusion (2.1) is maximally strict whenever f has exponential decay.

PROPOSITION 2.1. *Let $f \in L^\infty(\mathbb{R}_+, X)$, $f \neq 0$, and suppose that there exist $\varepsilon > 0$ and M such that*

$$\|f(t)\| \leqslant M\mathrm{e}^{-\varepsilon t}, \quad t \geqslant 0.$$

Then $\mathrm{sp}_+(f) = \emptyset$ and $\mathrm{sp}(g) = \mathbb{R}$ for all extensions $g \in L^\infty(\mathbb{R}, X)$ of f.

Proof. The Laplace transform $\hat{f}$ has a holomorphic extension to the region where $\operatorname{Re}\lambda > -\varepsilon$, and $\hat{f}(\lambda)$ is bounded for $\operatorname{Re}\lambda \geqslant -\frac{1}{2}\varepsilon$. It follows immediately that $\mathrm{sp}_+(f) = \emptyset$. Moreover, if $g \in L^\infty(\mathbb{R}, X)$ is an extension of f with $\mathrm{sp}(g) \neq \mathbb{R}$, then the Carleman transform $\hat{g}(\lambda)$ agrees with $\hat{f}(\lambda)$ for $\operatorname{Re}\lambda > 0$, and hence for $\operatorname{Re}\lambda > -\varepsilon$ by analytic continuation. Since $\hat{g}(\lambda)$ is bounded for $\operatorname{Re}\lambda \leqslant -\frac{1}{2}\varepsilon$, it follows that $\hat{g}$ extends to a bounded entire function, so $\hat{g}$ is constant. Since $\lim_{\lambda\to\infty}\hat{f}(\lambda) = 0$, $\hat{f} = 0$. Hence $f = 0$ almost everywhere. □

A function $f \in L^\infty(\mathbb{R}_+, X)$ is said to be *totally ergodic* if the Cesaro limits

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \mathrm{e}^{\mathrm{i}\eta t} f(s+t)\,\mathrm{d}t$$

exist, uniformly for $s \geqslant 0$, for every $\eta \in \mathbb{R}$. This is equivalent to requiring the existence of the Abel limits

$$\lim_{\alpha\to 0+} \alpha\hat{f}_s(\alpha+\mathrm{i}\eta)$$

uniformly in s, where $f_s(t) = f(s+t)$. Again, the limits exist uniformly and equal zero whenever $\eta \in \mathbb{R} \setminus \mathrm{sp}_+(f)$ (a proof of this is given in [9, proposition 3.3] with $\mathrm{sp}_+(f)$ replaced by the larger notion of spectrum used in [2]; we are grateful to Ralph Chill for informing us that this is also true for smaller notions of the half-line spectrum).

3. Slowly oscillating functions with countable spectrum

A function $u : \mathbb{R} \to X$ is said to be *weakly almost periodic* if $\phi \circ u \in \mathrm{AP}(\mathbb{R})$ for all $\phi \in X^*$. Any weakly almost periodic function $u : \mathbb{R} \to X$ is bounded, by the 'uniform boundedness principle'. Moreover, u is weakly continuous, so its range is weakly separable, and hence norm separable [14, p. 65]. By Pettis's theorem, u is strongly measurable. Thus the space $\mathrm{WAP}(\mathbb{R}, X)$ of all weakly almost periodic functions $u : \mathbb{R} \to X$ is a closed translation-invariant subspace of $L^\infty(\mathbb{R}, X)$. Moreover, if $\rho \in L^1(\mathbb{R})$ and $u \in \mathrm{WAP}(\mathbb{R}, X)$, then $\phi \circ (\rho * u) = \rho * (\phi \circ u) \in \mathrm{AP}(\mathbb{R})$ for all $\phi \in X^*$, so $\rho * u \in \mathrm{WAP}(\mathbb{R}, X)$.

Our terminology is consistent with [14], but it is important not to confuse the weakly almost periodic functions in this sense with the weakly almost periodic functions in the sense of Eberlein considered in [2] and elsewhere. Indeed, some authors use the terminology *scalarly almost periodic* for functions in the class $\mathrm{WAP}(\mathbb{J}, X)$.

The following simple lemma is probably well known.

LEMMA 3.1. *If* $u \in \mathrm{WAP}(\mathbb{R}, X)$, *then*

$$\|u\|_\infty = \sup_{t \geqslant \tau} \|u(t)\| \quad \textit{for all } \tau \in \mathbb{R}. \tag{3.1}$$

Proof. Note first that (3.1) holds for trigonometric polynomials and hence for $u \in \mathrm{AP}(\mathbb{R})$.

Now suppose that $u \in \mathrm{WAP}(\mathbb{R}, X)$, and let $\varepsilon > 0$. Take $t \in \mathbb{R}$ such that $\|u(t)\| > \|u\|_\infty - \varepsilon$ and $\phi \in X^*$ such that $\|\phi\| = 1$ and $|\langle u(t), \phi \rangle| > \|u\|_\infty - \varepsilon$. Since $\phi \circ u \in \mathrm{AP}(\mathbb{R})$, there exists $s \geqslant \tau$ such that $\|u\|_\infty - \varepsilon < |\langle u(s), \phi \rangle| \leqslant \|u(s)\|$. □

Recall [17, definition 9.6] that a function $u : \mathbb{J} \to X$ is said to be *slowly oscillating at infinity* if

$$\left.\begin{array}{c} \text{For all } \varepsilon > 0, \text{ there exist } a \in \mathbb{J} \text{ and } \delta > 0 \text{ such that} \\ \|u(t) - u(s)\| < \varepsilon \text{ whenever } |s - t| < \delta,\ s \geqslant a \text{ and } t \geqslant a. \end{array}\right\} \tag{3.2}$$

Equivalently, u is slowly oscillating at infinity if and only if $u = u_0 + u_1$, where u_1 is uniformly continuous and $\lim_{t\to\infty} u_0(t) = 0$ [11, lemma 1.6]. If u is bounded and slowly oscillating at infinity, then u_0 and u_1 may be chosen to be bounded.

THEOREM 3.2. *Let* $u \in L^\infty(\mathbb{R}, X)$, *and suppose that* $\mathrm{sp}(u)$ *is countable and* u *is slowly oscillating at infinity. Then there exists* $\tilde{u} \in \mathrm{WAP}(\mathbb{R}, X) \cap \mathrm{BUC}(\mathbb{R}, X)$ *such that* $u(t) = \tilde{u}(t)$ *almost everywhere.*

Proof. Choose $\rho_1 \in C_c^\infty(\mathbb{R})$ so that $0 \leqslant \rho_1$, $\rho_1(-t) = \rho_1(t)$, $\operatorname{supp} \rho_1 \subseteq [-1, 1]$, and

$$\int_{\mathbb{R}} \rho_1 = 1.$$

Let $\rho_n(t) = n\rho_1(nt)$ $(n \geqslant 2)$. Then $\rho_n * u \in \mathrm{BUC}(\mathbb{R}, X)$ and $\operatorname{sp}(\rho_n * u) \subseteq \operatorname{sp} u$. By Loomis's theorem, $\rho_n * u \in \mathrm{WAP}(\mathbb{R}, X)$. We will show that $(\rho_n * u)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{BUC}(\mathbb{R}, X)$.

Let $\varepsilon > 0$, and choose $a \in \mathbb{R}$ and $\delta > 0$ as in (3.2). Then, for $t \geqslant a + 1$ and n, $m \geqslant 1/\delta$,

$$\|(\rho_n * u)(t) - (\rho_m * u)(t)\| = \left\| \int_{-1}^{1} \rho_1(s)\left(u\left(t - \frac{s}{n}\right) - u\left(t - \frac{s}{m}\right)\right) \mathrm{d}s\right\| < \varepsilon.$$

It follows from lemma 3.1 that $\|\rho_n * u - \rho_m * u\|_\infty \leqslant \varepsilon$ whenever m, $n \geqslant 1/\delta$, as required.

Now let $\tilde{u} = \lim_{n\to\infty} \rho_n * u \in \mathrm{WAP}(\mathbb{R}, X) \cap \mathrm{BUC}(\mathbb{R}, X)$. For $\psi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \psi\, \tilde{u} = \lim_{n\to\infty} \int_{\mathbb{R}} \psi\, (\rho_n * u) = \lim_{n\to\infty} \int_{\mathbb{R}} (\rho_n * \psi)u = \int_{\mathbb{R}} \psi\, u.$$

Hence $\tilde{u} = u$ almost everywhere. □

COROLLARY 3.3. *Let $u \in L^\infty(\mathbb{R}, X)$, and suppose that u is slowly oscillating at infinity,* $\operatorname{sp}(u)$ *is countable and one of the following conditions is satisfied:*

(1) *X does not contain c_0;*

(2) *u has relatively weakly compact range;*

(3) *u is totally ergodic;*

(4) $\operatorname{sp} u$ *is discrete.*

Then there exists $\tilde{u} \in \mathrm{AP}(\mathbb{R}, X)$ such that $u(t) = \tilde{u}(t)$ almost everywhere.

Proof. In view of theorem 3.2, this follows from the corresponding results for $u \in \mathrm{BUC}(\mathbb{R}, X)$, [14, theorem 4, p. 92] (for conditions (1) and (2)), [18, § 3] (for condition (3)), [10] or [4] (for condition (4)) (see also [1, § 3] for conditions (1), (2) and (3)). □

While weakly almost periodic functions are not norm continuous in general (see [14, p. 75]), the argument of theorem 3.2 provides the following result.

PROPOSITION 3.4. *Let $u \in \mathrm{WAP}(\mathbb{R}, X)$ and suppose that u is slowly oscillating at infinity. Then $u \in \mathrm{BUC}(\mathbb{R}, X)$.*

Proof. For $\rho \in L^1(\mathbb{R})$, $\rho * u \in \mathrm{WAP}(\mathbb{R}, X) \cap \mathrm{BUC}(\mathbb{R}, X)$. By the argument of theorem 3.2, there exists $\tilde{u} \in \mathrm{BUC}(\mathbb{R}, X)$ such that $\tilde{u}(t) = u(t)$ almost everywhere. Since $\tilde{u}$ and u are weakly continuous, $\tilde{u}(t) = u(t)$ everywhere. □

It follows from corollary 3.3 (and can easily be seen directly) that a periodic function $f \in L^\infty(\mathbb{R}, X)$ is slowly oscillating at infinity if and only if f is continuous. Choosing a weakly continuous periodic function which is not norm continuous, it follows that the assumption of slow oscillation cannot be omitted from theorem 3.2, corollary 3.3 or proposition 3.4.

It is well known (and it follows easily from lemma 3.1) that a (weakly) almost periodic function $u : \mathbb{R} \to X$ such that $\lim_{t\to\infty} u(t)$ exists is constant. On the other hand, any almost periodic function is totally ergodic and has a Cesaro limit as $t \to \infty$. A concept which is intermediate between convergence and Cesaro convergence is the following notion of B-convergence which has proved to be a useful form of mean-convergence for studying Laplace transforms and Tauberian theorems [3,6]. For $u \in L^\infty(\mathbb{R}, X)$ and $u \in X$, we write $\text{B-lim}_{t\to\infty} u(t) = u_\infty$ if, for every $\delta > 0$,

$$\lim_{t\to\infty} \frac{1}{\delta} \int_t^{t+\delta} u(s)\,\mathrm{d}s = u_\infty.$$

Although we shall not need it here, the following form of Wiener's Tauberian theorem [17, theorem 9.7] clarifies this definition.

PROPOSITION 3.5. *Let $u \in L^\infty(\mathbb{R}, X)$, $u_\infty \in X$, and suppose that*

$$\lim_{t\to\infty} \frac{1}{\delta} \int_t^{t+\delta} u(s)\,\mathrm{d}s = u_\infty$$

holds for $\delta = \delta_1$ and $\delta = \delta_2$, where δ_1 and δ_2 are rationally independent. Then

$$\lim_{t\to\infty} (\rho * u)(t) = \left(\int_{\mathbb{R}} \rho\right) u_\infty \tag{3.3}$$

for all $\rho \in L^1(\mathbb{R})$. If u is slowly oscillating at infinity, then $\lim_{t\to\infty} u(t) = u_\infty$.

Proof. Let $\psi_\delta = (1/\delta)\mathbf{1}_{(0,\delta)}$, where $\mathbf{1}_{(0,\delta)}$ is the characteristic function of $(0, \delta)$. Then

$$\mathcal{F}\psi_\delta(s) = \begin{cases} \dfrac{1 - \mathrm{e}^{-\mathrm{i}s\delta}}{\mathrm{i}s\delta} & \text{if } s \neq 0, \\ 1 & \text{if } s = 0. \end{cases}$$

Thus $\mathcal{F}\psi_{\delta_1}$ and $\mathcal{F}\psi_{\delta_2}$ do not vanish simultaneously, so the translates of ψ_{δ_1} and ψ_{δ_2} form a total subset of $L^1(\mathbb{R})$, by Wiener's theorem [17, theorem 9.4]. Moreover, the set of all $\rho \in L^1(\mathbb{R})$ satisfying (3.3) is a closed translation-invariant subspace of $L^1(\mathbb{R})$, containing ψ_{δ_1} and ψ_{δ_2} by assumption, so it coincides with $L^1(\mathbb{R})$.

The final statement is proved in [17, theorem 9.7]. □

Now we show that B-convergence is incompatible with weak almost periodicity and with countable spectrum.

PROPOSITION 3.6. *Let $u \in L^\infty(\mathbb{R}, X)$, and suppose that* $\text{B-lim}_{t\to\infty}(\phi \circ u)(t)$ *exists for all $\phi \in X^*$. Suppose also that one of the following two conditions holds.*

(1) $u \in \mathrm{WAP}(\mathbb{R}, X)$.

(2) $\mathrm{sp}(u)$ *is countable.*

Then u is constant.

Proof. For $\delta > 0$ and $\phi \in X^*$, let

$$u_{\delta,\phi}(t) = \frac{1}{\delta} \int_t^{t+\delta} \langle u(s), \phi \rangle\,\mathrm{d}s.$$

Then $u_{\delta,\phi} \in \mathrm{BUC}(\mathbb{R})$, $\lim_{t\to\infty} u_{\delta,\phi}$ exists and $\mathrm{sp}(u_{\delta,\phi}) \subseteq \mathrm{sp}(u)$. In condition (1), $u_{\delta,\phi} \in \mathrm{AP}(\mathbb{R})$ since

$$u_{\delta,\phi} = \frac{1}{\delta} \int_0^\delta S(t)(\phi \circ u)\,\mathrm{d}t,$$

where $\{S(t) : t \in \mathbb{R}\}$ is the strongly continuous translation group on $\mathrm{AP}(\mathbb{R})$. In condition (2), $u_{\delta,\phi} \in \mathrm{AP}(\mathbb{R})$ by Loomis's theorem. Thus in each case, $u_{\delta,\phi}$ is constant. As $\delta \searrow 0$, $u_{\delta,\phi} \to \phi \circ u$ in the weak* topology, so $\phi \circ u$ is constant. By the Hahn–Banach theorem, u is constant. □

Recall [12] that the space $\mathrm{AAP}(\mathbb{R}_+)$ of *asymptotically almost periodic functions* on $\mathbb{R}_+$ is given by

$$\mathrm{AAP}(\mathbb{R}_+) = C_0(\mathbb{R}_+) \oplus \mathrm{AP}(\mathbb{R}_+),$$

where $C_0(\mathbb{R}_+)$ is the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{C}$ which vanish at infinity.

Corollary 3.7. *Let* $u : \mathbb{R}_+ \to X$ *be such that*

$$\phi \circ u \in \mathrm{AAP}(\mathbb{R}_+) \quad \textit{and} \quad \text{B-}\lim_{t\to\infty} (\phi \circ u)(t)$$

exists for all $\phi \in X^*$. *Then* $\lim_{t\to\infty}(\phi \circ u)(t)$ *exists for all* $\phi \in X^*$. *In particular, if* u *has relatively weakly compact range, then* $\lim_{t\to\infty} u(t)$ *exists weakly in* X.

Proof. Since $\phi \circ u \in \mathrm{AAP}(\mathbb{R}_+)$, $\phi \circ u = v_\phi + w_\phi$ for some $v_\phi \in C_0(\mathbb{R}_+)$ and $w_\phi \in \mathrm{AP}(\mathbb{R}_+)$. Then w_ϕ has an extension in $\mathrm{AP}(\mathbb{R})$ and

$$\text{B-}\lim_{t\to\infty} w_\phi(t) = \text{B-}\lim u_\phi(t).$$

By proposition 3.6, w_ϕ is a constant α_ϕ, so $\lim_{t\to\infty}(\phi \circ u)(t) = \alpha_\phi$. The final statement follows by observing that if $x \in X$ is a weak limit point of $(u(n))_{n\geqslant 1}$, then $\alpha_\phi = \phi(x)$. □

4. Individual solutions of Cauchy problems

Let A be a closed operator on X, $f \in L^1_{\mathrm{loc}}(\mathbb{J}, X)$, and $x \in X$. We consider the (first-order) inhomogeneous Cauchy problem (1.1). By a mild solution of (1.1), we mean a function $u \in L^1_{\mathrm{loc}}(\mathbb{J}, X)$ such that

$$\int_0^t u(s)\,\mathrm{d}s \in D(A)$$

for almost all $t \in \mathbb{J}$ and

$$u(t) = x + A\int_0^t u(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s \quad \text{a.e.}(t). \tag{4.1}$$

Lemma 4.1. *Let* u *be a mild solution of (1.1), and suppose that* $\{u(t) : t \in I\}$ *is relatively (weakly) compact for every bounded interval* I *in* $\mathbb{J}$. *Then there is a (weakly) continuous mild solution* $\tilde{u}$ *of (1.1) on* $\mathbb{J}$ *such that* $\tilde{u}(t) = u(t)$ *almost everywhere.*

Proof. There is a null subset N of $\mathbb{J}$ such that

$$\int_0^t u(s)\,\mathrm{d}s \in D(A)$$

and (4.1) holds for all $t \in \mathbb{J} \setminus N$. Let $t \in \mathbb{J}$ and (t_n) be a sequence in $\mathbb{J}$ converging to t. Then there exists a subsequence such that $y = \lim_{k\to\infty} u(t_{n_k})$ exists (weakly). Since A is closed, it follows that

$$\int_0^t u(s)\,\mathrm{d}s \in D(A)$$

and

$$y = x + A\int_0^t u(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s.$$

Thus the limit y is independent of the sequence (t_n) and subsequence (t_{n_k}), so $y = \lim_{s\to t} u(s)$ (weakly) and $y = u(t)$ if $t \in \mathbb{J} \setminus N$. Putting

$$\tilde{u}(t) := \lim_{s\to t} u(s) = x + A\int_0^t u(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s$$

gives the result. □

In the following, we assume that $f \in \mathcal{L}(\mathbb{J}, X)$, and we let $\mathbb{D} = \mathbb{C}_+$ in the case when $\mathbb{J} = \mathbb{R}_+$ and $\mathbb{D} = \mathbb{C} \setminus \mathrm{i}\mathbb{R}$ when $\mathbb{J} = \mathbb{R}$.

Let $u \in L^\infty(\mathbb{J}, X)$ be a mild solution of (1.1). Taking the Carleman or Laplace transform of (4.1) shows that $\hat{u}(\lambda) \in D(A)$ and

$$\hat{u}(\lambda) = \frac{x}{\lambda} + \frac{A\hat{u}(\lambda)}{\lambda} + \frac{\hat{f}(\lambda)}{\lambda} \tag{4.2}$$

for all $\lambda \in \mathbb{D}$ (see [9, proposition 5.1], [2, proposition 3.1]). Hence

$$\hat{u}(\lambda) = R(\lambda, A)x + R(\lambda, A)\hat{f}(\lambda), \quad \lambda \in \mathbb{D} \cap \rho(A). \tag{4.3}$$

PROPOSITION 4.2. *Let $u \in L^\infty(\mathbb{J}, X)$ be a mild solution of (1.1). Let $\eta \in \mathbb{R}$, and suppose that $\mathrm{i}\eta \notin \sigma_{\mathrm{ap}}(A)$, and that*

$$\lim_{\lambda\to \mathrm{i}\eta,\, \lambda\in\mathbb{D}} \hat{f}(\lambda) \quad \textit{exists.}$$

Then

$$\lim_{\lambda\to \mathrm{i}\eta,\, \lambda\in\mathbb{D}} \hat{u}(\lambda) \quad \textit{exists.}$$

Proof. First, we show that

$$\limsup_{\lambda\to \mathrm{i}\eta,\, \lambda\in\mathbb{D}} \|\hat{u}(\lambda)\| < \infty.$$

Otherwise, there is a sequence (λ_n) in $\mathbb{D}$ such that $\lambda_n \to \mathrm{i}\eta$ and $\|\hat{u}(\lambda_n)\| \to \infty$ as $n \to \infty$. Let

$$y_n = \frac{\hat{u}(\lambda_n)}{\|\hat{u}(\lambda_n)\|}.$$

Then

$$\lambda_n y_n - Ay_n = \frac{x}{\|\hat{u}(\lambda_n)\|} + \frac{\hat{f}(\lambda_n)}{\|\hat{u}(\lambda_n)\|} \to 0 \quad \text{as } n \to \infty.$$

This implies that $\mathrm{i}\eta \in \sigma_{\mathrm{ap}}(A)$, which contradicts our assumption.

Now we will show that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\hat{u}(\lambda) - \hat{u}(\mu)\| < \varepsilon$ whenever λ, $\mu \in \mathbb{D}$, $|\lambda - \mathrm{i}\eta| < \delta$ and $|\mu - \mathrm{i}\eta| < \delta$. Otherwise, there exist $\varepsilon > 0$ and sequences (λ_n) and (μ_n) in $\mathbb{D}$ such that $\lambda_n \to \mathrm{i}\eta$ and $\mu_n \to \mathrm{i}\eta$ as $n \to \infty$ but $\|\hat{u}(\lambda_n) - \hat{u}(\mu_n)\| \geqslant \varepsilon$ for all n. From (4.2),

$$(\lambda_n - \mu_n)\hat{u}(\lambda_n) + \mu_n(\hat{u}(\lambda_n) - \hat{u}(\mu_n)) - A(\hat{u}(\lambda_n) - \hat{u}(\mu_n)) = \hat{f}(\lambda_n) - \hat{f}(\mu_n).$$

From the previous paragraph, it follows that

$$\mu_n(\hat{u}(\lambda_n) - \hat{u}(\mu_n)) - A(\hat{u}(\lambda_n) - \hat{u}(\mu_n)) \to 0$$

as $n \to \infty$. This contradicts our assumption that $\mathrm{i}\eta \notin \sigma_{\mathrm{ap}}(A)$. Now Cauchy's criterion completes the proof. □

Now we have to state results separately for the cases $\mathbb{J} = \mathbb{R}$ and $\mathbb{J} = \mathbb{R}_+$.

First, consider the case when $\mathbb{J} = \mathbb{R}$. As observed in § 2, the continuous extension of $\hat{u}$ to open sets in $\mathrm{i}\mathbb{R}$ is automatically holomorphic. Thus proposition 4.2 shows that

$$\mathrm{sp}(u) \subseteq \mathrm{sp}(f) \cup \{\eta \in \mathbb{R} : \mathrm{i}\eta \in \sigma_{\mathrm{ap}}(A)\}. \tag{4.4}$$

THEOREM 4.3. *Let $\mathbb{J} = \mathbb{R}$, and let $u \in L^\infty(\mathbb{R}, X)$ be a mild solution of (1.1) which is slowly oscillating at infinity. Assume that $\sigma_{\mathrm{ap}}(A) \cap \mathrm{i}\mathbb{R}$ and $\mathrm{sp}(f)$ are countable. Then there exists $\tilde{u} \in \mathrm{WAP}(\mathbb{R}, X) \cap \mathrm{BUC}(\mathbb{R}, X)$ such that $u(t) = \tilde{u}(t)$ almost everywhere. If, in addition, any of conditions (1)–(3) of corollary 3.3 are satisfied, or if $\sigma_{\mathrm{ap}}(A) \cap \mathrm{i}\mathbb{R}$ and $\mathrm{sp}(f)$ are discrete, then $\tilde{u} \in \mathrm{AP}(\mathbb{R}, X)$.*

Proof. It follows from (4.4) that $\mathrm{sp}(u)$ is countable. Now theorem 3.2 and corollary 3.3 imply the result. □

Now we turn to the case when $\mathbb{J} = \mathbb{R}_+$. Then proposition 4.2 shows that

$$\mathrm{sp}_+(u) \subseteq \mathrm{sp}_+(f) \cup \{\eta \in \mathbb{R} : \mathrm{i}\eta \in \sigma_{\mathrm{ap}}(A)\}. \tag{4.5}$$

However, in this case the continuous extension of $\hat{u}$ to open sets may not be holomorphic. If $\hat{f}$ has a holomorphic extension to an open subset U of $\mathbb{C}_+ \cup \rho(A)$, then $\hat{u}$ also has a holomorphic extension to U given by (4.3) (see [2, proposition 3.1]). In the homogeneous case, this can be strengthened by a method similar to [8, theorem 1].

PROPOSITION 4.4. *Let $\mathbb{J} = \mathbb{R}_+$, let $u \in L^\infty(\mathbb{R}_+, X)$ be a mild solution of (1.1) with $f \equiv 0$, and let $\mathrm{i}\eta \in \mathrm{i}\mathbb{R} \setminus \sigma_{\mathrm{ap}}(A)$. Then $\hat{u}$ has a holomorphic extension to a neighbourhood of $\mathrm{i}\eta$ in $\mathbb{C}$.*

Proof. Let $Y = \overline{\mathrm{span}}\,(\{\hat{u}(\lambda) : \lambda \in \mathbb{C}_+\} \cup \{x\})$. Let A_Y be the part of A in Y, so that $D(A_Y) = \{y \in Y \cap D(A) : Ay \in Y\}$. Then A_Y is a closed operator on Y, although it may not be densely defined. By proposition 4.2,

$$y := \lim_{\lambda \to \mathrm{i}\eta,\, \lambda \in \mathbb{C}_+} \hat{u}(\lambda) \quad \text{exists.}$$

Since $\lambda\hat{u}(\lambda) - A\hat{u}(\lambda) = x$ for $\lambda \in \mathbb{C}_+$, we have $y \in D(A)$ and $\mathrm{i}\eta y - Ay = x$. Hence, $y \in D(A_Y)$ and $x \in \mathrm{Ran}(\mathrm{i}\eta I - A_Y)$. For $\lambda \in \mathbb{C}_+$, we have $\hat{u}(\lambda) \in D(A_Y)$ and

$$(\lambda - \mathrm{i}\eta)\hat{u}(\lambda) = x - (\mathrm{i}\eta\hat{u}(\lambda) - A\hat{u}(\lambda)) \in \mathrm{Ran}(\mathrm{i}\eta I - A_Y).$$

Thus $\hat{u}(\lambda) \in \mathrm{Ran}(\mathrm{i}\eta I - A_Y)$, so $\mathrm{i}\eta I - A_Y$ has dense range. Since $\mathrm{i}\eta \notin \sigma_{\mathrm{ap}}(A_Y)$, it follows that $\mathrm{i}\eta \in \rho(A_Y)$. For $\lambda \in \mathbb{C}_+ \cap \rho(A_Y)$, we have $\hat{u}(\lambda) = R(\lambda, A_Y)x$. Thus $R(\lambda, A_Y)x$ defines a holomorphic extension of $\hat{u}$ to a neighbourhood of $\mathrm{i}\eta$. □

A Tauberian theorem due to Chill [11, theorem 1.5], together with (4.5), yield the following result, even when holomorphic extensions may be absent.

THEOREM 4.5. *Let $\mathbb{J} = \mathbb{R}_+$, and let $u \in L^\infty(\mathbb{R}_+, X)$ be a solution of (1.1) which is slowly oscillating at infinity. Suppose that $\sigma_{\mathrm{ap}}(A) \cap \mathrm{i}\mathbb{R}$ and $\mathrm{sp}_+(f)$ are countable, and u is totally ergodic. Then $u = u_0 + u_1$, where $u_1 \in \mathrm{AP}(\mathbb{R}_+, X)$ and $\lim_{t\to\infty} u_0(t) = 0$.*

We remark here that in theorem 4.5 (and also theorem 4.3), in order to verify that u is totally ergodic, it suffices to establish that the Cesaro means of u exist uniformly when $\mathrm{i}\eta \in \sigma_{\mathrm{ap}}(A) \cup \mathrm{i}\,\mathrm{sp}_+(f)$.

Our methods are also applicable to second-order Cauchy problems,

$$\left.\begin{aligned} \ddot{u}(t) &= Au(t) + f(t), \quad t \in \mathbb{J}, \\ u(0) &= x, \quad \dot{u}(0) = y. \end{aligned}\right\} \tag{4.6}$$

A *mild solution* of (4.6) is a function $u \in L^1_{\mathrm{loc}}(\mathbb{J}, X)$ such that

$$\int_0^t (t-s)u(s)\,\mathrm{d}s \in D(A)$$

for almost all $t \in \mathbb{J}$ and

$$u(t) = x + ty + A\int_0^t (t-s)u(s)\,\mathrm{d}s + \int_0^t (t-s)f(s)\,\mathrm{d}s \quad \text{a.e.}(t). \tag{4.7}$$

Assume that $f \in \mathcal{L}(\mathbb{J}, X)$ and $u \in L^\infty(\mathbb{J}, X)$ is a mild solution of (4.6). Taking the Carleman or Laplace transform of (4.7) gives

$$\hat{u}(\lambda) = \frac{x}{\lambda} + \frac{y}{\lambda^2} + \frac{A\hat{u}(\lambda)}{\lambda^2} + \frac{\hat{f}(\lambda)}{\lambda^2}, \quad \lambda \in \mathbb{D}.$$

With only minor changes in the proof, proposition 4.2 remains valid with the assumption that $\mathrm{i}\eta \notin \sigma_{\mathrm{ap}}(A)$ replaced by $-\eta^2 \notin \sigma_{\mathrm{ap}}(A)$. Hence theorems 4.3 and 4.5 remain valid with the assumption that $\sigma_{\mathrm{ap}}(A) \cap \mathrm{i}\mathbb{R}$ is countable replaced by the assumption that $\sigma_{\mathrm{ap}}(A) \cap (-\infty, 0]$ is countable.

See [1, theorem 4.5] for a related result on second-order problems.

5. Individual solutions of Volterra equations

In this section, we describe how proposition 4.2, and consequently theorem 4.5, can be generalized to individual solutions of inhomogeneous Volterra equations on $\mathbb{R}_+$ (see the monograph of Prüss [16] for the background on Volterra equations).

Let $a \in L^1_{\rm loc}(\mathbb{R}_+)$ be exponentially bounded. We assume that $\hat{a}$ has a meromorphic extension to a map $\hat{a} : \mathbb{C}_+ \to \mathbb{C} \cup \{\infty\}$ and a continuous extension $\hat{a} : \mathbb{C}_+ \cup \mathrm{i}(\mathbb{R} \setminus \Sigma_0) \to \mathbb{C} \cup \{\infty\}$, where Σ_0 is a countable closed subset of $\mathbb{R}$. Let $g \in \mathcal{L}(\mathbb{R}_+, X)$.

Let $u \in L^\infty(\mathbb{R}_+, X)$ satisfy

$$\left.\begin{aligned} \int_0^t a(t-s)u(s)\,\mathrm{d}s \in D(A), \\ u(t) = g(t) + A\int_0^t a(t-s)u(s)\,\mathrm{d}s \end{aligned}\right\} \tag{5.1}$$

for almost all $t \in \mathbb{R}_+$. Taking Laplace transforms and using analytic continuation, we find that $\hat{u}(\lambda) \in D(A)$ for each $\lambda \in \mathbb{C}_+$, and

$$\left.\begin{aligned} \hat{u}(\lambda) = \hat{g}(\lambda) + \hat{a}(\lambda)A\hat{u}(\lambda), &\quad \hat{a}(\lambda) \neq \infty, \\ 0 = A\hat{u}(\lambda), &\quad \hat{a}(\lambda) = \infty. \end{aligned}\right\} \tag{5.2}$$

The situation considered in § 4 corresponds to the case when $a(t) = 1$ and

$$g(t) = x + \int_0^t f(s)\,\mathrm{d}s.$$

The role played in § 4 by $\sigma_{\rm ap}(A) \cap \mathrm{i}\mathbb{R}$ will now be taken by the following closed set Σ:

$$\Sigma := \begin{cases} \Sigma_0 \cup \{\eta \in \mathbb{R} : \hat{a}(\mathrm{i}\eta) = 0 \text{ or } \hat{a}(\mathrm{i}\eta) \neq 0,\ (\hat{a}(\mathrm{i}\eta))^{-1} \in \sigma_{\rm ap}(A)\} & \\ & \text{if } A \text{ is unbounded,} \\ \Sigma_0 \cup \{\eta \in \mathbb{R} : \hat{a}(i\eta) \neq 0,\ (\hat{a}(\mathrm{i}\eta))^{-1} \in \sigma_{\rm ap}(A)\} & \text{if } A \text{ is bounded.} \end{cases}$$

Here, and subsequently, $\infty^{-1} = 0$.

Theorem 5.1. *Let $u \in L^\infty(\mathbb{R}_+, X)$ be a solution of (5.1). Assume that Σ and $\mathrm{sp}_+(g)$ are countable, and u is totally ergodic and slowly oscillating at infinity. Then*

$$u = u_0 + u_1,$$

where $u_1 \in \mathrm{AP}(\mathbb{R}, X)$, $\lim_{t\to\infty} u_0(t) = 0$.

Proof. Let $\eta \in \mathbb{R} \setminus \Sigma$, $\eta \notin \mathrm{sp}_+(g)$.

First we show that

$$\limsup_{\lambda \to \mathrm{i}\eta,\, \lambda \in \mathbb{C}_+} \|\hat{u}(\lambda)\| < \infty.$$

Otherwise, there is a sequence (λ_n) in $\mathbb{C}_+$ such that $\lambda_n \to \mathrm{i}\eta$ and $\|\hat{u}(\lambda_n)\| \to \infty$ as $n \to \infty$. Let

$$y_n = \frac{\hat{u}(\lambda_n)}{\|\hat{u}(\lambda_n)\|}.$$

By (5.2),

$$\lim_{n\to\infty} \|y_n - \hat{a}(\lambda_n)Ay_n\| = \lim_{n\to\infty} \frac{\|\hat{g}(\lambda_n)\|}{\|\hat{u}(\lambda_n)\|} = 0. \tag{5.3}$$

There are two cases.

Case 1. $\hat{a}(\mathrm{i}\eta) = 0$. Since $\eta \notin \Sigma$, this occurs only when A is bounded, so

$$\lim_{n\to\infty} \|\hat{a}(\lambda_n) A y_n\| = 0,$$

while $\|y_n\| = 1$ for all n. This contradicts (5.3).

Case 2. $\hat{a}(\mathrm{i}\eta) \neq 0$. Then

$$\left\| \frac{y_n}{\hat{a}(\mathrm{i}\eta)} - A y_n \right\| \leqslant \left| \frac{1}{\hat{a}(\mathrm{i}\eta)} - \frac{1}{\hat{a}(\lambda_n)} \right| + \frac{\|y_n - \hat{a}(\lambda_n) A y_n\|}{|\hat{a}(\lambda_n)|} \to 0$$

as $n \to \infty$, by (5.3). Thus, $(\hat{a}(\mathrm{i}\eta))^{-1} \in \sigma_{\mathrm{ap}}(A)$, which contradicts the fact that $\eta \notin \Sigma$.

Next we will show that

$$\lim_{\lambda\to \mathrm{i}\eta,\, \lambda\in\mathbb{C}_+} \hat{u}(\lambda) =: \hat{u}(\mathrm{i}\eta) \quad \text{exists.}$$

Otherwise, there exist $\varepsilon > 0$ and sequences (λ_n) and (μ_n) in $\mathbb{C}_+$ such that $\lambda_n \to \mathrm{i}\eta$ and $\mu_n \to \mathrm{i}\eta$ as $n \to \infty$ and $\|\hat{u}(\lambda_n) - \hat{u}(\mu_n)\| \geqslant \varepsilon$ for all n. By (5.2),

$$\begin{aligned} \hat{u}(\lambda_n) - \hat{u}(\mu_n) - \hat{a}(\lambda_n) A(\hat{u}(\lambda_n) - \hat{u}(\mu_n)) \\ + (\hat{a}(\mu_n) - \hat{a}(\lambda_n)) A\hat{u}(\mu_n) = \hat{g}(\lambda_n) - \hat{g}(\mu_n), \end{aligned} \tag{5.4}$$

and

$$A\hat{u}(\mu_n) = \frac{\hat{u}(\mu_n)}{\hat{a}(\mu_n)} - \frac{\hat{g}(\mu_n)}{\hat{a}(\mu_n)}. \tag{5.5}$$

By the previous paragraph, $(\hat{u}(\mu_n))$ is bounded. If A is bounded, it is immediate that $(A\hat{u}(\mu_n))$ is bounded. If $\hat{a}(\mathrm{i}\eta) \neq 0$, it follows from (5.5) that $(A\hat{u}(\mu_n))$ is bounded. Since $\eta \notin \Sigma$, this establishes that $(A\hat{u}(\mu_n))$ is bounded in all cases. Since

$$\lim_{n\to\infty} \hat{a}(\mu_n) = \lim_{n\to\infty} \hat{a}(\lambda_n) = \hat{a}(\mathrm{i}\eta) \quad \text{and} \quad \lim_{n\to\infty} \hat{g}(\lambda_n) = \lim_{n\to\infty} \hat{g}(\mu_n) = \hat{g}(\mathrm{i}\eta),$$

it follows from (5.4) that

$$\lim_{n\to\infty} \|\hat{u}(\lambda_n) - \hat{u}(\mu_n) - \hat{a}(\lambda_n) A(\hat{u}(\lambda_n) - \hat{u}(\mu_n))\| = 0,$$

and hence that $(\hat{a}(\mathrm{i}\eta))^{-1} \in \sigma_{\mathrm{ap}}(A)$. This contradicts the assumption that $\eta \notin \Sigma$.

Now Chill's Tauberian theorem [11, corollary 1.7] gives the result. □

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