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Functional calculus, variational methods and Liapunov's theorem

By

WOLFGANG ARENDT, SHANGQUAN BU and MARKUS HAASE

Dedicated to Professor E. Lamprecht on the occasion of his 75th birthday.

Abstract. Given the generator $-A$ of a holomorphic semigroup on a Hilbert space H, we show that A is associated with a closed form if and only if $A + w \in BIP(H)$ for some $w \in \mathbb{R}$. Under this condition we also show that Liapunov's classical theorem is true, in the linear as well as the semilinear case.

0. Introduction. Let H be a complex Hilbert space and T a holomorphic C_0-semigroup on H with generator $-A$. If H is finite dimensional, then Liapunov's classical theorem says the following:

Assume that $\operatorname{Re} \lambda > 0$ for all $\lambda \in \sigma(A)$ (the spectrum of A). Then there exist $\varepsilon > 0$ and a scalar product on H such that $\|T(t)\| \leqq e^{-\varepsilon t} \quad (t \geqq 0)$.

We show that Liapunov's theorem holds in the infinite dimensional case if and only if $A + w$ has bounded imaginary powers for some $w \in \mathbb{R}$. This class of operators plays a crucial role in the theory of maximal regularity (see e.g. [16]). Using a result of Le Merdy [9], we show that it coincides precisely with the class of those operators which are associated with a closed form. The theory of closed forms is a classical subject of great importance for the investigation of elliptic operators (see [8], [18]). We also show a new perturbation result which is valid on every Banach space X. This is needed for our second main theorem, the extension of Liapunov's classical theorem on linearized stability to semilinear equations in infinite dimension.

1. Operators defined by forms. Let H be a complex Hilbert space with scalar product $(.|.)_H$. Let V be another Hilbert space which is continuously embedded into H with dense image. We use the short hand notation $V \underset{d}{\hookrightarrow} H$ for this situation. The norm of V is denoted by $\|.\|_V$. Let $a : V \times V \to \mathbb{C}$ be a sesquilinear form; i.e., $a(\cdot, v) : V \to \mathbb{C}$ and $\overline{a(v, \cdot)} : V \to \mathbb{C}$ are linear for all $v \in V$. Assume that a is continuous; i.e.,

(1.1) $$|a(u, v)| \leqq M\|u\|_V \|v\|_V \quad (u, v \in V)$$

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for some constant $M \geqq 0$. We say that a is *elliptic* (or more precisely H-*elliptic*) if there exist $\gamma > 0$, $\mu \in \mathbb{R}$ such that

$$\operatorname{Re} a(u,u) + \mu(u|u)_H \geqq \gamma \|u\|_V^2 \tag{1.2}$$

for all $u \in V$. Then we may associate an operator A on H with a in the following way. The domain $D(A)$ of A is given by

$$D(A) := \{u \in V : \exists v \in H \text{ such that } a(u,\varphi) = (v|\varphi)_H \text{ for all } \varphi \in V\} \tag{1.3}$$

and for $u \in D(A)$ we set

$$Au = v \tag{1.4}$$

where v is the unique element such that $a(u,\varphi) = (v|\varphi)_H$ for all $\varphi \in V$ (recall that V is dense in H). It is well-known that $-A$ is the generator of a holomorphic C_0-semigroup on H (see Theorem 1.2 below, or [18, Sec. 2.2 and Thm. 3.6.1]).

Definition 1.1. A *closed form* on H is a continuous, elliptic sesquilinear form $a : V \times V \to \mathbb{C}$ where $V \underset{d}{\hookrightarrow} H$. We call the operator A given by (1.3) and (1.4) *the operator associated with* a and write $a \sim A$.

One aim of this article is to describe those negative generators of holomorphic C_0-semigroups which are associated with some closed form.

Next we comment on a simple rescaling argument. Assume that A is associated with the closed form a, and denote by T the semigroup generated by $-A$. Let $\lambda \in \mathbb{C}$. Then $A + \lambda$ is associated with the closed form $a_\lambda : V \times V \to \mathbb{C}$ given by $a_\lambda(u,v) = a(u,v) + \lambda(u|v)_H$ and $-(A+\lambda)$ generates the semigroup $(e^{-\lambda t}T(t))_{t \geqq 0}$. It is possible to characterize those operators which are associated with a closed form by quasi-accretivity. We recall some notions. An operator A on H is called *accretive* if $\operatorname{Re}(Au|u)_H \geqq 0$ for all $u \in D(A)$. If in addition $I + A$ is surjective one says that A is m-*accretive*. By the Lumer-Phillips theorem, an operator A is m-accretive if and only if $-A$ generates a contractive C_0-semigroup on H. We say that A is *quasi-m-accretive* if there exists $w \in \mathbb{R}$ such that $A + w$ is m-accretive. Obviously, every operator associated with a closed form is quasi-m-accretive. The following characterization is a reformulation of some well-known results.

Theorem 1.2. *Let A be an operator on a Hilbert space H. The following assertions are equivalent:*

(i) *A is associated with a closed form;*
(ii) *there exists $\alpha \in (0, \frac{\pi}{2})$ such that $e^{i\alpha}A$ and $e^{-i\alpha}A$ are quasi-m-accretive;*
(iii) *there exists $w \geqq 0$ such that $-(A+w)$ generates a holomorphic C_0-semigroup which is contractive on a sector Σ_β for some $0 < \beta \leqq \frac{\pi}{2}$.*

Here we let

$$\Sigma_\beta := \{re^{i\alpha} : r > 0\ ,\ |\alpha| < \beta\}$$

where $0 < \beta \leqq \frac{\pi}{2}$.

This theorem is implicit in the monographs of Kato [8] and Tanabe [18]. We give the proof for convenience.

Proof. First observe that for a closed operator B in H, an angle β with $|\beta| < \pi/2$, and a real number w one has that $e^{i\beta}(B+w)$ is m-accretive if and only if $e^{i\beta}B + w(\cos\beta)$ is m-accretive. Moreover, if B is m-accretive, then B is densely defined.

Now assume that *(ii)* holds. Then for certain real numbers w, w' the operators $e^{i\alpha}A + w$ and $e^{-i\alpha}A + w'$ are both m-accretive and by taking the maximum of w and w' and shifting it to the right we can assume $0 \leqq w = w'$. The considerations above now show that $e^{\pm i\alpha}(A + (w/\cos\alpha))$ are m-accretive. From this it follows (see for example [1, Thm. 4.1] with $M = 1$ in the proof) that $-(A + (w/\cos\alpha))$ generates a holomorphic C_0-semigroup which is contractive on a whole sector around the positive real axis.

For the converse implication ((iii) $\Rightarrow$ (ii)) let $B := A + w$ and $(T(z))_{z\in\Sigma_\beta}$ be the holomorphic C_0-semigroup generated by $-B$. Let $0 < \alpha < \beta$. Then $T(te^{\pm i\alpha})_{t\geqq 0}$ are contraction semigroups by assumption. Their respective generators are $-e^{\pm i\alpha}B$ (cf. [6, p.101/102]). Thus $e^{\pm i\alpha}B$ are m-accretive by the Lumer-Phillips theorem. Hence $e^{\pm i\alpha}A + w\cos\alpha$ are m-accretive, from which (ii) follows.

Let us turn to the implication ((i) $\Rightarrow$ (ii)). Let a be a closed form in H and $A \sim a$. After rescaling we can assume that A satisfies (1.2) with $\mu = 0$; i.e., a is *coercive* (remember the observation at the beginning of the proof). In this case, $(\operatorname{Re} a)$ (i.e. the symmetric part of a) is an equivalent scalar product on V, hence by the Lax-Milgram Theorem, the operator $\mathbf{A} = (u \mapsto a(u, .)) : V \to V^*$ is an isomorphism. The operator $A : D(A) \to H$ being the part of $\mathbf{A}$ in H is therefore bijective with continuous inverse (look at $H \hookrightarrow V^* \overset{\mathbf{A}^{-1}}{\to} V \hookrightarrow H$.) Changing a to a_λ for $\operatorname{Re}\lambda \geqq 0$ the above arguments show that $0 \in \rho(A+\lambda)$. Furthermore, by the very definition of A and the coercivity of a one has $\operatorname{Re}(Au|u)_H = \operatorname{Re} a(u,u) \geqq \gamma\, \|u\|_V^2 \geqq 0$ for all $u \in D(A)$ from which it finally follows that A is m-accretive.

Now (1.1) and (1.2) imply that $|\operatorname{Im} a(u,u)| \leqq (M/\gamma)\operatorname{Re} a(u,u)$ for each $u \in V$ (remember that $\mu = 0$). Take an angle θ with $0 < |\theta| < \arctan(\gamma/M)$, hence $|\tan\theta| \leqq (1-\epsilon)(\gamma/M)$ for some $0 < \epsilon < 1$. Let $b(u,v) := e^{i\theta}a(u,v)$ for $u, v \in V$. It is easy to see, that b is a continuous, sesquilinear form on V which is even coercive with $\operatorname{Re} b(u,u) \geqq \epsilon(\cos\theta)\gamma\,\|u\|_V^2$. Of course, b is associated with the operator $e^{i\theta}A$. Summarizing the above considerations we have that for each $0 \leqq \alpha < \arctan(\gamma/M)$ the operators $e^{\pm i\alpha}A$ are m-accretive.

Finally we have to prove that (ii) implies (i). Rescaling reduces the problem to the case that $e^{\pm i\alpha}A$ are m-accretive. The proof of the equivalence ((ii) $\Leftrightarrow$ (iii)) shows that in fact $e^{i\theta}A$ is m-accretive for all $|\theta| \leqq \alpha$.

On $D(A)$ we define the scalar product $(.|.)_V$ to be the symmetric part of the sesquilinear form $((u,v) \mapsto ((A+I)u|v)_H)$, whence $\|u\|_V^2 = \operatorname{Re}(Au|u)_H + \|u\|_H^2$ for $u \in D(A)$. Furthermore we define $a := ((u,v) \mapsto (Au|v)_H) : D(A) \times D(A) \to \mathbb{C}$.

First observe that $(D(A), \|.\|_V) \overset{d}{\hookrightarrow} H$.

Next let us show that a is continuous on $D(A)$ with respect to $\|.\|_V$. Let $\beta := \frac{\pi}{2} - \alpha$. From the inequality $\operatorname{Re}(e^{i\theta}Au|u)_H \geqq 0$ for all $|\theta| \leqq \alpha$ one deduces easily that $|\operatorname{Im} a(u,u)| \leqq (\tan\beta)\operatorname{Re} a(u,u)$ which shows that $|a(u,u)| \leqq (1+\tan\beta)\operatorname{Re} a(u,u) \leqq (1+\tan\beta)\,\|u\|_V^2$.

Now let V be the (abstract) completion of $(D(A), \|.\|_V)$ and $\iota : V \to H$ the extension of the canonical embedding of $D(A)$ into H. Then ι is injective (cf. [8, p. 318]) whence $V \overset{d}{\hookrightarrow} H$. Extending a to V (keeping the symbol a for the extension), we have that $a : V \times V \to \mathbb{C}$ is a closed form in H; moreover, a_w is coercive for each $w \geqq 0$.

Finally we have to show that A is associated with a. Let $B \sim a$.

Then by construction $A \subset B$, and $B + I$ is m-accretive. But $A + I$ is m-accretive as well, whence it follows $A = B$. □

2. *BIP* and Perturbation. We interrupt the discussion of Section 1 in order to introduce a class of operators defined by some functional calculus. Let A be a densely defined *sectorial operator* of angle $\theta_1 \in (0, \pi)$ on a Banach space X; i.e., $\sigma(A) \subset \Sigma_{\theta_1}$ and

$$\|\lambda R(\lambda, A)\| \leqq M \quad (\lambda \in \mathbb{C} \setminus \Sigma_{\theta_1}) \tag{2.1}$$

for some $M \geqq 0$. Assume furthermore that $0 \in \varrho(A)$. Let $\theta_1 < \theta$. We define an $H^\infty(\Sigma_\theta)$-functional calculus in the following way. Let $\theta_1 < \theta_2 < \theta$ and

$$\Gamma(r) = \begin{cases} -re^{i\theta_2} & \text{if } r < 0 \\ re^{-i\theta_2} & \text{if } r \geqq 0\,. \end{cases}$$

We first assume that $\varphi \in H^\infty(\Sigma_\theta)$ satisfies

$$|\varphi(z)| \leqq c|z|^{-1} \quad (z \in \Sigma_\theta\,,\ |z| \geqq 1) \tag{2.2}$$

for some $c > 0$ and define in this case $\varphi(A)$ by Dunford's formula

$$\varphi(A) = \frac{1}{2\pi i} \int_\Gamma \varphi(\lambda) R(\lambda, A) d\lambda\,. \tag{2.3}$$

Then $\varphi(A) \in \mathscr{L}(X)$. Observe that for $x \in D(A)$ one has

$$\varphi(A)x = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\lambda)}{1+\lambda} R(\lambda, A)(I + A)x d\lambda\,. \tag{2.4}$$

This follows from Cauchy's theorem since $R(\lambda, A) - \frac{1}{1+\lambda} R(\lambda, A)(I + A) = \frac{1}{1+\lambda} I$. Now if $\varphi \in H^\infty(\Sigma_\theta)$ does not satisfy an estimate (2.2), then we define $\varphi(A)$ by (2.4) as a linear mapping from $D(A)$ into X. In particular, for $s \in \mathbb{R}$ we let $A^{is} = \varphi_s(A)$ where $\varphi_s(z) = z^{is}$.

Definition 2.1. Let A be an invertible, sectorial operator on X. One says that A *has bounded imaginary powers* and one writes $A \in BIP(X)$ if

$$\|A^{is}x\| \leqq c\|x\| \quad (x \in D(A)\,,\ |s| \leqq 1)$$

for some $c \geqq 0$.

We refer to [16], [17] and [9] for further information concerning this notion.

Remark 2.2 (Non-invertible operator). If A is injective with dense range $R(A)$ but possibly non-invertible, one usually defines

$$\varphi(A) = \frac{1}{2\pi i} \int_\Gamma \varphi(\lambda) \frac{\lambda}{(1+\lambda)^2} R(\lambda, A) d\lambda (I + A)^2 A^{-1} \tag{2.5}$$

on $D(A) \cap R(A)$. If A is invertible, this formula coincides with (2.4) by Cauchy's theorem, since $\frac{\lambda}{(1+\lambda)^2} R(\lambda, A)(I + A)^2 A^{-1} - \frac{1}{1+\lambda} R(\lambda, A)(I + A) = \frac{1}{(1+\lambda)^2}(I + A)A^{-1}$. □

For our purpose the class of all operators A such that $A + w \in BIP(X)$ for some $w \in \mathbb{R}$ will be of importance. It has been proved by Prüss-Sohr [17] (see also Monniaux [12]) that for $w > 0$ one has $A + w \in BIP(X)$ if $A \in BIP(X)$. We need the converse assertion. The following perturbation result will be useful.

Theorem 2.3. *Let A be a sectorial operator on X such that $0 \in \varrho(A)$ and $A \in BIP(X)$. Let $B \in \mathscr{L}(X)$ such that $0 \in \varrho(A+B)$ and $A+B$ is sectorial. Then $A+B \in BIP(X)$.*

P r o o f. We keep the notions above and we assume that $A+B$ is sectorial with the same angle (increasing the angle otherwise). We have to show that

$$\int_\Gamma \frac{\lambda^{is}}{1+\lambda} R(\lambda, A+B)d\lambda(I+A+B) \tag{2.6}$$

is uniformly bounded (on $s \in [-1, 1]$). We know that $\int_\Gamma \frac{\lambda^{is}}{1+\lambda} R(\lambda, A)d\lambda(I+A)$, and hence also

$$\int_\Gamma \frac{\lambda^{is}}{1+\lambda} R(\lambda, A)d\lambda(I+A+B) \tag{2.7}$$

is uniformly bounded. Thus it suffices to show that the difference of (2.6) and (2.7),

$$\int_\Gamma \frac{\lambda^{is}}{1+\lambda} R(\lambda, A)BR(\lambda, A+B)d\lambda(I+A+B) \tag{2.8}$$

is uniformly bounded. Note that

$$\int_\Gamma \lambda^{is} R(\lambda, A)BR(\lambda, A+B)d\lambda \tag{2.9}$$

is uniformly bounded. Since $I - \frac{1}{1+\lambda}(I+A+B) = \frac{1}{1+\lambda}(\lambda - A - B)$, the difference between (2.9) and (2.8) is

$$\int_\Gamma \frac{\lambda^{is}}{1+\lambda} R(\lambda, A)Bd\lambda\ , \tag{2.10}$$

which is uniformly bounded. Thus (2.8) is uniformly bounded, and the proof is complete. □

If A is sectorial, then $A+\omega$ is sectorial and invertible for all $\omega > 0$. The following converse implication will be needed for the proof of Theorem 4.1.

Corollary 2.4. *Let A be sectorial and invertible. If $A+w \in BIP(X)$ for some $w \in \mathbb{R}$, then $A \in BIP(X)$.*

R e m a r k 2.5. a) Corollary 2.4 is not valid if we omit the assumption that A is invertible even if A is a bounded operator on a separable Hilbert space. This follows from Le Merdy [11, Theorem 4.1, p. 54].
b) However, in Theorem 2.3 one can replace the hypothesis that $0 \in \varrho(A)$ by the weaker assumption that A has dense range. In fact, $A+I$ is invertible and $A+I \in BIP(X)$ by the result of Prüss-Sohr [17] mentioned above. Thus $A+B = A+I+(B-I) \in BIP(X)$ by Theorem 2.3. It is also possible to give a direct proof similar to the one above. □

We remark that Theorem 2.3 remains true if $B : D(A) \to X$ has the property that

$$\|BR(\lambda, A)\| \leqq c\frac{1}{|\lambda|^\beta} \quad (\lambda \in \mathbb{C}\setminus\Sigma_{\theta_1}\ ,\ |\lambda| \geqq 1) \tag{2.11}$$

or

$$\|R(\lambda, A)Bx\| \leqq c\frac{1}{|\lambda|^\beta}\|x\| \quad (\lambda \in \mathbb{C}\setminus\Sigma_{\theta_1}\ ,\ |\lambda| \geqq 1\ ,\ x \in D(A)) \tag{2.12}$$

for some $\beta > 0$. For example, if $-A$ generates an exponentially stable holomorphic C_0-semigroup and $BA^{-\alpha} \in \mathscr{L}(X)$ for $0 < \alpha < 1$, then (2.11) holds for $\beta = 1 - \alpha$. Finally, we mention that these results with the same proofs remain valid if we consider H^∞-functional calculus instead of property BIP.

For further different, but similar perturbation results we refer to Amann, Hieber and Simonett [2, Section 2].

3. *BIP* and forms. The discussion in Section 1 gave a satisfying answer to our question: which operators are associated with a closed form? However, we considered a fixed scalar product on the given Hilbert space. Now we will allow this scalar product also to vary. This will lead to a larger class of operators. At first we make precise what we mean by changing the scalar product. We assume throughout this section that H is a Hilbert space for some scalar product $(.|.)_H$. Another scalar product $(.|.)_1$ on H is called *equivalent* if the induced norm $\|x\|_1 = (x|x)_1^{1/2}$ is equivalent to the given norm $\|x\|_H = (x|x)_H^{1/2}$. Here is a different description, which is easy to prove.

Lemma 3.1. *Let $S \in \mathscr{L}(H)$ be invertible. Then*

$$(x \mid y)_1 := (Sx \mid Sy)_H \tag{3.1}$$

defines an equivalent scalar product on H. Moreover, each equivalent scalar product is of this form.

The notion of H-ellipticity, and hence of a closed form on H, is independent of the scalar product we choose on H. However, the associated operator does depend on it.

Let $a : V \times V \to \mathbb{C}$ be a closed form on H, where $V \underset{d}{\hookrightarrow} H$. Let $(.|.)_1$ be an equivalent scalar product on H. Then we may define the operator A on $(H, (.|.)_1)$ associated with the closed form a. In the following example we show that it may happen that for such an operator there exists no closed form with which A is associated if we consider the original scalar product. It will be convenient to use the following notion. Let A be an operator on H and $S \in \mathscr{L}(H)$ an isomorphism. We define the operator SAS^{-1} on H by letting

$$D(SAS^{-1}) := \{x \in H : S^{-1}x \in D(A)\}$$
$$(SAS^{-1})x = S(AS^{-1}x).$$

Two operators A and B are called *similar*, if there exists an isomorphism S such that $SAS^{-1} = B$. In view of Lemma 3.1 an operator A is associated with a closed form a on $(H, (.|.)_1)$ for some equivalent scalar product $(.|.)_1$ if and only if A is similar to an operator associated with a closed form b with respect to the original scalar product. This is easy to see.

E x a m p l e 3.2. Let A be an operator on H which is associated with a closed form. Assume that A has compact resolvent and $\dim H = \infty$. Consider the operator

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 2A \end{pmatrix}$$

on $H \times H$ with domain $D(A) \times D(A)$. Then also $\tilde{A}$ is associated with a closed form (as is easy to see). However, $\tilde{A}$ is similar to the operator

$$\tilde{B} = \begin{pmatrix} A & -4A \\ 0 & 2A \end{pmatrix}$$

on $H \times H$ with domain $D(\tilde{B}) = D(A) \times D(A)$ which is not quasi-accretive and hence not associated with a closed form.

Proof. Let $S = \begin{pmatrix} I & -4I \\ 0 & I \end{pmatrix}$. Then $S^{-1} = \begin{pmatrix} I & 4I \\ 0 & I \end{pmatrix}$ and $S\tilde{A}S^{-1} = \tilde{B}$. Assume that there exists $w \in \mathbb{R}$ such that $(w + \tilde{B})$ is accretive. Then in particular, for all $x \in D(A)$ one has,

$$\mathrm{Re}\,(-A\,x + 2wx \mid x)_H = \mathrm{Re}\,((\tilde{B} + w)\begin{pmatrix} x \\ x \end{pmatrix} \mid \begin{pmatrix} x \\ x \end{pmatrix})_{H \times H} \geqq 0.$$

Thus $A - 2w$ is dissipative. Since A has compact resolvent, there exists $\lambda > 0$ such that $\lambda + 2w \in \varrho(A)$. Thus $A - 2w$ is m-dissipative. Hence A generates a C_0-semigroup. Since $-A$ generates a holomorphic semigroup, this implies that A is bounded. Since A has compact resolvent we conclude that $\dim H < \infty$, contradicting our assumption. □

Now, using a remarkable characterization of $BIP(H)$ due to Le Merdy, we obtain the following characterization of those operators which are associated with a closed form after changing the scalar product on H.

Theorem 3.3. *Let A be an operator on H. Assume that $-A$ generates a holomorphic C_0-semigroup T. The following assertions are equivalent:*

(i) *There exists an equivalent scalar product $(.|.)_1$ and a closed form a on H such that A is associated with a on $(H, (.|.)_1)$.*
(ii) *There exists an equivalent scalar product $(.|.)_2$ on H such that A is quasi-m-accretive with respect to $(.|.)_2$.*
(iii) *There exists $w \in \mathbb{R}$ such that $A + w \in BIP(H)$.*

Proof. (i) $\Rightarrow$ (ii). This follows immediately from Theorem 2.1 and its proof.
(ii) $\Rightarrow$ (iii). Let $w \in \mathbb{R}$ such that $w + A$ is m-accretive with respect to $(.|.)_2$ and $0 \in \rho(A)$. Then $A + w$ has bounded imaginary powers by a well-known result (for a proof of this see [11, Thm. 4.5]).
(iii) $\Rightarrow$ (i). Replacing A by $A + w$ we can assume that $A \in BIP(H)$, $0 \in \rho(A)$ and that T is bounded on a sector (use Theorem 2.3). By [9, Theorem 1.1 and Corollary 4.7] there exists an equivalent scalar product $(.|.)_1$ such that T is contractive with respect to $(.|.)_1$ on a sector. Now (i) follows from Theorem 2.1 above. □

Next we give an example of an operator A such that $-A$ generates a holomorphic C_0-semigroup but A is not associated with a closed form whatever equivalent scalar product on H is chosen. Sectorial operators on Hilbert space with unbounded imaginary powers have first been constructed by Baillon-Clément [4], and independently by McIntosh-Yagi [13], and later by Venni [19]. We will use a recent simple example given by Le Merdy [10] which has the additional property that the resolvent is compact. As in [4] and [19], it is a diagonal operator with respect to some Schauder basis which is not unconditional. We refer to Duelli [5] for an account on properties of such diagonal operators and the associated semigroups. Let H be an infinite dimensional separable Hilbert space.

Theorem 3.4. *There exists a holomorphic C_0-semigroup T on H with generator $-A$ such that*

(a) *$T(t)$ is compact for all $t > 0$;*
(b) *A is not associated with a closed form on $(H, (.|.)_1)$ for any equivalent scalar product $(.|.)_1$ on H.*

Proof. Let $(e_j)_{j\in\mathbb{N}}$ be a Schauder basis of H which is not unconditional. Define $T(t)x = \sum_{j=1}^{\infty} e^{-2^j t} x_j e_j$ where $x = \sum_{j=1}^{\infty} x_j e_j$. Then by Venni [19, Theorem 3.2] (see also Duelli [5, 2.13]) T has a bounded, holomorphic extension to a sector. Denote by $-A$ the generator of T. Then A is invertible by [19, Lemma 2.4]. It is shown by Le Merdy [10] that $T(t)$ is compact $(t > 0)$ and that A is not similar to an accretive operator. Hence $A \notin BIP(H)$ by [9, Theorem 1.1]. Now (b) follows from Theorem 3.3 and Corollary 2.4. □

4. Liapunov's theorem. The purpose of this section is to prove the following theorem which extends Liapunov's classical result (see e.g. [7, Theorem 1, p. 145], or [3, 13.3]) to infinite dimension. Let H be a Hilbert space.

Theorem 4.1. *Let $-A$ be the generator of a holomorphic C_0-semigroup T on H such that $A + w \in BIP$ for some $w \in \mathbb{R}$. Assume that $\operatorname{Re}\lambda > 0$ for all $\lambda \in \sigma(A)$. Then there exist $\varepsilon > 0$ and a scalar product on H such that for the corresponding norm*

$$\|T(t)\| \leqq e^{-\varepsilon t} \quad (t \geqq 0)\ .$$

Proof. Since $\operatorname{Re}\lambda > 0$ for all $\lambda \in \sigma(A)$ and since the set $\{\mu \in \sigma(A) : \operatorname{Re}\mu \leqq 1\}$ is bounded, there exists $\varepsilon > 0$ such that $\operatorname{Re}\lambda \geqq 2\varepsilon$ for all $\lambda \in \sigma(A)$. It follows that $A - \varepsilon$ is sectorial. Thus $A - \varepsilon \in BIP(H)$ by Corollary 2.4. Now by Le Merdy's theorem [9, Theorem 1.1], there exists an equivalent scalar product on H such that

$$\|e^{-t(A-\varepsilon)}\| \leqq 1 \quad (t \geqq 0)\ . \quad \square$$

In a similar way we may extend the result on hyperbolic semigroups [3, (13.4)], [7, § 7.2] to the infinite dimensional case:

Theorem 4.2. *Let T be a holomorphic C_0-semigroup on H with generator $-A$. Assume that $A + w \in BIP(H)$ for some $w \in \mathbb{R}$.*

If T is hyperbolic (i.e., $i\mathbb{R} \subset \varrho(A)$), then $H = H_s \oplus H_u$ for closed subspaces H_s and H_u which are invariant under T and there exist $\varepsilon > 0$ and an equivalent scalar product on H such that for the corresponding norm

$$\|T(t)x\| \leqq e^{-\varepsilon t}\|x\| \quad (t \geqq 0) \text{ if } x \in H_s \tag{4.1}$$

and

$$\|T(t)x\| \geqq e^{\varepsilon t}\|x\| \quad (t \geqq 0) \text{ if } x \in H_u\ . \tag{4.2}$$

Proof. It is classical that H is the direct sum of invariant, closed subspaces H_s and H_u such that $\operatorname{Re}\lambda > 0$ for all $\lambda \in \sigma(A_s)$ and $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A_u)$ where $-A_s$ and $-A_u$ are the generators of $T_{|H_s}$ and $T_{|H_u}$, respectively. Moreover, A_u is a bounded operator. See e.g. [6, V. 1] for these results. Applying Theorem 4.1 to A_s and $-A_u$ we find $\varepsilon > 0$ and equivalent scalar products on H_s and H_u such that for the corresponding norms (4.1) holds and $\|T(-t)x\| \leqq e^{-\varepsilon t}\|x\|$ for $x \in H_u$. Hence $\|T(t)x\| = e^{\varepsilon t}e^{-\varepsilon t}\|T(t)x\| \geqq e^{\varepsilon t}\|T(-t)T(t)x\| = e^{\varepsilon t}\|x\|$ for $x \in H_u$, $t \geqq 0$. Defining the canonical direct-sum scalar product on H finishes the proof. □

We may also generalize a result by Vu and Yao [20, Theorem 9] for bounded generators. A C_0-semigroup is called *quasi-compact* if there exists $t_0 > 0$ such that $T(t_0) = K + N$ where K is compact and N has spectral radius $r(N) < 1$. See [14] and [20] for further information on this notion.

Theorem 4.3. *Let T be a holomorphic quasi-compact C_0-semigroup on a Hilbert space H with generator $-A$. Assume that $A + w \in BIP(H)$ for some $w \in \mathbb{R}$. If $\sup_{t \geqq 0}\|T(t)\| < \infty$, then there exists an equivalent scalar product on H such that $\|T(t)\| \leqq 1 \quad (t \geqq 0)$ for the corresponding norm.*

P r o o f. The proof of [20, Theorem 9] carries over to the situation considered here if we replace [20, Proposition 7] (valid for bounded generators) by our Theorem 4.1. □

5. Asymptotics stability for semilinear equations. Theorem 4.1 can be applied to prove a principle of linearized stability. We consider a simple case occuring frequently in applications. Let $-A$ be the generator of a holomorphic C_0-semigroup on a Hilbert space H. We assume that $A + w \in BIP(H)$ for some $w \in \mathbb{R}$. Let $f : H \to H$ be a locally Lipschitz continuous function; i.e., for each $r > 0$ there exists $L_r \geqq 0$ such that

$$\|f(x) - f(y)\| \leqq L_r\|x - y\| \text{ whenever } \|x\| \leqq r\ ,\ \|y\| \leqq r\ .$$

We consider the semilinear problem

$$\begin{cases} \dot{u}(t) + Au(t) = f(u(t)) \\ \qquad\quad u(0) = x\ . \end{cases} \tag{5.1}$$

Then for each $x \in H$ there is a unique mild solution $u \in C([0, t^+(x)); H)$ where $0 < t^+(x) \leqq \infty$ and, if $t^+(x) < \infty$ then $\lim\limits_{t \to t^+(x)} \|u(t)\| = \infty$ (see [15, Theorem 6.1.4]). Since on Hilbert space one has maximal regularity for the inhomogeneous problem (see e.g. [16, II.8.9 p. 231]), we have $u \in W^{1,2}((0, \tau); H)$ for each $0 < \tau < t^+(x)$. In particular, $u(t) \in D(A)$ and (5.1) holds a.e. Moreover, $\dot{u} \in L^2((0, \tau); H)$ for each $0 < \tau < t^+(x)$.

The solution of (5.1) is *stationary* if and only if $x \in D(A)$ and $Ax = f(x)$. We now prove Liapunov's theorem on asymptotic stability of stationary solutions in infinite dimension, keeping the hypotheses made above.

Theorem 5.1. *Let $\underline{x} \in D(A)$ such that $A\underline{x} = f(\underline{x})$. Assume that f is differentiable at $\underline{x}$ and that* $\operatorname{Re}\lambda > 0$ *for all $\lambda \in \sigma(A - Df(\underline{x}))$. Then there exist $\varepsilon > 0$, $\delta > 0$ and a scalar product on H such that for the corresponding norm the following holds: If $\|x - \underline{x}\| \leqq \delta$, then $t^+(x) = \infty$ and*

$$\|u(t) - \underline{x}\| \leqq e^{-\varepsilon t}\|x - \underline{x}\| \quad (t \geqq 0)\ . \tag{5.2}$$

P r o o f. By Theorem 4.1 and Theorem 2.3 there exist $\varepsilon > 0$ and an equivalent scalar product $(.|.)$ on H such that $A - Df(\underline{x}) - 2\varepsilon$ is accretive; i.e.,

$$-\operatorname{Re}(Ay \mid y) + \operatorname{Re}(Df(\underline{x})y \mid y) \leqq -2\varepsilon(y \mid y) \quad (y \in D(A))\ . \tag{5.3}$$

Choose $\delta > 0$ such that

$$\|f(x) - f(\underline{x}) - Df(\underline{x})(x - \underline{x})\| \leqq \varepsilon \|x - \underline{x}\| \tag{5.4}$$

whenenver $\|x - \underline{x}\| \leqq \delta$. Let $\|x - \underline{x}\| < \delta$ and let u be the solution of (5.1). Let $v = v(t) := u(t) - \underline{x}$ for $0 < t < t^+(x)$. Then by (5.3) and (5.4),

$$\begin{aligned}\|v\| \cdot \frac{d}{dt}\|v\| &= \frac{1}{2}\frac{d}{dt}\|v\|^2 = \operatorname{Re}(\dot{v}(t) \mid (v(t)) \\ &= \operatorname{Re}(-Au + f(u) \mid v) \\ &= \operatorname{Re}(-Av + Df(\underline{x})v + f(u) - f(\underline{x}) - Df(\underline{x})(u - \underline{x}) \mid v) \\ &\leqq -2\varepsilon\|v\|^2 + \|f(u) - f(\underline{x}) - Df(\underline{x})(u - \underline{x})\| \, \|v\| \\ &\leqq -\varepsilon\|v\|^2 \; a.e., \text{ whenever } t < t^+(x) \text{ and } \|v(t)\| \leqq \delta.\end{aligned}$$

Thus $\|v(t)\|$ is decreasing, and so $\|v(t)\| \leqq \delta$ for all $t < t^+(x)$. This implies $t^+(x) = \infty$ and $\frac{d}{dt}\|v(t)\| \leqq -\varepsilon\|v(t)\|$ a.e. for $t \geqq 0$. Thus $\|v(t)\| \leqq \|v(0)\|e^{-\varepsilon t}$ for all $t \geqq 0$ and (5.2) is proved. □

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Anschrift der Autoren:

Wolfgang Arendt
Abteilung Angewandte Analysis
Universität Ulm
89069 Ulm
Germany
arendt@mathematik.uni-ulm.de

Shangquan Bu
Department of Mathematical Science
University of Tsinghua
100084 Beijing
China
and
Abteilung Angewandte Analysis,
Universität Ulm
89069 Ulm
Germany
bu@mathematik.uni-ulm.de

Markus Haase
Abteilung Angewandte Analysis
Universität Ulm
89069 Ulm
Germany
haase@mathematik.uni-ulm.de