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# Does diffusion determine the body?

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**Abstract.** Kac's famous question "Can one hear the shape of a drum?", is equivalent to asking whether two domains in $\mathbb{R}^N$ are congruent whenever there exists a unitary operator intertwining the corresponding Dirichlet Laplacians. In the present paper we show that the answer is positive if the intertwining operator is supposed to be an order isomorphism instead of being unitary. This assumption signifies that positive solutions of the heat equation are mapped onto positive solutions. So one may rephrase the result by saying that diffusion determines the domain. This holds not only for Dirichlet but also for Neumann and Robin boundary conditions. However, for mixed boundary conditions (namely, periodic on one and Dirichlet on the other part of the boundary) a counterexample is given.

## 0. Introduction

In his famous paper "Can one hear the shape of a drum" M. Kac asked the following question: Let $\Omega_1, \Omega_2$ be two bounded domains in $\mathbb{R}^N$ (satisfying a suitable regularity condition) and consider the Laplacian $A_j$ with Dirichlet boundary conditions on $L^2(\Omega_j)$ $(j = 1, 2)$. Assume that $A_1$ and $A_2$ have the same series of eigenvalues. Does it follow that $\Omega_1$ and $\Omega_2$ are congruent? This problem can be formulated in terms of the semigroups $(e^{tA_j})_{t \geqq 0}$ generated by $A_j$ on $L^2(\Omega_j)$ in the following way. Consider an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $L^2(\Omega_1)$ consisting of eigenfunctions of $A_1$ and an orthonormal basis $\{f_n : n \in \mathbb{N}\}$ of $L^2(\Omega_2)$ of eigenfunctions of $A_2$. Let $U: L^2(\Omega_1) \to L^2(\Omega_2)$ be the unitary operator for which $Ue_n = f_n$ $(n \in \mathbb{N})$. Then the assumption that the spectra coincide is equivalent to the relation

$$Ue^{tA_1} = e^{tA_2}U \quad (t \geqq 0). \tag{0.1}$$

Thus, an equivalent formulation of Kac's question is: Assume that $U: L^2(\Omega_1) \to L^2(\Omega_2)$ is a unitary operator satisfying (0.1). Does it follow that $\Omega_1$ and $\Omega_2$ are congruent? In 1992 it has been shown by Gordon, Webb and Wolpert [GWW] that Kac's question has a negative answer, in general.

The purpose of this paper is to investigate whether congruence of $\Omega_1$ and $\Omega_2$ can be deduced from different assumptions on $U$. More precisely, instead of considering a unitary operator, we assume that $U$ is an order isomorphism; i.e., $U: L^2(\Omega_1) \to L^2(\Omega_2)$ is linear, bijective and satisfies

$$f \geqq 0 \text{ a.e.} \Leftrightarrow Uf \geqq 0 \text{ a.e.}$$

We show under this assumption that $\Omega_1$ and $\Omega_2$ are necessarily congruent if (0.1) holds.

Recall that for $f \in L^2(\Omega_j)$, the function

$$t \mapsto e^{tA_j}f: \mathbb{R}_+ \to L^2(\Omega_j)$$

is a solution of the diffusion equation determined by $A_j$, $j = 1, 2$. Thus, assumption (0.1) signifies that $U$ maps positive solutions to positive solutions of the diffusion equation. Hence our result might be rephrased by saying that diffusion determines the domain.

In our result we do not need that the boundary is smooth. Indeed, it turns out that the precise condition on the open set needed for the result is "regularity in capacity", a very weak notion of regularity which we study in detail.

We also obtain positive results for Neumann and Robin boundary conditions. In fact, we even show that diffusion determines the body and the boundary condition if we restrict ourselves to the three types of conditions: Dirichlet, Neumann and Robin.

However, it is remarkable that the result is not true for arbitrary boundary conditions which define a symmetric realization of the Laplacian on $L^2(\Omega)$. We give a counterexample where periodic boundary conditions are imposed on one part of the boundary and Dirichlet (or Neumann) boundary conditions on another part.

A different version of the result is obtained by replacing $L^2$ by $L^p$ for $p \neq 2$, $1 < p < \infty$ and order isomorphisms by isometric isomorphisms. In all cases, we show that the intertwining order isomorphism (or isometric isomorphism on $L^p$, $p \neq 2$) is of the form $Uf = c \cdot f \circ \tau$ $(f \in L^p)$ where $\tau$ is an isometric mapping from one domain to the other and $c$ a constant.

Meanwhile there exists a very concrete counterexample to Kac's question. Chapman [Ch], following Bérard [Be], constructs an intertwining isomorphism $U$ from $L^2(\Omega_1)$ onto $L^2(\Omega_2)$ where $\Omega_1$ and $\Omega_2$ are non-congruent polygons. In fact, $\Omega_1$ and $\Omega_2$ are presented as union of seven congruent triangles (put together in different ways). The operator $U$ is relatively simple. It comes from mapping triangles to triangles, reflection in the triangles and superposition of such operations (see [Ch] for a detailed description). Our result shows that the superposition is essential to make such an intertwining operator possible for non-congruent domains.

There is an abundant literature on inverse spectral geometry, in particular for the non-euclidean theory. We refer to Protter's article [Pr] and the proceedings [AL] edited by Andersson and Lapidus. Also, it should be mentioned that positive results for a very different kind of intertwining operators (viz, unitary Fourier integral operators) were obtained by Zelditch [Z] in the framework of Riemannian manifolds. For other types of inverse problems we refer to the monograph by Isakov [Is].

The present paper is restricted to the Euclidean case with emphasis on the treatment of different boundary conditions obtained by quadratic form methods (cf. Davies [Dav2]). It is organized as follows: In Section 1 we present the main result for Dirichlet boundary conditions. Moreover, we give counterexamples which are illuminating in context of the proofs given later on. Arbitrary realizations of the Laplacian in $L^p$ are considered in Section 2. The main results are contained in Section 3 where special boundary conditions are considered. Finally, in Section 4 isometric isomorphisms on $L^p$, $p \neq 2$, are considered as intertwining operators. Some extensions of the results presented here are given in [Are2].

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## 1. A typical result and a counterexample

This section has introductory character. We present the main result for Dirichlet boundary conditions. Further boundary conditions and refinements are given along with the proofs in Section 3. We also produce a counterexample showing that not all boundary conditions are allowed. This example illustrates well the difficulties to be overcome in the proofs and we may refer to it later on. Let $\Omega_1, \Omega_2$ be two open subsets of $\mathbb{R}^N$ and let $1 \leqq p < \infty$. A mapping $\tau \colon \Omega_2 \to \Omega_1$ is called an *isometry* if it is of the form

$$\tau(y) = By + b \quad (y \in \Omega_2), \tag{1.1}$$

where $B$ is an orthogonal matrix and $b \in \mathbb{R}^N$. The sets $\Omega_1$ and $\Omega_2$ are called *congruent*, if there exists an isometry $\tau$ from $\Omega_1$ onto $\Omega_2$.

**Definition 1.1.** Let $1 \leqq p \leqq \infty$. A linear mapping $U \colon L^p(\Omega_1) \to L^p(\Omega_2)$ is called an *order isomorphism* if $U$ is bijective and

$$Uf \geqq 0 \quad \text{if and only if} \quad f \geqq 0 \tag{1.2}$$

for all $f \in L^p(\Omega_1)$. Here $f \geqq 0$ stands for $f(x) \geqq 0$ a.e.

Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $1 \leqq p < \infty$. The definition of the Laplacian with Dirichlet boundary conditions in $L^p(\Omega)$ is standard and will be recalled in Section 3. It is an operator generating a positive contraction semigroup on $L^p(\Omega)$. Throughout the paper the term *semigroup* stands for $C_0$-semigroup. A semigroup $T = T(t)_{t \geqq 0}$ on $L^p(\Omega)$ is called *positive* if $T(t) \geqq 0$ for all $t \geqq 0$ (i.e. $T(t)f \geqq 0$ if $f \geqq 0$).

The following regularity condition on $\Omega$ expresses in a weak form that $\Omega$ lies on one side of the boundary. We say that $\Omega$ is *regular in capacity* if for all $x \in \partial\Omega$, $r > 0$,

$$\operatorname{cap}\big(B(x,r) \backslash \Omega\big) > 0, \tag{1.3}$$

where $B(x,r) = \{y \in \mathbb{R}^N : |x - y| < r\}$ is the ball of center $x$ and radius $r$. If $\Omega$ is *topological regular* (i.e. $\overset{\circ}{\bar{\Omega}} = \Omega$) then $\Omega$ is also regular in capacity, see Section 3 for further explanations. Now we can formulate our main result for Dirichlet boundary conditions.

**Theorem 1.2.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open sets which are regular in capacity. Assume that one of the sets is connected. Denote by $A_j$ the Laplacian with Dirichlet boundary conditions on $L^p(\Omega_j)$, $j = 1, 2$, where $1 < p < \infty$. Then the following conditions are equivalent*:

(i) *$\Omega_1$ and $\Omega_2$ are congruent.*

(ii) *There exist $1 < p < \infty$ and an order isomorphism $U: L^p(\Omega_1) \to L^p(\Omega_2)$ such that*

$$Ue^{tA_1} = e^{tA_2}U \quad (t \geqq 0). \tag{1.4}$$

(iii) *There exist $1 < p < \infty, p \neq 2$ and an isometric isomorphism $U: L^p(\Omega_1) \to L^p(\Omega_2)$ such that* (1.3) *holds.*

*In that case, U is of the form*

$$Uf = c \cdot f \circ \tau \quad (f \in L^p(\Omega_1)) \tag{1.5}$$

*where $\tau$ is an isometry from $\Omega_2$ onto $\Omega_1$ and $c > 0$ in the case* (ii), *$c = 1$ or $c = -1$ in the case* (iii).

Next we show by an example that Theorem 1.2 is no longer true if we consider periodic or a mixture of periodic and Dirichlet boundary conditions.

We need some preparation. Let $H$ be a Hilbert space with scalar product $(\,|\,)$. By a *positive form* we mean a pair $(a, V)$ where $V$ is a Hilbert space, continuously injected into $H$, such that $V$ is dense in $H$, and $a: V \times V \to \mathbb{R}$ is bilinear and continuous satisfying

$$a(u,u) \geqq 0 \quad \text{for all } u \in V, \tag{1.6}$$

$$a(u,v) = a(v,u) \quad \text{for all } u, v \in V, \tag{1.7}$$

$$a(u,u) + \|u\|_H^2 \geqq \alpha \|u\|_V^2 \quad (u \in V), \tag{1.8}$$

for some constant $\alpha > 0$. The space $V$ is called the *domain* of the form $a$.

The *operator A on H associated with a* is defined by

$$D(A) = \{u \in V : \exists v \in H \text{ such that } a(u,\varphi) = -(v|\varphi) \text{ for all } \varphi \in V\}, \quad Au = v.$$

The operator $A$ is selfadjoint and generates a contraction semigroup $(e^{tA})_{t \geqq 0}$ on $H$. See [Dav1], [RS], [Fu], [BH] for example.

**Lemma 1.3.** *Let $a_j$ be a positive form on a Hilbert space $H_j$ with domain $V_j$, $j = 1, 2$. Let $U: H_1 \to H_2$ be unitary. The following are equivalent*:

(i) $Ue^{tA_1} = e^{tA_2}U$ $(t \geqq 0)$.

(ii) $UV_1 = V_2$ *and* $a_2(Ux, Uy) = a_1(x, y)$ *for all* $x, y \in V_1$.

*Proof.* (i) $\Rightarrow$ (ii). It is well-known (cf. [Fu], §1.3) and easy to see by the spectral theorem that

$$\lim_{t\downarrow 0}\left(\frac{1}{t}(x - e^{tA_j}x) \mid y\right) = a_j(x|y) \tag{1.9}$$

for all $x, y \in V_j$ whereas

$$\lim_{t\downarrow 0}\frac{1}{t}(x - e^{tA_j}x, x) = \infty \tag{1.10}$$

if $x \in H_j \setminus V_j$. Using this one immediately obtains (ii) from (i).

(ii) $\Rightarrow$ (i). It follows from the definition of the associated operator that

$$UD(A_1) = D(A_2) \quad \text{and} \quad UA_1x = A_2Ux \quad \text{for all } x \in D(A_1).$$

This implies that

$$UR(\lambda, A_1) = R(\lambda, A_2)U \quad \text{for all } \lambda \in \varrho(A_1) \cap \varrho(A_2).$$

Since $e^{tA_j}$ is the strong limit of $\left(I - \frac{t}{n}A_j\right)^{-n}$ as $n \to \infty$, assertion (i) follows. □

We first give a 1-dimensional example which describes in a simple way what can happen.

**Example 1.4.** Let $\Omega = (-1, 1)$ and let $\tau$: $(-1, 1) \to (-1, 1)$ be given by

$$\tau(y) = \begin{cases} 1 - y & \text{if } y > 0, \\ -1 - y & \text{if } y < 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Then $\tau$ is isometric on $(-1, 0)$ and $(0, 1)$ but not on $(-1, 1)$. The operator

$$U: L^2(-1, 1) \to L^2(-1, 1)$$

given by $Uf = f \circ \tau$ is a unitary order isomorphism.

Consider the positive form $a$ on $L^2(-1, 1)$ given by

$$a(u, v) = \int_{-1}^{1} u'v'$$

with domain

$$V := \{u \in W^{1,2}(-1, 1): u(-1) = u(1)\}.$$

Let $A$ be the associated operator. Then it follows easily from Lemma 1.3 that

$$Ue^{tA} = e^{tA}U \quad (t \geqq 0).$$

But $U$ is not given as composition by an isometry. □

**Remark 1.5.** It is easy to see that

$$D(A) = \{u \in W^{2,2}(-1,1)\colon u(1) = u(-1),\ u'(1) = u'(-1)\}, \quad Au = u''.$$

Next we modify the example in order to produce non-congruent domains.

**Example 1.6.** Let $h \in C^1[-1,1]$ such that $h(-1) = h(1) = h(0) = 1$ and $h(x) \geqq 1$ for all $x \in [-1,1]$. Let

$$\Omega_2 = \{(x,y)\colon -1 < x < 1,\ 0 < y < h(x)\}.$$

Define $\tau\colon \Omega_2 \to \mathbb{R}^2$ by

$$\tau(x,y) = \begin{cases} (1-x, y) & \text{if } x > 0, \\ (0, y) & \text{if } x = 0, \\ (-1-x, y) & \text{if } x < 0, \end{cases}$$

and let $\Omega_1 = \tau(\Omega_2)$. We may choose $h$ in such a way that $\Omega_1$ and $\Omega_2$ are not congruent.

Denote by $W^{1,2}(\Omega_j) = \{f \in L^2(\Omega_j)\colon D_x f \in L^2(\Omega_j), D_y f \in L^2(\Omega_j)\}$ the first Sobolev space, where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$.

Define the positive form $a_j$ on $L^2(\Omega_j)$ by

$$a_j(u,v) = \int_{\Omega_j} \nabla u \nabla v \, dx$$

with domain

$$\begin{aligned} V_j = \big\{u \in W^{1,2}(\Omega_j)\colon (Tu)(-1,y) = (Tu)(1,y) \text{ if } 0 < y < 1, \\ (Tu)\big(x, h(x)\big) = 0 \text{ if } x \in (-1,1) \text{ and } (Tu)(x,0) = 0 \text{ if } x \in (-1,1)\big\} \end{aligned}$$

where $T\colon W^{1,2}(\Omega_j) \to L^2(\partial\Omega_j, \mathscr{H}^1)$ denotes the trace operator [EG], p. 133. Let $A_j$ be the operator associated with $a_j$. It follows from the definition that $\mathscr{D}(\Omega_j) \subset D(A_j)$ and $A_j u = \Delta u$ for all $u \in D(A_j)$ $(j = 1,2)$. Define the operator $U\colon L^2(\Omega_1) \to L^2(\Omega_2)$ by $Uf = f \circ \tau$. Then $U$ is unitary and an order isomorphism. Using integration by parts [EG], 4.3, Theorem 1 (ii), p. 133, one sees that

$$f \circ \tau \in V_2 \quad \text{if } f \in V_1 \tag{1.11}$$

and

$$D_x(f \circ \tau) = -(D_x f) \circ \tau, \quad D_y(f \circ \tau) = D_y f \circ \tau. \tag{1.12}$$

This implies that $UV_1 = V_2$ and

$$a_2(Uf, Ug) = a_1(f, g) \quad \text{for all } f, g \in V_1.$$

So by Lemma 1.3 we deduce that

$$e^{tA_2}U = Ue^{tA_1} \quad (t \geqq 0). \quad \square$$

Of course, in Example 1.6 we may glue together the boundaries $\{(-1, y): 0 < y < 1\}$ and $\{(1, y): 0 < y < 1\}$ and obtain manifolds with boundaries which are isomorphic.

A different example is obtained by identifying some sides of rectangles:

**Example 1.7.** The Laplacian defined on a rectangle and an $L$-shaped domain, with Neumann (or Dirichtlet) boundary conditions on part of the boundary and identifying boundary conditions on the other part, are intertwined by a unitary order isomorphism.

a) More precisely, let $N = 2$, $\Omega_1 = (0,3) \times (0,1)$, $\Omega_2 = \Omega_2' \cup (1,2) \times \{1\}$ with

$$\Omega_2' = (0,2) \times (0,1) \cup (1,2) \times (1,2).$$

Then $\Omega_1$ and $\Omega_2$ are connected open sets which are not congruent. Let

$$\Omega_1' = \big((0,2) \times (0,1)\big) \cup \big((2,3) \times (0,1)\big).$$

Thus $L^2(\Omega_j) = L^2(\Omega_j')$ $(j = 1,2)$. Define the bijective mapping $\tau: \Omega_2' \to \Omega_1'$ by

$$\tau(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in (0,2) \times (0,1), \\ (x+1, y-1) & \text{if } (x, y) \in (1,2) \times (1,2). \end{cases}$$

Then $\tau$ is isometric on each component of $\Omega_2'$. The mapping $Uf = f \circ \tau$ a.e. defines a unitary order isomorphism $U$ from $L^2(\Omega_1)$ onto $L^2(\Omega_2)$.

Define the closed form $(a_j, V_j)$ on $L^2(\Omega_j)$ by

$$a_j(u, v) = \int_{\Omega_j} \nabla u \nabla v \quad (u, v \in V_j,\ j = 1,2)$$

with

$$V_1 = \{u \in W^{1,2}(\Omega_1): (Tf)(x,1) = (Tf)(x+1,0) \text{ for all } x \in (1,2)\},$$

$$V_2 = \{u \in W^{1,2}(\Omega_2): (Tf)(1,y) = (Tf)(2,y-1) \text{ for all } y \in (1,2)\},$$

where $T: W^{1,2}(\Omega_j) \to L^2(\partial\Omega_j)$ $(j = 1,2)$ is the trace operator. Let $A_j$ be the operator associated with $a_j$ $(j = 1,2)$. Then it follows from Lemma 1.3 that

$$Ue^{tA_1} = e^{tA_2}U \quad (t \geqq 0).$$

In fact, using integration by parts [EG], p. 133 one sees that $UV_1 = V_2$ and $D_x Uf = UD_x f$, $D_y Uf = UD_y f$ for all $f \in V_1$, where $D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$. This implies that

$$a_2(Uf, Ug) = a_1(f, g) \quad \text{for all } f, g \in V_1.$$

b) Similarly, one could choose Dirichlet boundary conditions (instead of Neumann) on all sides of $\Omega_j$ besides the sides $(1,2) \times \{1\}$ and $(2,3) \times \{0\}$ which are identified in the case of $\Omega_1$ and the sides $\{1\} \times (1,2)$ and $\{2\} \times (1,2)$ which are identified in the case of $\Omega_2$.

## 2. General realizations of the Laplacian

In this section we characterize order isomorphisms intertwining arbitrary realizations of the Laplacian. We will show that they are given by composition of "local isometries". The results obtained here are the first step for the main result in Section 3 where special boundary conditions are considered. Only for some, in fact the most familiar, boundary conditions, "global" isometries are obtained.

Let $\Omega \subset \mathbb{R}^N$ be a bounded and open set. By $\mathscr{D}(\Omega)$ we denote the space of all test functions and by $\mathscr{D}(\Omega)'$ the space of all distributions. As usual we identify $L^1_{\mathrm{loc}}(\Omega)$ with a subspace of $\mathscr{D}(\Omega)'$. In particular, for $f \in L^1_{\mathrm{loc}}(\Omega)$ the Laplacian $\Delta f$ of $f$ is always defined as an element of $\mathscr{D}(\Omega)'$.

**Definition 2.1.** Let $1 < p < \infty$. An operator $A$ on $L^p(\Omega)$ is called a *realization of the Laplacian* in $L^p(\Omega)$ if

(a) $\mathscr{D}(\Omega) \subset D(A)$ and

(b) $Af = \Delta f$ in $\mathscr{D}(\Omega)'$ for all $f \in D(A)$.

It should be emphasized that the definition above does depend on the open set $\Omega$ and not only on the space $L^p(\Omega)$. We make this more precise.

If $F \subset \mathbb{R}^N$ is a Lebesgue measurable set we denote by $|F|$ its Lebesgue measure. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open sets. We consider $L^p(\Omega_j)$ as a subspace of $L^p(\mathbb{R}^N)$ extending functions defined on $\Omega_j$ by 0 to $\mathbb{R}^N \backslash \Omega_j$. Then

$$L^p(\Omega_1) = L^p(\Omega_2) \quad \text{if and only if} \quad |\Omega_1 \triangle \Omega_2| = 0, \tag{2.1}$$

where $\Omega_1 \triangle \Omega_2 = (\Omega_1 \backslash \Omega_2) \cup (\Omega_2 \backslash \Omega_1)$ denotes the symmetric difference. Now assume that $|\Omega_1 \triangle \Omega_2| = 0$. Let $A$ be a realization of the Laplacian in $L^p(\Omega_1)$. Then $A$ might not be a realization of the Laplacian in $L^p(\Omega_2)$. However, it is, if $\Omega_2 \subset \Omega_1$.

Let $A_j$ be an operator on a Banach space $E_j$, $j = 1, 2$ and let $U: E_1 \to E_2$ be linear, continuous and invertible. We say that $U$ *intertwines* $A_1$ and $A_2$ if

$$UD(A_1) = D(A_2) \quad \text{and} \quad A_2 Ux = UA_1 x \tag{2.2}$$

for all $x \in D(A_1)$. Note that this is equivalent to

$$(2.3) \qquad Ue^{tA_1} = e^{tA_2}U \quad (t \geqq 0)$$

whenever $A_1$ and $A_2$ generate semigroups.

The purpose of this section is to prove the following:

**Theorem 2.2.** *Let $\Omega_1, \Omega_2$ be two open subsets of $\mathbb{R}^N$, $1 < p < \infty$. Let $A_j$ be a realization of the Laplacian in $L^p(\Omega_j)$ $(j = 1,2)$. Let $U$: $L^p(\Omega_1) \to L^p(\Omega_2)$ be an order isomorphism intertwining $A_1$ and $A_2$. Then there exist open sets $\Omega_j' \subset \Omega_j$ satisfying $|\Omega_j \backslash \Omega_j'| = 0$ $(j = 1,2)$, a homeomorphism $\tau$ from $\Omega_2'$ onto $\Omega_1'$ which is isometric on each component of $\Omega_2'$ and a mapping $h$: $\Omega_2' \to (0,\infty)$ which is constant on each component of $\Omega_2'$ such that*

$$(2.4) \qquad (Uf)(y) = \begin{cases} h(y)f\big(\tau(y)\big) & (y \in \Omega_2'), \\ 0 & (y \in \Omega_2 \backslash \Omega_2') \end{cases}$$

*for all $f \in \mathscr{D}(\Omega_1)$.*

Examples 1.4, 1.5 and 1.6 show that $\Omega_j'$ does not need to be connected even if $\Omega_j$ is.

For the proof of Theorem 2.2 we need several auxiliar results.

**Proposition 2.3.** *Let $\Omega \subset \mathbb{R}^N$ be open and connected and let $\tau \in C^1(\Omega, \mathbb{R}^N)$ be a mapping such that $(D\tau)(x)$ is orthogonal for all $x \in \Omega$. Then $\tau$ is an isometry.*

*Proof.* a) Let $B \subset \Omega$ be a ball, $x, y \in B$. Then

$$\begin{aligned} |\tau(x) - \tau(y)| &= \left| \int_0^1 \frac{d}{dt} \tau\big(x + t(y-x)\big)\, dt \right| \\ &= \left| \int_0^1 D\tau\big(x + t(y-x)\big)(y-x)\, dt \right| \\ &\leqq |y - x|. \end{aligned}$$

b) Applying a) locally to $\tau^{-1}$ we conclude that for each $a \in \Omega$ there exists $r > 0$ such that $B(a,r) \subset \Omega$ and $|\tau(x) - \tau(y)| = |x - y|$ for all $x, y \in B(a,r)$.

c) Let $B(a,r) \subset \Omega$ such that $|\tau(x) - \tau(y)| = |x - y|$ for all $x, y \in B(a,r)$. We show that there exists $0 < \varepsilon < r$ such that $\tau$ is an isometry on $B(a,\varepsilon)$. We can assume $a = 0$. Replacing $\tau$ by $\tau - \tau(0)$ we can assume that $\tau(0) = 0$. Since

$$(x|y) = \frac{1}{2}(|x|^2 + |y|^2 - |x-y|^2)$$

we have $\big(\tau(x)\,|\,\tau(y)\big) = (x|y)$ for all $x, y \in B(a,r)$. This implies that

$$|\tau(x+y) - \tau(x) - \tau(y)|^2 = 0$$

whenever $x, y, x + y \in B(0, r)$ and

$$\tau(\lambda x) = \lambda \tau(x) \quad \text{if } x, \lambda x \in B(0, r), \lambda \in \mathbb{R}.$$

Thus $\tau$ is linear in a neighborhood of 0.

d) We have shown that $\tau$ is locally an isometry, i.e. $(D\tau)(x)$ is locally constant and so constant since $\Omega$ is connected. Let $B = (D\tau)(x)$. Let $\Psi(x) = \tau(x) - Bx$. Then $(D\Psi)(x) = 0$ for all $x \in \Omega$. Thus $\Psi$ is constant. □

**Proposition 2.4.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open sets and let $U: \mathscr{D}(\Omega_1) \to C^\infty(\Omega_2)$ be a linear mapping satisfying*

(2.5) $$f \geqq 0 \quad \textit{implies} \quad Uf \geqq 0;$$

(2.6) $$f \cdot g = 0 \quad \textit{implies} \quad (Uf) \cdot (Ug) = 0;$$

(2.7) $$\Delta Uf = U\Delta f \quad (f \in \mathscr{D}(\Omega_1)).$$

*Then there exists an open set $\Omega_2' \subset \Omega_2$, a mapping $\tau: \Omega_2' \to \Omega_1$ which is isometric on each component of $\Omega_2'$, a function $h: \Omega_2' \to (0, \infty)$ which is constant on each component of $\Omega_2'$ such that for all $f \in \mathscr{D}(\Omega_1)$,*

(2.8) $$(Uf)(y) = \begin{cases} h(y) f\big(\tau(y)\big) & (y \in \Omega_2'), \\ 0 & (y \in \Omega_2 \backslash \Omega_2'). \end{cases}$$

*Proof.* a) The set $\Omega_2' := \{y \in \Omega_2 \colon \exists f \in \mathscr{D}(\Omega_1), (Uf)(y) \neq 0\}$ is open. For $y \in \Omega_2'$ define the linear form $S_y$ on $\mathscr{D}(\Omega_1)$ given by $S_y(f) = (Uf)(y)$. Since $S_y$ is positive, there exists a Borel measure $\mu_y$ such that $S_y(f) = \int_\Omega f \, d\mu_y$ (see [DL], p. 567). It follows from (2.6) that the support of $\mu_y$ is a singleton. Thus, there exist $\tau(y) \in \Omega_1$, $h(y) > 0$ such that $S_y(f) = h(y) f\big(\tau(y)\big)$. We have shown that $U$ is of the form (2.8).

b) We show that $\tau$ is continuous. If not, we find $y, y_n \in \Omega_2'$, $\varepsilon > 0$ such that $\lim y_n = y$ but $|\tau(y_n) - \tau(y)| \geqq \varepsilon$ $(n \in \mathbb{N})$. Let $f \in \mathscr{D}(\Omega_1)$ such that $f\big(\tau(y)\big) = 1$ but $f\big(\tau(y_n)\big) = 0$ $(n \in \mathbb{N})$. Then $Uf$ is not continuous, a contradiction.

b) We show that $h \in C^\infty(\Omega_2')$. Let $\omega \subset \Omega_2'$ be open, bounded such that $\bar{\omega} \subset \Omega_2'$. Then $\tau(\bar{\omega})$ is compact. Choose $f \in \mathscr{D}(\Omega_1)$ such that $f = 1$ on $\tau(\bar{\omega})$. Then $Uf = h$ on $\omega$. Thus $h \in C^\infty(\omega)$.

c) Let $j \in \{1, \ldots, N\}$. Choose $f \in \mathscr{D}(\Omega_1)$ such that $f(x) = x_j$ on $\tau(\bar{\omega})$. Then $(Uf)(y) = h(y)\tau_j(y)$ for $y \in \omega$. Thus $\tau_j \in C^\infty(\Omega_2')$ where $\tau = (\tau_1, \ldots, \tau_N)$.

d) It follows from (2.7) that

(2.9) $$\Delta(h \cdot f \circ \tau) = h \cdot (\Delta f) \circ \tau \quad \text{on } \Omega_2'$$

for all $f \in \mathscr{D}(\Omega_1)$. Choose $f \in \mathscr{D}(\Omega_1)$ such that $f = 1$ on $\tau(\bar{\omega})$. Then it follows from (2.9) that $\Delta h = 0$ on $\omega$. Since $\omega \subset \Omega_2'$ is an arbitrary open relatively compact subset of $\Omega_2'$, we conclude that

$$(2.10) \qquad (\Delta h)(y) = 0 \quad \text{for all } y \in \Omega_2'.$$

Since for $f, g \in C^2(\Omega_2')$,

$$\Delta(f \cdot g) = (\Delta f) \cdot g + 2\nabla f \cdot \nabla g + f \Delta g,$$

we deduce from (2.9) and (2.10),

$$(2.11) \qquad 2\nabla h \cdot \nabla(f \circ \tau) + h\Delta(f \circ \tau) = h(\Delta f) \circ \tau \quad \text{on } \Omega_2'$$

for all $f \in \mathscr{D}(\Omega_1)$.

Let $j \in \{1, \dots, N\}$ and let $f \in \mathscr{D}(\Omega_1)$ such that $f(x) = x_j$ on $\omega$ (where $\omega$ is chosen as in b)). Then $f \circ \tau = \tau_j$ on $\omega$. We deduce from (2.11) that

$$(2.12) \qquad 2\nabla h \cdot \nabla \tau_j + h\Delta \tau_j = 0 \quad \text{on } \Omega_2' \quad \text{for } j = 1, \dots, N.$$

e) For $f \in \mathscr{D}(\Omega_1)$ we compute on $\Omega_2'$,

$$\begin{aligned}
D_j(f \circ \tau) &= \sum_m (D_m f) \circ \tau \cdot D_j \tau_m, \\
D_j^2(f \circ \tau) &= \sum_m \sum_k (D_k D_m f) \circ \tau \cdot D_j \tau_k \cdot D_j \tau_m \\
&\quad + \sum_m (D_m f) \circ \tau \cdot D_j^2 \tau_m, \\
\Delta(f \circ \tau) &= \sum_m \sum_k (D_k D_m f) \circ \tau \cdot \nabla \tau_k \nabla \tau_m \\
&\quad + \sum_m (D_m f) \circ \tau \cdot \Delta \tau_m.
\end{aligned}$$

Thus, in virtue of (2.12) the left hand side of (2.11) becomes

$$\begin{aligned}
&2\sum_j D_j h \sum_m (D_m f) \circ \tau \cdot D_j \tau_m + h \cdot \Delta(f \circ \tau) \\
&\quad = 2\sum_m (D_m f) \circ \tau \cdot \nabla h \nabla \tau_m + h \cdot \sum_m (D_m f) \circ \tau \cdot \Delta \tau_m \\
&\qquad + h \cdot \sum_m \sum_k (D_k D_m f) \circ \tau \cdot \nabla \tau_k \nabla \tau_m \\
&\quad = h \cdot \sum_m \sum_k (D_k D_m f) \circ \tau \cdot \nabla \tau_k \nabla \tau_m.
\end{aligned}$$

Hence (2.11) yields

$$\sum_m \sum_k (D_k D_m f) \circ \tau \cdot \nabla\tau_k \nabla\tau_m = (\Delta f) \circ \tau \quad \text{on } \Omega_2' \tag{2.13}$$

for all $f \in \mathscr{D}(\Omega_1)$. Choosing $f \in \mathscr{D}(\Omega_1)$ such that $f(y) = \frac{1}{2} y_i^2$ on $\tau(\omega)$, we deduce from (2.13)

$$\nabla\tau_i \cdot \nabla\tau_i = 1 \quad (i = 1, \dots, N).$$

Choosing $f \in \mathscr{D}(\Omega_1)$ such that $f(y) = y_i y_j$, where $i \neq j$, we obtain from (2.13) that

$$\nabla\tau_i \cdot \nabla\tau_j = 0 \quad (i \neq j).$$

We have shown that $(D\tau)(y)$ is orthogonal for all $y \in \Omega_2'$. Thus, by Proposition 2.3, $\tau$ is an isometry on each component of $\Omega_2'$.

In particular, $\Delta\tau_j = 0$ $(j = 1, \dots, N)$. Thus by (2.12), $\nabla h \cdot \nabla\tau_j = 0$ on $\Omega_2'$, $j = 1, \dots, N$. Since $(D\tau)(y)$ is orthogonal for all $y \in \Omega_2'$, it follows that $\nabla h = 0$ on $\Omega_2'$. This implies that $h$ is constant on each component of $\Omega_2'$. □

**Remark 2.5.** a) Consider the situation described in Proposition 2.4. Let $\omega \subset \Omega_2'$ be a component. Denote by $\bar\tau$ the isometric extension of $\tau_{|_\omega}$. Then

$$\bar\tau(\partial\omega \cap \Omega_2) \subset \partial\Omega_1. \tag{2.14}$$

In fact, let $y_0 \in \partial\omega \cap \Omega_2$. Assume that $\bar\tau(y_0) \in \Omega_1$. Let $f \in \mathscr{D}(\Omega_1)$ be equal to 1 in a neighborhood of $\bar\tau(y_0)$. Let $y_n \in \omega$ such that $\lim_{n\to\infty} y_n = y_0$. Then $\lim_{n\to\infty} \tau(y_n) = \bar\tau(y_0)$. Note that $h$ is equal to a constant $c > 0$ on $\omega$. It follows from (2.8) that $(Uf)(y_n) = c > 0$ for $n$ sufficiently large. On the other hand, by (2.8), $(Uf)(y_0) = 0$ since $y_0 \in \Omega_2 \backslash \Omega_2'$. This contradicts the continuity of $Uf$.

b) Conversely, let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open, $\Omega_2' \subset \Omega_2$ open, $\tau\colon \Omega_2' \to \Omega_1$ isometric on each component of $\Omega_2'$ satisfying (2.14). Let $h\colon \Omega_2' \to (0, \infty)$ be constant on each component of $\Omega_2'$. Then (2.8) defines a linear mapping $U\colon \mathscr{D}(\Omega_1) \to C^\infty(\Omega_2)$ satisfying (2.5), (2.6) and (2.7).

*Proof.* Let $f \in \mathscr{D}(\Omega_1)$. It is clear that $g = Uf$ is of class $C^\infty$ on each component of $\Omega_2'$. Let $y_0 \in \Omega_2 \backslash \Omega_2'$. Choose $0 < \varepsilon < \operatorname{dist}(\operatorname{supp} f, \partial\Omega_1)$ such that $B(y_0, \varepsilon) \subset \Omega_2$. We show that $g = 0$ on $B(y_0, \varepsilon)$. In fact, let $y \in B(y_0, \varepsilon) \cap \Omega_2'$. Let $\omega$ be the component of $\Omega_2'$ such that $y \in \omega$. Let $t_1 = \inf\{t \in [0, 1]\colon y_0 + t(y - y_0) \in \omega\}$, $y_1 = y_0 + t_1(y - y_0)$. Then

$$y_1 \in \partial\omega \cap B(y_0, \varepsilon) \subset \partial\omega \cap \Omega_2.$$

Let $\bar\tau$ be the isometric extension of $\tau_{|_\omega}$. Then by (2.14), $\bar\tau(y_1) \in \partial\Omega_1$. Hence

$$\operatorname{dist}\big(\tau(y), \partial\Omega_1\big) \leqq |\tau(y) - \bar\tau(y_1)| = |y - y_1| < \varepsilon.$$

Hence $g(y) = h(y) f\big(\tau(y)\big) = 0$. We have shown that $g \in C^\infty(\Omega_2)$. The other properties are clear. □

c) In Proposition 2.4 it might happen that $\Omega_1$ and $\Omega_2$ are connected, but $\Omega_2'$ has infinite many components even if $U$ extends to an order isomorphism on $L^p$. To give an example, let $\Omega'$ be the union of all cubes

$$Q_n = \{(x, y) \in \mathbb{R}^2 \colon 2^{-n} < x, y < 2^{-n+1}\},$$

$n \in \mathbb{N}$. Define $\tau \colon \Omega' \to \Omega'$ as a rotation of angle $\pi/2$ on $Q_n$ if $n$ is even, and a rotation on $Q_n$ of angle $-\pi/2$ if $n$ is odd. Then (2.14) holds (with $\Omega_1 = \Omega_2 = \Omega'$) and

$$Uf(y) = \begin{cases} f(\tau(y)) & (y \in \Omega'), \\ 0 & (y \in \Omega \backslash \Omega') \end{cases}$$

defines a mapping from $\mathscr{D}(\Omega)$ into $C^\infty(\Omega)$ satisfying (2.5), (2.6) and (2.7). □

We need some regularity results for the Laplacian. Let $\Omega \subset \mathbb{R}^N$ be open, $1 < p < \infty$. We consider the Sobolev spaces $W^{k,p}(\Omega)$ and $W^{k,p}_{\text{loc}}(\Omega)$ $(k \in \mathbb{N} \cup \{0\})$. The following regularity result is well-known (see [DL], §3, Prop. 8, [Ru], Theorem 8.12 for $p = 2$, for general $p$ it follows from [GT], Theorem 9.11, p. 235).

**Proposition 2.6.** (a) $\bigcap_{k \in \mathbb{N}} W^{k,p}_{\text{loc}}(\Omega) = C^\infty(\Omega)$.

(b) *Let* $u, f \in L^p_{\text{loc}}(\Omega)$. *Assume that* $\Delta u = f$ *in* $\mathscr{D}(\Omega)'$. *If* $f \in W^{k,p}_{\text{loc}}(\Omega)$, *then*

$$u \in W^{k+2,p}_{\text{loc}}(\Omega).$$

From this we deduce immediately:

**Lemma 2.7.** *Let $A$ be a realization of the Laplacian in $L^p(\Omega)$ where $1 < p < \infty$. Then* $\bigcap_{k \in \mathbb{N}} D(A^k) \subset C^\infty(\Omega)$.

*Proof.* By Proposition 2.6 (a), it suffices to show that $D(A^k) \subset W^{2k,p}_{\text{loc}}(\Omega)$. This is trivial for $k = 0$. Assuming it for $k \in \mathbb{N}_0$, let $f \in D(A^{k+1})$. Then

$$\Delta f = Af \in D(A^k) \subset W^{2k,p}_{\text{loc}}(\Omega)$$

by the inductive hypothesis. Hence $f \in W^{2k+2,p}_{\text{loc}}(\Omega)$ by Proposition 2.6 (b). □

*Proof of Theorem* 2.2. a) Since $U$ intertwines $A_1$ and $A_2$, it follows that

$$UD(A_1^k) = D(A_2^k) \quad (k \in \mathbb{N}). \tag{2.15}$$

Since $A_1$ is a realization of the Laplacian in $L^p(\Omega_1)$, we have $\mathscr{D}(\Omega_1) \subset \bigcap_{k \in \mathbb{N}} D(A_1^k)$. Hence by (2.15), $U\mathscr{D}(\Omega_1) \subset \bigcap_{k \in \mathbb{N}} D(A_2^k)$. It follows from Lemma 2.7 that

$$UD(\Omega_1) \subset C^\infty(\Omega_2). \tag{2.16}$$

Since $U$ is an order isomorphism, we have $U(f \wedge g) = Uf \wedge Ug$ and $|Uf| = U|f|$ for all $f, g \in L^p(\Omega_1)$, where $(f \wedge g)(x) = \inf\{f(x), g(x)\}$ a.e. In particular, $f \cdot g = 0$ implies $|f| \wedge |g| = 0$, and hence $|Uf| \wedge |Ug| = U|f| \wedge U|g| = 0$. Thus, the restriction of $U$ to $\mathscr{D}(\Omega_1)$

satisfies assumption (2.5), (2.6) and (2.7) of Proposition 2.4. So we find $\Omega_2'$, $h$, $\tau$ as in Proposition 2.4 such that (2.8) holds. Since $U$ is surjective and $\mathscr{D}(\Omega_1)$ dense in $L^p(\Omega_1)$, it follows that $|\Omega_2 \backslash \Omega_2'| = 0$.

Next we show that $\tau$ is injective. Let $\omega_1$ and $\omega_2$ be two different components of $\Omega_2'$. It suffices to show that $\tau(\omega_1) \cap \tau(\omega_2) = \emptyset$. Assume that $\omega := \tau(\omega_1) \cap \tau(\omega_2) \neq \emptyset$. Then $\omega$ is an open subset of $\Omega_1$. Let $\tau_j$ be the isometry which coincides with $\tau_{|_{\omega_j}}$, $\omega_j' = \tau_j^{-1}(\omega)$, $j = 1, 2$. Then $\omega_1'$, $\omega_2'$ are non-empty open subsets of $\Omega_2'$ such that $\omega_1' \cap \omega_2' = \emptyset$ and $\omega_j' \subset \omega_j$ $(j = 1, 2)$. Let $h = c_1 > 0$ on $\omega_1'$ and $h = c_2 > 0$ on $\omega_2'$. Let $g = Uf$ with $f \in \mathscr{D}(\Omega_1)$. Then for $y \in \omega_1'$,

$$g(y) = c_1 f\big(\tau_1(y)\big) = \frac{c_1}{c_2} c_2 f\big(\tau_2 \tau_2^{-1} \tau_1(y)\big) = \frac{c_1}{c_2} g\big(\tau_2^{-1} \tau_1(y)\big).$$

Hence $g(y) = \dfrac{c_1}{c_2} g\big(\tau_2^{-1}\tau_1(y)\big)$ for all $y \in \omega_2'$ and all $g$ in the image of $U$. This is impossible, since $U\mathscr{D}(\Omega_1)$ is dense in $L^p(\Omega_2)$.

Let $\Omega_1' = \tau(\Omega_2')$. Then $\Omega_1'$ is open and $\tau$ is a homeomorphism of $\Omega_2'$ onto $\Omega_1'$. It is clear that $\tau$ is measure preserving. Thus, it follows from (2.8) that

$$(Uf)(y) = h(y) f\big(\tau(y)\big) \quad \text{a.e.} \tag{2.17}$$

on $\Omega_2$ for all $f \in L^p(\Omega_1)$. Since $U$ is injective, it follows from (2.17) that

$$|\Omega_1 \backslash \Omega_1'| = |\Omega_1 \backslash \tau(\Omega_2')| = 0. \quad \square$$

## 3. Boundary conditions

Let $\Omega \subset \mathbb{R}^N$ be an open set. We consider realizations of the Laplacian by three different types of boundary conditions:

(D) $u_{|_{\partial\Omega}} = 0$ (Dirichlet);

(N) $\dfrac{\partial u}{\partial \nu}_{|_{\partial\Omega}} = 0$ (Neumann);

(R) $\left(\dfrac{\partial u}{\partial \nu} + \beta u\right)_{|_{\partial\Omega}} = 0$ (Robin).

We show that intertwining order isomorphisms exist only if the boundary conditions are the same and the domains are congruent.

**Definition 3.1.** An operator $A$ on $L^2(\Omega)$ is called a *symmetric realization of the Laplacian in* $L^2(\Omega)$ if $A$ is associated with a positive form $(a, V)$ on $L^2(\Omega)$ satisfying the following two conditions:

(3.1) $V$ is a closed subspace of $W^{1,2}(\Omega)$ containing $\mathscr{D}(\Omega)$

and

$$a(u, \varphi) = \int_\Omega \nabla u \nabla \varphi \quad \text{for all } u \in V, \varphi \in \mathscr{D}(\Omega). \tag{3.2}$$

The operator $A$ is called a *submarkovian symmetric realization of the Laplacian in* $L^2(\Omega)$ if in addition the following two Beurling-Deny criteria are satisfied:

$$u \in V \Rightarrow u^+, u^- \in V \quad \text{and} \quad a(u^+, u^-) \geqq 0 \tag{3.3}$$

and

$$0 \leqq u \in V \Rightarrow 1 \wedge u \in V \quad \text{and} \quad a(1 \wedge u, 1 \wedge u) \leqq a(u, u). \tag{3.4}$$

It follows from conditions (3.1) and (3.2) that a symmetric realization of the Laplacian in $L^2(\Omega)$ is also a realization of the Laplacian in the sense of Definition 2.1. Conditions (3.3) and (3.4) imply that the semigroup $(e^{tA})_{t \geqq 0}$ generated by $A$ is positive and contractive in the sense of $L^p(\Omega)$ for all $p \in [1, \infty]$. In particular, for $1 \leqq p < \infty$, there exist positive contraction semigroups $T_p = \big(T_p(t)\big)_{t \geqq 0}$ on $L^p(\Omega)$ which are consistent (i.e.

$$T_p(t) f = T_q(t) f \quad \text{for } f \in L^p \cap L^q,\ t \geqq 0 \tag{3.5}$$

whenever $1 \leqq p, q < \infty$) such that

$$T_2(t) = e^{tA} \quad (t \geqq 0)$$

(see [Dav2], [BH], Chap. I, [Fu], [RS] for example). We call $T_p$ the *extension* of $(e^{tA})_{t \geqq 0}$ in $L^p(\Omega)$ and denote by $A_p$ its generator.

An operator on $L^p(\Omega)$ defined in this way via a positive form satisfying (3.1)–(3.4) is called a *symmetric submarkovian realization of the Laplacian* in $L^p(\Omega)$, $1 < p < \infty$. This is justified by the following:

**Lemma 3.2.** *The operator $A_p$ satisfies*

(a) $\mathscr{D}(\Omega) \subset D(A_p)$

*and*

(b) $A_p f = \Delta f$ *in* $\mathscr{D}(\Omega)'$ *for all* $f \in D(A_p)$;

*i.e. $A_p$ is a realization of the Laplacian in $L^p(\Omega)$ in the sense of Definition* 2.1.

*Proof.* Recall, if $S$ is a semigroup on a Banach space $X$ with generator $B$, then for $x, y \in X$ one has $x \in D(B)$, $Bx = y$ if and only if

$$S(t)x - x = \int_0^t S(s) y \, ds \quad (t \geqq 0).$$

Let $u \in \mathscr{D}(\Omega)$, $\Delta u = v$. Then

$$T_p(t) u - u = \int_0^t T_p(s) v \, ds \quad (t \geqq 0) \tag{3.6}$$

holds for $p = 2$. Hence (3.6) also holds for $p \in (1, \infty)$ by consistency. Thus $u \in D(A_p)$ and $A_p u = v = \Delta u$.

Next we show that $A_p u = \Delta u$ for all $u \in D(A_p)$. For that consider the space

$$F = \{f \in L^p \cap D(A_2) \colon A_2 f \in L^p\}.$$

Then $F$ is invariant under $T_p$ and dense in $L^p(\Omega)$ since $\mathscr{D}(\Omega) \subset F$. Thus $F$ is a core of $A_p$. Since $A_p f = A_2 f = \Delta f$ for all $f \in F$, the claim follows. □

Now the examples we are interested in are defined in the following way.

**Example 3.3** (Dirichlet boundary conditions). Let $V := W_0^{1,2}(\Omega)$ be the closure of $\mathscr{D}(\Omega)$ in $W^{1,2}(\Omega)$ and let

$$a(u, v) = \int \nabla u \nabla v = \int \sum_{j=1}^{N} D_j u D_j v.$$

Then (3.1)–(3.4) are satisfied (see [Dav2]). The operator on $L^2(\Omega)$ associated with $a$ by the procedure above is called the *Laplacian with Dirichlet boundary conditions*, or simply the *Dirichlet Laplacian*, on $L^p(\Omega)$.

**Example 3.4** (Neumann boundary conditions). Choosing $V = W^{1,2}(\Omega)$ and

$$a(u, v) = \int \nabla u \cdot \nabla v$$

we call the associated operator on $L^p(\Omega)$ the *Laplacian with Neumann boundary conditions*, or simply *Neumann Laplacian*, on $L^p(\Omega)$. Conditions (3.1)–(3.4) are satisfied (see [Dav2], [BH], Chapt. 1, [Fu] or [RS]).

**Example 3.5** (Robin boundary conditions). Assume that $\Omega$ is bounded with Lipschitz boundary. Denote by $\sigma$ the surface measure on $\partial\Omega$ (i.e. the $(N-1)$-dimensional Hausdorff measure, cf. [EG]). Let $0 \leqq \beta \in L^\infty(\partial\Omega, d\sigma)$. Let $V = W^{1,2}(\Omega)$ and

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} uv\beta \, d\sigma.$$

Here, in the right integral, we identify $u$ and $v$ with their traces on $\partial\Omega$ (see [EG]). Then the positive form $a$ satisfies (3.1)–(3.4) (see also [AtE], [Dan1], [Dan2]). We call the operator associated with this form *the Laplacian with Robin boundary conditions* on $L^p(\Omega)$ associated with $\beta$. Note that we refind the Neumann Laplacian if $\beta = 0$.

**Example 3.6** (Partial periodic boundary conditions). Also the forms occuring in Examples 1.4, 1.5 and 1.6 satisfy assumptions (3.1)–(3.4).

In order to treat the Dirichlet Laplacian we need some preparation. By

$$\operatorname{cap}(A) = \inf\{\|u\|^2_{H^1(\mathbb{R}^N)} \colon u \in H^1(\mathbb{R}^N), u \geqq 1 \text{ in a neighborhood of } A\}$$

we define the *capacity* of a subset $A$ of $\mathbb{R}^N$ (cf. [BH], [EG], [Fu]). We use the following notation and facts.

A property is said to hold *quasi everywhere* (q.e.) if it holds outside a set of capacity 0. Let $\Omega \subset \mathbb{R}^N$ be open. A function $f: \Omega \to \mathbb{R}$ is called *quasi continuous* if for every $\varepsilon > 0$ there exists an open set $O \subset \Omega$ of capacity $\mathrm{cap}(O) < \varepsilon$ such that $f$ is continuous on $\Omega \backslash O$. If $f$ is quasi-continuous and $f(x) = 0$ a.e., then $f = 0$ q.e. (see [BH], Prop. 8.1.6). For every $f \in H^1(\mathbb{R}^N)$ there exists a quasi-continuous function $\tilde{f}: \Omega \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ a.e. (see [BH], Prop. 8.2.1 or [EG], 4.8). It follows that $\tilde{f}$ is uniquely determined q.e.

We can identify $W_0^{1,2}(\Omega)$ with a closed subspace of $W^{1,2}(\mathbb{R}^N)$ in the following way:

$$W_0^{1,2}(\Omega) = \{f \in W^{1,2}(\mathbb{R}^N): \tilde{f}(x) = 0 \text{ q.e. on } \mathbb{R}^N \backslash \Omega\} \tag{3.7}$$

(see e.g. [AM], Theorem 1.1, [Den], p. 143, [He], Theorem 3.1, p. 241 or [Fu], Example 3.3.2, p. 81).

**Lemma 3.7.** *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ *be open. If* $\mathrm{cap}(\Omega_1 \backslash \Omega_2) > 0$, *then there exists*

$$\varphi \in \mathscr{D}(\Omega_1) \backslash W_0^{1,2}(\Omega_2).$$

*Proof.* Let $K_n \subset K_{n+1}$ be compact such that $\bigcup_{n \in \mathbb{N}} K_n = \Omega_1$. Then

$$\mathrm{cap}(\Omega_1 \backslash \Omega_2) = \lim_{n \to \infty} \mathrm{cap}(K_n \backslash \Omega_2)$$

(by [BH], Prop. 8.1.3). Hence there exists $n \in \mathbb{N}$ such that $\mathrm{cap}(K_n \backslash \Omega_2) > 0$. Let $\varphi \in \mathscr{D}(\Omega_1)$ such that $\varphi \geqq 1$ on $K_n$. Then $\varphi \notin W_0^{1,2}(\Omega_2)$ by (3.7). □

In view of (3.7) we deduce from Lemma 3.7 the following. Recall, that we identify $W_0^{1,2}(\Omega)$ with a subspace of $L^2(\mathbb{R}^N)$.

**Proposition 3.8.** *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ *be two open sets. Then* $W_0^{1,2}(\Omega_1) = W_0^{1,2}(\Omega_2)$ *if and only if* $\mathrm{cap}(\Omega_1 \triangle \Omega_2) = 0$.

**Corollary 3.9.** *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ *be open such that* $|\Omega_1 \triangle \Omega_2| = 0$ *and hence* $L^2(\Omega_1) = L^2(\Omega_2)$. *Denote by* $A_j$ *the Dirichlet Laplacian on* $L^2(\Omega_j)$ $(j = 1, 2)$. *Then* $A_1 = A_2$ *if and only if* $\mathrm{cap}(\Omega_1 \triangle \Omega_2) = 0$.

We need the following result:

**Proposition 3.10.** *Let* $\Omega$ *be an open connected set. Let* $\Omega' \subset \Omega$ *be open such that* $\mathrm{cap}(\Omega \backslash \Omega') = 0$. *Then* $\Omega'$ *is connected.*

We use an argument from the theory of positive semigroups (cf. [Na]) to prove this. A positive semigroup $T = \big(T(t)\big)_{t \geqq 0}$ on $L^p(\Omega)$ $(1 \leqq p < \infty)$ is called *irreducible* if there does not exist any non-trivial closed ideal of $L^p(\Omega)$ which is invariant under $T$. Here, a *closed ideal* of $L^p(\Omega)$ is a subspace $J$ of the form

$$J = \{f \in L^p(\Omega) \colon f(x) = 0 \text{ a.e. on } \Omega \backslash S\}$$

where $S \subset \Omega$ is a Borel set.

**Lemma 3.11.** *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $T$ be the semigroup generated by the Dirichlet Laplacian on $L^2(\Omega)$. Then $T$ is irreducible if and only if $\Omega$ is connected.*

*Proof.* If $\Omega$ is connected, then $T$ is irreducible by [Dav2], Theorem 3.3.5 or [Are2], Theorem 1.5. Conversely, assume that $\Omega$ is the disjoint union of two open sets $\Omega_1$ and $\Omega_2$. Let $J = \{f \in L^2(\Omega) \colon f = 0 \text{ a.e. on } \Omega_2\}$. Let $A$ be the generator of $T$. We show that $T(t)J \subset J$ $(t \geqq 0)$. For this, it suffices to show that $R(\lambda, A)J \subset J$ for all $\lambda > 0$. Let $u = R(\lambda, A)v$, where $v \in J$; i.e.

$$\int_\Omega \nabla u \nabla \varphi + \lambda \int_\Omega u\varphi = \int_\Omega v\varphi$$

for all $\varphi \in \mathscr{D}(\Omega)$. Let $u_1 = u \cdot 1_{\Omega_1}$. Then $u_1 \in H_0^1(\Omega)$ and $\nabla u_1 = \nabla u \cdot 1_{\Omega_1}$. So

$$\int_\Omega \nabla u_1 \nabla \varphi + \lambda \int_\Omega u\varphi = \int_\Omega v\varphi$$

for all $\varphi \in \mathscr{D}(\Omega)$. Hence $(\lambda - A)u_1 = v$. Thus $u = u_1$. $\square$

*Proof of Proposition* 3.10. Since $\operatorname{cap}(\Omega \backslash \Omega') = 0$ we have $L^2(\Omega) = L^2(\Omega')$ and the semigroup generated by the Dirichlet Laplacian with respect to $\Omega$ and to $\Omega'$ coincide. Thus the latter is irreducible and the claim follows from Lemma 3.11. $\square$

Next we introduce some regularity properties of an open set $\Omega$ which all express in some weak form that $\Omega$ lies only on one side of $\partial\Omega$. Recall that an open set $\Omega \subset \mathbb{R}^N$ is called *topological regular* if $\mathring{\bar{\Omega}} = \Omega$. It is easy to see that this is equivalent to saying that

$$B(z,r) \backslash \Omega \text{ has non-empty interior for all } z \in \partial\Omega, r > 0.$$

**Definition 3.12.** An open set $\Omega \subset \mathbb{R}^N$ is called *regular in measure* if

$$|B(z,r) \backslash \Omega| > 0 \quad \text{for all } z \in \partial\Omega, r > 0.$$

The set $\Omega$ is called *regular in capacity* if

$$\operatorname{cap}\big(B(z,r) \backslash \Omega\big) > 0 \quad \text{for all } z \in \partial\Omega, r > 0.$$

Finally, we say that $\Omega$ is *locally connected at the boundary* if for all $z \in \partial\Omega$ there exists $r_0 > 0$ such that $B(z,r) \cap \Omega$ is connected for all $r \in (0, r_0)$.

It is clear that topological regularity implies regularity in measure and regularity in measure implies regularity in capacity. The set

$$\Omega = \{x \in \mathbb{R}^2 \colon |x| < 1\} \backslash \{(a,0) \colon 0 \leqq a < 1\}$$

is regular in capacity but not in measure.

The third property in Definition 3.12 is independent of the others. For example, $\Omega = \mathbb{R}^2 \backslash \{0\}$ is locally connected but not regular in capacity. On the other hand $\Omega = \mathbb{R}^2 \backslash \{(x, y): x^2 + y^2 \leqq 1, x \cdot y \geqq 0\}$ is topological regular but not locally connected at the boundary (at $z = 0$).

**Proposition 3.13.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open.*

(a) *Assume that $\Omega_1$ and $\Omega_2$ are regular in measure. If $|\Omega_1 \triangle \Omega_2| = 0$, then $\Omega_1 = \Omega_2$.*

(b) *Assume that $\Omega_1$ and $\Omega_2$ are regular in capacity. If* $\mathrm{cap}(\Omega_1 \triangle \Omega_2) = 0$, *then $\Omega_1 = \Omega_2$.*

*Proof.* (a) Let $|\Omega_1 \triangle \Omega_2| = 0$. Assume that there exists $x \in \Omega_1 \backslash \Omega_2$. Then $x \in \overline{\Omega_2}$. Let $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \Omega_1$. Since $\Omega_2$ is regular in measure we have $|B(x, \varepsilon) \backslash \Omega_2| > 0$. Hence $|\Omega_1 \backslash \Omega_2| > 0$, a contradiction.

(b) This is completely analogous. □

Now we can prove the main result for Dirichlet boundary conditions.

**Theorem 3.14.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open and regular in measure. Assume that $\Omega_2$ is connected. Let $1 < p < \infty$ and denote by $A_1$ the Dirichlet Laplacian on $L^p(\Omega_1)$. Let $A_2$ be a symmetric submarkovian realization of the Laplacian in $L^p(\Omega_2)$. Assume that $U: L^p(\Omega_1) \to L^p(\Omega_2)$ is an order isomorphism satisfying*

$$Ue^{tA_1} = e^{tA_2} U \quad (t \geqq 0). \tag{3.8}$$

*Then $A_2$ is the Dirichlet Laplacian on $L^p(\Omega_2)$ and $U$ is of the form*

$$Uf = c \cdot f \circ \tau \quad (f \in L^p(\Omega_1)) \tag{3.9}$$

*where $\tau$ is an isometry from $\Omega_2$ onto $\Omega_1$ and $c > 0$.*

*Proof.* By Theorem 2.2 there exist open sets $\Omega_j' \subset \Omega_j$ satisfying $|\Omega_j \backslash \Omega_j'| = 0$, $j = 1, 2$, a homeomorphism $\tau: \Omega_2' \to \Omega_1'$ which is isometric on each component of $\Omega_2'$ and $h: \Omega_2' \to (0, \infty)$ which is constant on each component of $\Omega_2'$ such that (2.4) holds. Moreover, $U$ is given by (2.17) on $L^p(\Omega_1)$. It is easy to see that

$$\|U^{-1}\|^{-1} \leqq h(y) \leqq \|U\|$$

for all $y \in \Omega_2'$. Since $\tau$ is measure preserving $U$ has an extension as an order isomorphism from $L^q(\Omega_1)$ onto $L^q(\Omega_2)$ for all $1 < q < \infty$. Thus we can assume that $p = 2$.

a) We show that $\mathrm{cap}(\Omega_2 \backslash \Omega_2') = 0$. It follows from (3.8) and [Pa], (6.9), p. 70 that

$$(I - A_2)^{-1/2} U = U(I - A_1)^{-\frac{1}{2}}. \tag{3.10}$$

Denote by $V = (I - A_2)^{-1/2} L^2(\Omega_2)$ the form domain of $A_2$.

Since $(I - A_1)^{-\frac{1}{2}} L^2(\Omega_1) = W_0^{1,2}(\Omega_1)$ it follows from (3.10) that

$$UW_0^{1,2}(\Omega_1) = V.$$

Now by the closed graph theorem, $U$ induces a continuous operator from $W_0^{1,2}(\Omega_1)$ into $V$. Suppose that $\mathrm{cap}(\Omega_2\backslash\Omega_2') > 0$. Then there exists a compact subset $K$ of $\Omega_2$ such that $\mathrm{cap}(K\backslash\Omega_2') > 0$ (cf. proof of Lemma 3.7). Let $\varphi \in \mathscr{D}(\Omega_2)$ such that $\varphi \geqq 1$ on $K$. Since $\mathscr{D}(\Omega_1)$ is dense in $W_0^{1,2}(\Omega_1)$, there exist $f_n \in \mathscr{D}(\Omega_1)$ such that $g_n = Uf_n$ converges to $\varphi$ in $H^1(\Omega_2)$. Then $h_n = \varphi g_n$ converges to $\varphi^2$ in $H^1(\Omega_2)$. Taking a subsequence, if necessary, we can assume that $h_n$ converges to $\varphi^2$ q.e. But $h_n(y) = 0$ and $\varphi^2(y) \geqq 1$ for all $y \in K\backslash\Omega_2'$. This is a contradiction. We have shown that $\mathrm{cap}(\Omega_2\backslash\Omega_2') = 0$.

b) Now it follows from Proposition 3.10 that $\Omega_2'$ is connected. Thus $\tau$ is an isometry from $\Omega_2'$ onto $\Omega_1'$ and $h$ is a constant. The set $\tau(\Omega_2)$ is regular in measure since $\Omega_2$ is. Moreover, $|\Omega_1 \triangle \tau(\Omega_2)| = 0$ (since $|\Omega_j\backslash\Omega_j'| = 0$ and $\tau(\Omega_2') = \Omega_1'$). It follows from Proposition 3.13 that $\Omega_1 = \tau(\Omega_2)$.

Since (3.9) holds a.e. for all $f \in \mathscr{D}(\Omega_1)$ with $c = h$, it is also true on $L^p(\Omega_1)$ by density. We can assume $c = 1$. Then $U$ is unitary. From the special form of $U$ one sees that $UW_0^{1,2}(\Omega_1) = W_0^{1,2}(\Omega_2)$ and $\int_{\Omega_1} \nabla f \nabla g = \int_{\Omega_2} \nabla(Uf)\nabla(Ug)$ for all $f, g \in W_0^{1,2}(\Omega_1)$. Now it follows from Lemma 1.3 that $A_2$ is the Dirichlet Laplacian. □

By the following example we show that the roles of $A_1$ and $A_2$ in Theorem 3.14 cannot be exchanged; i.e., the theorem is false, in general, if we assume that $\Omega_1$ is connected but $\Omega_2$ is not.

**Example 3.15.** Let $\Omega_1 = (0,2)$, $\Omega_2 = (0,1) \cup (2,3)$. Observe that $\Omega_1$ and $\Omega_2$ are both regular in measure (and even topologically regular). Let $\Omega_1' = \Omega_1\backslash\{1\}$ and define the homeomorphism $\tau\colon \Omega_2 \to \Omega_1'$ by

$$\tau(y) = \begin{cases} y & \text{if } y \in (0,1), \\ y-1 & \text{if } y \in (2,3). \end{cases}$$

Then $Uf = f \circ \tau$ defines a unitary order isomorphism $U$ from $L^2(\Omega_1)$ onto $L^2(\Omega_2)$. Let $A_1$ be the Dirichlet Laplacian and consider the positive form $(a_2, V_2)$ on $L^2(\Omega_2)$ given by

$$V_2 = \{f \in H^1(\Omega_2)\colon f(1) = f(2),\ f(0) = f(3) = 0\}, \quad a_2(f,g) = \int_{\Omega_2} f'g'.$$

Let $A_2$ be the operator associated with $a_2$. Then $A_2$ is a symmetric submarkovian realization of the Laplacian in $L^2(\Omega_2)$. It follows from Lemma 1.3 that (3.8) holds. □

The following example shows that Theorem 3.14 does not hold in general if $\Omega_2$ is not regular in measure.

**Example 3.16.** Let $\Omega_1 = \{x \in \mathbb{R}^2\colon |x| < 1\}$ and $\Omega_2 = \Omega_1\backslash\{(a,0)\colon 0 \leqq a < 1\}$. Note that $\Omega_2$ is connected and regular in capacity but not regular in measure. Since $\Omega_2 \subset \Omega_1$, $|\Omega_1\backslash\Omega_2| = 0$, we have $L^2(\Omega_1) = L^2(\Omega_2)$. Let $U$ be the identity operator and let $A_1 = A_2$ be the Dirichlet Laplacian on $L^2(\Omega_1)$ (with respect to $\Omega_1$ in both cases). Then $A_2$ is a symmetric submarkovian realization of the Laplacian on $L^2(\Omega_2)$. Condition (3.8) is trivially satisfied, but $\Omega_1$ and $\Omega_2$ are not congruent. □

However, if we already know that $A_2$ is the Dirichlet Laplacian on $L^2(\Omega_2)$, the regularity condition in Theorem 3.1 can be relaxed.

**Corollary 3.17.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be two open sets which are regular in capacity. Assume that $\Omega_2$ is connected. Let $1 < p < \infty$ and denote by $A_j$ the Dirichlet Laplacian on $L^p(\Omega_j)$ $(j = 1, 2)$. Let $U\colon L^p(\Omega_1) \to L^p(\Omega_2)$ be an order isomorphism such that the commutator condition* (3.8) *is satisfied. Then there exist an isometry $\tau$ from $\Omega_2$ onto $\Omega_1$ and a constant $c > 0$ such that $Uf = c \cdot f \circ \tau$ $(f \in L^p(\Omega_1))$.*

*Proof.* We proceed as in the proof of Theorem 3.14 to deduce that $U$ is of the form $Uf = c \cdot f \circ \tau$ $(f \in L^2(\Omega_1))$, where $c > 0$ and $\tau$ is an isometry from $\Omega_2'$ onto $\Omega_1'$. Moreover, $\operatorname{cap}(\Omega_2 \backslash \Omega_2') = 0$ as before. Since $|\Omega_1 \backslash \Omega_1'| = 0$ it follows that $L^2(\Omega_1') = L^2(\Omega_1)$. It follows from Lemma 1.3 and the assumption (3.8) that $A_1$ is the Dirichlet Laplacian on $L^2(\Omega_1')$ (with respect to $\Omega_1'$). So Corollary 3.9 implies that $\operatorname{cap}(\Omega_1 \backslash \Omega_1') = 0$. Consequently, $\operatorname{cap}\big(\Omega_1 \triangle \tau(\Omega_2)\big) = 0$. Since $\Omega_1$ and $\tau(\Omega_2)$ are regular in capacity, it follows from Proposition 3.13 that $\Omega_1 = \tau(\Omega_2)$. □

One can associate to every open set an open set which is regular in capacity without changing the Dirichlet Laplacian.

**Proposition 3.18.** *Let $\Omega \subset \mathbb{R}^N$ be open. Then there exists a unique open set $\tilde{\Omega} \supset \Omega$ which is regular in capacity and satisfies $\operatorname{cap}(\tilde{\Omega} \backslash \Omega) = 0$. In particular, $L^2(\Omega) = L^2(\tilde{\Omega})$ and the Dirichlet Laplacians with respect to $\Omega$ and $\tilde{\Omega}$ coincide. Moreover, $\Omega$ is connected if and only if $\tilde{\Omega}$ is connected.*

*Proof.* Let $\tilde{\Omega}$ be the union of all balls $B(x, r)$ satisfying $\operatorname{cap}\big(B(x, r) \backslash \Omega\big) = 0$. Then $\tilde{\Omega}$ is clearly open and contains $\Omega$. We show that $\operatorname{cap}(\tilde{\Omega} \backslash \Omega) = 0$. Let $K_n \subset K_{n+1}$ be compact sets such that $\bigcup_{n \in \mathbb{N}} K_n = \tilde{\Omega}$. Let $n \in \mathbb{N}$. Then there exist finitely many balls

$$B(x_i, r_i), \quad i = 1, \ldots, m,$$

covering $K_n$ and satisfying $\operatorname{cap}\big(B(x_i, r_i) \backslash K_n\big) = 0$. Thus, using the usual properties of capacity [BH], Proposition 8.1.3, we conclude that

$$\operatorname{cap}(K_n \backslash \Omega) \leqq \sum_{i=1}^{m} \operatorname{cap}\big(B(x_i, r_i) \backslash K_n\big) = 0$$

and hence $\operatorname{cap}(\tilde{\Omega} \backslash \Omega) = \lim_{n \to \infty} \operatorname{cap}(K_n \backslash \Omega) = 0$. Next we show that $\tilde{\Omega}$ is regular in capacity. In fact, let $B(x, r)$ be a ball such that $\operatorname{cap}\big(B(x, r) \backslash \tilde{\Omega}\big) = 0$. Then

$$\operatorname{cap}\big(B(x, r) \backslash \Omega\big) \leqq \operatorname{cap}\big(B(x, r) \backslash \tilde{\Omega}\big) + \operatorname{cap}(\tilde{\Omega} \backslash \Omega) = 0.$$

Thus $x \in \tilde{\Omega}$ by definition of $\tilde{\Omega}$. In order to show uniqueness let $\Omega_1 \supset \Omega$ be open, regular in capacity such that $\operatorname{cap}(\Omega_1 \backslash \Omega) = 0$. Then $\operatorname{cap}(\Omega_1 \backslash \tilde{\Omega}) \leqq \operatorname{cap}(\Omega_1 \backslash \Omega) = 0$ and

$$\operatorname{cap}(\tilde{\Omega} \backslash \Omega_1) \leqq \operatorname{cap}(\tilde{\Omega} \backslash \Omega) = 0.$$

It follows from Proposition 3.14 that $\Omega_1 = \tilde{\Omega}$. The remaining assertion follows from Proposition 3.9 and 3.11. □

For non-regular open sets Corollary 3.17 now obtains the following form.

**Corollary 3.19.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open, $\Omega_2$ connected. Let the remaining assumptions of Corollary* 3.17 *be satisfied. Then $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are congruent.*

Next we consider Neumann boundary conditions.

**Theorem 3.20.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be two open sets which are regular in measure. Assume that $\Omega_2$ is connected and $\Omega_1$ is locally connected at the boundary. Let $1 < p < \infty$ and let $A_1$ be the Neumann Laplacian on $L^p(\Omega_1)$ and $A_2$ a symmetric submarkovian realization of the Laplacian in $L^p(\Omega_2)$. Let $U\colon L^p(\Omega_1) \to L^p(\Omega_2)$ be an order isomorphism such that*

$$Ue^{tA_1} = e^{tA_2}U \quad (t \geqq 0).$$

*Then there exists an isometry $\tau$ from $\Omega_2$ onto $\Omega_1$ and a constant $c > 0$ such that $Uf = c \cdot f \circ \tau$ $(f \in L^p(\Omega_1))$. Moreover, $A_2$ is the Neumann Laplacian on $L^p(\Omega_2)$.*

*Proof.* a) By Theorem 2.2 there exist open sets $\Omega_j' \subset \Omega_j$ such that

$$|\Omega_j \backslash \Omega_j'| = 0 \quad (j = 1, 2)$$

and a homeomorphism $\tau$ from $\Omega_2'$ onto $\Omega_1'$ which is isometric on each component of $\Omega_2'$ and a mapping $h\colon \Omega_2' \to (0, \infty)$ which is constant on each component of $\Omega_2'$ such that (2.4) holds. As in the proof of Theorem 3.14 we can assume that $p = 2$. Moreover, it follows that

$$UH^1(\Omega_1) = V \tag{3.11}$$

where $V$ is the form domain of $A_2$.

b) We will show that $\Omega_2' = \Omega_2$. For this let $\omega$ be a component of $\Omega_2'$. We claim that $\partial\omega \cap \Omega_2 = \emptyset$. Since $\Omega_2$ is connected, this implies $\omega = \Omega_2$ and so $\Omega_2' = \Omega_2$. The mapping $\tau$ is isometric on $\omega$. Denote by $\bar{\tau}$ the isometric extension of $\tau_{|_\omega}$ to $\mathbb{R}^N$. Recall from (2.14) that

$$\bar{\tau}(\partial\omega \cap \Omega_2) \subset \partial\Omega_1. \tag{3.12}$$

Now assume that there exists $y_0 \in \partial\omega \cap \Omega$. Let $z_0 := \bar{\tau}(y_0)$. Let $\varepsilon > 0$ be such that $B(y_0, \varepsilon) \subset \Omega_2$ and such that $B(z_0, \varepsilon) \cap \Omega_1$ is connected. We claim that

$$\tau\big(\omega \cap B(y_0, \varepsilon)\big) = \Omega_1 \cap B(z_0, \varepsilon). \tag{3.13}$$

In fact, $\bar{\tau}\big(B(y_0, \varepsilon)\big) = B(z_0, \varepsilon)$ since $\bar{\tau}$ is an isometry. So $\tau\big(\omega \cap B(y_0, \varepsilon)\big)$ is a non-empty open subset of $\Omega_1 \cap B(z_0, \varepsilon)$. In order to prove (3.13), it suffices to show that $\tau\big(\omega \cap B(y_0, \varepsilon)\big)$ is relatively closed in $\Omega_1 \cap B(z_0, \varepsilon)$.

Let $y_n \in \omega \cap B(y_0, \varepsilon)$ such that $x = \lim_{n\to\infty} \tau(y_n) \in \Omega_1 \cap B(z_0, \varepsilon)$. Then

$$y := \lim_{n\to\infty} y_n = \bar{\tau}^{-1}(x)$$

exists and $|y - y_0| = |\bar{\tau}(y) - \bar{\tau}(y_0)| = |x - z_0| < \varepsilon$. Thus $y \in B(y_0, \varepsilon) \subset \Omega_2$. Assume that $y \notin \omega$. Then $y \in \partial\omega \cap \Omega_2$. Hence by (3.12), $\bar{\tau}(y) = x \in \partial\Omega_1$, a contradiction. Thus $y \in \omega \cap B(y_0, \varepsilon)$ and $\tau(y) = x$. Now (3.13) is proved. Next recall from (2.17) that

(3.14) $$(Uf)(y) = h(y)f(\tau(y)) \text{ a.e.}$$

for all $f \in L^p(\Omega_1)$. Let $f \in \mathcal{D}(\mathbb{R}^N)$ such that $\operatorname{supp} f \subset B(z_0, \varepsilon)$ and $f = 1$ on $B\left(z_0, \frac{\varepsilon}{2}\right)$. Then $f_{|\Omega_1} \in W^{1,2}(\Omega_1)$. Let $g = U\left(f_{|\Omega_1}\right)$. Then $g$ is in the form domain $V$ and so in $W^{1,2}(\Omega_2)$. Moreover, by (3.14), $g(y) = h(y)f(\tau(y)) = h(y) = c > 0$ a.e. on $\omega \cap B\left(y_0, \frac{\varepsilon}{2}\right)$. Let $y \in \Omega_2' \backslash \omega$. Then, in view of (3.13), $\tau(y) \notin B(z_0, \varepsilon)$. Hence by (3.14), $g(y) = 0$ a.e. on $B\left(y_0, \frac{\varepsilon}{2}\right) \backslash \omega$. Now define $k \in W^{1,2}(B)$, $B = B\left(z_0, \frac{\varepsilon}{2}\right)$, by $k(x) = g(\bar{\tau}^{-1}(x))$. Then $k = c$ a.e. on $B \cap \Omega_1$ and $k = 0$ a.e. on $B \backslash \Omega_1$. Since $\Omega_1$ is regular in measure, it follows that $k$ is discontinuous at each $x \in \partial\Omega_1 \cap B$. But $k$ has a quasicontinuous representative (because it has an extension in $W^{1,2}(\mathbb{R}^N)$). Hence $\operatorname{cap}(\partial\Omega_1 \cap B) = 0$. It follows that $W_0^{1,2}(B) = W_0^{1,2}(B \backslash \partial\Omega_1)$. But, since $\Omega_1$ is regular in measure, one has $|B \backslash \Omega_1| > 0$. Since $|\partial\Omega_1 \cap B| = 0$, it follows that $B \backslash \overline{\Omega}_1 \neq \emptyset$. Thus $B \backslash \partial\Omega_1$ is not connected. This contradicts Proposition 3.10. We have finished the proof that $\Omega_2' = \Omega_2$.

c) Thus $\tau \colon \Omega_2 \to \Omega_1'$ is an isometry and $h$ is a positive constant $c$. Since $\Omega_1' = \tau(\Omega_2)$ and $\Omega_1$ are regular in measure and since $|\Omega_1 \backslash \Omega_1'| = 0$ we conclude from Proposition 3.13 that $\Omega_1 = \Omega_1'$. We can assume $c = 1$. Then it follows from Lemma 1.3 that $A_2 = UA_1U^{-1}$ is the Neumann Laplacian on $L^2(\Omega_2)$. □

The same proof applies to Robin boundary conditions yielding the following result.

**Theorem 3.21.** *Let $\Omega_1, \Omega_2$ be two bounded open subsets of $\mathbb{R}^N$ with Lipschitz boundary. Assume that $\Omega_2$ is connected. Let $1 < p < \infty$ and let $0 \leqq \beta_1 \in L^\infty(\partial\Omega_1, d\sigma)$. Let $A_1$ be the Laplacian with Robin boundary condition on $L^p(\Omega_1)$ associated with $\beta_1$. Let $A_2$ be a symmetric submarkovian realization of the Laplacian in $L^p(\Omega_2)$. Let $U \colon L^p(\Omega_1) \to L^p(\Omega_2)$ be an order isomorphism such that*

$$Ue^{tA_1} = e^{tA_2}U \quad (t \geqq 0).$$

*Then $U$ is of the form*

(3.15) $$Uf = c \cdot f \circ \tau \quad (f \in L^p(\Omega_1))$$

*where $\tau$ is an isometry from $\Omega_2$ onto $\Omega_1$ and $c > 0$. Moreover, $A_2$ is the Laplacian with Robin boundary condition associated with $\beta_2 = \beta_1 \circ \tau$.*

*Proof.* As in the proof of the previous theorem one sees that $U$ is of the form (3.15). To prove that $A_2$ is of the special form we can assume $c = 1$ and $p = 2$ (as we did already). It follows from Lemma 1.3, that $A_2$ is associated with the form $b$ given on the form domain $UW^{1,2}(\Omega_1)$ by

$$\begin{aligned} b(Uf, Ug) &= a(f, g) \\ &= \int_{\Omega_1} \nabla f \nabla g + \int_{\partial\Omega_1} fg\beta_1 \, d\sigma. \end{aligned}$$

But $UW^{1,2}(\Omega_1) = W^{1,2}(\Omega_2)$ and we find for $f, g \in W^{1,2}(\Omega_2)$,

$$\begin{aligned} b(f,g) &= a(f \circ \tau^{-1}, g \circ \tau^{-1}) \\ &= \int_{\partial\Omega_1} \nabla(f \circ \tau^{-1}) \nabla(g \circ \tau^{-1}) + \int_{\partial\Omega_1} f \circ \tau^{-1} g \circ \tau^{-1} \beta_1 \, d\sigma \\ &= \int_{\partial\Omega_2} \nabla f \nabla g + \int_{\partial\Omega_2} f g \beta_1 \circ \tau \, d\sigma. \end{aligned}$$

Thus, by Lemma 1.3, $A_2$ is the Laplacian with Robin boundary conditions associated with $\beta_1 \circ \tau$. □

**Remark 3.22.** If $p = 2$ then we can omit the assumption that $A_2$ be submarkovian in Theorems 3.14, 3.20 and 3.21.

## 4. Intertwining isometric isomorphisms

In Theorem 3.14, if $p = 2$, then we are not allowed to replace "order isomorphism" by "isometric isomorphism", i.e. "unitary". This is exactly what the counterexample to Kac's question shows, see Introduction. However, things are different, if $p \neq 2$. Isometries on $L^p$ are of special nature if $p \neq 2$. This follows from the following lemma [Ro], p. 416.

**Lemma 4.1.** *Let $1 \leqq p < \infty$, $p \neq 2$ and let $f, g \in L^p$. Then*

$$\|f+g\|^p + \|f-g\|^p = 2\|f\|^p + 2\|g\|^p \tag{4.1}$$

*if and only if $f \cdot g = 0$ a.e.*

We deduce from this the following:

**Proposition 4.2.** *Let $U: L^p(\Omega_1) \to L^p(\Omega_2)$ be an isometric isomorphism where $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ are open and $1 \leqq p < \infty$, $p \neq 2$. Then there exists a unique order isomorphism $|U|: L^p(\Omega_1) \to L^p(\Omega_2)$ such that*

$$|Uf| = |U|f \quad (f \in L^p(\Omega_1),\ f \geqq 0). \tag{4.2}$$

*Moreover,*

$$|U^{-1}| = |U|^{-1}. \tag{4.3}$$

*Proof.* It follows from (4.1) that $U$ is disjointness preserving; i.e.,

$$f \cdot g = 0 \text{ a.e.} \Leftrightarrow (Uf)(Ug) = 0 \text{ a.e.} \tag{4.4}$$

for all $f, g \in L^p(\Omega_1)$. Now the claim follows from [Are1], Section 1. In this case one could also use Lamperti's theorem [Ro], p. 416, together with [Ro], Theorem 17, p. 410. □

**Proposition 4.3.** *Let $U: L^p(\Omega_1) \to L^p(\Omega_2)$ be an isometric isomorphism where $1 \leqq p < \infty$, $p \neq 2$. Let $S_j \in \mathscr{L}(L^p(\Omega_j))$ be a positive operator $(j = 1, 2)$ such that $S_2 U = U S_1$. Then*

$$S_2|U| = |U|S_1.$$

*Proof.* Let $0 \leqq f \in L^p(\Omega_1)$. Then by (4.2),

$$|U|S_1 f = |US_1 f| = |S_2 Uf| \leqq S_2|Uf| = S_2|U|f.$$

Hence

$$|U|S_1 \leqq S_2|U|. \tag{4.5}$$

The same argument applied to $U^{-1}$ gives $|U^{-1}|S_2 \leqq S_1|U^{-1}|$. Hence by (4.3),

$$|U|^{-1}S_2 \leqq S_1|U|^{-1}$$

which implies $S_2|U| \leqq |U|S_1$. Together with (4.5), this proves the claim. □

Now, if $1 < p < \infty$, $p \neq 2$, we may replace order isomorphism by isometric isomorphism in Theorem 3.15, Corollary 3.16, Theorem 3.18 and Theorem 3.19. In addition, the constant $c$ in the conclusion is 1 or $-1$. We give an explicit formulation in the case of Dirichlet boundary conditions.

**Theorem 4.4.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open and regular in capacity. Assume that $\Omega_2$ is connected. Denote by $A_j$ the Dirichlet Laplacian on $L^p(\Omega_j)$, $j = 1, 2$, where $1 < p < \infty$, $p \neq 2$. Assume that*

$$Ue^{tA_1} = e^{tA_2}U \quad (t \geqq 0). \tag{4.6}$$

*Then there exists an isometry $\tau$ from $\Omega_2$ onto $\Omega_1$ such that*

$$Uf = c \cdot f \circ \tau \quad \textit{for all } f \in L^p(\Omega_1)$$

*where $c = 1$ or $c = -1$.*

*Proof.* It follows from Proposition 4.3 that $|U|e^{tA_1} = e^{tA_2}|U|$ $(t \geqq 0)$. By Corollary 3.16, there exist an isometry $\tau$ from $\Omega_2$ onto $\Omega_1$ and a constant $c_0 > 0$ such that

$$|U|f = c_0 \cdot f \circ \tau \quad (f \in L^p(\Omega_1)).$$

Since $|U|$ is isometric, we have $c_0 = 1$. Note that $U\mathcal{D}(\Omega_1) \subset C^\infty(\Omega_2)$ by (2.13). Let $y \in \Omega_2$. Then $|(Uf)(y)| \leqq (|U|\,|f|)(y) = |f(\tau(y))|$ for all $f \in \mathcal{D}(\Omega_1)$. This implies that

$$(Uf)(y) = h(y)f(\tau(y))$$

for all $f \in \mathcal{D}(\Omega_1)$ where $h(y) \in \{-1, 1\}$. By the proof of Proposition 2.6, the function $h$ is continuous. This implies that $h$ is a constant $c \in \{1, -1\}$. Thus $Uf = c \cdot f \circ \tau$ for all $f \in \mathcal{D}(\Omega_1)$. By density, we deduce that $Uf = c \cdot f \circ \tau$ a.e. for all $f \in L^p(\Omega_1)$. □

Now the proof of Theorem 1.2 is complete.

Concluding, we give another formulation of this theorem, which might illuminate somehow modelisation of the heat equation in a functional analytic framework.

Let $\Omega \subset \mathbb{R}^N$ be open and non-empty, $1 \leqq p < \infty$. Then the Banach lattice $L^p(\Omega)$ is isomorphic to $E := L^p(0,1)$. Here we call two Banach lattices *isomorphic* if there exists an order isomorphism from one onto the other. Given a positive semigroup $T$ on $L^p(\Omega)$ we may consider it as well as a positive semigroup on $E = L^p(0,1)$. Then the question arises whether we can refind the set $\Omega$.

For $N \in \mathbb{N}$ let $O_N$ be the set of all non-empty connected open subsets of $\mathbb{R}^N$ which are regular in capacity identifying two sets if they are congruent. For $1 < p < \infty$ we denote by $\mathcal{S}_{p+}$ the set of all positive semigroups on $E$. We identify two semigroups $T_1 = \big(T_1(t)\big)_{t \geqq 0}$ and $T_2 = \big(T_2(t)\big)_{t \geqq 0}$ if there exists an order isomorphism $U$ on $E$ such that

$$T_2(t)U = UT_1(t) \quad (t \geqq 0). \tag{4.7}$$

To each $\Omega \in O_N$ we associate the semigroup $T_\Omega$ generated by the Dirichlet Laplacian on $L^p(\Omega)$ but considered as an element of $\mathcal{S}_{p+}$. Then our result says that $\Omega \mapsto T_\Omega$ is injective.

Similarly, $L^p(\Omega)$ is isomorphic to $E = L^p(0,1)$ as a Banach space. Here we call two Banach spaces *isomorphic* if there exists an isometric isomorphism which maps one space onto the other. For $1 < p < \infty$, $p \neq 2$, denote by $\mathcal{S}_p$ the set of all semigroups on $E$ identifying two semigroups $T_1$ and $T_2$ if (4.7) holds for some isometric isomorphism $U$ on $E$. Then $\Omega \mapsto T_\Omega$ is injective as mapping from $O_N$ into $\mathcal{S}_p$.

An inspection of the proof of Proposition 2.4 shows that also the dimension can be identified. More precisely, let $\Omega_j \subset \mathbb{R}^{N_j}$ be open, connected and regular in capacity where $j = 1, 2$. Assume that $T_{\Omega_1}$ and $T_{\Omega_2}$ are equivalent as semigroup on $E$ (in the sense of Banach lattices or of Banach spaces). Then $N_1 = N_2$ and $\Omega_1$ and $\Omega_2$ are congruent.

Of course, for bounded domains, it is well-known that the spectrum alone does determine the dimension (see [Ka]).

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