Different Domains Induce Different Heat Semigroups on $C_0(\Omega)$

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ABSTRACT Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be two open, connected sets which are regular in the sense of Wiener. Denote by $\Delta_0^{\Omega_j}$ the Laplacian on $C_0(\Omega_j)$, $j = 1, 2$. Assume that there exists a non-zero linear mapping $U : C_0(\Omega_1) \to C_0(\Omega_2)$ such that

(a) $|Uf| = U|f| \qquad (f \in C_0(\Omega_1))$ and

(b) $Ue^{t\Delta_0^{\Omega_1}} = e^{t\Delta_0^{\Omega_2}} U \qquad (t \geq 0)$.

Then it is shown that Ω_1 and Ω_2 are congruent. This result complements [2] where the Laplacian on L^p was considered and U was supposed to be bijective.

0 INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be an open set. By Δ_2^Ω we denote the Dirichlet Laplacian on $L^2(\Omega)$. This is a self-adjoint operator generating a contraction semigroup $(e^{t\Delta_2^\Omega})_{t \geq 0}$. If Ω is bounded, then Δ_2^Ω has compact resolvent and hence $L^2(\Omega)$ has an orthogonal basis $\{e_n : n \in \mathbb{N}\}$ consisting of eigenvectors of Δ_2^Ω ; i.e.

$$-\Delta_2^\Omega e_n = \lambda_n e_n ,$$

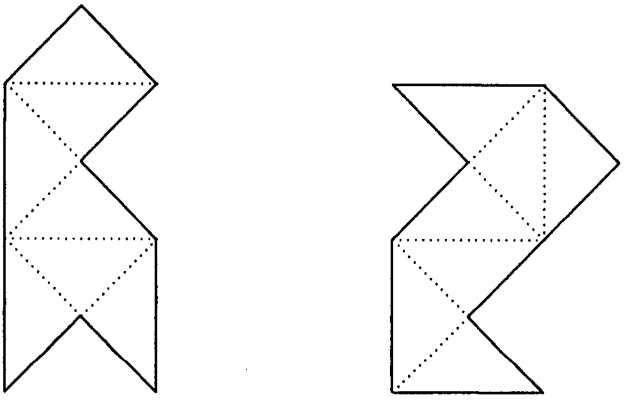
$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$, $\lim_{n \to \infty} \lambda_n = \infty$. Now let Ω_1 and Ω_2 be two bounded open sets. We say that Ω_1 and Ω_2 are **isospectral** if the operators $\Delta_2^{\Omega_1}$ and $\Delta_2^{\Omega_2}$ have the same sequence of eigenvalues. It was Marc Kac, who asked the following question in his famous paper [12] of 1962:

Question (Marc Kac): Let Ω_1, Ω_2 be two bounded open, connected sets in $\mathbb{R}^N$ which are of class C^∞ . Assume that Ω_1 and Ω_2 are isospectral. Does it follow that Ω_1 and Ω_2 are congruent?

If we have in mind that the sequence of eigenvalues is just the same as the proper frequencies of the body, one can indeed reformulate this question by asking *"Can you hear the shape of a drum?"*, which is precisely the title of Kac's paper.

It was known already to Kac that the answer is negative if we consider the Laplace Beltrami operator on a compact manifold instead of domains in $\mathbb{R}^N$. In the Euclidean case, a counterexample to Kac's question was given by Urakawa [21] if dimension $N \geq 4$. For $N \geq 2$ finally, a counterexample was given by Gordon, Webb and Wolpert [10] in 1992.

Today, very elementary descriptions of examples are given (see Berard [4], Chapman [6]). For example, in dimension 2, seven triangles may be put together in the two different ways shown below to produce two polygones which are isospectral but not congruent.



However, it seems that so far no Euclidean counterexample with smooth boundary is known. Thus Kac's question in the precise form he asked it, is still open.

There is another way to look at isospectral sets. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open and bounded. Then Ω_1 and Ω_2 are isospectral if and only if there exists a unitary operator $U : L^2(\Omega_1) \to L^2(\Omega_2)$ such that

$$e^{t\Delta_2^{\Omega_2}} U = U e^{t\Delta_2^{\Omega_1}} \qquad (t \geq 0) . \tag{0.1}$$

In fact, we may define U by mapping the orthonormal basis diagonalizing $\Delta_2^{\Omega_1}$ onto the one which diagonalizes $\Delta_2^{\Omega_2}$.

Let $f \in L^2(\Omega_1)$. Then the function $u(t,x) = (e^{t\Delta_2^{\Omega_1}} f)(x)$ is the unique solution of the heat equation:

$$\left.\begin{array}{l} u \in C^\infty((0,\infty) \times \Omega_1) \\ u(t,\cdot) \in H_0^1(\Omega_1) \\ u_t(t,x) = \Delta u(t,x) \qquad (t > 0 \ , \ x \in \Omega_1) \\ \lim_{t \downarrow 0} u(t,\cdot) = f \text{ in } L^2(\Omega_1) \ . \end{array}\right\} \tag{0.2}$$

Thus (0.1) is equivalent to saying that U maps solutions of (0.2) to solutions.

Now observe that $u(t,x) \geq 0$ a.e. if $f \geq 0$. Moreover, if we think of heat conduction or diffusion as a physical model, then only positive solutions have a physical meaning. So it is natural to consider mappings U which map positive solutions to positive solutions.

By an **order isomorphism** we unterstand a bijective linear mapping $U : L^2(\Omega_1) \to L^2(\Omega_2)$ satisfying

$$Uf \geq 0 \text{ if and only if } f \geq 0 \tag{0.3}$$

for all $f \in L^2(\Omega_1)$.

If in (0.1) we replace the unitary operator U by an order isomorphism U , then the following result holds [2, Corollary 3.17].

THEOREM 0.1 *Let Ω_1, Ω_2 be two open connected sets which are regular in capacity. If there exists an order isomorphism $U : L^2(\Omega_1) \to L^2(\Omega_2)$ such that (0.1) holds, then Ω_1 and Ω_2 are congruent.*

Having in mind the previous interpretation, we may reformulate Theorem 0.1 by saying that diffusion determines the domain. We refer to [2] for the proof of Theorem 0.1 and to Section 2 - 4 for further explanations, in particular the notion of regularity in capacity.

The aim of the present paper is to extend Theorem 0.1 in two ways. First of all we will prove that it holds even if we do no longer assume that U is onto. Secondly, we will establish an analogous result where L^2 is replaced by a space of continuous function.

For our arguments it will be essential that the semigroup generated by the Dirichlet Laplacian on $C_0(\Omega)$ is irreducible. We will prove this in Section 1 by using that the semigroup actually consists of classical solutions; i.e. that (0.2) holds. Using a classical maximum principle we then obtain irreducibility.

1 CLASSICAL SOLUTIONS OF THE HEAT EQUATION AND STRICT POSITIVITY

In this section we show that the heat equation always has classical solution due to interiour elliptic regularity. This is not new (cf. [5, IX 6]), but we use this to prove irreducibility with help of the classical strict maximum principle for parabolic equations. This is an alternative much more elementary way in comparison with the use of lower Gaussian bounds (see Davies [7, Theorem 3.3.5]). In addition, we obtain not only irreducibility in L^2 but also in $C_0(\Omega)$ which is stronger and will be used in Section 2.

Let $\Omega \subset \mathbb{R}^N$ be an open set. We consider realizations of the Laplacian on $L^2(\Omega)$ which generate differentiable positive semigroups. Our aim is to show that such a semigroup is automatically strictly positive. The concrete example we will consider later is the Laplacian with Dirichlet boundary conditions.

Let $T = (T(t))_{t \geq 0}$ be a semigroup on a Banach space X with generator A . Then for each $k \in \mathbb{N}$, the space $D(A^k)$ is a Banach space for the norm

$$\|x\|_{A^k} = \|x\| + \|Ax\| + \ldots + \|A^k x\| \ . \tag{1.1}$$

The semigroup is called **differentiable** if $T(t)x \in D(A)$ for all $t > 0$, $x \in X$. In that case one has

$$T(\cdot)x \in C^k((0,\infty)\ ;\ D(A^m)) \tag{1.2}$$

for all $k \in \mathbb{N}$, $m \in \mathbb{N}$, $x \in X$ (see e.g. Pazy [16]).

An operator A on $L^2(\Omega)$ is called a **realization of the Laplacian**, if

$$Af = \Delta f \qquad (\text{in } \mathcal{D}(\Omega)')$$

for all $f \in D(A)$. We now show that a differentiable semigroup T whose generator A is a realization of the Laplacian in $L^2(\Omega)$ governs a classical solution of the heat equation. More precisely, we have the following:

THEOREM 1.1 *Let $T=(T(t))_{t\geq 0}$ be a differentiable semigroup on $L^2(\Omega)$ whose generator is a realization of the Laplacian. Given $f \in L^2(\Omega)$ let*

$$u(t,x) = (T(t)f)(x) \qquad (t > 0 \ , \ x \in \Omega) \ .$$

Then $u \in C^\infty((0,\infty)\times\Omega)$ and

$$u_t(t,x) = \Delta u(t,x) \qquad (t > 0 \ , \ x \in \Omega) \ . \tag{1.3}$$

We need the following two results on regularity which are easily proved with help of the Fourier transform (see e.g., [18, Theorem 8.12.]).

By $H^k(\Omega)$ we denote the k-th Sobolev Space; i.e.

$$H^k(\Omega) = \{f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega) \text{ if } |\alpha| \leq k\} \ ,$$

where $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{N}_0^N$ denotes a multi-index and $|\alpha| = \sum_{j=1}^N \alpha_j$ its order, $D^\alpha = D_1^{\alpha_1}\dots D_N^{\alpha_N}$, $D_j = \frac{\partial}{\partial x_j}$, $j = 1,\dots N$, $\mathbb{N}_0 = \mathbb{N}\cup\{0\} = \{0,1,2,\dots\}$. The local Sobolev spaces are defined by $H^k_{loc}(\Omega) = \{f \in L^2_{loc}(\Omega) : D^\alpha f \in L^2_{loc}(\Omega)$ if $|\alpha| \leq k\}$. Here we consider, in the usual way, $L^2_{loc}(\Omega)$ as a subspace of $\mathcal{D}(\Omega)'$ and D^α as an operator on $\mathcal{D}(\Omega)'$. We set $H^0(\Omega) = L^2(\Omega)$, $H^0_{loc}(\Omega) = L^2_{loc}(\Omega)$ for consistency.

LEMMA 1.2 *Let $k,m \in \mathbb{N}_0$, $k > \frac{N}{2}$. Then $H^{k+m}_{loc}(\Omega) \subset C^m(\Omega)$. In particular, $\bigcap_{k\in\mathbb{N}} H^k_{loc}(\Omega) = C^\infty(\Omega)$.*

Here we let $C(\Omega) = C^0(\Omega)$ be the space of all continuous functions on Ω with values in $\mathbb{C}$ and $C^k(\Omega)$ is the space of all functions which are k-times continuously differentiable.

LEMMA 1.3 *Let $u,f \in L^2_{loc}(\Omega)$ such that $\Delta u = f$ in $\mathcal{D}(\Omega)'$. Then $u \in H^2_{loc}(\Omega)$. Moreover, if $f \in H^k_{loc}(\Omega)$, then $u \in H^{k+2}_{loc}(\Omega)$.*

It is not difficult to see that for $k \in \mathbb{N}_0$,

$$H^{k+1}_{loc}(\Omega) = \{f \in H^1_{loc}(\Omega) : D_j f \in H^k_{loc}(\Omega) \ , \ j = 1,\dots,N\} \ . \tag{1.4}$$

Proof of Theorem 1.1. Let $\omega \subset \Omega$ be open, bounded such that $\bar\omega \subset \Omega$. Since A is a realization of the Laplacian, it follows from Lemma 1.2 that $D(A^k) \subset H^{2k}_{loc}(\Omega)$

for all $k \in \mathbb{N}$. Let $p > \frac{N}{2}$: Then for $2k > p$, $H^{2k}_{loc}(\Omega) \subset C^{2k-p}(\Omega)$. Let $\omega \subset \Omega$ be open and bounded such that $\bar{\omega} \subset \Omega$. For $g : \Omega \to \mathbb{C}$, we denote by $j(g)$ the restriction of g to $\bar{\omega}$. It follows from the closed graph theorem that j defines a bounded operator from $D(A^k)$ into $C^{2k-p}(\bar{\omega})$. Hence by (1.2), for $f \in L^2(\Omega)$ we have $j \circ T(\cdot)f \in C^m((0,\infty)\ ,\ C^{2k-p}(\bar{\omega}))$ for all k , $m \in \mathbb{N}$, $2k > p$. This implies that $u(\cdot\ ,\ \cdot) \in C^m((0,\infty) \times \omega)$. □

We recall the classical strict parabolic maximum principle, see e.g. [8, V § 5, 3.4, p. 1081].

PROPOSITION 1.4 *Let* $\tau > 0$. *Let* $\Omega \subset \mathbb{R}^N$ *be open and connected. Let*

$$u \in C^2((0,\tau) \times \Omega) \cap C([0,\tau] \times \bar{\Omega}) \text{ such that}$$
$$u_t(t,x) = \Delta u(t,x) \qquad t \in (0,\tau)\ ,\ x \in \Omega\ .$$

Assume that there exist $x_0 \in \Omega$, $t_0 \in (0,\tau]$ *such that*

$$u(t_0,x_0) = \max_{\substack{t\in(0,\tau] \\ x\in\Omega}} u(t,x)\ .$$

Then u *is constant.*

From this we can now deduce that every semigroup generated by a realization of the Laplacian is automatically strictly positive whenever it is positive.

THEOREM 1.5 *Let* $\Omega \subset \mathbb{R}^N$ *be open and connected. Assume that* $T = (T(t))_{t\geq 0}$ *is a differentiable positive semigroup whose generator* A *is a realization of the Laplacian. Then* $T(t)f \in C^\infty(\Omega)$ *for all* $t > 0$, $f \in L^2(\Omega)$; *and if* $0 \leq f \in L^2(\Omega)$, $f \neq 0$, *then*

$$(T(t)f)(x) > 0 \tag{1.5}$$

for all $x \in \Omega$, $t > 0$.

Proof. It follows from Theorem 1.1 that the function u given by $u(t,x) = (T(t)f)(x)$ is in $C^\infty((0,\infty) \times \Omega)$ and satisfies the heat equation (1.3). Assume that $f \geq 0$. Then $u(t,x) \geq 0$ for all $t > 0$, $x \in \Omega$ by hypothesis. Assume that there exists $t_0 > 0$, $x_0 \in \Omega$ such that $u(t_0,x_0) = 0$. Let ω be open bounded, connected such that $\bar{\omega} \subset \Omega$. The strict maximum principle applied to $-u$ shows that $u(t,x) = 0$ for all $t \in (0,t_0]$, $x \in \omega$. Now a simple connectedness argument shows that $u(t,x) = 0$ for all $t \in (0,t_0]$, $x \in \Omega$. Since $f = \lim_{t\downarrow 0} u(t,\cdot)$ in $L^2(\Omega)$ it follows that $f = 0$. □

REMARK 1.6 (Irreducibility) A positive semigroup T on a Banach lattice E is called **irreducible** if for all $f \in E_+$, $f \neq 0$ and all $\varphi \in E'_+$, $\varphi \neq 0$ there exists $t > 0$ such that $\langle T(t)f\ ,\ \varphi\rangle > 0$. If in addition T is holomorphic then it is automatically true that $\langle T(t)f\ ,\ \varphi\rangle > 0$ for all $t > 0$ (see [15, C-III Theorem 3.2]. Theorem 1.5 implies in particular that the semigroup T considered here is irreducible.

REMARK 1.7 Of course there do exist realizations of the Laplacian which do not generate a positive semigroup. For example, let A on $L^2(0,1)$ be given by $D(A) = \{f \in H^2(0,1) : f(0) = -f(1)\ ,\ f'(0) = -f'(1)\}$, $Af = f''$. Then A generates a semigroup which is not positive (cf. [1, 3.4], [15, p. 255]).

2 INTERTWINING LATTICE HOMOMORPHISMS ON L^2

Let $\Omega \subset \mathbb{R}^N$ be an open set. By Δ_2^Ω we denote the Dirichlet Laplacian on $L^2(\Omega)$; i.e.

$$D(\Delta_2^\Omega) = \{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\}\ ,\ \Delta_2^\Omega f = \Delta f\ .$$

Then Δ_2^Ω is a form negative operator which generates a positive contraction semigroup $T = (e^{t\Delta_2^\Omega})_{t \geq 0}$ on $L^2(\Omega)$.

By $cap(F) = \inf\{\|u\|_{H^1}^2 : u \geq 1$ on a neighborhood of $F\}$ we denote the **capacity** of a subset F of $\mathbb{R}^N$. Then cap defines an outer measure on $\mathbb{R}^N$. An open subset Ω of $\mathbb{R}^N$ is called **regular in capacity** if

$$cap(B(z,r) \setminus \Omega) > 0$$

for all $z \in \partial\Omega$, $r > 0$. Note that Ω is regular in capacity whenever it is **topologically regular**, i.e. $\mathring{\bar{\Omega}} = \Omega$. But also the set

$$\Omega = B(0,1) \setminus \{(x_1, 0) : x_1 \geq 0\} \subset \mathbb{R}^2$$

is regular in capacity.

The aim of this section is to prove the following result which extends Theorem 0.1 mentioned in the introduction.

THEOREM 2.1 *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open, connected and regular in capacity. Assume that there exists a linear operator $U : L^2(\Omega_1) \to L^2(\Omega_2)$ such that $U \neq 0$ and*

(a) $|Uf| = U|f| \qquad (f \in L^2(\Omega_1))$;

(b) $Ue^{t\Delta_2^{\Omega_1}} = e^{t\Delta_2^{\Omega_2}} U \qquad (t \geq 0)$.

Then Ω_1 is congruent to Ω_2 . More precisely, there exist an isometry $\tau : \mathbb{R}^N \to \mathbb{R}^N$ and a constant $c > 0$ such that $\tau(\Omega_2) = \Omega_1$ and

$$(Uf)(y) = cf(\tau(y)) \qquad (y \in \Omega_2) \tag{2.1}$$

for all $f \in L^2(\Omega_1)$.

Here a mapping $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is called an **isometry** if there exist an orthogonal matrix B and a vector $b \in \mathbb{R}^N$ such that $\tau(y) = By + b$ for all $y \in \mathbb{R}^N$. Two open sets Ω_1 and Ω_2 are called **congruent** if there exists an isometry such that $\tau(\Omega_2) = \Omega_1$. In that case it is easy to see that

$$Uf = f \circ \tau$$

defines a unitary operator satisfying (a) and (b).

Note that the regularity assumption in Theorem 2.1 cannot be omitted. This is made clear in the following remark.

REMARK 2.2 Let $\Omega \subset I\!R^N$ be an open set. Then there exists a unique open set $\tilde{\Omega}$ which is regular in capacity such that $\tilde{\Omega} \supset \Omega$ and $cap(\tilde{\Omega} \setminus \Omega) = 0$. This implies in particular that $L^2(\Omega) = L^2(\tilde{\Omega})$ and $\Delta_2^{\Omega} = \Delta_2^{\tilde{\Omega}}$ (see [2] for the proofs). Now, if Ω is not regular in capacity, then we have $\Omega \neq \tilde{\Omega}$, and (a) and (b) are satisfied for U the identity operator.

Proof of Theorem 2.1. It follows as in [2, (2.16)] that $U\mathcal{D}(\Omega_1) \subset C^\infty(\Omega_2)$ and that

$$(Uf)(y) = \begin{cases} h(y)f(\tau(y)) & y \in \Omega_2' \\ 0 & y \in \Omega_2 \setminus \Omega_2' \end{cases} \tag{2.2}$$

for all $f \in \mathcal{D}(\Omega_1)$, where $\Omega_2' \subset \Omega_2$ is open, $\tau : \Omega_2' \to \Omega_2$ isometric on each component of Ω_2 and $h : \Omega_2' \to (0, \infty)$ is constant on each component of Ω_2'. Moreover, as in [2, (3.10)] one sees that U induces a continuous operator from $H_0^1(\Omega_1)$ into $H_0^1(\Omega_2)$. Now let $0 < f \in \mathcal{D}(\Omega_1)$, $g = e^{t\Delta_2^{\Omega_2}} Uf$. Then $g \in C^\infty(\Omega_2)$ and $g(y) > 0$ for all $y \in \Omega_2$.

On the other hand $g = Ue^{t\Delta_2^{\Omega_1}} f$ in $H_0^1(\Omega_2)$. Let $k = e^{t\Delta_2^{\Omega_1}} f$. Then $k \in H_0^1(\Omega_1)$. Let $k_n \in \mathcal{D}(\Omega_1)$ such that $k_n \to k$ in $H_0^1(\Omega_1)$. Then $Uk_n \to Uk$ in $H_0^1(\Omega_2)$ and so q.e. after extraction of a suitable subsequence. But $Uk = g$. Hence $Uk_n \to g$ q.e. Since $Uk(y) = 0$ for $y \in \Omega_2 \setminus \Omega_2'$, it follows that $g(y) = 0$ q.e. on $\Omega_2 \setminus \Omega_2'$. Since g is strictly positive it follows that $cap(\Omega_2 \setminus \Omega_2') = 0$. By [2, Proposition 3.10], it follows that Ω_2' is connected. Thus τ is an isometry and h equal to a constant $c > 0$. It follows from (2.1) and density that

$$(Uf)(y) = cf(\tau(y)) \tag{2.3}$$

a.e. on Ω_2 for all $f \in L^2(\Omega_1)$. Note that $\Omega_3 := \tau(\Omega_2)$ is an open subset of Ω_2 which is regular in capacity. If suffices to show that $cap(\Omega_1 \setminus \Omega_3) = 0$ in order to deduce that $\tau(\Omega_2) = \Omega_1$. Consider the mapping $V : L^2(\Omega_2) \to L^2(\Omega_3)$ given by $Vg = c^{-1} g \circ \tau^{-1}$. Let $W : L^2(\Omega_1) \to L^2(\Omega_3)$ be given by $W = V \circ U$. Then

$$We^{t\Delta_2^{\Omega_1}} = e^{t\Delta_2^{\Omega_3}} W \qquad (t \geq 0)$$

and

$$(Wf)(x) = f(x)$$

for all $x \in \Omega_3$ and all $f \in \mathcal{D}(\Omega_1)$. But $Wf \in H_0^1(\Omega_3)$. Hence $f = 0$ on $\Omega_1 \setminus \Omega_3$ q.e. for all $f \in \mathcal{D}(\Omega_1)$. This implies that $cap(\Omega_1 \setminus \Omega_3) = 0$, by [2, (3.7)]. □

3 THE CONGRUENCE PROBLEM WITH RESPECT TO $C_0(\Omega)$

Let $\Omega \subset I\!R^N$ be an open non-empty set. We consider the Dirichlet Laplacian Δ_0^{Ω} on

$$C_0(\Omega) := \{f \in C(\Omega) : \text{ for all } \varepsilon > 0 \text{ there exists } K \subset \Omega \text{ compact such that } f(x) = 0 \text{ whenever } x \in \Omega \setminus K\};$$

i.e. the operator Δ_0^Ω is given by

$$D(\Delta_0^\Omega) = \{f \in C_0(\Omega) : \Delta f \in C_0(\Omega)\} \ , \ \Delta_0^\Omega f = \Delta f \ .$$

Here Δf is a distribution. Since $C_0(\Omega) \subset L^1_{loc}(\Omega) \subset \mathcal{D}(\Omega)'$, the definition has a sense. Note that $D(\Delta_0^\Omega) \not\subset C^2(\Omega)$ since the Laplacian does not satisfy maximal regularity on spaces of continuous functions.

It has been investigated in [3] under which conditions Δ_0^Ω is generator of a C_0 -semigroup. The result uses classical notions of Potential Theory.

DEFINITION 3.1 *a) Let* $z \in \partial\Omega$. *A* **barrier** *is a function* $w \in C(\overline{\Omega \cap B})$ *such that* $\Delta w \leq 0$ *in* $\mathcal{D}(\Omega \cap B)'$, $w(z) = 0$, $w(x) > 0$ *for all* $x \in (\overline{\Omega \cap B}) \setminus \{z\}$ *where* $B = B(z, r)$ *is a ball centered at* z .
b) Ω *ist* **regular** *(in the sense of Wiener) if at each point* $z \in \partial\Omega$ *there exists a barrier.*

Every open subset of $\mathbb{R}$ is regular. In higher dimension, if the boundary of Ω is locally Lipschitz, then Ω is regular. But, for example, if $N \geq 2$, then for every $x \in \Omega$, $\Omega \setminus \{x\}$ is not regular. A bounded open set Ω is regular if and only if the Dirichlet problem is well-posed; i.e. for all $\varphi \in \partial\Omega$ there exists $u \in C(\bar{\Omega})$ such that $u_{|\partial\Omega} = \varphi$ and

$$\Delta u = 0 \text{ in } \mathcal{D}(\Omega)' \ .$$

Now we describe when Δ_0^Ω generates a semigroup (by which we always mean a C_0-semigroup) . The following characterization is proved in [3, Section 3].

THEOREM 3.2 *Let* $\Omega \subset \mathbb{R}^N$ *be open. The following conditions are equivalent.*

(i) Ω *is regular (in the sense of Wiener).*

(ii) $\varrho(\Delta_0^\Omega) \neq \emptyset$;

(iii) Δ_0^Ω *generates a holomorphic semigroup.*

In that case, the semigroup $T(t) = e^{t\Delta_0^\Omega}$ *generated by* Δ_0^Ω *is positive and contractive.*

Moreover, T *is consistent with the semigroup* $(e^{t\Delta_2^\Omega})_{t\geq 0}$ *on* $L^2(\Omega)$; *i.e. for* $f \in C_0(\Omega) \cap L^2(\Omega)$ *we have*

$$e^{t\Delta_2^\Omega} f = e^{t\Delta_0^\Omega} f \quad (t > 0) \ . \tag{3.1}$$

It follows from Theorem 1.5 that the semigroup $(e^{t\Delta_0^\Omega})_{t\geq 0}$ is strictly positive:

THEOREM 3.3 *Let* $\Omega \subset \mathbb{R}^N$ *be open and regular (in the sense of Wiener). Let* $0 < f \in C_0(\Omega)$. *Then*

$$(e^{t\Delta_0^\Omega} f)(x) > 0 \textit{ for all } x \in \Omega \ , \ t > 0 \ . \tag{3.2}$$

Here we use the notation

$$f \geq 0 \ :\Leftrightarrow \ f(x) \geq 0 \ \text{ for all } \ x \in \Omega \ ;$$
$$f > 0 \ :\Leftrightarrow \ f \geq 0 \ \text{ and } \ f \neq 0 \ .$$

Next we consider two open subsets Ω_1, Ω_2 of $I\!R^N$. A linear operator $U: C_0(\Omega_1) \to C_0(\Omega_2)$ is called a **lattice homomorphism** if

$$|Uf| = U|f| \text{ for all } f \in C_0(\Omega) , \tag{3.3}$$

where $|f|(x) = |f(x)| \quad (x \in \Omega_1)$.

If U is an **order isomorphism** (i.e. U is bijective, $U \geq 0$ and $U^{-1} \geq 0$), then U is clearly a lattice homomorphism (see [19] for more details). Note that every lattice homomorphism U is **disjointness preserving**, i.e.

$$f \cdot g = 0 \text{ implies } (Uf) \cdot (Ug) = 0 \tag{3.4}$$

for all $f, g \in C_0(\Omega_1)$.

In fact, if $f \cdot g = 0$, then $\inf\{|f|, |g|\} = 0$. This implies $\inf\{U|f|, U|g|\} = 0$. Hence

$$|Uf| \cdot |Ug| = (U|f|) \cdot (U|g|) = 0 .$$

Now we prove the first result on congruence. The space $C_0(\Omega)$ is easier to handle than L^p-spaces since point evaluations are continuous.

PROPOSITION 3.4 *Let $\Omega_1, \Omega_2 \subset I\!R^N$ be open and connected. Let $U: C_0(\Omega_1) \to C_0(\Omega_2)$ be a bounded operator such that*

(a) $f \cdot g = 0$ implies $(Uf) \cdot (Ug) = 0$ for all $f, g \in \mathcal{D}(\Omega_1)$;

(b) for all $y \in \Omega_2$ there exists $f \in C_0(\Omega_1)$ such that $(Uf)(y) \neq 0$;

(c) $\Delta Uf = U\Delta f$ for all $f \in \mathcal{D}(\Omega_1)$.

Then there exist a constant $c \in \mathbb{C} \backslash \{0\}$ and an isometry $\tau: I\!R^N \to I\!R^N$ satisfying $\tau(\Omega_2) = \Omega_1$ such that

$$(Uf)(y) = cf(\tau(y)) \quad (y \in \Omega_2)$$

for all $f \in C_0(\Omega_2)$. In particular, Ω_1 and Ω_2 are congruent.

For the proof we need classical regularity properties of the Laplacian. They will be used in the following form.

LEMMA 3.5 *Let $\Omega \subset I\!R^N$ be open, $k \in I\!N \cup \{0\}$, $g \in C(\Omega)$. If $\Delta g \in C^k(\Omega)$, then $g \in C^{k+1}(\Omega)$.*

Proof. We recall that for a distribution $u \in \mathcal{D}(\Omega)'$, if $\Delta u \in L^p(\Omega)$ for some $p > N$, then $u \in C^1(\Omega)$ (see e.g. [8, Chapter II, § 3, Proposition 6]). Thus the assertion of the lemma holds for $k = 0$. Assume that it holds for some $k \in I\!N \cup \{0\}$. Assume that $\Delta g \in C^{k+1}(\Omega)$. Then $g \in C^1(\Omega)$ (by the case $k = 0$) and $\Delta D_j g = D_j \Delta g \in C^k(\Omega)$. Hence $D_j g \in C^{k+1}(\Omega)$ by the inductive hypothesis $(j = 1, \ldots, N)$. We have shown that $g \in C^{k+2}(\Omega)$. □

Proof of Proposition 3.4. 1. We show that $U\mathcal{D}(\Omega_1) \subset C^\infty(\Omega_2)$. Let $k \in I\!N \cup \{0\}$. Then

$$U\mathcal{D}(\Omega_1) \subset C^k(\Omega_2) \tag{3.5}$$

for $k = 0$. Assume that (3.5) holds for some k . Let $f \in \mathcal{D}(\Omega_1)$. Then $\Delta U f = U \Delta f \in C^k(\Omega_2)$. Hence $Uf \in C^{k+1}(\Omega_2)$ by the lemma.
2. Let $y \in \Omega_2$. Then $\varphi(f) = (Uf)(y)$ defines a functional $\varphi \in C_0(\Omega_1)' \setminus \{0\}$. It follows from assumption b) that supp φ is a singleton. Thus, there exist $\tau(y) \in \Omega_1$ and $h(y) \in \mathbb{C} \setminus \{0\}$ such that $\varphi(f) = h(y) f(\tau(y))$ for all $f \in C_0(\Omega_1)$.
3. Now the proof of [2, Proposition 2.4] shows that h is constant $h \equiv c \neq 0$ and $\tau : \Omega_2 \to \Omega_1$ is an isometry. We denote its isometric extension to $\mathbb{R}^N$ still by τ . It remains to show that $\tau(\Omega_2) = \Omega_1$.
4. We show that $\tau(\partial\Omega_2) \subset \partial\Omega_1$. Let $y_0 \in \partial\Omega_2$. Assume $\tau(y_0) \in \Omega_1$. Take $y_n \in \Omega_2$ such that $\lim_{n\to\infty} y_n = y_0$. Choose $f \in \mathcal{D}(\Omega_2)$ such that $f(\tau(y_0)) = 1$. Then $\lim_{n\to\infty} (Uf)(y_n) = \lim_{n\to\infty} cf(\tau(y_n)) = c \neq 0$. This contradicts the fact that $Uf \in C_0(\Omega_2)$.
5. The set $\tau(\Omega_2)$ is open. Since by 4., $\partial(\tau(\Omega_2)) = \tau(\partial\Omega_2) \subset \partial\Omega_1$, it follows that $\tau(\Omega_2)$ is relatively closed in Ω_1 . Since Ω_1 is connected, we conclude that $\tau(\Omega_2) = \Omega_1$. □

Condition b) cannot be omitted in Proposition 3.6. In order to see this, it suffices to choose $\Omega_1 = (0,1) \subset \mathbb{R} = \Omega_2$ and to take for U the embedding from $C_0(0,1)$ into $C_0(\mathbb{R})$.

It is surprising that we can omit b) if we strengthen the intertwining condition slightly. For that we will suppose that $\Delta_0^{\Omega_1}$ and $\Delta_0^{\Omega_2}$ are generators. We recall the following easy description of intertwining operators.

PROPOSITION 3.6 *Let A_j be the generator of a semigroup T_j on a Banach space E_j , $j = 1, 2$. Let $U \in \mathcal{L}(E_1, E_2)$. The following are equivalent:*

(i) $UT_1(t) = T_2(t)U \quad (t \geq 0)$;

(ii) $UD(A_1) \subset D(A_2)$ *and* $A_2 U x = U A_1 x$ *for all* $x \in D(A_1)$.

Assuming Wiener regularity and the intertwining property we can now show that condition b) of Proposition 3.4 is automatically satisfied. The key argument is strict positivity in the sense of Theorem 3.3.

THEOREM 3.7 *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open, connected and regular (in the sense of Wiener). Let $U : C_0(\Omega_1) \to C_0(\Omega_2)$, be a lattice homomorphism, $U \neq 0$, such that*

$$Ue^{t\Delta_0^{\Omega_1}} = e^{t\Delta_0^{\Omega_2}} U \quad (t \geq 0) . \tag{3.6}$$

Then there exist a constant $c > 0$ and an isometry $\tau : \mathbb{R}^N \to \mathbb{R}^N$ satisfying $\tau(\Omega_2) = \Omega_1$ such that

$$(Uf)(y) = cf(\tau(y)) \quad (y \in \Omega_2)$$

for all $f \in C_0(\Omega_1)$.

REMARK 3.8 By Proposition 3.6, condition (3.6) is equivalent to saying that for $v, f \in C_0(\Omega_1)$

$$\Delta v = f \text{ in } \mathcal{D}(\Omega_1)' \Rightarrow \Delta U v = Uf \text{ in } \mathcal{D}(\Omega_2)' . \tag{3.7}$$

which is stronger than condition (c) of Proposition 3.4.

Proof. By the second part of the proof of Proposition 3.4 the operator U is of the form

$$(Uf)(y) = \begin{cases} h(y)f(\tau(y)) & y \in \Omega_2' \\ 0 & y \in \Omega_2 \setminus \Omega_2' \end{cases}$$

where $\Omega_2' = \{y \in \Omega_2 : \exists\ f \in C_0(\Omega_1)\ ,\ Uf(y) \neq 0\}$, $h : \Omega_2' \to (0,\infty)$ and $\tau : \Omega_2' \to \Omega_1$ are functions. Let $0 < f \in C_0(\Omega_1)$ such that $Uf > 0$. By Theorem 3.3 we have

$$(Ue^{t\Delta_0^{\Omega_1}}f)(y) = (e^{t\Delta_0^{\Omega_2}}Uf)(y) > 0$$

for all $y \in \Omega_2$, where $t > 0$. This implies that $\Omega_2 = \Omega_2'$. Thus condition b) in Proposition 3.6 is satisfied and the claim follows. □

We conclude this section showing by a counterexample that Theorem 3.7 is not true if we replace Dirichlet boundary conditions by periodic boundary conditions, even if U is an order isomorphism.

EXAMPLE 3.9 Consider the Banach lattice $E = \{f \in C[-1,1] : f(-1) = f(1)\}$ with supremum norm and let A be the operator on E given by $D(A) = \{f \in C^2[-1,1] : f^{(k)}(-1) = f^{(k)}(1)$ for $k = 0,1,2\}$, $Af = f''$. Then A generates a semigroup T . Let $U : E \to E$ be given by

$$(Uf)(y) = \begin{cases} f(1-y) & \text{if} \quad 0 \leq y \leq 1 \\ f(-1-y) & \text{if} \quad -1 \leq y < 0\ . \end{cases}$$

Then $UT(t) = T(t)U \quad (t \geq 0)$ even though U is not composition by an isometry.

Proof. The operator given $Bf = f'$, $D(B) = \{f \in E \cap C^1[-1,1] : f'(-1) = f'(1)\}$ generates the shift group on E . Hence $A = B^2$ generates a holomorphic semigroup T . One easily sees that $UD(A) = D(A)$ and $AUx = UAx \quad (x \in D(A))$. □

4 APPENDIX: REGULARITY IN THE SENSE OF WIENER AND REGULARITY IN CAPACITY

Let Ω be an open set in $I\!R^N$. We say that Ω is **regular in measure** (resp., **in capacity**) if for all $z \in \partial\Omega$ and all $r > 0$, $|\Omega \setminus B(z,r)| > 0$ (resp., $cap(\Omega \setminus B(z,r)) > 0$) . Here $|F|$ denotes the Lebesgue measure of a measurable set F in $I\!R^N$. It is clear that **toplogical regularity** (i.e. $\overset{\circ}{\bar\Omega} = \Omega$) implies regularity in measure, and regularity in measure implies regularity in capacity.

EXAMPLE 4.1 Let $B = B(0,1)$ be the euclidean unit ball in $I\!R^2$. Then $\Omega = B \setminus \{(x_1, 0) : 0 \leq x_1 < 1\}$ is regular in capacity but not regular in measure.

The reason why these notions of regularity are introduced in [2] is the following. Consider $L^2(\Omega)$ as a subspace of $L^2(I\!R^N)$ by extending functions by 0 out of

Ω. Then $L^2(\Omega_1) = L^2(\Omega_2)$ if and only if $|\Omega_1 \triangle \Omega_2| = 0$. This in turn, implies that $\Omega_1 = \Omega_2$ if Ω_1 and Ω_2 are regular in measure. Here $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ are open sets.

If $|\Omega_1 \triangle \Omega_2| = 0$, then $\Delta_2^{\Omega_1} = \Delta_2^{\Omega_2}$ if and only if $cap(\Omega_1 \triangle \Omega_2) = 0$, and this in turn implies that $\Omega_1 = \Omega_2$ whenever Ω_1 and Ω_2 are regular in capacity (see [2]).

REMARK 4.2 The condition *"regular in measure"* did occur in different context (under different name). It seems to be a crucial condition for smooth approximation in Sobolev spaces (cf. [20]). For example, if $\Omega \subset \mathbb{R}^2$ is open, bounded and star-shaped, then regularity in measure is sufficient for $C^\infty(\overline{\Omega})$ being dense in $W^{k,p}(\Omega)$ $(k \in \mathbb{N}\ ,\ 1 \leq p < \infty)$, see [20, Theorem B]. It is also a necessary condition in special situations (see [20, Theorem C and Example 1]).

Next we show that regularity in the sense of Wiener implies regularity in capacity. Our proof is based on the results of [3].

PROPOSITION 4.3 *Let $\Omega \subset \mathbb{R}^N$ be an open set which is regular in the sense of Wiener. Then Ω is regular in capacity.*

Proof. Let $\tilde{\Omega}$ be open, regular in capacity such that $cap(\tilde{\Omega} \setminus \Omega) = 0$ (see [2, Proposition 3.18]). Then $L^2(\tilde{\Omega}) = L^2(\Omega)$ and $\Delta_2^{\tilde{\Omega}} = \Delta_2^{\Omega}$. Now assume that $\Omega \neq \tilde{\Omega}$. Choose $z \in \partial\Omega \cap \tilde{\Omega}$. Let $0 < f \in C_0(\Omega) \cap L^2(\Omega)$. Then by [3, (3.3)] $(e^{t\Delta_0^\Omega} f) = e^{t\Delta_2^\Omega} f$. It follows from Theorem 1.5 that $e^{t\Delta_2^{\tilde{\Omega}}} f \in C^\infty(\tilde{\Omega})$ and $(e^{t\Delta_2^{\tilde{\Omega}}} f)(z) > 0$. But $e^{t\Delta_2^\Omega} f \in C^\infty(\Omega)$ and $e^{t\Delta_2^\Omega} f = e^{t\Delta_2^{\tilde{\Omega}}} f$. This contradicts that $e^{t\Delta_2^\Omega} f \in C_0(\Omega)$. □

There is a remarkable criterion due to Wiener which describes regularity. Assume that $N \geq 3$. Then Ω is regular in the sense of Wiener if and only if

$$\sum_{j=1}^{\infty} 2^{j(N-2)}\ cap\ (B(z, 2^{-j}) \setminus \Omega) = \infty \tag{4.1}$$

for every point $z \in \partial\Omega$.

Thus Wiener's criterion is a quantitative version of regularity in capacity. One can also see from Wiener's criterion that regularity implies regularity in capacity (note however that here $N \geq 3$). Every open set in $\mathbb{R}^3$ which satisfies the exterior cone condition (meaning that for each $z \in \partial\Omega$ there exists a cone in $\mathbb{R}^3 \setminus \Omega$ with vertex z) is regular. But there exist cusps which are not regular (see [14, p. 288]). Such a cusp gives an example of an open set in $\mathbb{R}^3$ (or higher dimension) which is regular in capacity but not in the sense of Wiener.

In dimension $N = 2$ the situation is more complicated. In fact, it is known that $\Omega \subset \mathbb{R}^2$ is regular whenever for each $z \in \partial\Omega$ there exists a continuous, injective function $f : [0,1] \to \mathbb{R}^2 \setminus \Omega$ such that $f(0) = z$ (see [11, p. 173]).

Here is an example of a set in $\mathbb{R}^2$ which is not regular in the sense of Wiener but regular in capacity (and even topologically regular).

PROPOSITION 4.4 *Let B be the open unit ball in $\mathbb{R}^2$ and let*

$$\Omega = B \setminus (\bigcup_{n=1}^{\infty} \overline{B}(a_n, r_n) \cup \{0\}) \tag{4.2}$$

where $a_n = (\alpha_n, 0)$, $\alpha_n > 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\alpha_n + r_n > 0$ are chosen such that the closed balls $\overline{B}(a_n, r_n)$ are disjoint. Thus Ω is open and regular in capacity (and even topologically). However, one can choose $r_n > 0$ such that Ω is not regular.

We are grateful to Charles Batty for the following probabilistic proof. Some preparation concerning Brownian motion and potential theory is needed (see [17] for example). Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set. By $\{B_t : t \geq 0\}$ we denote the Brownian motion and by P^x the Wiener measure $(x \in \mathbb{R}^N)$. Then regularity can be characterized in terms of Brownian motion in the following way (see [17]).

PROPOSITION 4.5 *The set Ω is regular if and only if*

$$P^x[\exists\ \tau > 0 : B(s) \in \Omega\ \forall\ s \in (0, \tau)] = 0 \tag{4.3}$$

for all $x \in \partial\Omega$.

REMARK 4.6 a) Condition (4.3) says that Brownian motion starting at $x \in \partial\Omega$ has to leave immediately Ω with a positive probability (equivalently with probability 1).
b) To see the relation with the Dirichlet problem we mention that for $f \in C(\partial\Omega)$

$$u(x) = E^x[f(B_{\tau_\Omega})] \qquad (x \in \Omega) \tag{4.4}$$

defines a harmonic function on Ω . Here $\tau_\Omega = \inf\{t > 0 : B_t \notin \Omega\}$ is the first exit time. If (4.3) is satisfied, then $\lim_{\substack{x\to z\\ x\in\Omega}} u(x) = f(z)$ for all $z \in \partial\Omega$. Thus u is the solution of the Dirichlet problem. □

We mention that for $N \geq 2$ and $x \in \mathbb{R}^N \setminus \{0\}$, $t > 0$

$$P^\circ[B(s) = x \text{ for some } 0 \leq s \leq t] = 0\ . \tag{4.5}$$

Now we can prove the proposition.

Proof of Proposition 4.5. We fix a sequence $\alpha_n \downarrow 0$. Let $a_n = (\alpha_n, 0)$. Let $t > 0$. Then $f_n(r) = P^\circ[B(s) \in B(a_n, r)$ for some $s \leq t]$ is decreasing in $r > 0$, and by (4.5), $\lim_{r\downarrow 0} f_n(r) = 0$. This allows us to choose $r_n > 0$ satisfying the requirements of the proposition and such that $f_n(r_n) < 2^{-n}$ for all $n \in \mathbb{N}$. Thus $P^\circ[B(s) \in \bigcup_{n\in\mathbb{N}} B(a_n, r_n)$ for some $s \leq t] \leq \sum_{n=1}^{\infty} P^\circ[B(s) \in B(x_n, r_n)$ for some $s \leq t] < 1$. Consequently, $P^\circ[\exists\ \tau > 0 : B(s) \in \Omega$ for all $s \in (0, \tau)] \geq P^\circ[B(s) \in \Omega$ for all $s \in (0, t)] > 0$. Thus (4.3) is violated. □

Of course, the example is also valid in higher dimensions than 2.

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