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Vector-valued holomorphic functions revisited

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Dedicated to Professor Heinz König on the occasion of his seventieth birthday.

Abstract. Let $\Omega \subset \mathbb{C}$ be open, X a Banach space and $W \subset X'$. We show that every $\sigma(X, W)$-holomorphic function $f : \Omega \to X$ is holomorphic if and only if every $\sigma(X, W)$-bounded set in X is bounded. Things are different if we assume f to be locally bounded. Then we show that it suffices that $\varphi \circ f$ is holomorphic for all $\varphi \in W$, where W is a separating subspace of X' to deduce that f is holomorphic. Boundary Tauberian convergence and membership theorems are proved. Namely, if boundary values (in a weak sense) of a sequence of holomorphic functions converge/belong to a closed subspace on a subset of the boundary having positive Lebesgue measure, then the same is true for the interior points of Ω, uniformly on compact subsets. Some extra global majorants are requested. These results depend on a distance Jensen inequality. Several examples are provided (bounded and compact operators; Toeplitz and Hankel operators; Fourier multipliers and small multipliers).

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0 Introduction

Vector-valued holomorphic functions are very useful, for example, in the theory of one-parameter semigroups or in spectral theory. But even for proving theorems about scalar-valued holomorphic functions, it is sometimes a useful trick to consider functions with values in a Banach space (we give two examples in Sect. 2).

In practice, one does not verify holomorphy of a vector-valued function by checking the property of the definition. It is much easier to prove weak holomorphy, in most cases. Our first goal is to investigate how weak the definition of holomorphy is allowed to be in order to imply strong holomorphy.

More precisely, let $\Omega \subset \mathbb{C}$ be a non-empty open set, X a Banach space and let W be a norming subspace of X'. Let $f : \Omega \to X$ be a function such that $\varphi \circ f$ is holomorphic for all $\varphi \in W$. If f is locally bounded, then it is well-known that f is holomorphic. In the first part of this paper we investigate what happens if f is not assumed to be locally bounded. First of all we show that f is still holomorphic on a dense open subset of Ω (Theorem 1.8). If W has the property that every $\sigma(X, W)$-bounded subset is norm bounded, then f is automatically locally bounded and hence holomorphic. If W does not have this property the main result of Sect. 1, Theorem 1.5, shows that there always exists a non-holomorphic function $f : \Omega \to X$ such that $\varphi \circ f$ is holomorphic for all $\varphi \in W$. Our argument also yields a short proof of the following result due to Wrobel [W]: Whenever Y is a Banach space and $j : X \to Y$ is a linear continuous injection such that $j(X)$ is not closed, then there exists $f : \Omega \to X$ such that $j \circ f$ is holomorphic but f is not.

In a second part of the paper we give a short proof of Vitali's theorem, based on the uniqueness theorem and the weak characterization of vector-valued holomorphic functions mentioned above. Notice that the vector-valued version of Vitali's theorem plays an important role in semigroup theory (see e.g. [AEH, Theorem 4.2] or [O, Theorem 2.4]). In contrast to the scalar case, it cannot be derived from Montel's theorem (which is not valid in infinite dimension), and, so far, there is only a quite complicated proof by Hille-Phillips [HP, Theorem 3.14.1]. Our argument gives a series of (vector-valued) Vitali's theorems, each-one corresponding to a uniqueness theorem for holomorphic functions. In Sect. 3 we use Vitali's theorem to improve considerably the criterion mentioned above: A locally bounded function $f : \Omega \to X$ is holomorphic whenever $\varphi \circ f$ is holomorphic for all $\varphi \in W$, where $W \subset X'$ separates X. This had been formulated as an open problem by Wrobel [Wr]. A complicated proof of this fact is given by Grosse-Erdmann [GE].

In Sect. 4 we use the same technique which leads to Vitali's theorem to prove a boundary Tauberian convergence theorem: A bounded sequence of holomorphic functions on the disc converges on the disc whenever the boundary functions converge on a subset of positive measure of the torus.

Various more general versions of this theorem are proved in Sect. 5, where we use a direct method (instead of the short quotient-method used before). Some restriction of the growth near the boundary is needed. The natural condition is described by the Nevanlinna norm. In the scalar case the classical prototype of the results we obtain is the Khinchin-Ostrowski theorem [Pr, II.7.1]. Omitting some technical details (for instance some extra hypotheses (H1) - (H4) in Section 5), the results can be described as follows. Considering a subspace $E \subset X$ and supposing that it is closed for the same weak topology $\sigma(X, W)$, for which a Smirnov class function $f \in \mathcal{N}^+(\mathbb{D}, X)$ has boundary limits, we prove a distance Jensen inequality

$$\log(\mathrm{dist}(f(z), E)) \leq \int_{\mathbb{T}} P_z(t) \log(\mathrm{dist}(f(t), E))\, dm(t)$$

for $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. As for the classical Khinchin - Ostrowski theorem, this implies convergence and membership as mentioned above: if Smirnov type functions $f_n \in \mathcal{N}^+(\mathbb{D}, X)$ with uniformly bounded Nevanlinna characteristics converge/belong to the space E on a subset $S \subset \mathbb{T}$ of positive Lebesgue measure, then they do the same in the entire unit disc $\mathbb{D}$. Examples show that, in general, nothing can be said about the boundary values on the rest of the boundary $\mathbb{T} \backslash S$.
For many applications (for example, for $X = l^\infty$, $E = c$; $X = \mathcal{L}(Y, Z)$, $E = S_\infty(Y, Z)$), the hypothesis that E is $\sigma(X, W)$ closed is too restrictive. Replacing it by a kind of weak approximation property, we prove the same convergence and membership theorem (Theorem 5.10). In particular, the theorem holds for the aforementioned pairs X, E. In fact, stronger theorems of Khruschev type [Kh] theorems can be proved for vector-valued functions in a similar way, but here we restrict ourselves to the simple analogue of the classical Khinchin - Ostrowski theorem.

1 Weakly holomorphic functions

Let X be a Banach space. A subspace W of X' is called **almost norming** if

$$q_W(x) \;=\; \sup\{|\varphi(x)| : \varphi \in W\,,\; \|\varphi\| \leq 1\}$$

defines an equivalent norm on X; the subspace W is called **norming** if $q_W(x) = \|x\|$ for all $x \in X$.

Lemma 1.1 *The following are equivalent.*

(i) *W is almost norming;*
(ii) *for every $\varphi \in X'$ there exists a bounded net in W converging to φ for $\sigma(X', X)$.*

Proof. (i)⇒(ii). Denote by B and B' the unit balls in X and X', respectively. We can assume that $\|\quad\| = q_W$; i.e., $B = W_1^\circ$ where $W_1 = W \cap B'$ and $\circ$ denotes the polar with respect to the duality $\langle X, X' \rangle$. By the bipolar theorem, $\overline{W_1}^{\sigma(X',X)} = W_1^{\circ\circ} = B^\circ = B'$.

(ii) $\Rightarrow$*(i)*. We have to show that the set $H := \{x \in X : q_W(x) \leq 1\}$ is norm bounded. Let $\varphi \in X'$. By assumption, there exists a net $(\varphi_i)_{i \in I}$ in W such that $c := \sup \|\varphi_i\| < \infty$ and $\sigma(X', X) - \lim_i \varphi_i = \varphi$. Thus $|\varphi(x)| = \lim_i |\varphi_i(x)| \leq c$ for all $x \in H$. We have shown that $\sup_{x \in H} |\varphi(x)| < \infty$ for all $\varphi \in X'$. By the uniform boundedness principle this implies that H is bounded. □

A subset W of X' is called **separating** if for all $x \in X \setminus \{0\}$ there exists $\varphi \in W$ such that $\varphi(x) \neq 0$. We discuss this notion in the following remark.

Remark 1.2 Let W be a subspace of X'.

a) W is separating if and only if W is $\sigma(X', X)$-dense in X'.

b) If W is norm-dense in X', then W is almost norming. This follows from Lemma 1.1.

c) If W is almost norming, then W is separating.

d) Assume that $X = Y'$ is a dual space and $W \subset Y$ (seen as a subspace of X' by evaluation). Then the following are equivalent:

(i) W is separating;
(ii) W is norm-dense in Y;
(iii) W is almost norming;
(iv) W is norming.

This is immediate from the Hahn-Banach theorem. In particular, if X is reflexive, then the four properties: separating, being norm-dense , almost norming and norming, are equivalent. Taking Y non-reflexive and $W = Y$ we obtain an example of a norming subspace of X' which is not norm-dense.

e) Let $X = H^\infty(\mathbb{D})$ where $\mathbb{D}$ is the unit disc and let W be the space of all linear combinations of the Dirac measures $\delta_{1/n} \quad (n = 2, 3, \cdots)$. Then W is almost norming. In fact, W is separating by the uniqueness theorem. Moreover, we can identify $H^\infty(\mathbb{D})$ with the space $F = \{f \in L^\infty(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n = -1, -2, \cdots\} = \{f \in L^\infty(\mathbb{T}) : \int_{\mathbb{T}} f(z)g(z)dz = 0$ for all $g \in G\}$ where G is the closed subspace of $L^1(\mathbb{T})$ generated by the function $e_n \ (n = 1, 2 \cdots)$, $e_n(z) = z^{-n}$. Thus $F = E'$ where $E = L^1(\mathbb{T})/G$ via the duality $\langle f, g + G \rangle = \int_{\mathbb{T}} f(z)g(z)dm(z)$ where $dm(z)$ is the normalized Lebesgue measure on the torus $\mathbb{T}$. Let $g_n(z) = \frac{1}{z - 1/n} \quad (n = 2, 3, \cdots)$. Then $g_n + G = \delta_{1/n}$ by Cauchy's integral formula. Thus W can be considered as a subspace of the predual E of $H^\infty(\mathbb{D})$ and the claim follows from d).

f) A separating subspace is not almost norming in general. We give an example: Let $\sigma \subset \mathbb{D}$ be a countable subset of the unit disc whose closure contains $\mathbb{T}$, which does not carry any non trivial measure orthogonal to complex polynomials $\mathcal{P}$ and which satisfies

$$\inf\{|b_\lambda(\mu)| : \lambda \neq \mu,\ \lambda, \mu \in \sigma\} = 0\,,$$

where b_λ stands for the elementary Blaschke factor, $b_\lambda(z) = (\lambda - z)/(1 - \bar{\lambda}z)$. (In fact, it is well-known that there exist even **Blaschke sequences** σ, that is $\Sigma_{\lambda\in\sigma}(1 - |\lambda|) < \infty$, satisfying these conditions, see [N2, Chap. 7, Sect. 3], for example). Let $X = \ell^1(\sigma)$, $X' = \ell^\infty(\sigma)$ and $W = \mathcal{P}|_\sigma$. Then, the manifold W is separating: if $x \in X$ and

$$0 = \langle x, z^n \rangle = \Sigma_{\lambda\in\sigma} x(\lambda)\lambda^n$$

for all $n \geq 0$, then $x = 0$. To show that W is not almost norming, observe that $\|f_{|\sigma}\|_{X'} = \sup_\sigma |f| = \sup_{\mathrm{T}} |f| = \|f\|_\infty$ for every polynomial $f \in \mathcal{P}$. Let $x_{\lambda,\mu} \in X$, $x_{\lambda,\mu} = \chi_{\{\lambda\}} - \chi_{\{\mu\}}$, where $\lambda \neq \mu$ and $\lambda, \mu \in \sigma$. Then,

$$q_W(x_{\lambda,\mu}) = \sup\{|f(\lambda) - f(\mu)| : f \in \mathcal{P}\ ,\ \|f\|_\infty \leq 1\}\,.$$

By the Schwarz lemma, we have $|f(\lambda) - f(\mu)| \leq |b_\lambda(\mu)| \cdot |1 - \overline{f(\lambda)}f(\mu)|$, and hence $q_W(x_{\lambda,\mu}) \leq 2|b_\lambda(\mu)|$. Therefore, $\inf\{q_W(x_{\lambda,\mu}) : \lambda, \mu \in \sigma, \lambda \neq \mu\} = 0$, whereas $\|x_{\lambda,\mu}\|_X = 2$ for all $\lambda \neq \mu$. So, the norms $\|\cdot\|_X$ and q_W are not equivalent, and W is not almost norming.

h) On a more abstract level, Davis and Lindenstrauss [DL] proved that the following two assertions are equivalent:[1]

(i) X' contains a separating subspace which is not quasi norming;
(ii) $\dim X''/X = \infty$. □

The following result is a consequence of Cauchy's integral formula and can be found in [K, p. 139]. It will be extended to the case where W is simply separating in Sect. 3.

Theorem 1.3 *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to X$ be locally bounded. Assume that there exists a norming subspace W of X' such that $\varphi \circ f$ is holomorphic for all $\varphi \in W$. Then f is holomorphic.*

Our aim is to show that the assumption of local boundedness in Theorem 1.3. cannot be omitted. For this we need the following definition.

We say, a subspace W of X' **determines boundedness**, if every sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\sup_{n\in\mathbb{N}} |\varphi(x_n)| < \infty$ for all $\varphi \in W$ is bounded. In other words, W determines boundedness if and only if every $\sigma(X, W)$-bounded subset of X is norm bounded.

Remark 1.4 a) If $W \subset X'$ determines boundedness, then W is norming.
b) $W = X'$ determines boundedness by the uniform boundedness principle.
c) If $X = Y'$ and $W = Y \subset X'$, then W determines boundedness.
d) Let $X = \mathcal{L}(E, F)$, E, F Banach spaces. Then $W = E \otimes F'$ (the

[1] We are grateful to Dirk Werner for this reference.

algebraic tensor product) determines boundedness. Here we consider the duality $\langle x \otimes y', T\rangle = \langle Tx, y'\rangle$ ($x \in E, y' \in F'$, $T \in \mathcal{L}(E,F)$).
e) Assume that X is continuously embedded into a Banach space Z (i.e., $X \subset Z$ and $\|x\|_Z \leq \text{const}\cdot\|x\|_X$ for all $x \in X$). Let $W = \{\varphi_{|X} : \varphi \in Z'\}$. Then W determines boundedness if and only if X is closed in Z (i.e. $\| \quad \|_Z$ defines an equivalent norm on X). This is easy to see.
f) Let $X = \ell^p$, $1 \leq p \leq \infty$, W the set of all finitely supported sequences. Then W is norming but does not determine boundedness.
g) Let $X = L^p(\Omega) \quad (1 \leq p \leq \infty)$, $\Omega \subset \mathbb{R}^N$ open. The space $C_c(\Omega)$ of all continuous functions with compact support is norming but does not determine boundedness.
h) Let $X = C[0,1]$, then $W = \text{lin}\,\{\delta_t : t \in [0,1]\} \subset X'$ is almost norming, but does not determine boundedness.

Let $\Omega \subset \mathbb{C}$ be open. Let $W \subset X'$ and $f : \Omega \to X$ such that $\varphi \circ f$ is holomorphic for all $\varphi \in W$. If W determines boundedness, then it follows from Theorem 1.3 that f is holomorphic. The following main result of this section shows that for this conclusion, the hypothesis that W determines boundedness, is also nesessary. By $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ we denote the unit disc.

Theorem 1.5 *Let X be a Banach space and W a subspace of X' which does not determine boundedness. Then there exists a function $f : \mathbb{D} \to X$ which is not holomorphic such that $\varphi \circ f$ is holomorphic for all $\varphi \in W$.*

Proof. Consider the segments

$$L_k = \left\{ re^{i\pi/2k} : \frac{1}{2k} \leq r \leq 1 \right\}$$

$k = 1, 2 \cdots$ and open neighborhoods V_k of L_k such that $V_i \cap V_j = \emptyset$ for $i \neq j$ and such that $\mathbb{C} \setminus (L_k \cup (\overline{\mathbb{D}} \setminus V_k))$ is connected. By Runge's theorem [Ru, 13.7, p. 290] there exist polynomials f_k such that

$$|f_k(z)| \geq k \quad (z \in L_k)$$
$$|f_k(z)| \leq \frac{1}{k^2} \quad (z \in \bar{\mathbb{D}} \setminus V_k)\ .$$

Let $b_k = \sup\limits_{|z|\leq 1} |f_k(z)|$.

Assume that W does not determine boundedness. Then there exist $x_n \in X$ such that

$$\sup_{n\in\mathbb{N}} |\varphi(x_n)| < \infty \ \text{ for all } \ \varphi \in W$$

but $\lim_{n\to\infty} \|x_n\| = \infty$. Taking a subsequence, if necessary, we can assume that $\sum_{k=1}^{\infty} \frac{b_k}{\|x_k\|} < \infty$. Define $f : \mathbb{D} \to X$ by $f(z) = \sum_{k=1}^{\infty} f_k(z) \cdot \frac{x_k}{\|x_k\|}$.

Since for all $\varphi \in W$,

$$\left| f_k(z) \frac{\varphi(x_k)}{\|x_k\|} \right| \leq \frac{b_k}{\|x_k\|} \cdot \sup_{k\in\mathbb{N}} |\varphi(x_k)| \quad (z \in \bar{\mathbb{D}}) ,$$

the function $\varphi \circ f$ is holomorphic on $\mathbb{D}$.

On the other hand, choose $z_k \in L_k$, such that $\lim_{k\to\infty} z_k = 0$. Then

$$\begin{aligned} \|f(z_k)\| &\geq |f_k(z_k)| - \left\| \sum_{j\neq k} f_j(z_k) \frac{x_j}{\|x_j\|} \right\| \\ &\geq k - \sum_{j=1}^{\infty} \frac{1}{j^2} . \end{aligned}$$

Thus $\lim_{k\to\infty} \|f(z_k)\| = \infty$. Consequently, f is not continuous in 0. □

As an immediate consequence of Theorem 1.5 and Example 1.4 e) we obtain the following

Theorem 1.6 *Let X be a Banach space which is continuously embedded into another Banach space Z. If X is not closed in Z, then there exists a function $f : \mathbb{D} \to X$ which not holomorphic, such that $f : \mathbb{D} \to Z$ is holomorphic.*

Theorem 1.6 is due to I. Globevnik [G] for the case $X = \ell^p$, $Z = \ell^q$ $1 \leq p < q \leq \infty$ and to Wrobel [Wr] for the general case, however with a more complicated proof.

Remark 1.7 In Theorem 1.6 the space Z can be replaced by a Fréchet space (and this is also proved by Wrobel [Wr]). In fact, let X be continuously injected into a Fréchet space Z. Assume that X ist not closed in Z, i.e., the norm of X is not continuous on Z. Let $\{p_k : k = 1, 2\cdots\}$ be a sequence of continuous seminorms on Z defining the topology of Z. Then for each $n \in \mathbb{N}$ there exists $x_n \in X$ such that $\|x_n\| \geq n$ but $p_k(x_n) \leq 1$, $k = 1, 2\cdots n$. Let $\varphi \in Z'$. Then there exist $m \in \mathbb{N}$, $c \geq 0$ such that $|\varphi(x)| \leq c \max_{i=1\cdots m} p_i(x)$. It follows that $\sup_{n\in\mathbb{N}} |\varphi(x_n)| < \infty$. Hence $W := \{\varphi_{|X} : \varphi \in Z'\} \subset X'$ is not determining boundedness (in X). By Theorem 1.5, there exists a non-holomorphic function $f : \mathbb{D} \to X$ such that $\varphi \circ f$ is holomorphic for all $\varphi \in W = Z'$. Thus f is holomorphic considered as a function with values in Z (by Jarchow [J, p. 362]). □

If in Theorem 1.3 we omit local boundedness we have seen that f is no longer holomorphic in general. However, the following result, which is inspired by Osgood's theorem [R2, p. 130] shows that f is always holomorphic on a dense open set.

Theorem 1.8 *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to X$ be a function such that $\varphi \circ f$ is holomorphic for all $\varphi \in W$, where $W \subset X'$ is an almost norming subspace. Then there exists a dense open subset $\Omega_\circ$ of Ω such that f is holomorphic on $\Omega_\circ$.*

Proof. We can assume that

$$\|x\| = \sup\{|\varphi(x)| : \varphi \in W\ ,\ \|\varphi\| \leq 1\}\ .$$

Then the function $\|f(\cdot)\| : \Omega \to \mathbb{R}_+$ is lower semicontinuous as supremum of continuous functions. In particular, the sets

$$A_n := \{z \in \Omega : \|f(z)\| \leq n\}$$

are closed $(n \in \mathbb{N})$. Since $\bigcup_{n\in\mathbb{N}} A_n = \Omega$, it follows from Baire's theorem that for every $z \in \Omega\ ,\ r > 0$ such that $D(z,r) \subset \Omega$ there exists $n(z,r) \in \mathbb{N}$ such that $U_{z,r} := D(z,r) \cap \overset{\circ}{A}_{n(z,r)} \neq \emptyset$. Thus, the union $\Omega_\circ$ of all such sets $U_{z,r}$ is open and dense in Ω. Moreover, since for $w \in U_{z,r}\ ,\ \|f(w)\| \leq n(z,r)$, the function f is locally bounded on $\Omega_\circ$. It follows from Theorem 1.3 that f is holomorphic in $\Omega_\circ$. □

2 Tauberian convergence theorems

In this section we consider sequences of holomorphic functions which converge on a subset of a domain Ω. We look for additional properties which ensure convergence on the entire domain. Such results are of Tauberian type (even though their Abelian counterpart is trivial in this case: it is the assertion that convergence on Ω implies convergence on a subset of Ω). An important example is Vitali's theorem, where subsets admitting a limit point in Ω are considered and local boundedness is a possible additional property. In the scalar case, it seems that, so far, the easiest proof of Vitali's theorem is given with help of Montel's theorem (see the proof in [R2, p. 129] and the historical remarks [R2, p. 138]). However, Montel's theorem does no longer hold in the vector-valued case if the underlying Banach space is infinite dimensional. It is surprising that Vitali's theorem is still valid. In fact, Lindelöf's (quite technical) direct proof goes through and is presented in the vector-valued case in Hille-Phillips [HP, p. 104 - 105]. Here we give an easy direct proof based on Theorem 1.3 and the uniqueness theorem (Theorem 2.2).

It is remarkable that this functional analytic proof uses vector-valued holomorphic functions even in the scalar case. The classical result is known for sequences. For the application in Sect. 3 we formulate it more generally for nets.

Theorem 2.1 (Vitali) *Let Ω be an open, connected subset of $\mathbb{C}$. Let $(f_i)_{i\in I}$ be a net of holomorphic functions on Ω with values in X which is locally bounded (i.e., for all $z \in \Omega$ there exists a neighborhood on which $(f_i)_{i\in I}$ is bounded).*

Then the following assertions are equivalent.

(i) The net $(f_i(z))_{i\in I}$ converges uniformly on all compact subsets of Ω to holomorphic function $f : \Omega \to X$;
(ii) the set $\Omega_0 := \{z \in \Omega : \lim_i f_i(z)$ exists$\}$ has an accumulation point in Ω;
(iii) there exists $z_0 \in \Omega$ such that $\lim_i f_i^{(k)}(z_0)$ exists for all $k \in \mathbb{N}$.

For the proof we need the following uniqueness result which is an immediate consequence of the classical theorem.

Theorem 2.2 (Uniqueness Theorem). *Let $\Omega \subset \mathbb{C}$ be a connected open set and let $f : \Omega \to X$ be holomorphic. Let $Y \subset X$ be a closed subspace. Assume that*

(a) the set $\Omega_0 = \{z \in \Omega : f(z) \in Y\}$ has an accumulation point in Ω; or
(b) there exists $z_0 \in \Omega$ such that $f^{(k)}(z_0) \in Y$ for all $k = 0, 1, 2 \ldots$.

Then $f(z) \in Y$ for all $z \in \Omega$.

Proof. If $Y = \{0\}$, this follows from the Hahn Banach theorem and the classical uniqueness result. In the general case, apply this remark to $q \circ f$ where $q : X \to X/Y$ is the quotient mapping. □

Proof of Theorem 2.1 (ii)⇒(i). Define $F : \Omega \to \ell^\infty(I, X)$ by $F(z) = (f_i(z))_{i\in I}$. It follows from Theorem 1.3 that F is holomorphic. The space $c := \{(y_i)_{i\in I} \in \ell^\infty(I, X) : \lim_i y_i$ exists$\}$ is closed in $\ell^\infty(I, X)$ and $F(z) \in c$ for all $z \in \Omega_0$. It follows from the Uniqueness Theorem 2.2 that $F(z) \in c$ for all $z \in \Omega$. Thus $f(z) = \lim_i f_i(z)$ exists for all $z \in \Omega$. Notice that $\Phi(y) = \lim_i y_i \quad (y = (y_i)_{i\in I} \in c)$ defines a bounded operator Φ from c into X. It follows that $f = \Phi \circ F : \Omega \to X$ is holomorphic.

It follows from Cauchy's integral formula that every locally bounded family of holomorphic functions is equicontinuous on every compact subset of Ω (see e.g.[Ru, 14.6, formula (3)]) so that convergence is uniform on each compact subset of Ω.

(iii) $\Rightarrow$ (i). We define F as before. Then the hypothesis implies that $F^{(k)}(z_0) \in c$ for all $k = 0, 1, \ldots$. Thus, by the uniqueness Theorem 2.2, $F(z) \in c$ for all $z \in \Omega$. The implication (i) $\Rightarrow$ (ii) is trivial. With help of Cauchy's integral formula one sees that (i) implies (iii). □

One immediately deduces the following restricted version of Montel's theorem from Theorem 2.1.

Corollary 2.3 *Let $(f_n)_{n\in\mathbb{N}}$ be a locally bounded sequence of holomorphic functions defined on an open, connected subset Ω of $\mathbb{C}$ with values in X. Assume that*

(a) there exists $z_0 \in \Omega$ such that the set $\{f_n^{(k)}(z_0) : n \in \mathbb{N}\}$ is relatively compact in X for all $k \in \mathbb{N}$;

or

(b) the set $\Omega_0 := \{z \in \Omega : \{f_n(z) : n \in \mathbb{N}\}$ is relatively compact in $X\}$ has an accumulation point in Ω.

Then there exists a subsequence which converges to a holomorphic function uniformly on compact subsets of Ω.

Proof. In the case *(a)*, by a diagonalization argument, one finds a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ such that $\lim\limits_{m\to\infty} f_{n_m}^{(k)}(z_0)$ exists for all $k \in \mathbb{N}$.
In the case *(b)* one takes a sequence $(z_k)_{k\in\mathbb{N}}$ converging in Ω_0 and finds (by a diagonalization argument) a subsequence such that $\lim\limits_{m\to\infty} f_{n_m}(z_k)$ exists for all $k \in \mathbb{N}$.

In both cases, it follows from Theorem 2.1 that $(f_{n_m})_{m\in\mathbb{N}}$ converges uniformly on compact subsets of Ω to a holomorphic function. □

In a way similar to the proof of Theorem 2.1 one obtains the following vector-valued version of Blaschke's convergence theorem [R2, p. 129] using Blaschke's identity theorem [R2, 4.3.2] or [Du, p. 18].

Theorem 2.4 *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of holomorphic functions defined on the unit disc $\mathbb{D}$ with values in X. Let $a_j \in \mathbb{D}$ $(j \in \mathbb{N})$ such that $\sum\limits_{j=1}^{\infty}(1 - |a_j|) = \infty$. Assume that $\lim\limits_{n\to\infty} f_n(a_j)$ exists for all $j \in \mathbb{N}$. Then $(f_n)_{n\in\mathbb{N}}$ converges to a holomorphic function uniformly on each compact subset of $\mathbb{D}$.* □

From the identity theorem [R2, 4.3.4 (b), p. 88] we deduce in a similar way the following:

Corollary 2.5 *Let $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z \rangle 0\}$ and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of holomorphic functions defined on Ω with values in X. Assume* *that*

$$\lim_{n\to\infty} f_n(k)$$

exists for all $k \in \mathbb{N}$. *Then* $(f_n)_{n\in\mathbb{N}}$ *converges to a holomorphic function, uniformly on all compact subsets of* Ω.

3 Locally bounded functions

In this section we prove positive counterparts to Theorem 1.5 and 1.6 for locally bounded functions. Moreover, we consider holomorphic extensions of functions. We first establish an improvement of Theorem 1.3. Recall that a subset W of X' is called **separating** if for every $x \in X\backslash\{0\}$ there exists $\varphi \in W$ such that $\varphi(x) \neq 0$. In the following, $\Omega \subset \mathbb{C}$ is open and X is a Banach space.

Theorem 3.1 *Let* $f : \Omega \to X$ *be a locally bounded function such that* $\varphi \circ f$ *is holomorphic for all* $\varphi \in W$ *where* $W \subset X'$ *is separating. Then* f *is holomorphic.*

Proof. Let $H = \{\varphi \in X' : \varphi \circ f \text{ is holomorphic}\}$. Then H is a subspace of X' containing W. It follows that H is $\sigma(X', X)-$dense. Let $H_1 = \{\varphi \in H : \|\varphi\| \leq 1\}$. It follows from Vitali's theorem that H_1 is $\sigma(X', X) -$ closed. Now the Krein-Šmulyan theorem [S, IV.6] or [P, p. 73] implies that H is $\sigma(X', X) -$ closed. Thus $H = X'$ and the claim follows from Theorem 1.3. □

Remark 3.2 In virtue of the maximum principle, the assumption of local boundedness in Theorem 3.1 can be relaxed in the following way: Let $f : \Omega \to X$ be a function such that $\varphi \circ f$ is holomorphic for all $\varphi \in W$, where W is separating. Assume that for every $z \in \Omega$ there exists a compact set $K \subset \Omega$ such that $z \in \overset{\circ}{K}$ (the interior of K) and f is bounded on ∂K. Then f is holomorphic.

In fact, keeping the notation of the proof above, by the maximum principle, the family $\{\varphi \circ f : \varphi \in H_1\}$ is locally bounded. So we can apply Vitali's theorem. □

Remark 3.3 (Fréchet spaces). Theorem 3.1 remains valid if X is a Fréchet space. In fact, the Krein-Šmulyan theorem remains true in that case [S, p. 152] and a function is holomorphic if it is weakly holomorphic [J, p. 362]. See Grosse-Erdmann [GE] for further information concerning holomorphic functions with values in a locally convex space.

Corollary 3.4 *Let* Y *be a Banach space and* $j : Y \to X$ *a bounded, injective, linear operator. Let* $f : \Omega \to Y$ *be a locally bounded function. If* $j \circ f$ *is holomorphic, then* f *is holomorphic.*

Proof. The space $W = j'X'$ is separating. So the result follows from Theorem 3.1. □

Corollary 3.4 is formulated as an open problem by Wrobel [Wr] in 1982. A very complicated proof of Corollary 3.4 as well as Theorem 3.1 is presented by Grosse-Erdmann [GE]. He also gives a detailed account and many interesting references on the history of vector-valued functions.

Frequently, it is easy to show that a given vector-valued holomorphic function has weak holomorphic extensions. It is useful to have a criterion which allows one to deduce strong holomorphic extensions from this. Some very general results of this kind are proved by Gramsch [Gra1], [Gra2] for locally convex spaces. Here we give a simple result on Banach spaces X whose proof is based on the Uniqueness Theorem 2.2 again.

Theorem 3.5 *Let $\Omega \subset \mathbb{C}$ be open and connected. Let $A \subset \Omega$ be a set having a limit point in Ω and let $f : A \to X$ be a function such that $\varphi \circ f$ has a holomorphic extension to Ω for all $\varphi \in W$, where W is a closed almost norming subspace of X'. Then f has a holomorphic extension from Ω into X.*

We need an auxiliar result whose proof is based on the Banach Steinhaus theorem. By X^* we denote the algebraic dual of X.

Lemma 3.6 *Let X be a Banach space, and let $\Omega \subset \mathbb{C}$ be open and connected. Let $h : \Omega \to X^*$ be $\sigma(X^*, X)$ holomorphic. Assume that the set $A = \{z \in \Omega : h(z) \in X'\}$ has a limit point in Ω. Then $h(z) \in X'$ for all $z \in A$ and $h : \Omega \to X'$ is holomorphic.*

Proof. Let $z_0 \in \Omega$ such that there exist $z_k \in A$, $z_k \neq z_0$, $\lim_{k\to\infty} z_k = z_0$. Let $r\rangle 0$ such that $D(z_0, r) := \{z \in \mathbb{C} : |z - z_\circ| < r\} \subset \Omega$. We show that $D(z_0, r) \subset A$. Using that Ω is connected, by a standard argument this implies that $\Omega = A$.

Since $\langle x, h(\cdot)\rangle$ is holomorphic on Ω for all $x \in X$, there exist $a_n \in X^*$ such that $\langle x, h(z)\rangle = \sum_{n=0}^{\infty} \langle x, a_n\rangle (z - z_0)^n$ for all $z \in D(z_0, r)$, $x \in X$.

We show by induction that $a_m \in X'$ for all $m \in \mathbb{N}$. Let $m = 0$. Since $h(z_k) \in X'$ and $\lim_{k\to\infty} \langle x, h(z_k)\rangle = \langle x, a_0\rangle$ for all $x \in X$, it follows from the Banach Steinhaus theorem that $a_0 \in X'$. Now let $m \in \mathbb{N}$ such that $a_n \in X'$ for all $n = 0, 1, \ldots, m-1$. Let $\langle x, f(z)\rangle = \sum_{n=m}^{\infty} \langle x, a_n\rangle (z-z_0)^{n-m}$ $\quad (z \in D(z_0, r))$. Then $f : D(z_0, r) \to X^*$ is $\sigma(X^*, X)$-holomorphic. For $z \in D(z_0, r)$ one has $\langle x, f(z)\rangle = (z - z_0)^{-m}\{\langle x, h(z)\rangle - \sum_{n=0}^{m-1} \langle x, a_n\rangle (z - z_0)^n\}$ for all $x \in X$. Thus $f(z_k) \in X'$ for all $k \in \mathbb{N}$. Since $\langle x, a_m\rangle =$

$\lim_{k\to\infty} \langle x, f(z_k)\rangle$ for all $x \in X$, it follows from the Banach Steinhaus theorem that $a_m \in X'$. □

Proof of Theorem 3.5. For $x' \in W$ there exists a unique holomorphic function $h_{x'} : \Omega \to \mathbb{C}$ such that $h_{x'}(z) = \langle f(z), x'\rangle$ for $z \in A$. Thus $h_{x'}(z)$ is linear in $x' \in W$; $i.e.$ there exists $h : \Omega \to W^*$ such that $\langle h(z), x'\rangle = h_{x'}(z)$ for all $z \in \Omega,\ x' \in W$. Since $h(z) \in X \subset W'$ for all $z \in A$, it follows from the lemma that $h(z) \in W'$ for all $z \in \Omega$. Thus $h : \Omega \to W'$ is holomorphic. Since X is a closed subspace of W' (via evaluation) and $h(z) \in X$ for $z \in A$, it follows from the uniqueness theorem (Theorem 2.2) that $h(z) \in X$ for all $z \in \Omega$. □

If the space W is not complete one has to impose a norm condition. The following corollary is an extension of Theorem 3.1.

Corollary 3.7 *Let $\Omega \subset \mathbb{C}$ be open and connected. Let $A \subset \Omega$ have a limit point in Ω and let $h : A \to X$ be a function. Assume that there exist $c \geq 0$ and a separating subspace W of X' such that $\varphi \circ h$ has a holomorphic extension $H_\varphi : \Omega \to \mathbb{C}$ for all $\varphi \in W$ such that*

$$|H_\varphi(z)| \leq c\|\varphi\| \qquad (\varphi \in W, \quad z \in \Omega)\,. \tag{3.1}$$

Then h has a holomorphic extension to Ω with values in X.

Proof. Let $F = \{\varphi \in X' :$ there exists a holomorphic extension H_φ of $\varphi \circ h\}$. Then F is a subspace of X' containing W. Thus F is $\sigma(X', X)$-dense. Let $F_1 = \{\varphi \in F : \|\varphi\| \leq 1\}$. It follows from Vitali's theorem that F_1 is $\sigma(X', X)$-closed. Thus F is closed by the Krein-Šmulyan theorem. We have proved that $F = X'$. Now the claim follows from Theorem 3.5. □

Corollary 3.8 *Let Y be a Banach space continuously embedded into X. Let $f : \Omega \to X$ be holomorphic. Assume that for each $z \in \Omega$ there exists an open bounded set $\omega \subset \Omega$ such that $z \in \omega,\ \bar{\omega} \subset \Omega\ ,\ f(v) \in Y$ for all $v \in \partial\omega$, and $\sup_{v\in\partial\omega} \|f(v)\|_Y < \infty$. Then $f(z) \in Y$ for all $z \in \Omega$ and f is holomorphic if it is considered as a function with values in Y.*

Proof. By the maximum principle we can use Corollary 3.7 with $W = \{\varphi_{|_Y} : \varphi \in X'\} \subset Y'$. □

Corollary 3.9 *Let (S, Σ, μ) be a measure space, $1 \leq p, q \leq \infty$. Let $f : \Omega \to L^p := L^p(S, \Sigma, \mu)$ be holomorphic. Assume that for each $z \in \Omega$ there exists an open bounded set $\omega \subset \mathbb{C}$ such that $z \in \omega\ ,\ \bar{\omega} \subset \Omega\ ,\ f(v) \in L^q$ for all $v \in \partial\omega$ and $\sup_{v\in\partial\omega} \|f(v)\|_{L^q} < \infty$. Then $f(z) \in L^q$ for all $z \in \Omega$ and f is holomorphic as a mapping with values in L^q.*

Proof. Apply Corollary 3.8 to $X = L^p + L^q$ and $Y = L^q$. □

4 Boundary Tauberian convergence theorems

In the theorems of Vitali type considered in Sect. 2 it was assumed that a sequence of functions converges in a small set in the interior of the domain in order to deduce convergence on the entire domain. Here we consider as hypothesis convergence on a small subset of the boundary. We use the same simple method as in Sect. 2. In Sect. 5 another approach will be presented which leads to more general results in the sense that less control on the growth close to the boundary is demanded.

For simplicity we restrict ourselves to the unit disc $\mathbb{D}$. In the following X denotes a Banach space. We need some preparation before formulating the theorem. Let $f \in H^1(\mathbb{D}, X)$; $i.e.$, $f : \mathbb{D} \to X$ is holomorphic and $\sup\limits_{0 \le r < 1} \int_0^{2\pi} \|f(re^{i\theta})\| d\theta < \infty$. By $\mathcal{B}$ we denote the Borel algebra on the torus $\mathbb{T} = \partial D$. There exists a vector measure $\mu : \mathcal{B} \to X$ such that

$$f(z) = \int\limits_{\mathbf{T}} P_z(t) d\mu(t) \qquad (z \in \mathbb{D}) \tag{4.1}$$

where $P_z(t) = \frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}$ $(z = re^{i\theta}\ ,\ t = e^{i\alpha})$ is the **Poisson kernel** (see [Du, p. 2] for the scalar case and [BuD], [Bl] for the vector-valued case). We call μ **a vector measure representing** f. One says that X has the **analytic Radon Nikodým property** (ARN property, for short), if for each $f \in H^1(\mathbb{D}, X)$ there exists $\varphi \in L^1(\mathbb{T}, X)$ such that $\mu(A) = \int_A \varphi(t)\, dm(t)$ for all $A \in \mathcal{B}$ (where μ is the vector measure representing f via (4.1) and dm denotes the normalized Lebesgue measure on $\mathbb{T}$). In that case,

$$f(z) = \int_{\mathbf{T}} P_z(t)\varphi(t)\, dm(t) \qquad (z \in \mathbb{D}) \tag{4.2}$$

and

$$\varphi(t) = \lim_{r \to 1} f(rt) \qquad (t \in \mathbb{T}\ a.e.) \tag{4.3}$$

as well as

$$\lim_{r \to 1} \int\limits_S \|f(rt) - \varphi(t)\|\, dm(t) = 0. \tag{4.4}$$

We call φ the **boundary function** of f and set $f(t) = \varphi(t) \quad (t \in \mathbb{T})$.

All L^p-spaces $(1 \le p < \infty)$ and all reflexive Banach spaces have the ARN property, but the space c_0 (of all sequences converging to 0) does not. A Banach lattice has the ARN property if and only if $c_0 \not\subset X$ (see [BuD]).

Definition 4.1 *Let $f \in H^1(\mathbb{D}, X)$ and let $S \subset \mathbb{T}$ be a Borel set of positive measure. We say that f* **has a boundary function on S** *if there exists a* *function* $\varphi \in L^1(S, X)$ *such that* $\lim\limits_{r\to 1} \int\limits_S \|f(rt) - \varphi(t)\|\, dm(t) = 0$. *In that case we call* φ *the* **boundary function** *of* f *on* S *and set* $f(t) = \varphi(t) \quad (t \in S)$.

Lemma 4.2 *Let* $f \in H^1(\mathbb{D}, X)$ *be represented by a vector measure* μ *via (4.1). Let* $S \in \mathcal{B}$ *have positive Lebesgue measure and assume that* f *has a boundary function on* S. *Then*

$$\mu(A) = \int_A f(t)\, dm(t)$$

for all $A \in \mathcal{B}$ *such that* $A \subset S$.

Proof. Recall that in the scalar case the Poisson means $g * P_r$ converge to the function $g \in L^1(\mathbb{T})$ for the norm of $L^1(\mathbb{T})$ and almost everywhere as $r \to 1$. Applying this to $g = 1_A$ we obtain for $A \in \mathcal{B}$, $\lim\limits_{r\to 1} \int_{\mathrm{T}} P_{rt}(s) 1_A(t)\, dm(t)$ $= \lim\limits_{r\to 1} \int_{\mathrm{T}} P_{rs}(t) 1_A(t)\, dm(t) = 1_A(s)$ for almost all $s \in \mathbb{T}$. Moreover, $\left|\int_{\mathrm{T}} P_{rt}(s) 1_A(t)\, dm(t)\right| \le 1 \quad (s \in \mathbb{T})$. Now let $A \in \mathcal{B}$, $A \subset S$. Using the dominated convergence theorem, Fubini's theorem and applying the above remark to $\varphi \circ f$, we obtain for $\varphi \in X'$,

$$\begin{aligned}
\langle \mu(A) \, , \, \varphi \rangle &= \lim_{r\to 1} \int_{\mathrm{T}} \int_{\mathrm{T}} P_{rt}(s) 1_A(t)\, dm(t)\, d\langle \mu(s), \varphi \rangle \\
&= \lim_{r\to 1} \int_{\mathrm{T}} \int_{\mathrm{T}} P_{rt}(s) d\langle \mu(s), \varphi \rangle 1_A(t)\, dm(t) \\
&= \lim_{r\to 1} \int_{\mathrm{T}} \langle f(rt), \varphi \rangle 1_A(t)\, dm(t) \\
&= \int_{\mathrm{T}} \langle f(t), \varphi \rangle 1_A(t)\, dm(t) \; = \; \left\langle \int_A f(t)\, dm(t) \; , \; \varphi \right\rangle .
\end{aligned}$$

Since $\varphi \in X'$ is arbitrary, this proves the claim. □

Now we can prove the following boundary Tauberian convergence theorem.

Theorem 4.3 *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of holomorphic functions on* $\mathbb{D}$ *with values in* X. *Assume that*

$$\sup_{0<r<1} \int_0^{2\pi} \sup_{n\in\mathbb{N}} \|f_n(re^{i\theta})\| d\theta < \infty \,. \tag{4.5}$$

Assume that for all $n \in \mathbb{N}$, f_n *has a boundary function on a Borel set* $S \subset \mathbb{T}$ *of a positive measure. Then, if the sequence of boundary functions converges* *in* $L^1(S,X)$*, then the sequence* $(f_n(z))_{n\in\mathbb{N}}$ *converges in* $\mathbb{D}$ *uniformly on compact subsets of* $\mathbb{D}$*. In particular, if* $\lim_{n\to\infty} f_n = 0$ *in* $L^1(S,X)$*, then* $\lim_{n\to\infty} f_n(z) = 0$ *uniformly on compact subsets of* $\mathbb{D}$*.*

Proof. Notice that by Cauchy's integral formula, condition (4.5) implies that the sequence is locally bounded. We let $\ell^\infty(X)$ be the space of all bounded sequences with values in X and $c(X)$ the subspace of all convergent sequences. Let $F(z) = (f_n(z))_{n\in\mathbb{N}}$. It follows from Theorem 3.1 that $F \in H^1(\mathbb{D}, \ell^\infty(X))$. Let $\mu : \mathcal{B} \to \ell^\infty(X)$ be a measure representing F ; $i.e.$ $F(z) = \int_{\mathbb{T}} P_z(t)d\mu(t) \quad (z \in \mathbb{D})$. For $A \in \mathcal{B}$ we have $\mu(A) = (\mu_n(A))_{n\in\mathbb{N}} \in \ell^\infty(X)$. Then μ_n is an X-valued measure on $\mathcal{B}$ and $f_n(z) = \int_{\mathbb{T}} P_z(t)d\mu_n(t) \quad (z \in \mathbb{D})$. It follows from Lemma 4.2 that $\mu_n(A) = \int_A f_n(t)\, dm(t)$ whenever $A \in \mathcal{B}$, $A \subset S$. The assumption implies that $\mu(A) \in c(X)$ for all $A \in \mathcal{B}$ with $A \subset S$. Now let $\psi \in (\ell^\infty(X))'$ be a functional vanishing on $c(X)$. Then $\psi \circ F \in H^1(\mathbb{D})$ and $(\psi \circ F)(z) = \int_{\mathbb{T}} P_z(t)d(\psi \circ \mu)(t)$ for $z \in \mathbb{D}$. Then by Lemma 4.2., for $A \in \mathcal{B}$, $A \subset S$, $\int_A (\psi \circ F)(t)\,\mathrm{dt} = \psi(\mu(A)) = 0$. Thus $(\psi \circ F)(t) = 0$ $a.e.$ on S. It follows from the scalar boundary uniqueness theorem (see [Du, p. 17] or Corollary 5.5. below) that $\psi \circ F(z) = 0$ for all $z \in \mathbb{D}$. Now the Hahn-Banach theorem implies that $F(z) \in c(X)$ for all $z \in \mathbb{D}$. □

Remark 4.4 If in Theorem 4.3 no boundary function exists one may assume that $(\mu_n(A))_{n\in\mathbb{N}}$ converges for all $A \in \mathcal{B}$, $A \subset S$, where μ_n is a measure representing f_n, and the same conclusion holds.

5 Vector-valued Khinchin-Ostrowski theorems

In this section we complete the idea to use the uniqueness phenomena to establish "Tauberian convergence theorems" by vector-valued analogues of the classical Khinchin - Ostrowski theorem, see [Pr], Sect. 2 of Chap. 2. A theorem of this kind is already proved in Sect. 4 above, namely Theorem 4.3. Here we give sharper results using some versions of Jensen's inequality for vector-valued functions. The main difference between Theorem 4.3 and the approach of this section is that now we will work with more general (weaker) boundary growth conditions, as well as with a weaker type of boundary limits. The latter is particularly important for applications to operator valued functions, because, typically, in spectral theory, the boundary functions do not exist for the norm convergence. On the other hand, they frequently exist for the weak operator topology (WOT). This is why we start this section introducing some notions concerning locally convex spaces.

Basic notation. In what follows, we consider a vector space *X* endowed with a weak topology $\sigma = \sigma(X, W)$ generated by a set *W* of linear forms on *X*, which separates X. In particular, *X* can be a Banach space and $W \subset X'$. By a *seminorm generated by* $\sigma(X, W)$ we mean a function of the form

$$q(x) = q_w(x) = \sup\{|\langle x, \varphi\rangle| : \varphi \in w\},$$

which is supposed to be finite for all $x \in X$, where $w \subset W$ is a subset of *W*. The strong topology $s\sigma = s\sigma(X, W)$, generated by $\sigma(X, W)$, corresponds to the family of all finite seminorms q_w.

In particular, if *X* is a Banach space, and $W \subset X'$ is a norming subspace, then $s\sigma(X, W)$ is the usual norm topology of *X*.

Denote by $Hol(\mathbb{D}, X) = Hol_\sigma(\mathbb{D}, X)$ the vector space of X-valued functions on $\mathbb{D}$ which are $s\sigma(X, W)$ locally bounded and $\sigma(X, W)$ holomorphic. The above theory (Theorem 3.1) implies that every such function *f* is holomorphic in the strong topology $s\sigma(X, W)$. We endow $Hol_\sigma(\mathbb{D}, X)$ with the topology of uniform $s\sigma(X, W)$ convergence on compact subsets of the disc $\mathbb{D}$. We always suppose that

(H1) X is boundedly sequentially $\sigma(X, W)$ *complete; i.e. if* $(x_n)_{n\geq 1}$ *is bounded in* $s\sigma(X, W)$ *and the limits* $\lim_n \langle x_n, \varphi\rangle$ *exist for all* $\varphi \in W$*, then there exists* $x \in X$ *such that* $x = (\sigma) - \lim_n x_n$.

For a *countably generated W*, in the sense that

(H2) W is separable with respect to the strong topology of $(X, s\sigma)'$,

these conditions guarantee the existence almost everywhere on $\mathbb{T}$ of *weak* $\sigma(X, W)$*-boundary values* (radial or angular)

$$f(t) = \lim_{r\to 1} f(rt)$$

for any $s\sigma(X, W)$-bounded function $f \in Hol_\sigma(\mathbb{D}, X)$. For instance, this is the case for the space of bounded linear operators $X = \mathcal{L}(Y, Z)$ from a separable Banach space *Y* to a weakly sequentially complete Banach space *Z* with separable dual space Z' endowed with the weak operator topology $\sigma(X, W)$, where $W = \{y \otimes \psi : y \in Y, \psi \in Z'\}$ and $\langle T, y \otimes \psi\rangle = \langle Ty, \psi\rangle$ for $T \in \mathcal{L}(Y, Z)$.

In what follows, we consider vector-valued Hardy and Nevanlinna classes in $\mathbb{D}$; basic references for these are [SzNF] and [RR], and for corresponding scalar classes [D], [Pr] and [Z]. Namely, $H^p_\sigma(\mathbb{D}, X)$ denotes the weak Hardy space of all functions $f \in Hol_\sigma(\mathbb{D}, X)$ such that $\langle f(\cdot), \varphi)\rangle \in H^p(\mathbb{D})$ for every $\varphi \in W$, and $H^p(\mathbb{D}, X)$ stands for the (strong) Hardy space of all $f \in Hol_\sigma(\mathbb{D}, X)$ such that

$$\sup_{0\leq r<1} \int_{\mathbf{T}} q(f(rt))^p dm(t) < \infty$$

for all $q \in s\sigma(X,W)$. Similar notations are used for Nevanlinna classes $\mathcal{N}_\sigma(\mathbb{D},X)$ and $\mathcal{N}(\mathbb{D},X)$, and for Smirnov classes $\mathcal{N}_\sigma^+(\mathbb{D},X)$ and $\mathcal{N}^+(\mathbb{D},X)$, respectively. For instance, the first one is defined as the vector space of all functions $f \in Hol_\sigma(\mathbb{D},X)$ satisfying

$$\mathcal{N}(\langle f,\varphi\rangle) =: \sup_{0\le r<1} \int_{\mathbb{T}} \log^+ |\langle f(rt),\varphi\rangle| dm(t) \ < \infty$$

for every $\varphi \in W$, and $\mathcal{N}_\sigma^+(\mathbb{D},X)$ consists of all functions $f \in \mathcal{N}_\sigma(\mathbb{D},X)$ such that

$$\lim_{r\to 1} \int_{\mathbb{T}} \log^+ |\langle f(rt),\varphi\rangle| dm(t) = \int_{\mathbb{T}} \log^+ |\langle f(t),\varphi\rangle| dm(t)$$

for every $\varphi \in W$. See [Pr] and [Z] for properties of scalar Nevanlinna and Smirnov classes. Note that

$$\mathcal{N}(\langle f,\varphi\rangle) \le \sup_{0\le r<1} \int_{\mathbb{T}} |\langle f(rt),\varphi\rangle| dm(t) = \|\langle f(\cdot),\varphi\rangle\|_{H^1}$$

and hence $H_\sigma^p(\mathbb{D},X) \subset \mathcal{N}_\sigma^+(\mathbb{D},X)$ for every $p > 0$. By the strong Nevanlinna class $\mathcal{N}(\mathbb{D},X)$ we mean the space of functions $f \in Hol_\sigma(\mathbb{D},X)$ satisfying

$$\mathcal{N}(q(f(\cdot))) =: \sup_{0\le r<1} \int_{\mathbb{T}} \log^+ |q(f(rt))| dm(t) \ < \infty$$

for every $q \in s\sigma(X,W)$, and $\mathcal{N}^+(\mathbb{D},X) = \mathcal{N}(\mathbb{D},X) \cap \mathcal{N}_\sigma^+(\mathbb{D},X)$

Clearly, the above conclusion on the existence of the boundary function is true for functions from $H_\sigma^p(\mathbb{D},X)$ and $\mathcal{N}_\sigma(\mathbb{D},X)$ (the latter space is the largest one).

Now, we make some remarks on boundary functions in the case of a Banach space *X*. Namely, it is easy to see from Banach-Steinhaus type arguments, that for $f \in \mathcal{N}(\mathbb{D},X)$, boundary limits $f(t) = \lim_{r\to 1} f(rt)$ exist a.e. on $\mathbb{T}$ for the stronger topology $\sigma(X,\overline{W})$,where $\overline{W}$ stands for the norm closure of W in X'. For instance, consider $X = \mathcal{L}(Y,Z)$ endowed with the WOT topology, i.e. $\sigma(X,W)$, where *W* is the set of finite rank operators $A \in \mathcal{L}(Z,Y)$, $A = \sum_{k=1}^n \langle \cdot, z_k'\rangle y_k$, $z_k' \in Z'$, $y_k \in Y$ acting on $\mathcal{L}(Y,Z)$ by the usual trace duality,

$$\langle T,A\rangle = Trace(TA) = \sum_{k=1}^{n} \langle Ty_k, z_k'\rangle \ .$$

Then, for $f \in \mathcal{N}(\mathbb{D},X)$, the limits $f(t) = \lim_{r\longrightarrow 1} f(rt)$ exists (under the above hypotheses on *Y* and *Z*) not only for the WOT topology, but also for the ultra-weak $\sigma(X, S_1(Z,Y))$ operator topology.

In what follows, we always suppose hypotheses *(H1)* and *(H2)*.
We start with the following version of Jensen's inequality.

Lemma 5.1 *Let* $f \in \mathcal{N}^+_\sigma(\mathbb{D}, X)$ *, and* $q \in s\sigma(X, W)$*. Then,*

$$\log|\langle f(z), \varphi\rangle| \leq \int_{\mathbf{T}} P_z(t) \log|\langle f(t), \varphi\rangle| dm(t) \tag{5.1}$$

for every $z \in \mathbb{D}$ *and* $\varphi \in W$*, where* P_z *stands for the Poisson kernel,* $P_z(t) = (1 - |z|^2)/|1 - z\bar{t}|^2$ *for* $t \in \mathbb{T}$*. Moreover,*

$$\log(q(f(z))) \leq \int_{\mathbf{T}} P_z(t) \log(q(f(t)) dm(t) \text{ for every } z \in \mathbb{D} . \tag{5.2}$$

Finally, for every compact subset $K \subset \mathbb{D}$ *there exists a constant* C_K *such that*

$$\begin{aligned} \log(q(f(z))) &\leq C_K \cdot \mathcal{N}(q \circ f) + \textstyle\int_S P_z(t) \log(q(f(t)))dm(t) \\ &\leq C_K \cdot \mathcal{N}(q \circ f) + \sup_{t \in S} \log(q(f(t))) \end{aligned} \tag{5.3}$$

for all it $z \in K$ *and all measurable subsets* $S \subset \mathbb{T}$ *.*

Proof. First observe, that $f(t),\ t \in \mathbb{T}$ always means the weak boundary value as defined above. In order to prove (5.1) take $\varphi \in W$ and apply the scalar Jensen inequality to the function $z \mapsto \langle f(z), \varphi\rangle$, $z \in \mathbb{D}$.
Next we show (5.2). For any $q = q_w \in s\sigma(X, W)$ and $\varphi \in w \subset W$, we obviously have $|\langle f(t), \varphi\rangle| \leq q(f(t))$, and hence

$$\log|\langle f(z), \varphi\rangle| \leq \int_{\mathbf{T}} P_z(t) \log(q(f(t)))dm(t).$$

Passing to the supremum over all $\varphi \in w$, we obtain the inequality.
Finally we prove (5.3). Let $q = q_w$ and $\varphi \in w$. From previous inequalities, we have

$$\begin{aligned} \log|\langle f(z), \varphi\rangle| &\leq \int_{\mathbf{T}\setminus S} P_z(t) \log|\langle f(t), \varphi\rangle| dm(t) \\ &\quad + \int_S P_z(t) \log|\langle f(t), \varphi\rangle| dm(t) \\ &\leq \int_{\mathbf{T}\setminus S} P_z(t) \log^+|\langle f(t), \varphi\rangle| dm(t) \\ &\quad + \int_S P_z(t) \log(q(f(t))dm(t) \\ &\leq C_K \int_{\mathbf{T}} \log^+|\langle f(t), \varphi\rangle| dm(t) + \sup_{t \in S} \log(q(f(t)), \end{aligned}$$

where $z \in \mathbb{D}$ and $C_K = \sup\{P_z(t) : z \in K, t \in \mathbb{T}\}$. Since

$$\int_{\mathbb{T}} \log^+ |\langle f(t), \varphi \rangle| dm(t) \le \sup_{0 \le r < 1} \int_{\mathbb{T}} \log^+ |\langle f(rt), \varphi \rangle| dm(t)$$
$$\le \sup_{0 \le r < 1} \int_{\mathbb{T}} \log^+ q(f(rt)) dm(t) = \mathcal{N}(q \circ f)$$

for every $\varphi \in w$, we obtain the claimed inequality. □

Remark 5.2. Of course, we also have

$$\log |\langle f(z), \varphi \rangle| \le C_K \cdot \mathcal{N}(\langle f, \varphi \rangle) + \sup_{t \in S} \log |\langle f(t), \varphi \rangle|$$

for every $z \in K$ and $\varphi \in W$.

Corollary 5.3 *Let X be a Banach space and let $W \subset X'$ be norming. Let $f \in \mathcal{N}_\sigma^+(\mathbb{D}, X)$ for $\sigma = \sigma(X, W)$. Then*

$$\log \|f(z)\| \le \int_{\mathbb{T}} P_z(t) \log \|f(t)\| dm(t)$$

and

$$\log \|f(z)\| \le C_K \cdot \mathcal{N}(\|f(\cdot)\|) + \sup_{t \in S} \log \|f(t)\|$$

for every $z \in K$ and every measurable subset S of $\mathbb{T}$. □

Corollary 5.4 *If $f \in \mathcal{N}_\sigma^+(\mathbb{D}, X)$ and $f(t) = 0$ on a set $S \subset \mathbb{T}$ of positive Lebesgue measure, then $f = 0$.* □

Khinchin - Ostrowski type theorems. Now, we apply vector-valued Jensen inequalities to prove ”boundary Tauberian convergence theorems”. For scalar-valued functions, they are known as theorems of Khinchin - Ostrowski type. The classical Khinchin - Ostrowski theorem says the following. Let $S \subset \mathbb{T}$ be a closed subset of the unit circle $\mathbb{T}$, $m(S) > 0$, and let f_n be polynomials such that $\sup_n \mathcal{N}(f_n) < \infty$ and $\lim_n f_n(t) = 0$ uniformly on S; then $\lim_n f_n(z) = 0$ uniformly on compact subsets of the disc $\mathbb{D}$, see [Pr], Sect. 7 of Chap. 2, for the proof.

The following vector-valued version of the Khinchin - Ostrowski theorem is almost immediate from Lemma 5.1.

Theorem 5.5 *Let X and $s\sigma(X, W)$ be as above (see Basic notation), and $S \subset \mathbb{T}$, $m(S) > 0$. Let $f_n \in \mathcal{N}_\sigma^+(\mathbb{D}, X)$ be functions such that*

(i) $\sup_n \mathcal{N}(q \circ f_n) < \infty$ *for every $q \in s\sigma(X, W)$, and assume that*
(ii) the boundary values $(f_n(t))_{n \ge 1}$ converge everywhere on S for the strong $s\sigma(X, W)$ topology.

Then, there exists $f \in Hol_\sigma(\mathbb{D}, X)$ *such that* $\lim_n f_n(z) = f(z)$ *for the* $s\sigma(X, W)$ *topology uniformly on compact subsets of the disc* $\mathbb{D}$ *.*

The same is true if we replace the strong topology by the $\sigma(X, W)$ *weak topology everywhere.*

Proof. If necessary, replacing *S* by a smaller set $S' \subset S$, with $m(S') > 0$ and using the (scalar) Egorov theorem, we can assume that $(f_n(\cdot))_{n\geq 1}$ converges uniformly on S.

Applying Lemma 5.1, we obtain

$$\log(q(f_n(z) - f_m(z))) \leq C_K \cdot \mathcal{N}(q \circ f_n + q \circ f_m)) + \sup_{t \in S} \log(q(f_n(t) - f_m(t)))$$

for $z \in K$, *K* being a compact subset of $\mathbb{D}$. Hence $(f_n(z))_{n\geq 1}$ is a Cauchy sequence for the $s\sigma(X, W)$ topology, and so, has a limit $f(z) = \lim f_n(z)$ in *X* (completeness follows from *(H1)* of the *Basic notation* above). Clearly, the limit is uniform on compact subsets, and hence $f \in Hol_\sigma(\mathbb{D}, X)$. □

Remarks 5.6 Theorem 5.5 remains valid if boundedness of the $H^1(\mathbb{D}, X)$ norm is assumed instead of boundedness of the Nevanlinna characteristics $W(q \circ f_n)$.

Moreover, convergence to the zero function holds true under the following hypothesis which is weaker than *(ii)*:

$$\lim_n \int_S \log(q \circ f_n) dm(t) = -\infty$$

for every $q \in s\sigma(X, W)$.

It is also worth mentioning that, for scalar functions, there exist very advanced generalizations of the Khinchin - Ostrowski theorem. Namely, the growth of f_n can be controlled by a given majorant λ (instead of condition *(i)* of Theorem 5.5), but, in turn, the nowhere dense closed set *S* should satisfy a kind of finite λ - entropy hypothesis; see [Kh] for exhausting results. These results also can be generalized to vector-valued functions.

For other comments see Remarks 5.11 and 5.13, and Examples 5.14 - 5.19 below.

Distance inequalities and membership in a subspace. Now, we consider some applications of Lemma 5.1 to *X* -valued functions which approximate a given closed subspace $E \subset X$. In a sense, the point is to project the above results to the quotient space X/E, and conversely, to consider their liftings from X/E to *X*. The difficulty is that for many interesting examples, as $E = c \subset X = l^\infty$ or $E = S_\infty \subset X = \mathcal{L}(Y, Z)$ (see Examples below), the subspace *E* is not $\sigma(X, W)$ closed. In this general setting, we are unable to prove a complete analogue of Lemma 5.1. However, we prove a (more rough) version of it, sufficient for many examples and sufficient to derive the Khinchin-Ostrowski type convergEnce theorem, and the boundary uniqueness theorem (see 5.8, 5.9, 5.10 and 5.12 below).

We start with a simple situation, where the complete analogue of the preceding theory exists for quotients. Namely, consider the following hypothesis *(H3)*,

(H3) E is a $\sigma(X,W)$ closed subspace of a Banach space X, and $W \cap E^{\perp}$ is a norming for X/E.

Recall that, as usual, we identify the dual space $(X/E)'$ with the annihilator $E^{\perp} = \{\varphi \in X' : \varphi|E = 0\}$. The $\sigma(X,W)$ closedness of *E* implies that $W \cap E^{\perp}$ is separating for X/E. *Particular cases where hypothesis (H3) is satisfied* are the following:

a) $W = X'$, *E* is an arbitrary closed subspace of a Banach space *X* (such that assumptions *(H1)* and *(H2)* are satisfied). Particular cases are:
b) *X* is a separable reflexive Banach space, *E* is an arbitrary closed subspace;
c) *X* is a dual space, $X = (X_*)'$, $W = X_*$, and *E* is a $\sigma(X, X_*)$ closed subspace of *X*; in particular,
d) $X = \mathcal{L}(Y,Z)$, where Y, Z are two Hilbert spaces, $W = X_* = S_1(Z,Y)$ the predual space of trace class operators, and *E* is an ultra-weak (i.e. $\sigma(X,W)$) closed subspace of *X*.

Lemma 5.7 *Let $E \subset X$ be a pair satisfying the hypotheses (H1) - (H3), and $f \in \mathcal{N}_{\sigma}^{+}(\mathbb{D}, X)$. Then,*

$$\log(\operatorname{dist}(f(z),E)) \leq \int_{\mathbb{T}} P_z(t) \log(\operatorname{dist}(f(t),E))dm(t)$$

for every $z \in \mathbb{D}$, where $\operatorname{dist}(x,E) = inf_{y\in E}\|x-y\|$.

Moreover, for every compact subset $K \subset \mathbb{D}$ there exists a constant C_K such that

$$\begin{aligned}\log(\operatorname{dist}(f(z),E)) \leq\ & C_K \cdot \mathcal{N}(\|f(\cdot)\|) \\ & + \int_S P_z(t) \log(\operatorname{dist}(f(t),E))dm(t) \\ \leq\ & C_K \cdot \mathcal{N}(\|f(\cdot)\|) + \sup_{t\in S} \log(\operatorname{dist}(f(t),E))\end{aligned}$$

for all $z \in K$ and all measurable subsets $S \subset \mathbb{T}$.

Proof. By hypothesis *(H3)*,

$$\operatorname{dist}(x,E) = \|x\|_{X/E} = \sup\{|\langle x,\varphi\rangle| : \varphi \in W;\ \varphi|E = 0;\ \|\varphi\| \leq 1\}.$$

Now, the lemma follows from Lemma 5.1 applied to the quotient space X/E instead of *X*, and to the projection of *f* to X/E instead of *f*. □

As in the case of the whole space *X*, we can derive the following corollaries.

Corollary 5.8 *Let* $f \in \mathcal{N}_\sigma^+(\mathbb{D}, X)$ *and* $\int_{\mathbb{T}} \log(\text{dist}(f(t), E))dm(t) = -\infty$. *Then* $f(z) \in E$ *for all* $z \in \mathbb{D}$. □

Theorem 5.9 *Let X, E be spaces satisfying (H3) and let* $S \subset \mathbb{T}$ *be a measurable set such that* $m(S) > 0$. *Let* $f_n \in \mathcal{N}_\sigma^+(\mathbb{D}, X)$ *such that:*

(i) $\sup_n \mathcal{N}(\|f_n(\cdot)\|) < \infty$, *and*
(ii) $\lim_n \text{dist}(f_n(t), E) = 0$ *almost everywhere on S.*

Then, there exist $g_n \in Hol_\sigma(\mathbb{D}, E)$ *such that* $\lim_n(f_n(z) - g_n(z)) = 0$ *uniformly on compact subsets of the disc* $\mathbb{D}$ *; in particular,* $\lim_n f_n(z) = 0$ *if* $E = \{0\}$, *and a subsequence of* $(f_n)_{n\geq 1}$ *converges to a limit* $g \in Hol(\mathbb{D}, E)$ *if* $dimE < \infty$.

Proof. By Lemma 5.7, $\lim_n \text{dist}(f_n(z), E) = 0$ uniformly on compact subsets of the disc $\mathbb{D}$. By a Gleason theorem [Gl] applied to the canonical map $\pi : X \longrightarrow X/E$, there exist functions $g_n \in Hol_\sigma(\mathbb{D}, E)$ such that

$$\sup_{z\in K} \|f_n(z) - g_n(z)\| \leq C_K \cdot \sup_{z\in K} \|f_n(z)\|_{X/E}$$

for every compact set $K \subset \mathbb{D}$. The main assertion follows.

The last property is a consequence of the Montel theorem applied to $(g_n)_{n\geq 1}$. □

Approximation property. Now, our aim is to adapt the preceding results to obtain uniqueness and convergence theorems for some cases not fitting into the above hypotheses, like $E = c \subset X = l^\infty$ or $E = S_\infty \subset X = \mathcal{L}(Y, Z)$. To this end, we need the following hypothesis:

(H4) X is a Banach space, and E its closed subspace satisfying the following approximation property: there exists a bounded sequence $(T_n)_{n\geq 1}$ *of linear operators,* $T_n \in \mathcal{L}(X)$, *such that* $T_nX \subset E$, $T_n'W \subset W$ *and* $\lim_n \|T_nx - x\| = 0$ *for every* $x \in E$.

Observe that, in particular, every T_n of hypothesis *(H4)* is $\sigma(X, W)$ continuous. Hypothesis *(H4)* is verified if there exists a projection onto *E* which is $\sigma(X, W)$ continuous. For several examples and applications see below.

Theorem 5.10 *Let X and E satisfy (H4), and let* $S \subset \mathbb{T}$ *be measurable such that* $m(S) > 0$. *Let* $f_n \in \mathcal{N}_\sigma^+(\mathbb{D}, X)$ *be functions satisfying*

(i) $\sup_n \mathcal{N}(\|f_n(\cdot)\|) < \infty$, *and*
(ii) $f_n|S$ *is separable-valued and* $\lim_n \text{dist}(f_n(t), E) = 0$ *almost everywhere on S.*

Then, the conclusions of Theorem 5.9 hold true.

Proof. Using the notation of *(H4)*, we observe that $T_m f_n \in \mathcal{N}_\sigma^+(\mathbb{D}, X)$ and $(T_m f_n)(t) = T_m(f_n(t))$ a.e.. Applying Lemma 5.1 to $T_m f_n - f_n$, we obtain

$$\log \|T_m(f_n(z)) - f_n(z)\| \leq \int_{\mathbb{T}} P_z(t) \log \|T_m f_n(t) - f_n(t)\| dm(t).$$

The restrictions $f_n|S$ being separable-valued are limits of step functions, and hence satisfy Lusin's theorem: there exists a closed subset $S' \subset S$ of positive measure such that all $f_n|S'$ are norm continuous. By a theorem of R.Bartle and L.Graves [BG] applied to the surjective projection $\pi : X \longrightarrow X/E$, there exist continuous functions $g_n : S' \longrightarrow X$, and a constant $A > 0$ such that

$$h_n(t) =: f_n(t) - g_n(t) \in E$$

for $t \in S'$, and

$$\|g_n(t)\| \leq A \cdot \mathrm{dist}(f_n(t), E)$$

for $t \in S'$.

Since $\lim_n \mathrm{dist}(f_n(t), E) = 0,\ t \in S'$ and $\lim_m \|T_m h_n(t) - h_n(t)\| = 0,\ t \in S'$, by Egorov's theorem, we can choose a subset of positive measure $S'' \subset S'$ such that all these limits are uniform on S''. Hence, we obtain

$$\|f_n(t) - T_m f_n(t)\| \leq \|h_n(t) - T_m h_n(t)\| + \|(I - T_m) g_n(t)\| \tag{5.4}$$

for all $n,\ m$ and $t \in S''$.

Now, we take $\epsilon > 0$ and N such that $\mathrm{dist}(f_n(t), E) < \epsilon$ for $t \in S''$ and $n > N$. Then, for every n, take $m = m(n)$ such that $\|T_m h_n(t) - h_n(t)\| < \epsilon$ for $t \in S''$. Denoting $B = \sup_m \|T_m\|$, we obtain

$$\|T_m f_n(t) - f_n(t)\| < \epsilon + (1 + B) A \cdot \epsilon$$

for $t \in S''$. We finish the proof as in Lemma 5.1: choose a compact $K \subset \mathbb{D}$ and an appropriate constant C_K, and write

$$\begin{aligned} &\log \|T_m(f_n(z)) - f_n(z)\| \\ &\quad \leq \int_{\mathbb{T} \setminus S''} P_z(t) \log^+ \|T_m f_n(t) - f_n(t)\| dm(t) \\ &\qquad + \int_{S''} P_z(t) \log \|T_m f_n(t) - f_n(t)\| dm(t) \\ &\quad \leq C_K (1 + B) \mathcal{N}(\|f_n(\cdot)\|) + \log(\epsilon + A(1 + B)\epsilon). \end{aligned}$$

Since $T_m X \subset E$, this implies $\log(\mathrm{dist}(f_n(z), E)) \leq const + \log(\epsilon + A(1 + B)\epsilon)$ for $z \in K,\ n > N$ and for an appropriate constant. This means that $\lim_n \mathrm{dist}(f_n(z), E) = 0$ uniformly on compacts.

The rest of the proof is the same as for Theorem 5.9. □

Remarks 5.11 Using (5.1) in a similar way, we can obtain the following "Jensen inequality": given $f \in \mathcal{N}^+_\sigma(\mathbb{D}, X)$ with separable boundary values $f(t),\ t \in \mathbb{T}$, and a compact $K \subset \mathbb{D}$, there exists a constant *C* (depending on *K*, $\mathcal{N}(\|f(\cdot)\|)$ and *A*, *B* of the preceding proof only) such that

$$\log(\mathrm{dist}(f(z), E)) \leq C + \int_{\mathbb{T}} P_z(t) \log(\mathrm{dist}(f(t), E)) dm(t)$$

for $z \in K$.

Moreover, using the well-known fact that for $f \in \mathcal{N}^+(\mathbb{D}, X)$ the functions $t \longmapsto \log^+ \|f(rt)\|$, $0 \leq r < 1$ form an equipotentially absolutely continuous family (in particular, this is the case if $f \in H^1(\mathbb{D}, X)$; see [Pr] and [Z], Ch.7 for the scalar case), we can derive the same inequality but with a constant *C depending on A and B only.*

Corollary 5.12 *Let X, E be spaces satisfying (H4) and let $S \subset \mathbb{T}$ be a measurable set such that $m(S) > 0$. Let $f \in \mathcal{N}^+_\sigma(\mathbb{D}, X)$ be a function, which is separable-valued on S such that $f(t) \in E$ for all $t \in S$. Then, $f(z) \in E$ for every $z \in \mathbb{D}$.*

In fact, setting $f_n = f$ for every *n*, we obtain from the theorem $\mathrm{dist}(f(z), E) = 0$ for all $z \in \mathbb{D}$. □

Remarks 5.13 Remarks similar to 5.6 are valid in the context of Theorem 5.10 and Corollary 5.12 as well. Let us also mention that Theorem 5.10 and Corollary 5.12 do not imply any convergence to *E* or the membership in *E*, $f(t) \in E$, on the rest of the boundary $t \in \mathbb{T} \backslash S$. Of course, $f(t) \in E$ for any $t \in \mathbb{T}$ for which the radial limit exists in a stronger sense, namely for the same topology, for which *E* is closed; see below for several examples.

Examples and counterexamples. We start with a few general remarks on possible applications of the theorems above. First of all, we observe the following: the weaker is the topology for which boundary values exist, the more interesting could be applications of Khinchin-Ostrowski type theorems, especially in their quotient form of 5.7 - 5.9 and 5.10. On the other hand, the most important source of vector-valued holomorphic functions is probably spectral theory, in its both settings, the general Banach algebra and Banach space setting. In these theories, holomorphic functions are usually operator valued, and the existence of boundary values for the norm convergence of $\mathcal{L}(X, Y)$ is a rare exception. On the other hand, WOT and U - WOT boundary values exist, for example, for any bounded, or even Nevanlinna holomorphic function on any separable Hilbert space.

Thus, we can say that consequences of the preceding theorems based on axiom *(H4)*, and hence on norm closed subspaces *E* and weak $\sigma(X, W)$

boundary values, are more rare than those based on axiom *(H3)*, where the subspace *E* is closed for the same topology, which is used for obtaining the boundary values. And, of course, counterexamples of the type presented in 5.14 and 5.15 below are impossible under hypothesis *(H3)*.

Below, Examples 5.14 - 5.17 are of the first kind, that is based on *(H4)*, and Example 5.18 is of the second one.

Example 5.14 Bounded and convergent sequences. Let $X = l^\infty,\ E = c$ (or $E = c_0$), and W be the set of finitely supported sequences (or $W = l^1$). Verifying axiom *(H4)*, we can take as T_n the standard truncation operators $T_n x = y$ with $y_k = x_k,\ 1 \le k \le n$ and $y_k = GLIM(x)$ for $k > n$, where $GLIM$ stands for the Banach generalized limit of a sequence $x \in l^\infty$.

Applying Corollary 5.12, we obtain the following (scalar) Khinchin-Ostrowski theorem: *if* $f = (f_n)_{n\ge 1}$ *is a sequence of scalar holomorphic functions,* $f_n \in \mathcal{N}^+(\mathbb{D})$, *such that*

$$\int_T \log^+(\sup_n |f_n(rt)|)dm(t) \le \ const$$

for all $0 \le r < 1$, *and if* $(f_n(t))_{n\ge 1}$ *converges for* $t \in S$, *where* $S \subset \mathbb{T}$ *is measurable satisfying* $m(S) > 0$, *then* $(f_n(z))_{n\ge 1}$ *converges for all* $z \in \mathbb{D}$ *(and* $\lim_n f_n(z) = 0$ *for all* $z \in \mathbb{D}$ *if* $\lim_n f_n(t) = 0$ *for all* $t \in S$*).* □

It is worth mentioning that applying directly Theorem 5.5 we obtain the same conclusion under the weaker hypothesis $\sup_n \int_T \log^+ |f_n(rt)|dm(t) \le \ const$. Moreover, the same is true, of course, for vector-valued functions $f_n \in \mathcal{N}_\sigma(\mathbb{D}, X)$, see Theorem 5.5 above.

Counterexample. Corollary 5.12 (or Theorem 5.10) does not imply any particular behaviour of boundary functions on the complementary part of the boundary $\mathbb{T}\backslash S$. Indeed, take outer functions f_n with moduli

$$|f_n| = \chi_{T\backslash S} + \frac{1}{n}\chi_S,$$

where $S \subset \mathbb{T}$ is a non-trivial arc of $\mathbb{T}$. Then $f = (f_n)_{n\ge 1} \in H^\infty(l^\infty)$, $\lim_n f_n(t) = 0$ for $t \in S$, and hence $\lim_n f_n(z) = 0$ for $z \in \mathbb{D}$ (which is also obvious from a direct computation), but $|f_n(t)| = 1$ for all *n* and $t \in \mathbb{T}\backslash S$. □

Example 5.15 Bounded and compact operators. Let $X = \mathcal{L}(Y, Z)$ and $E = S_\infty(Y, Z)$, the ideal of compact operators, where $Y,\ Z$ are Banach spaces. Assume that *Z* satisfies the classical countable approximation property: there exists a sequence of finite rank operators $P_n : Z \longrightarrow Z$ such that $\lim_n P_n u = u$ for all $u \in Z$. Setting $T_n(A) = P_n A$, $A \in \mathcal{L}(Y, Z)$, we obtain property *(H4)* for the WOT and U-WOT topologies on X.

Consequently, *every $\mathcal{N}_\sigma^+(\mathbb{D}, \mathcal{L}(Y,Z))$ function whose WOT boundary limits are compact on a subset $S \subset \mathbb{T}$ of positive Lebesgue measure, takes compact values in the unit disc $\mathbb{D}$. In general, this is not the case for the rest of the boundary values on $\mathbb{T} \backslash S$*, see a counterexample below. □

Having in mind convergence and uniqueness theorems similar to 5.7-5.9 and 5.10, 5.12, it is curious to recall that for the case of Hilbert spaces *Y*, *Z*, every function $f \in H^\infty(\mathcal{L}(Y,Z))$ is a bordered resolvent of an operator. By this we mean the following consequence of model theory (see [SzNF]): there exist contractive operators $A,\ B,\ C$ on corresponding Hilbert spaces and a constant $\lambda > 0$ such that

$$f(z) = f(0) + \lambda z A(I - zB)^{-1} C \text{ for every } z \in \mathbb{D} .$$

Counterexample. A simple construction of a counterexample mentioned above is already contained in the previous subsection 5.14. Indeed, we can interpret a bounded sequence as a diagonal operator on the space l^2; such an operator is compact if and only if the sequence tends to zero. Hence, taking $f \in H^\infty(l^\infty)$ from the previous counterexample, we get a bounded operator-valued holomorphic function whoes WOT boundary values on *S* are compact (as well as the values in $\mathbb{D}$), but which are unitary on $\mathbb{T} \backslash S$. It is curious to note, that a special paper [Y] is devoted to a quite complicated construction of a counterexample of this kind. □

Example 5.16 Multipliers and small multipliers. Another possibly interesting example represents the pair of spaces $X = Mult(\mathcal{F}L^p)$ and $E = mult(\mathcal{F}L^p)$, that is the spaces of all Fourier multipliers and small Fourier multipliers on L^p. *Refering to [H] and [BS], recall that* $\varphi \in Mult(L^p(\mathbb{R}^n))$ if $\|\mathcal{F}^{-1}\varphi\mathcal{F}g\|_p \leq C\|g\|_p$ for $g \in \mathcal{S}_0$, i.e. for every compactly supported smooth function *g* on $\mathbb{R}^n$; the best possible *C* is the multiplier norm of φ. By definition, $\varphi \in mult(L^p(\mathbb{R}^n))$ if φ is approximable for the multiplier norm by functions $\varphi_n \in \mathcal{S}_0,\ n \geq 1$. Appropriate modifications of these definitions exist for every commutative locally compact group instead of $\mathbb{R}^n$. Recall that for $p = 1$ and $p = \infty$ one has $Mult = \mathcal{M}(\mathbb{R}^n)$ (all bounded complex Borel measures on $\mathbb{R}^n$), $mult = L^1(\mathbb{R}^n)$, and for $p = 2$, $Mult = L^\infty(\mathbb{R}^n)$, $mult = C_0(\mathbb{R}^n)$.

As is well-known, see [H] (and also [N1] for the case of the group $\mathbb{Z}$), the Fejér averages of Fourier transforms (or Fourier series for the case of a discrete group) are contractive operators on $X = Mult(L^p(\mathbb{R}^n))$. Hence, these averages converge to the function φ if and only if $\varphi \in E = mult(L^p(\mathbb{R}^n))$. Here condition *(H4)* is satisfied, with $\sigma(X, W)$ standing for the WOT topology of $Mult(L^p(\mathbb{R}^n))$.

Hence, Corollary 5.12 is applicable: *if $f \in \mathcal{N}_\sigma^+(\mathbb{D}, Mult(L^p(\mathbb{R}^n)))$ and $f(t) \in mult(L^p(\mathbb{R}^n))$ for $t \in S$, where $S \subset \mathbb{T}$ is a measurable subset of positive measure, then $f(z) \in mult(L^p(\mathbb{R}^n))$ for $z \in \mathbb{D}$.* □

Example 5.17 Continuity at a point. Let *K* be a metric space, μ a locally finite Borel measure on *K*, and $p_0 \in supp(\mu)$. Further, let $X = L^\infty(K, \mu)$, $W = L^1(K, \mu)$ and $f \in \mathcal{N}_\sigma^+(\mathbb{D}, X)$, so that $f = f(z, p)$ is a function of two variables on $\mathbb{D} \times K$. *If there exists a measurable subset $S \subset \mathbb{T}$ of positive measure such that the boundary values $f(t, \cdot)$ are continuous at a point p_0 for $t \in S$, then the same continuity holds for all $f(z, \cdot),\ z \in \mathbb{D}$.*

Indeed, taking as $E \subset L^\infty(K, \mu)$ the subspace of all functions continuous at p_0, we only need to check hypothesis *(H4)* (and then apply Corollary 5.12). To this end we set

$$T_n f(p) = \frac{1}{\mu(B_n)} \int_{B_n} f d\mu$$

for $p \in B_n = \{p : \operatorname{dist}(p, p_0) < 1/n\}$, and $T_n f(p) = f$ outside of B_n. Clearly, $T_n \in \mathcal{L}(L^\infty(K, \mu)),\ \|T_n\| = 1$ and $\lim_n T_n f = f$ if (and only if) $f \in E$.

The other requests of *(H4)* are obvious. □

It is clear that we can also obtain continuity on a set $K_0 \subset K$ (applying the above reasoning point by point for $p_0 \in K_0$). In particular, this gives another proof for a partial case of Example 5.16: $K = K_0 = \mathbb{T}$, $\mu = m$.

Example 5.18 Triangular, Toeplitz, Hankel, and other operators. Here, we choose as subspace $E \subset \mathcal{L}(l^2)$ in 5.7 - 5.12 one of the following sets: the set of all bounded operators on l^2 having 1) lower (respectively, upper) triangular matrices; 2) Toeplitz matrices; 3) Hankel matrices. We can add, $E = \{\mathcal{A}\}'$, the commutator of a given set of operators $\mathcal{A} \subset \mathcal{L}(Y, Z)$. All these subspaces are U - WOT closed, and hence satisfy *(H3)*. A particular case is (d). In all these cases, counterexamples similar to above 5.14 and 5.15 are impossible: if $f(t) \in E$ on a set of positive Lebesgue measure, then $f(t) \in E$ for all $t \in \mathbb{T}$.

Example 5.19 Radical and Volterra operators. One further consequence of the preceding theory is related to radicals of commutative Banach algebras. Namely, let *X* be a commutative Banach algebra and assume that *E* its radical. It is norm closed, and - in the case when we can verify *(H4)* - we can apply 5.10 and 5.12.

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