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The operator-valued Marcinkiewicz multiplier theorem and maximal regularity

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Abstract. Given a closed linear operator on a UMD-space, we characterize maximal regularity of the non-homogeneous problem

$$u' + Au = f$$

with periodic boundary conditions in terms of R-boundedness of the resolvent. Here A is not necessarily generator of a C_0-semigroup. As main tool we prove an operator-valued discrete multiplier theorem. We also characterize maximal regularity of the second order problem for periodic, Dirichlet and Neumann boundary conditions.

Classical theorems on L^p-multipliers are no longer valid for operator-valued functions unless the underlying space is isomorphic to a Hilbert space (see Sect. 1 for precise statements of this result). However, recent work of Clément-de Pagter-Sukochev-Witvliet [CPSW], Weis [W1], [W2] and Clément-Prüss [CP] show that the right notion in this context is *R-boundedness* of sets of operators. This condition is strictly stronger than boundedness in operator norm (besides in the Hilbert space) and may be defined with help of the Rademacher functions. And indeed, Weis [W1] showed that Mikhlin's classical theorem on Fourier multipliers on $L^p(\mathbb{R}; X)$ holds if boundedness is replaced by R-boundedness (see [CP] for another proof based on results of Clément-de Pagter-Sukochev and Witvliet [CPSW]).

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Motivation of these investigations are regularity problems for differential equations in Banach spaces. Given the generator A of a holomorphic C_0-semigroup, the problem of *maximal regularity* of the inhomogeneous problem

$$P_0 \quad \begin{cases} u'(t) = Au(t) + f(t) \ t \in [0,1] \\ u(0) = 0 \end{cases}$$

with Dirichlet boundary conditions obtained much attention since the pioneering articles of Da Prato-Grisvard [DG] and Dore-Venni [DV]. And indeed, it is now possible to characterize maximal regularity of the problem P_0 in terms of R-boundedness of the resolvent (see Weis [W1], [W2] and Clément-Prüss [CP]). In the present article we study maximal regularity of the inhomogeneous problem with periodic boundary conditions

$$P_{per} \quad \begin{cases} u'(t) = Au(t) + f(t) \ \ t \in [0,2\pi] \\ u(0) = u(2\pi) \ . \end{cases}$$

Now it is no longer natural to suppose that A is the generator of a C_0-semigroup. We merely assume that A is a closed operator on a UMD-space. One of our main results (Theorem 2.3) says that P_{per} is strongly L^p-well-posed for $1 < p < \infty$ if and only if the set $\{k(ik - A)^{-1} : k \in \mathbb{Z}\}$ is R-bounded.
In order to treat the periodic case we need a multiplier theorem in the discrete case. Our main result of Sect. 1 is an operator-valued version of the Marcinkiewicz theorem which is very easy to formulate. It turns out that this discrete multiplier theorem is not only suitable for the treatment of the periodic problem P_{per} but gives an alternative approach to maximal regularity for P_0 (Sect. 5). It is possible to deduce our discrete multiplier theorem from a more complicated version by Štraklj and Weis [SW] whose formulation and proof are quite involved. So we prefer to give a direct and easy proof in Sect. 1.

Even though it became clear now that R-boundedness of resolvents is the right notion for maximal regularity, it is not easy to verify this condition in concrete cases. In Sect. 4 we show how R-boundedness of $|s|^\theta(is-A)^{-1}$ for $\theta \in (0,1)$ can be deduced from boundedness of $|s|(is - A)^{-1} \quad (s \in \mathbb{R})$. This is used to prove that the mild solutions of P_{per} are Hölder continuous. Again this result is true for arbitrary closed operators. We need some preparation to clarify the notion of mild solution in Sect. 3. If A generates a C_0-semigroup T, then it can be defined with help of the variation of constant formula, and by a result of Prüss [Pr], mild well-posedness of P_{per} is equivalent to $(I - T(2\pi))$ being invertible. We show in Sect. 3 that this in turn is equivalent to $((ik - A)^{-1})_{k\in\mathbb{Z}}$ being an L^p-multiplier. An analogous continuous version of this is proved by Latushkin and Shvydkoy [LS] in recent work.

Finally, in Sect. 6 we characterize strong L^p-well-posedness of the second order Cauchy problem with periodic, Dirichlet and Neumann boundary conditions in terms of R-boundedness. Again the results are valid for arbitrary closed operators.

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1. The operator-valued Marcinkiewicz multiplier theorem

Let X be a complex Banach space. We consider the Banach space $L^p(0, 2\pi; X)$ with norm

$$\|f\|_p := \left(\int_0^{2\pi} \|f(t)\|^p dt \right)^{\frac{1}{p}}$$

where $1 \le p < \infty$. For $f \in L^p(0, 2\pi; X)$ we denote by

$$\hat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt$$

the *k-th Fourier coefficient of f*, where $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, $x \in X$ we let $e_k(t) = e^{ikt}$ and $(e_k \otimes x)(t) = e_k(t)x \quad (t \in \mathbb{R})$. Then for $x_k \in X$, $k = -m, -m+1, \dots, m$,

$$f = \sum_{k=-m}^{m} e_k \otimes x_k$$

is the trigonometric polynomial given by

$$f(t) = \sum_{k=-m}^{m} e^{ikt} x_k \quad (t \in \mathbb{R}) \, .$$

Then $\hat{f}(k) = 0$ if $|k| > m$. The space $\mathbf{T}(X)$ of all trigonometric polynomials is dense in $L^p(0, 2\pi; X)$. In fact, let $f \in L^p(0, 2\pi; X)$. Then by Fejer's theorem, one has

(1.1) $$f = \lim_{n \to \infty} \sigma_n(f)$$

in $L^p(0, 2\pi; X)$ where

$$\sigma_n(f) := \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes \hat{f}(k) \, .$$

As an immediate consequence we note the *Uniqueness Theorem*. Let $f \in L^1(0, 2\pi; X)$.

a) If $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$, then $f(t) = 0 \quad t-$a.e.
b) If $\hat{f}(k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$, then $f(t) = \hat{f}(0) \quad t-$a.e.

Let X, Y be Banach spaces and let $\mathcal{L}(X, Y)$ be the set of all bounded linear operators from X to Y. If $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is a sequence, we consider the associated linear mapping

$$M : \mathbf{T}(X) \to \mathbf{T}(Y)$$

given by

$$M\left(\sum_k e_k \otimes x_k\right) = \sum_k e_k \otimes M_k x_k \ .$$

We say that the sequence $(M_k)_{k\in\mathbb{Z}}$ is an L^p*-multiplier*, if there exists a constant C such that

$$\left\|\sum_k e_k \otimes M_k x_k\right\|_p \leq C \left\|\sum_k e_k \otimes x_k\right\|_p$$

for all trigonometric polynomials $\sum\limits_k e_k \otimes x_k$.

Proposition 1.1. *Let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ be a sequence, then the following two statements are equivalent*

(i) $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier.
(ii) For each $f \in L^p(0, 2\pi; X)$ there exists $g \in L^p(0, 2\pi; Y)$ such that

$$\hat{g}(k) = M_k \hat{f}(k) \quad \textit{for all} \ \ k \in \mathbb{Z} \ .$$

In that case there exists a unique operator $M \in \mathcal{L}(L^p(0, 2\pi; X), L^p(0, 2\pi; Y))$ such that

$$(Mf)\hat{}(k) = M_k \hat{f}(k) \quad (k \in \mathbb{Z}) \tag{1.2}$$

for all $f \in L^p(0, 2\pi; X)$. We call M the operator associated with $(M_k)_{k\in\mathbb{Z}}$. *One has*

$$Mf = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes M_k \hat{f}(k) \tag{1.3}$$

in $L^p(0, 2\pi; Y)$ for all $f \in L^p(0, 2\pi; X)$.

Proof. $(i) \Rightarrow (ii)$. Define $\tilde{M} : \mathbf{T}(X) \to \mathbf{T}(Y)$ by $\tilde{M}(\sum e_k \otimes x_k) = \sum e_k \otimes M_k x_k$. Then by the assumption, $\tilde{M}$ has a unique extension $M \in \mathcal{L}(X,Y)$. Then (1.3) follows by continuity. Clearly, (1.3) implies (1.2).
$(ii) \Rightarrow (i)$. Define $Mf = g$ with $\hat{g}(k) = M_k \hat{f}(k) \quad (k \in \mathbb{Z})$. Then the uniqueness theorem and closed graph theorem show that $M \in \mathcal{L}(L^p(0,2\pi;X), L^p(0,2\pi;Y))$. □

Let $1 \leq q < \infty$. Denote by r_j the j-th Rademacher function on $[0,1]$. For $x \in X$ we denote by $r_j \otimes x$ the vector-valued function $t \mapsto r_j(t)x$.

Definition 1.2. A family $\mathbf{T} \subset \mathcal{L}(X,Y)$ is called *R-bounded* if there exist $c_q \geq 0$ such that

$$\left\| \sum_{j=1}^{n} r_j \otimes T_j x_j \right\|_{L^q(0,1;X)} \leq c_q \left\| \sum_{j=1}^{n} r_j \otimes x_j \right\|_{L^q(0,1;X)} \tag{1.4}$$

for all $T_1, \ldots, T_n \in \mathbf{T}$, $x_1, \cdots, x_n \in X$ and $n \in \mathbb{N}$, where $1 \leq q < \infty$. By Kahane's inequality [LT, Theorem 1.e.13] if such constant c_q exists for some $q \in [1,\infty)$, there also exists such constant for all $q \in [1,\infty)$. We denote by $R_q(\mathbf{T})$ the smallest constant c_q such that (1.4) holds. Sometimes we say that $\mathbf{T}$ *is R-bounded in* $\mathcal{L}(X,Y)$ to be more precise.

The concept of R-boundedness (read *Rademacher boundedness* or *randomized boundedness*) was introduced by Bourgain [Bo]. It is fundamental to recent work of Clément-de Pagter-Sukochev-Witvliet [CPSW], Weis [W1], [W2], Štrkalj-Weis [SW] and Clément-Prüss [CP]. We will use several basic results of [CPSW].

Now we can formulate the following multiplier theorem which is the discrete analog of the operator-valued version of Mikhlin's theorem due to Weis [W1] (see also [CP]).

Theorem 1.3 (Marcinkiewicz operator-valued multiplier theorem). *Let X, Y be UMD-spaces. Let $M_k \in \mathcal{L}(X,Y) \quad (k \in \mathbb{Z})$. If the sets $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ and $\{M_k : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k \in \mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.*

We need the following definition. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1.4. An *unconditional Schauder composition* of X is a family $\{\Delta_k : k \in \mathbb{N}_0\}$ of projection in $\mathcal{L}(X)$ such that

(a) $\Delta_k \Delta_\ell = 0$ if $k \neq \ell$
(b) $\sum_{k=0}^{\infty} \Delta_{\pi(k)} x = x$ for all $x \in X$ and for each permutation $\pi : \mathbb{N}_0 \to \mathbb{N}_0$.

The basic example for our purposes is the following

Example 1.5. Let X be a UMD-space. For $k \in \mathbb{N}$ define $\Delta_k f = \sum_{2^k \le |m| < 2^{k+1}} e_m \otimes \hat{f}(m)$ and $\Delta_0 f = e_{-1} \otimes \widehat{f}(-1) + e_0 \otimes \widehat{f}(0) + e_1 \otimes \widehat{f}(1)$, $(f \in L^p(0, 2\pi; X))$. Then $(\Delta_k)_{k \in \mathbb{N}_0}$ is an unconditional Schauder decomposition of $L^p(0, 2\pi; X)$.

The proof of this result is due to Bourgain [Bo]. However, in the case where $X = L^p(\Omega)$, $1 < p < \infty$, it can be deduced from the scalar case which is a central part of classical Littlewood-Paley theory (see [EG]).

The following lemma due to Clément-de Pagter-Sukochev-Witvliet [CPSW, Theorem 3.4] gives a sufficient condition for multipliers with respect to an unconditional Schauder decomposition.

Proposition 1.6. *Let $(\Delta_k)_{k \in \mathbb{N}_0}$ be an unconditional Schauder decomposition of a Banach space X. Let $\{T_k : k \in \mathbb{N}_0\} \subset \mathcal{L}(X)$ be an R-bounded sequence such that*

$$T_k \Delta_k = \Delta_k T_k$$

for all $k \in \mathbb{N}$. Then

$$Tx = \sum_{k=0}^{\infty} T_k \Delta_k x$$

converges for all $x \in X$ and defines an operator $T \in \mathcal{L}(X)$.

Besides Proposition 1.6 we need the following properties for the proof of the multiplier theorem.

Lemma 1.7 (Kahane's contraction principle [LT]). *One has*

$$\left\| \sum_{j=1}^{m} r_j \otimes \lambda_j x_j \right\|_p \le 2 \max_{j=1,\ldots,m} |\lambda_j| \left\| \sum_{j=1}^{m} r_j \otimes x_j \right\|_p$$

for all $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$, $x_1, \ldots, x_m \in X$.

Lemma 1.8. *Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then $\mathbf{S} \cdot \mathbf{T} = \{S \cdot T : S \in \mathbf{S}, T \in \mathbf{T}\}$ is R-bounded and*

$$R_p(\mathbf{S}T) \le R_p(\mathbf{S}) \cdot R_p(\mathbf{T}) \, .$$

This is easy to see.

Lemma 1.9 ([CPSW, Lemma 3.2]). *If $\mathbf{S} \subset \mathcal{L}(X, Y)$ is R-bounded, then*

$$R_p(co\mathbf{S}) \le 2 R_p(\mathbf{S})$$

$(1 \le p < \infty)$ *where*

$$co\mathbf{S} = \left\{ \sum_{j=1}^{m} \lambda_j S_j : S_j \in \mathbf{S}\ ,\ \lambda_j \in \mathbb{C}\ ,\ \sum_{j=1}^{m} |\lambda_j| \le 1, m \in \mathbb{N} \right\}\ .$$

If X is a UMD-space, then by a result of Burkholder [Bur],

$$R\left(\sum_{k=-N}^{N} e_k \otimes x_k\right) = \sum_{k=0}^{N} e_k \otimes x_k$$

defines an L^p-multiplier R for $1 < p < \infty$, which is called the *Riesz-projection*.

We need the following easy consequence.

Lemma 1.10. *Let X be a UMD-space and $1 < p < \infty$. Define the projections P_ℓ on $L^p(0, 2\pi; X)$ by*

$$P_\ell(\sum_{k\in\mathbb{Z}} e_k \otimes x_k) = \sum_{k\ge\ell} e_k \otimes x_k\ .$$

Then the set $\{P_\ell : \ell \in \mathbb{Z}\}$ is R-bounded in $\mathcal{L}(L^p(0, 2\pi; X))$.

Proof. For $n \in \mathbb{Z}$ let $S_n \in \mathcal{L}(L^p(0, 2\pi; X))$ be $S_n f = e_{-n} \cdot f$. Then $P_\ell = S_{-\ell} R S_\ell$. Since R is a bounded operator, it suffices to show that the set $\{S_\ell : \ell \in \mathbb{Z}\}$ is R-bounded in $\mathcal{L}(L^p(0, 2\pi; X))$. This is an easy consequence of Kahane's contraction principle. □

Proof of Theorem 1.3. a) We assume that $X = Y$. Let $Z = \{f \in L^p(0, 2\pi; X) : \hat{f}(k) = 0$ for all $k < 0\}$. Since the Riesz-projection is bounded it suffices to show that for some constant $C > 0$

$$\left\| \sum_{k=0}^{N} e_k \otimes M_k x_k \right\|_p \le C \left\| \sum_{k=0}^{N} e_k \otimes x_k \right\|_p$$

whenever $x_0, \ldots, x_N \in X$. Define $Q_n \in \mathcal{L}(Z)$ by

$$Q_n f = \sum_{2^{n-1} \le k < 2^n} e_k \otimes \hat{f}(k) \ \text{ for } n \in \mathbb{N}$$

and $Q_0 f = e_0 \otimes \hat{f}(0)$. It follows from Example 1.5 and the boundedness of the Riesz-projection, that the sequence $(Q_n)_{n\in\mathbb{N}_0}$ is an unconditional Schauder decomposition of Z. For each $k \in \mathbb{N}_0$ define $A_k \in \mathcal{L}(Z)$ by $(A_k f)(t) = M_k f(t) \quad (t \in [0, 2\pi])$. It follows from the assumption and Fubini's Theorem that the sets $\{A_k : k \in \mathbb{Z}\}$ and $\{k(A_{k+1} - A_k) : k \in \mathbb{Z}\}$ are R-bounded in $\mathcal{L}(Z)$.

Now let $f = \sum_{k\geq 0} e_k \otimes x_k \in Z$ be a trigonometric polynomial. Let

$$
\begin{aligned}
Tf &= \sum_{k\geq 0} e_k \otimes M_k x_k \\
&= \sum_{n\geq 0} Q_n \left(\sum_{k\geq 0} e_k \otimes M_k x_k \right) \\
&= \left(\sum_{n\geq 1} \left[\sum_{k=2^{n-1}}^{2^n-1} (P_k - P_{k+1}) A_k \right] \cdot Q_n \right) \sum_{k\geq 0} e_k \otimes x_k \\
&\quad + (P_0 - P_1) Q_0 \sum_{k\geq 0} e_k \otimes x_k \ .
\end{aligned}
$$

Thus

$$
\begin{aligned}
T &= \sum_{n\geq 1} \left[\sum_{k=2^{n-1}}^{2^n-1} A_k (P_k - P_{k+1}) \right] Q_n + (P_0 - P_1) A_0 Q_0 \\
&= \sum_{n\geq 1} A_{2^{n-1}} P_{2^{n-1}} Q_n - \sum_{n\geq 1} A_{2^n-1} P_{2^n} Q_n \\
&\quad + \sum_{n\geq 1} \left[\sum_{k=2^{n-1}+1}^{2^n-1} (A_k - A_{k-1}) P_k \right] Q_n + (P_0 - P_1) A_0 Q_0
\end{aligned}
$$

Since $(Q_n)_{n\in\mathbb{N}_0}$ is an unconditional Schauder decomposition of Z by Proposition 1.6 and Lemma 1.8, $\sum_{n\geq 1} A_{2^{n-1}} P_{2^{n-1}} Q_n$, $\sum_{n\geq 1} A_{2^n-1} P_{2^n} Q_n$ and $(P_0 - P_1) A_0 Q_0$ define bounded linear operators on Z. In order to estimate the third term observe that $\sum_{k=2^{n-1}+1}^{2^n-1} \frac{1}{k-1} \leq 1$. Hence by Lemma 1.9 and 1.8,

$$
\begin{aligned}
& R_p \left(\left\{ \sum_{k=2^{n-1}+1}^{2^n-1} P_k (A_k - A_{k-1}) : n \in \mathbb{N} \right\} \right) \\
&= R_p \left(\left\{ \sum_{k=2^{n-1}+1}^{2^n-1} \frac{1}{k-1} (k-1)(A_k - A_{k-1}) P_k : n \in \mathbb{N} \right\} \right) \\
&\leq 2 R_p (\{ (k-1)(A_k - A_{k-1}) P_k : k \in \mathbb{N} \} \\
&\leq 2 R_p \{ k (A_{k+1} - A_k) : k \in \mathbb{Z} \} \cdot R_p (\{ P_k : k \in \mathbb{Z} \}) < \infty \ .
\end{aligned}
$$

This finishes the proof if $X = Y$.

b) Now we consider the general case. Since X and Y are UMD-spaces, also

$X \oplus Y$ is a UMD-space. Define $\tilde{M}_k \in \mathcal{L}(X \oplus Y)$ by $\tilde{M}_k(x, y) = (0, M_k x)$. It follows from the case a) that $(\tilde{M}_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier. It is easy to see that this implies that $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier $(1 < p < \infty)$. □

Next we show that, conversely, R-boundedness is a necessary condition for L^p-multipliers.

Proposition 1.11. *Let X be a Banach space and let $(M_k)_{k\in\mathbb{Z}}$ be an L^p-multiplier, where $1 \le p < \infty$. Then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

Proof. By Kahane's contraction principle we have for $\eta_j \in \mathbb{R},\ x_j \in X$

$$\begin{aligned}\left\|\sum_j r_j \otimes e^{i\eta_j} x_j\right\|_{L^p(0,1;X)} &\le 2\left\|\sum_j r_j \otimes x_j\right\|_{L^p(0,1;X)}\\ &= 2\left\|\sum_j r_j \otimes e^{-i\eta_j} e^{i\eta_j} x_j\right\|_{L^p(0,1;X)}\\ &\le 4\left\|\sum_j r_j \otimes e^{i\eta_j} x_j\right\|_{L^p(0,1;X)}.\end{aligned}$$

By assumption there exists $c \ge 0$ such that

$$\left\|\sum_k e_k \otimes M_k x_k\right\|_{L^p(0,2\pi;X)} \le c\left\|\sum_k e_k \otimes x_k\right\|_{L^p(0,2\pi;X)}.$$

Hence for all $t \in [0, 2\pi]$

$$\left\|\sum_j r_j \otimes M_j x_j\right\|_{L^p(0,1;X)} \le 2\left\|\sum_j r_j \otimes e_j(t) M_j x_j\right\|_{L^p(0,1;X)}.$$

Integrating over $t \in [0, 2\pi]$ yields

$$\begin{gathered}2\pi\left\|\sum_j r_j \otimes M_j x_j\right\|^p_{L^p(0,1;X)} \le\\ 2^p \int_0^{2\pi}\int_0^1 \left\|\sum_j r_j(s) e_j(t) M_j x_j\right\|^p ds dt =\\ 2^p \int_0^1\int_0^{2\pi} \left\|\sum_j e_j(t) r_j(s) M_j x_j\right\|^p dt ds \le\end{gathered}$$

$$2^p c^p \int_0^1 \left\| \sum_j e_j(t) r_j(s) x_j \right\|^p dtds =$$

$$2^p c^p \int_0^{2\pi} \left\| \sum_j r_j \otimes e_j(t) x_j \right\|^p_{L^p(0,1;X)} dt \leq$$

$$2^p c^p 2^p \int_0^{2\pi} \left\| \sum_j r_j \otimes x_j \right\|^p_{L^p(0,1;X)} dt =$$

$$2^p c^p 2^p 2\pi \| \sum_j r_j \otimes x_j \|^p_{L^p(0,1;X)} .$$

Thus $R_p\{M_k : k \in \mathbb{Z}\} \leq 4c$. □

We conclude this section by several comments about optimality of the operator-valued Marcinkiewicz theorem above (Theorem 1.3).

First of all we remark that on a Hilbert space X each bounded sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ is an L^2-multiplier. This follows from the fact that the Fourier transform given by

$$f \in L^2(0, 2\pi; X) \mapsto (\hat{f}(k))_{k\in\mathbb{Z}} \in \ell^2(X)$$

is an isometric isomorphism if X is a Hilbert space. On the other hand, if X is not isomorphic to a Hilbert space then there always exists a bounded sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$, which may even be chosen lacunary, such that $(M_k)_{k\in\mathbb{Z}}$ is not an L^2-multiplier.

This phenomenon had been discovered by G. Pisier (unpublished). We want to explain this in more detail and give some extensions showing in particular that Theorem 1.3 holds merely on Hilbert spaces if R-boundedness is replaced by boundedness.

A sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ is called *lacunary* if $M_k = 0$ for all $k \in \mathbb{Z} \setminus \{\pm 2^m : m \in \mathbb{N}_0\} \cup \{0\}$. We recall the following inequality due to Pisier [Pi1]: for $1 \leq p < \infty$, there exist $\alpha, \beta > 0$ such that

$$\alpha \left\| \sum_j r_j \otimes x_j \right\|_{L^p(0,1;X)} \leq \left\| \sum_j e_{2^j} \otimes x_j \right\|_{L^p(0,2\pi;X)} \leq \beta \left\| \sum_j r_j \otimes x_j \right\|_{L^p(0,1;X)}$$

which holds for all $x_j \in X$, where X is an arbitrary Banach space.

Recall that a Banach space X is *of type* $1 \le p \le 2$ if, there exists $C > 0$ such that for $x_1, x_2, \cdots, x_n \in X$, we have

$$\left\| \sum_{j=1}^{n} r_j \otimes x_j \right\|_{L^2(0,1;X)} \le C \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{1/p} .$$

X is *of cotype* $2 \le q \le \infty$ if, there exists $C' > 0$ such that for $x_1, x_2, \cdots,$ $x_n \in X$, we have

$$(\sum_{j=1}^{n} \|x_j\|^q)^{1/q} \le C' \| \sum_{j=1}^{n} r_j \otimes x_j \|_{L^2(0,1;X)}.$$

(with the usual modification if $q = \infty$) [Pi2] (see also [LT]). It is well known that every Banach space is of type 1 and of cotype ∞, and for every measure space (Ω, Σ, μ) and for every $1 \le p < \infty$, the space $L^p(\Omega, \Sigma, \mu)$ is of type $Min(2,p)$ and of cotype $Max(2,p)$. Kwapien has shown that a Banach space X is isomorphic to a Hilbert space if and only if X is of type 2 and of cotype 2 [Kw] (see also [LT, p. 73, 74]). Finally, a Banach space is said to have a non trivial type if it is of type p for some $1 < p \le 2$.

Proposition 1.12. *Let X be a Banach space and $1 < p < \infty$. Then the following assertions are equivalent:*

(i) X has a non trivial type;
(ii) for every Banach space Y, each lacunary R-bounded sequence in $\mathcal{L}(X,Y)$ defines an L^p-multiplier;
(iii) each lacunary R-bounded sequence in $\mathcal{L}(X)$ defines an L^p-multiplier.

Proof. $(i) \Rightarrow (ii)$. Assume that X has a non trivial type and let Y be a Banach space. By Lemma 6 of [Le] (see also [Pi2]), there exists $C > 0$ such that for $f \in L^p(0, 2\pi; X)$,

$$\left\| \sum_{n \ge 0} e_{2^n} \otimes \hat{f}(2^n) \right\|_{L^p(0,2\pi;X)} \le C \|f\|_{L^p(0,2\pi;X)}$$

$$\left\| \sum_{n \ge 0} e_{-2^n} \otimes \hat{f}(-2^n) \right\|_{L^p(0,2\pi;X)} \le C \|f\|_{L^p(0,2\pi;X)}.$$

Let $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X,Y)$ be a lacunary R-bounded sequence. By Pisier's inequality,

$$\left\| \sum_{k \in \mathbb{Z}} e_k \otimes M_k \hat{f}(k) \right\|_{L^p(0,2\pi;Y)}$$

$$\leq \left\|\sum_{k\geq 0} e_k \otimes M_k \hat{f}(k)\right\|_{L^p(0,2\pi;Y)} + \left\|\sum_{k<0} e_k \otimes M_k \hat{f}(k)\right\|_{L^p(0,2\pi;Y)}$$

$$\leq \beta \left(\left\|\sum_{k\geq 0} r_{k+1} \otimes M_{2^k} \hat{f}(2^k)\right\|_{L^p(0,1;Y)} \right.$$

$$\left. + \left\|\sum_{k\geq 0} r_{k+1} \otimes M_{-2^k} \hat{f}(-2^k)\right\|_{L^p(0,1;Y)} \right) + \|M_0\| \|f\|_{L^p(0,2\pi;X)}$$

$$\leq \beta R_p(M_k : k \in \mathbb{Z}) \left(\left\|\sum_{k\geq 0} r_{k+1} \otimes \hat{f}(2^k)\right\|_{L^p(0,1;X)} \right.$$

$$\left. + \left\|\sum_{k\geq 0} r_{k+1} \otimes \hat{f}(-2^k)\right\|_{L^p(0,1;X)} \right) + \|M_0\| \|f\|_{L^p(0,2\pi;X)}$$

$$\leq \alpha^{-1} \beta R_p(M_k : k \in \mathbb{Z}) \left(\left\|\sum_{k\geq 0} e_{2^k} \otimes \hat{f}(2^k)\right\|_{L^p(0,2\pi;X)} \right.$$

$$\left. + \left\|\sum_{k\geq 0} e_{-2^k} \otimes \hat{f}(-2^k)\right\|_{L^p(0,2\pi;X)} \right) + \|M_0\| \|f\|_{L^p(0,2\pi;X)}$$

$$\leq (2\alpha^{-1}\beta C R_p(M_k : k \in \mathbb{Z}) + \|M_0\|) \|f\|_{L^p(0,2\pi;X)} \, .$$

This shows that $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier.
$(ii) \Rightarrow (iii)$ is trivial by taking $Y = X$.
$(iii) \Rightarrow (i)$. Assume that every lacunary R-bounded sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ defines an L^p-multiplier. Define $M_k = I$ if $k = 2^n$ for some $n \in \mathbb{N}_0$ and $M_k = 0$ otherwise. Then $(M_k)_{k\in\mathbb{Z}}$ is lacunary and R-bounded, by assumption there exists $C' > 0$ such that for $f \in L^p(0, 2\pi; X)$,

$$\left\|\sum_{n\geq 0} e_{2^n} \otimes \hat{f}(2^n)\right\|_{L^p(0,2\pi;X)} \leq C' \|f\|_{L^p(0,2\pi;X)}.$$

This implies that the closed subspace of $L^p(0, 2\pi; X)$ generated by $\{e_{2^n} \otimes x_n : n \in \mathbb{N}_0, \ x_n \in X\}$ is complemented in $L^p(0, 2\pi; X)$. By Lemma 6 of [Le] X has a non trivial type. □

If X is isomorphic to a Hilbert space, then a subset $\mathbf{T}$ of $\mathcal{L}(X)$ is R-bounded if and only if it is bounded. Actually the following more general proposition holds. The authors are indebted to C. Le Merdy and G. Pisier for communicating them this result. We include the short proof for completeness.

Proposition 1.13. *Let X and Y be Banach spaces. Then the following assertions are equivalent:*

(i) X is of cotype 2 and Y is of type 2;
(ii) Each bounded subset in $\mathcal{L}(X,Y)$ is R-bounded.

Proof. Assume that X is of cotype 2 and Y is of type 2. Let $M \subset \mathcal{L}(X,Y)$ be a bounded subset and let $C, C' > 0$ be the constants in the definitions of type and cotype. Then for $T_1, T_2, \cdots, T_n \in M,\ x_1, x_2, \cdots, x_n \in X$, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n} r_j \otimes T_j x_j\right\|_{L^2(0,1;Y)} &\leq C(\sum_{j=1}^{n} \|T_j x_j\|^2)^{1/2} \\ &\leq C sup_{T\in M}\|T\| \left(\sum_{j=1}^{n} \|x_j\|^2\right)^{1/2} \\ &\leq CC' sup_{T\in M}\|T\| \left\|\sum_{j=1}^{n} r_j \otimes x_j\right\|_{L^2(0,1;X)} .\end{aligned}$$

This shows that M is R-bounded.

Conversely, assume that each bounded set in $\mathcal{L}(X,Y)$ is R-bounded. Let $e \in X$, $e^* \in X^*$ such that $\langle e, e^*\rangle = \|e\| = \|e^*\| = 1$.
Then the set $\mathbf{T} = \{e^* \otimes y : y \in Y, \|y\| \leq 1\}$ is R-bounded, by assumption. Let $y_1, \ldots, y_m \in Y$. Then $T_j = e^* \otimes \frac{y_j}{\|y_j\|} \in \mathbf{T}$. Hence

$$\begin{aligned}\left\|\sum_{j=1}^{m} r_j \otimes y_j\right\|_{L^2(0,1;Y)} &= \left\|\sum_{j=1}^{m} r_j \otimes \|y_j\right\| \cdot T_j e\|_{L^2(0,1;Y)} \\ &\leq R_2(\mathbf{T}) \left\|\sum_{j=1}^{m} r_j \otimes \right\| y_j\| \cdot e\|_{L^2(0,1;X)} \\ &= R_2(\mathbf{T}) \left(\sum_{j=1}^{m} \|y_j\|^2\right)^{\frac{1}{2}} .\end{aligned}$$

This shows that Y is of type 2. In order to prove that X is of cotype 2, let $\alpha := R_2(\{x^* \otimes f : x^* \in X^*, \|x^*\| \le 1\})$ which is finite by assumption, where $f \in Y$, $\|f\| = 1$ is fixed. Let $x_1, \ldots, x_m \in X$. Choose $x_j^* \in X^*$ such that $\|x_j^*\| = 1$ and $\langle x_j^*, x_j\rangle = \|x_j\|$. Let $S_j = x_j^* \otimes f$. Then

$$\begin{aligned}\left(\sum_{j=1}^m \|x_j\|^2\right)^{1/2} &= \left\|\sum_{j=1}^m r_j \otimes \|x_j\|\right\| \cdot f\|_{L^2(0,1;Y)} \\ &= \left\|\sum_{j=1}^m r_j \otimes S_j x_j\right\|_{L^2(0,1;Y)} \\ &\le \alpha \left\|\sum_{j=1}^m r_j \otimes x_j\right\|_{L^2(0,1;X)} .\end{aligned}$$

This proves that X is of cotype 2. □

In view of Proposition 1.13 we may now formulate the following interesting special case of the Marcinkiewicz multiplier theorem.

Corollary 1.14. *Let* $X = L^{p_1}(\Omega, \Sigma, \mu)$, $Y = L^{p_2}(\Omega, \Sigma, \mu)$ *where* $1 < p_1 \le 2 \le p_2 < \infty$ *and* (Ω, Σ, μ) *is a measure space. Then each bounded sequence* $(M_k)_{k\in\mathbb{Z}}$ *in* $\mathcal{L}(X, Y)$ *satisfying* $\sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty$ *is an* L^p*-multiplier for each* $1 < p < \infty$.

Proposition 1.13 shows that in Proposition 1.12 R-boundedness may not be replaced by boundedness, unless X is a Hilbert space. More precisely, the following holds.

Proposition 1.15. *Let* X *be a Banach space and* $1 < p < \infty$. *The following assertions are equivalent:*

(i) X *is isomorphic to a Hilbert space;*
(ii) Each bounded lacunary sequence in $\mathcal{L}(X)$ *is an* L^p*-multiplier.*

Proof. $(i) \Rightarrow (ii)$. Let X be a Banach space isomorphic to a Hilbert space and $(M_k)_{k\in\mathbb{Z}}$ be a bounded lacunary sequence. Then $(M_k)_{k\in\mathbb{Z}}$ is R-bounded by Proposition 1.13 as X is of type 2 and of cotype 2 (or by direct verification). Proposition 1.12 shows that the sequence is an L^p-multiplier.
$(ii) \Rightarrow (i)$. It follows from the assumption and Proposition 1.11 that each bounded sequence in $\mathcal{L}(X)$ is R-bounded. By Proposition 1.13 and Kwapien's Theorem, this implies that X is isomorphic to a Hilbert space. □

Remark 1.16. Using Proposition 1.12 and the same argument as in the proof of Proposition 1.15, we can easily establish the following: If X has a non trivial type, then X is of cotype 2 and Y is of type 2 if and only if each bounded lacunary sequence in $\mathcal{L}(X,Y)$ is an L^p-multiplier.

The following proposition shows that we can not replace the R-boundedness in Theorem 1.3 by boundedness in operator norm unless the underlying Banach spaces X is of cotype 2 and Y is of type 2 (when $X = Y$, this is equivalent to say that X is isomorphic to a Hilbert space).

Proposition 1.17. *Let X and Y be UMD-spaces. Then the following assertions are equivalent:*

(i) X is of cotype 2 and Y is of type 2;
(ii) There exists $1 < p < \infty$ such that each sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ satisfying $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$ and $\sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty$ is an L^p-multiplier.

Proof. $(i) \Rightarrow (ii)$. Assume that X is of cotype 2 and Y is of type 2, then by Proposition 1.13 each bounded subset in $\mathcal{L}(X,Y)$ is actually R-bounded, so the result follows from Theorem 1.3.

$(ii) \Rightarrow (i)$. Assume that for some $1 < p < \infty$, each sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ satisfying $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$ and $\sup_{k\in\mathbb{Z}} \|k(M_{k+1}-M_k)\| < \infty$ defines an L^p-multiplier. Let $(M_k)_{k\geq 0} \subset \mathcal{L}(X,Y)$ be a bounded sequence. Define $(N_n)_{n\in\mathbb{Z}} \in \mathcal{L}(X,Y)$ by

$$N_n = \begin{cases} 0 & \text{if } n \leq 0 \\ M_k & \text{if } n = 2^k \text{ for some } k \geq 0 \\ M_k + \frac{n-2^k}{2^k}(M_{k+1} - M_k) & \text{if } 2^k \leq n < 2^{k+1} \text{ for some } k \geq 0 \,. \end{cases}$$

Then one can easily verify that

$$\sup_{n\in\mathbb{Z}} \|N_n\| = \sup_{k\geq 0} \|M_k\| < \infty$$

$$\sup_{n\in\mathbb{Z}} \|n(N_{n+1} - N_n)\| \leq 4 \sup_{k\geq 0} \|M_k\| < \infty.$$

Therefore the sequence $(N_n)_{n\in\mathbb{Z}}$ is an L^p-multiplier by assumption. By Proposition 1.11 this implies that the sequence $(N_n)_{n\in\mathbb{Z}}$ is R-bounded, in particular the sequence $(M_k)_{k\geq 0}$ is R-bounded. We deduce from this that each bounded subset in $\mathcal{L}(X,Y)$ is actually R-bounded, By Proposition 1.13, this implies that X is of cotype 2 and Y is of type 2. □

Finally, we remark that in the scalar case more general conditions are known to be sufficient in Theorem 1.3. Let $(M_k)_{k\in\mathbb{Z}}$ be a bounded scalar sequence. Instead of assuming that $k(M_{k+1} - M_k)$ is bounded, it suffices to assume that

$$\sup_{j\in\mathbb{N}} \sum_{2^j \le |k| < 2^{j+1}} |M_{k+1} - M_k| < \infty$$

in order to deduce that $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$ (see [EG, Chapter 8]). This is the classical Marcinkiewicz multiplier theorem. Štrkalj and Weis [SW] give an operator-valued version of this result, where the absolute value is replaced by a certain norm $\| \ \|_{\mathbf{T}}$ which is defined as the gauge function of an R-bounded set $\mathbf{T}$ in $\mathcal{L}(X)$. Actually, our Theorem 1.3 can be deduced from the results in [SW], but the proofs given there are more complicated. They depend in particular on the work by Zimmermann [Zi].

2. Strong L^p-well posedness of the periodic problem

We first introduce periodic Sobolev spaces. Let X be a Banach space.

Lemma 2.1. *Let $1 \le p < \infty$ and let $u, u' \in L^p(0, 2\pi; X)$. The following are equivalent:*

(i) $\int_0^{2\pi} u'(t)dt = 0$ *and there exists $x \in X$ such that*

$$u(t) = x + \int_0^t u'(s)ds \quad \textit{a.e. on } [0, 2\pi] \ ;$$

(ii) $(u')\hat{}(k) = ik\hat{u}(k) \quad (k \in \mathbb{Z})$.

Proof. $(i) \Rightarrow (ii)$. Let $k \in \mathbb{Z} \setminus \{0\}$. Integration by parts yields,

$$\hat{u}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \int_0^t u'(s)dsdt = \frac{1}{ik}\widehat{u'}(k) \ .$$

Since $\widehat{u'}(0) = \frac{1}{2\pi} \int_0^{2\pi} u'(s)ds = 0$, assertion *(ii)* is proved.

$(ii) \Rightarrow (i)$. Let $v(t) = \int_0^t u'(s)ds$. Since $\widehat{u'}(0) = 0$ one has $v(2\pi) = 0$. As above one has $\hat{v}(k) = \frac{1}{ik}\widehat{u'}(k) = \hat{u}(k)$ for $k \in \mathbb{Z} \setminus \{0\}$. Thus $u - v$ is a constant function. □

If $u \in L^p(0,2\pi;X)$, then there exists at most one $u' \in L^p(0,2\pi;X)$ such that the equivalent conditions of Lemma 2.1 are satisfied. We denote by $H^{1,p}_{per}$ the space of all $u \in L^p(0,2\pi;X)$ such that there exists $u' \in L^p(0,2\pi;X)$ such that the equivalent conditions of Lemma 2.1 are satisfied. If $u \in H^{1,p}_{per}$, it follows from *(i)* that u has a unique continuous representative. We always identify u with this continuous function. Thus we have

$$u(t) = u(0) + \int_0^t u'(s)ds \quad (t \in [0,2\pi])$$

and $u(0) = u(2\pi)$ for all $u \in H^{1,p}_{per}$.

Next we describe multipliers mapping $L^p(0,2\pi;X)$ into $H^{1,p}_{per}$.

Lemma 2.2. *Let* $1 \leq p < \infty$. *Let* $M_k \in \mathcal{L}(X)$ $(k \in \mathbb{Z})$. *The following assertions are equivalent:*

(i) $(M_k)_{k\in\mathbb{Z}}$ *is an* L^p*-multiplier such that the associated operator* $M \in \mathcal{L}(L^p(0,2\pi;X))$ *maps* $L^p(0,2\pi;X)$ *into* $H^{1,p}_{per}$*;*
(ii) $(kM_k)_{k\in\mathbb{Z}}$ *is an* L^p*-multiplier.*

Proof. $(i) \Rightarrow (ii)$. Let $f \in L^p(0,2\pi;X)$. By assumption, there exists $g \in H^{1,p}_{per}$ such that $\hat{g}(k) = M_k\hat{f}(k) \quad (k \in \mathbb{Z})$. Hence $\widehat{g'}(k) = ikM_k\hat{f}(k)(k \in \mathbb{Z})$.
$(ii) \Rightarrow (i)$. Let $f \in L^p(0,2\pi;X)$. By assumption, there exists $v \in L^p(0,2\pi;X)$ such that $ikM_k\hat{f}(k) = \hat{v}(k)$ for all $k \in \mathbb{Z}$. In particular, $\int_0^{2\pi} v(t)dt = 2\pi\hat{v}(0) = 0$. Let $w(t) = \int_0^t v(s)ds$. Then for $k \in \mathbb{Z}\setminus\{0\}$, $\hat{w}(k) = \frac{1}{ik}\hat{v}(k) = M_k\hat{f}(k)$. Let $u = w + e_0 \otimes (M_0\hat{f}(0) - \hat{w}(0))$. Then $u(t) = x + \int_0^t v(s)ds$ where $x = u(0)$. Hence $u \in H^{1,p}_{per}$. Moreover, $\hat{u}(k) = \hat{w}(k) = M_k\hat{f}(k)$ for $k \in \mathbb{Z}\setminus\{0\}$ and $\hat{u}(0) = M_0\hat{f}(0)$. □

Now let A be a closed operator on X. For $1 \leq p < \infty$, $f \in L^p(0,2\pi;X)$, we consider the problem

$$P_{per}\begin{cases} u'(t) = Au(t) + f(t) \ t \in (0,2\pi) \\ u(0) = u(2\pi)\ . \end{cases}$$

By a *strong* L^p*-solution* we understand a function $u \in H^{1,p}_{per}$ such that $u(t) \in D(A)$ and $u'(t) = Au(t) + f(t)$ for almost all $t \in [0,2\pi]$.

Theorem 2.3 (Strong L^p-well-posedness). *Assume that X is a UMD-space. Let $1 < p < \infty$. Then the following assertions are equivalent:*

(i) *For each $f \in L^p(0, 2\pi; X)$ there is a unique strong L^p-solution of P_{per};*
(ii) *$i\mathbb{Z} \subset \varrho(A)$ and $(kR(ik, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier;*
(iii) *$i\mathbb{Z} \subset \varrho(A)$ and the sequence $(kR(ik, A))_{k\in\mathbb{Z}}$ is R-bounded.*

We say that P_{per} is strongly L^p-well-posed *if these equivalent conditons hold.*

Proof. $(i) \Rightarrow (ii)$. Let $k \in \mathbb{Z}$. Let $y \in X$, $f = e_k \otimes y$. There exists $u \in H^{1,p}_{per}$ such that $u' = Au + f$. Taking Fourier transforms on both sides we obtain that $\hat{u}(k) \in D(A)$ and $ik\hat{u}(k) = \widehat{u'}(k) = A\hat{u}(k) + \hat{f}(k) = A\hat{u}(k) + y$. Thus $(ik - A)$ is surjective. If $(ik - A)x = 0$, then $u(t) = e_k \otimes x$ defines a periodic solution of $u' = Au$. Hence $u = 0$ by the assumption of uniqueness. We have shown that $(ik - A)$ is bijective. Since A is closed we deduce that $ik \in \varrho(A)$.
Next we show that $(kR(ik,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. Let $f \in L^p(0, 2\pi; X)$. By assumption, there exists a unique $u \in H^{1,p}_{per}$ such that $u' = Au + f$. Taking Fourier transforms, we deduce that $\hat{u}(k) \in D(A)$ and $ik\hat{u}(k) = A\hat{u}(k) + \hat{f}(k)$; i.e., $\hat{u}(k) = R(ik, A)\hat{f}(k)$ for all $k \in \mathbb{Z}$. Consequently, $\widehat{u'}(k) = ik\hat{u}(k) = ikR(ik, A)\hat{f}(k)$ for all $k \in \mathbb{Z}$. This proves the claim.
$(ii) \Rightarrow (i)$. Let $f \in L^p(0, 2\pi; X)$. By Lemma 2.2 there exists $u \in H^{1,p}_{per}$ such that $\hat{u}(k) = R(ik, A)\hat{f}(k)$ for all $k \in \mathbb{Z}$. Since $AR(ik, A) = ikR(ik, A) - I$, the sequence $(AR(ik, A))_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ is an L^p-multiplier. Observe that A^{-1} is an isomorphism of X onto $D(A)$ (seen as a Banach space with the graph norm). Hence, $(R(ik, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier in $\mathcal{L}(X, D(A))$. This shows that $u \in L^p(0, 2\pi; D(A))$. Since $\widehat{u'}(k) = ik\hat{u}(k) = ikR(ik, A)\hat{f}(k) = AR(ik, A)\hat{f}(k) + \hat{f}(k) = A\hat{u}(k) + \hat{f}(k)$ for all $k \in \mathbb{Z}$, one has $u' = Au + f$ by the uniqueness theorem; i.e., u is a strong solution of P_{per}. It remains to show uniqueness. If $u \in H^{1,p}_{per} \cap L^p(0, 2\pi; D(A))$ such that $u'(t) = Au(t) \quad (t \in (0, 2\pi))$, then $\hat{u}(k) \in D(A)$ and $ik\hat{u}(k) = A\hat{u}(k)$. Since $ik \in \varrho(A)$ this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$.
$(ii) \Rightarrow (iii)$ follows from Proposition 1.11.
$(iii) \Rightarrow (ii)$. Let $M_k = ikR(ik, A)$. We show that the set $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ is R-bounded. Then *(ii)* follows Theorem 1.3. One has

$$\begin{aligned} k(M_{k+1} - M_k) &= \\ k(i(k+1)R(i(k+1), A) - ikR(ik, A)) &= \\ ikR(i(k+1), A)((k+1)(ik - A) - k(i(k+1) - A))R(ik, A) &= \\ ikR((i(k+1), A)(-A)R(ik, A) &= \end{aligned}$$

$$ikR(i(k+1), A)(I - ikR(ik, A)) .$$

Since the product of R-bounded sequences is R-bounded, the claim follows. □

Corollary 2.4. *Let X be a UMD-space. If there exists $p \in (1, \infty)$ such that P_{per} is strongly L^p-well-posed then P_{per} is strongly L^p-well-posed for all $p \in (1, \infty)$.*

We conclude this section mentioning analogue non-discrete results. In one of the first investigation on maximal regularity, Mielke [Mi] studies strong L^p-well-posedness on the entire line. This leads to ”bisectorial operators” which are not necessarily generators of C_0-semigroups. Mielke proves p-independance of maximal regularity and gives a characterization on Hilbert spaces. Further results on UMD-spaces are obtained recently by Schweiker [Sch].

3. Mild solutions

Let A be a closed operator on a Banach space X. Let $f \in L^1(0, 2\pi; X)$. A function $u \in C([0, 2\pi]; X)$ is called a *mild solution* of the problem P_{per} (see Sect. 2) if $u(0) = u(2\pi)$ and

$$\begin{cases} \int_0^t u(s)ds \in D(A) \text{ and} \\ u(t) - u(0) = A \int_0^t u(s)ds + \int_0^t f(s)ds \end{cases} \tag{3.1}$$

for all $t \in [0, 2\pi]$. It is clear that every strong L^p-solution is a mild solution. Conversely, if u is a mild solution and $u \in H^{1,p}_{per}$, then u is a strong L^p-solution. We want to describe mild solutions in terms of the Fourier coefficients.

For this we need the following lemma.

Lemma 3.1. *Let $f, g \in L^p(0, 2\pi; X)$, where $1 \leq p < \infty$. Then the following are equivalent.*

(i) $f(t) \in D(A)$ *and* $Af(t) = g(t)$ *a.e.;*
(ii) $\hat{f}(k) \in D(A)$ *and* $A\hat{f}(k) = \hat{g}(k)$ *for all* $k \in \mathbb{Z}$.

Proof. $(i) \Rightarrow (ii)$. This follows from the closedness of A (cf. [ABHN, Proposition 1.1.7]).
$(ii) \Rightarrow (i)$. There exists n_ℓ converging to ∞ as $\ell \to \infty$ such that $\sigma_{n_\ell}(f)(t) \to f(t)$ a.e. and $\sigma_{n_\ell}(g)(t) \to g(t)$ a.e. as $\ell \to \infty$ (cf. (1.1)). Since $\sigma_{n_\ell}(f)(t) \in D(A)$ and $A\sigma_{n_\ell}(f)(t) = \sigma_{n_\ell}(g)(t)$ for all $t \in [0, 2\pi]$, the claim follows from the closedness of A. □

Proposition 3.2. *Let $u \in C([0,2\pi],X)$ such that $u(0) = u(2\pi)$. Assume that $\overline{D(A)} = X$. Then u is a mild solution of P_{per} if and only if*

$$\hat{u}(k) \in D(A) \ \text{ and } \ (ik - A)\hat{u}(k) = \hat{f}(k) \ \text{ for all } \ k \in \mathbb{Z}\,. \tag{3.2}$$

Proof. 1. Assume that u is a mild solution. Letting $t = 2\pi$ in (3.1) we see that $\hat{u}(0) \in D(A)$ and $-A\hat{u}(0) = \hat{f}(0)$. Consider the functions $v(t) = \int_0^t u(s)ds$ and $g(t) = u(t) - u(0) - \int_0^t f(s)ds$. Then by Lemma 3.1, $\hat{v}(k) \in D(A)$ and $A\hat{v}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$. For $k \neq 0$ we have $\hat{v}(k) = -\frac{1}{ik}\hat{u}(0) + \frac{1}{ik}\hat{u}(k)$, and $\hat{g}(k) = \hat{u}(k) + \frac{1}{ik}\hat{f}(0) - \frac{1}{ik}\hat{f}(k)$. Since $-A\hat{u}(0) = \hat{f}(0)$ we obtain (3.2).
2. Conversely, assume that (3.2) holds. Let $x^* \in D(A^*)$. By [ABHN, Proposition B.10], it suffices to show that

$$\int_0^t \langle u(s)\,,\ A^*x^*\rangle ds = \langle u(t)\,,\ x^*\rangle - \langle u(0)\,,\ x^*\rangle - \int_0^t \langle f(s)\,,\ x^*\rangle ds\,.$$

Consider the function $w(s) = \langle u(s),\, A^*x^*\rangle + \langle f(s),\, x^*\rangle$. Then $\hat{w}(k) = ik\langle \hat{u}(k), x^*\rangle$ for all $k \in \mathbb{Z}$ by assumption (3.2). Consider $g(t) = \int_0^t w(s)ds - \langle u(t),\, x^*\rangle$. Then for $k \in \mathbb{Z} \setminus \{0\}$, $\hat{g}(k) = -\frac{1}{ik}\hat{w}(0) + \frac{1}{ik}\hat{w}(k) - \langle \hat{u}(k), x^*\rangle = 0$ since $\hat{w}(0) = 0$. It follows from the Uniqueness Theorem that g is constant; i.e. $g(t) = g(0) = -\langle u(0), x^*\rangle$ for all $t \in [0,2\pi]$. This is precisely what we claimed. □

As a corollary we obtain the following characterization of uniqueness of mild solutions of P_{per}. By $\sigma_p(A)$ we denote the set of all eigenvalues of A.

Corollary 3.3. *The following assertions are equivalent:*

(i) For all $f \in L^1(0,2\pi;X)$ there exists at most one mild solution of P_{per};
(ii) $i\mathbb{Z} \cap \sigma_p(A) = \emptyset$.

Next we want to characterize well-posedness of P_{per} in the mild sense.

Proposition 3.4. *Assume that $\overline{D(A)} = X$. Let $1 \le p < \infty$. Assume that for all $f \in L^p(0,2\pi;X)$ there exists a unique mild solution of P_{per}. Then $i\mathbb{Z} \subset \varrho(A)$ and $(R(ik,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. As in the proof of Theorem 2.3 one sees that $i\mathbb{Z} \subset \varrho(A)$. Let $f \in L^p(0,2\pi;X)$. Let u be the mild solution of P_{per}. It follows from (3.2) that $\hat{u}(k) = R(ik,A)\hat{f}(k)$ for all $k \in \mathbb{Z}$. Now the claim follows from Proposition 1.1. □

We do not know whether the converse of Proposition 3.4 is true in general. Given f, $u \in L^p(0,2\pi;X)$ such that $\hat{u}(k) = R(ik,A)\hat{f}(k) \quad (k \in \mathbb{Z})$, the problem is to show that u is continuous.

This is the case if A generates a C_0-semigroup. In that case, mild solutions can be described differently [ABHN, Proposition 3.1.16]:

Lemma 3.5. *Let T be a C_0-semigroup with generator A. Let $\tau > 0$, $f \in L^1(0,\tau;X)$, $u \in C([0,\tau],X)$, $x \in X$. The following assertions are equivalent:*

(i) $\int_0^t u(s)ds \in D(A)$ *and*

$$u(t) - x = A\int_0^t u(s)ds + \int_0^t f(s)ds \text{ a.e.}$$

(ii) $u(t) = T(t)x + T * f(t) \quad (t \in [0,\tau])$, *where* $T * f(t) = \int_0^t T(t-s)f(s)ds$.

Now we obtain the following characterization of mild L^p-well-posedness.

Theorem 3.6. *Let A be the generator of a C_0-semigroup T and let $1 \le p < \infty$. Then the following are equivalent:*

(i) *For all $f \in L^p(0,2\pi;X)$ there exists a unique mild solution u of P_{per};*
(ii) *$i\mathbb{Z} \subset \varrho(A)$ and $(R(ik,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier;*
(iii) $1 \in \varrho(T(2\pi))$.

Proof. $(i) \Rightarrow (ii)$ is Proposition 3.4.
$(ii) \Rightarrow (i)$. It will be convenient to identify $L^p(0,2\pi;X)$ with $L^p_{2\pi}(\mathbb{R},X)$ of all 2π-periodic X-valued functions f such that the restriction of f on $[0,2\pi]$ is p-integrable. Let $f \in L^p_{2\pi}(\mathbb{R},X)$, $f_n = \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e_k \otimes \hat{f}(k)$. Then $f_n \in C^\infty(\mathbb{R},X)$ is 2π-periodic and $\lim_{n\to\infty} f_n = f$ in $L^p(0,2\pi;X)$. Let $u_n = \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e_k \otimes R(ik,A)\hat{f}(k)$. The hypothesis implies that $u = \lim_{n\to\infty} u_n$ exists in $L^p(0,2\pi;X)$. The function u_n is in $C^\infty(\mathbb{R},D(A))$ and $u_n'(t) = Au_n(t) + f_n(t)$ $(t \in \mathbb{R})$. We find a subsequence such that $u_{n_\ell}(r) \to u(r)$ and $f_{n_\ell}(r) \to f(r)$ a.e. as $\ell \to \infty$. Fix $r_0 \le 0$ such that $\lim_{\ell\to\infty} u_{n_\ell}(r_0) = u(r_0)$. Let $v_n(t) = u_n(t + r_0)$. Then $v_n'(t) = Av_n(t) + f_n(t + r_0)$. It follows that

$$v_n(t) = T(t)u_n(r_0) + \int_0^t T(s)f_n(t + r_0 - s)ds$$

for all $t \geq 0$. The right hand side converges to $v(t) := T(t)u(r_0) + \int_0^t T(s)f(t + r_0 - s)ds$ in X for all $t \geq 0$ as $n \to \infty$. Observe that $v : \mathbb{R}_+ \to X$ is continuous. Since v_n is 2π-periodic, also v is 2π-periodic. Since $u_{n_\ell} \to u$ a.e. we have $u(t+r_0) = v(t) \quad (t \geq 0)$ a.e. Changing u on a set of measure 0, we may assume that $u(t+r_0) = v(t)$ for all $t \geq 0$. In particular, taking $t = -r_0$, we have $u(0) = \lim_{\ell\to\infty} v_{n_\ell}(-r_0) = \lim_{\ell\to\infty} u_{n_\ell}(0)$. Thus we may choose $r_0 = 0$ in the above argument and deduce that $u(t) = T(t)u(0) + \int_0^t T(s)f(t - s)ds$. Since $u(2\pi) = u(0)$, it follows from Lemma 3.5 that u is a mild solution of P_{per}. Uniqueness of the solution follows from (3.2) by the Uniqueness Theorem.
$(iii) \Rightarrow (i)$. Let $f \in L^p(0, 2\pi; X)$. Choose $x = (I - T(2\pi))^{-1}(T * f)(2\pi)$ and $u(t) = T(t)x + (T * f)(t)$. Then $u(0) = u(2\pi)$ and u is a mild solution of P_{per} by Lemma 3.5.
$(i) \Rightarrow (iii)$ follows from Prüss [Pr]. □

Next we show that the condition in Theorem 3.6 cannot be replaced by the weaker condition that $(R(is, A))_{s\in\mathbb{R}}$ be R-bounded. In other words, the well-known characterization of negative type on Hilbert space by boundedness of the resolvent on the right half plane [ABHN, Theorem 5.2.1] due to Prüss [Pr] is not true on L^p-spaces for $p \neq 2$ even if boundedness is replaced by the stronger assumption of R-boundedness.

Example 3.7. There exists the generator A of a C_0-semigroup T on the space $X = L^p(0, \infty)$, where $2 < p < \infty$, such that $s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\} < 0$ and such that the set $\{R(\lambda, A) : \mathrm{Re}\lambda \geq 0\}$ is R-bounded . But $(R(ik, A))_{k\in\mathbb{Z}}$ is not an L^q-multiplier for any $q \in [1, \infty)$.

Proof. Let $2 < p < \infty$. In [Ar] (see also [ABHN, Example 5.1.11]) a positive C_0-semigroup T on $Y := L^p(0, \infty) \cap L^2(0, \infty)$ is constructed whose generator A satisfies $s(A) < 0$ but T has type $\omega(T) = 0$. Since T is positive, this implies that $1 = e^{t\omega(T)} = r(T(t)) \in \sigma(T(t))$. Thus $\{R(ik, A) : k \in \mathbb{Z}\}$ is not an L^q-multiplier for any $q \in [1, \infty)$. We show that still, $\{R(ik, A) : k \in \mathbb{Z}\}$ is R-bounded. In fact, by [LT, Remark on p. 191 and Section 2.f], the space Y is isomorphic to $L^p(0, \infty)$ as a Banach space. By [LT, 1.d.7 (ii)], it follows that Y is a p-concave Banach lattice. Since $R(\lambda, A)f = \int_0^\infty e^{-\lambda t}T(t)f dt$ one has

$$|R(\lambda, A)f| \leq R(0, A)|f| \ \text{ for all } f \in Y$$

whenever $\mathrm{Re}\lambda \geq 0$. Now it follows from Maurey's result [LT, Theorem 1.d.6] that the set $\{R(\lambda, A) : \mathrm{Re}\lambda \geq 0\}$ is R-bounded. Since Y is isomorphic to $L^p(0, \infty)$ all claims are proved. □

4. Hölder continuous solution

In this section we show that P_{per} has a unique Hölder continuous solution whenever the resolvent decreases fast enough on the imaginary axis. For $0 < \alpha < 1$ we denote by $C^\alpha([0, 2\pi]; X)$ the space of all continuous functions $f : [0, 2\pi] \to X$ such that

$$\|f(t) - f(s)\| \leq c|t - s|^\alpha \quad (s, t \in [0, 2\pi])$$

for some $c \geq 0$. The following is the main result of this section.

Theorem 4.1. *Let A be a closed operator on a UMD-space X such that $i\mathbb{Z} \subset \varrho(A)$. Assume that*

$$\sup_{n\in\mathbb{Z}} |n|^\theta \|R(in, A)\| < \infty \tag{4.1}$$

where $3/4 < \theta \leq 1$. Let $\frac{1}{4\theta-3} < p < \infty$, $0 < \alpha < 4\theta - 3 - \frac{1}{p}$. Then for each $f \in L^p(0, 2\pi; X)$ there exists a unique mild solution u of P_{per}. Moreover, $u \in C^\alpha([0, 2\pi]; X)$.

We need the following lemma.

Lemma 4.2. *Let $\varrho > 0$, $\lambda_1 > 0$, $0 \leq \theta \leq 1$. Define inductively $\lambda_{n+1} = \lambda_n + \varrho\lambda_n^\theta$. Then for $\gamma > 1 - \theta$ the series $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^\gamma}$ converges.*

Proof. Let $\alpha > 0$ such that

$$\frac{\theta + \gamma - 1}{\gamma} = \frac{\gamma - (1 - \theta)}{\gamma} > \frac{\alpha}{1 + \alpha} .$$

Let $\delta > 0$, $\varepsilon > 0$ such that $\gamma\varrho - \varepsilon > \delta(1 + \alpha) + \varepsilon$. Let for $n \in \mathbb{N}$,

$$\begin{aligned} A_n &= \lambda_n^\gamma \\ B_{n+1} &= B_n + (\gamma\varrho - \varepsilon) B_n^{\frac{\theta+\gamma-1}{\gamma}} , \qquad B_1 = 1 \\ C_{n+1} &= C_n + (\delta(1 + \alpha) + \varepsilon) C_n^{\frac{\alpha}{1+\alpha}} , \ C_1 = 1 \\ D_n &= (1 + n\delta)^{1+\alpha} . \end{aligned}$$

Then

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \lim_{n\to\infty} D_n = \infty .$$

One has

$$\begin{aligned} D_{n+1} &= (1+n\delta)^{1+\alpha}\left(1+\frac{\delta}{1+n\delta}\right)^{1+\alpha} \\ &= (1+n\delta)^{1+\alpha}\left(1+\frac{\delta(1+\alpha)}{1+n\delta}+o\left(\frac{1}{1+n\delta}\right)\right) \\ &= D_n+\delta(1+\alpha)D_n^{\frac{\alpha}{1+\alpha}}+o\left(D_n^{\frac{\alpha}{1+\alpha}}\right) \\ &\le D_n+(\delta(1+\alpha)+\varepsilon)D_n^{\frac{\alpha}{1+\alpha}} \end{aligned}$$

when $n \ge n_0$ for some $n_0 \in \mathbb{N}$. Let $n_1 \in \mathbb{N}$ such that $D_{n_0} \le C_{n_1}$. One shows by induction that $D_{n_0+m} \le C_{n_1+m}$ for all $m \in \mathbb{N}$. Since $\sum \frac{1}{D_n} < \infty$ we conclude that $\sum\limits_{n=1}^{\infty} \frac{1}{C_n} < \infty$. Choose $n_2 \in \mathbb{N}$ such that $C_1 \le B_{n_2}$. Since $(\delta(1+\alpha)+\varepsilon) \le \gamma\varrho-\varepsilon$ and $\frac{\alpha}{1+\alpha} \le \frac{\theta+\gamma-1}{\gamma}$ it follows that $C_{1+m} \le B_{n_2+m}$ for all $m \in \mathbb{N}$. Consequently $\sum\limits_{m=1}^{\infty} \frac{1}{B_m} < \infty$. Similarly as above one has

$$\begin{aligned} A_{n+1} &= \lambda_n^{\gamma}(1+\varrho\lambda_n^{\theta-1})^{\gamma} \\ &= \lambda_n^{\gamma}(1+\varrho\gamma\lambda_n^{\theta-1}+o(\lambda_n^{\theta-1})) \\ &\ge A_n+(\gamma\varrho-\varepsilon)A_n^{\frac{\theta+\gamma-1}{\gamma}} \end{aligned}$$

for $n \ge n_3$ if n_3 is large enough. Choose n_4 such that $B_1 \le A_{n_4}$. Then it follows that $B_{n+1} \le A_{m+n_4}$ for all $m \in \mathbb{N}$. Since $\sum \frac{1}{B_m} < \infty$, it follows that $\sum \frac{1}{A_m} < \infty$. □

Proposition 4.3. *Let* $s_0 \ge 1, 1/2 < \theta \le 1$. *Assume that* $\{is : s \in \mathbb{R}\,,\ |s| \ge s_0\} \subset \varrho(A)$ *and*

$$\sup_{|s|\ge s_0} |s|^{\theta}\|R(is,A)\| < \infty\ . \tag{4.2}$$

Then the set $\{|s|^{\beta}R(is,A) : |s| \ge s_0\}$ *is* R*-bounded whenever* $0 < \beta < 2\theta-1$.

Proof. Let $c \ge 1$ be larger than the supremum in (4.2). Let $\lambda_0 \ge s_0$ such that $\lambda_0 - \frac{1}{2c}\lambda_0^{\theta} \ge s_0$. By Taylor's formula we have

$$R(i\lambda,A) = R(i\lambda_0,A)\sum_{k=0}^{\infty}(i\lambda_0-i\lambda)^k R(i\lambda_0,A)^k$$

whenever $\lambda \in I(\lambda_0) := [\lambda_0 - \frac{1}{2c}\lambda_0^{\theta}, \lambda_0 + \frac{1}{2c}\lambda_0^{\theta}]$. Hence for $1 \le q < \infty$,

$$R_q\{\lambda^{\theta}R(i\lambda,A) : \lambda \in I(\lambda_0)\}$$

$$\leq \|R(i\lambda_0, A)\| \sum_{k=0}^{\infty} R_q\{\lambda^\theta R(i\lambda_0, A)^k (\lambda - \lambda_0)^k : \lambda \in I(\lambda_0)\}$$
$$\leq \frac{c}{\lambda_0^\theta} \sum_{k=0}^{\infty} \|R(i\lambda_0, A)\|^k 2 \sup\{|\lambda|^\theta |\lambda - \lambda_0|^k : \lambda \in I(\lambda_0)\}$$
$$\leq \frac{c}{\lambda_0^\theta} \sum_{k=0}^{\infty} (\frac{c}{\lambda_0^\theta})^k \cdot 2 \left(\lambda_0 + \frac{1}{2c}\lambda_0^\theta\right)^\theta \left(\frac{1}{2c}\lambda_0^\theta\right)^k$$
$$= 4c\left(1 + \frac{1}{2c}\lambda_0^{\theta-1}\right)^\theta \leq 4c\left(1 + \frac{1}{2c}\right)^\theta \leq 8c\,.$$

Define λ_n inductively by $\lambda_{n+1} = \lambda_n + \frac{1}{2c}\lambda_n^\theta$. Then

$$R_q\{|s|^\beta R(is, A) : s \geq \lambda_0\} \leq \sum_{n\geq 0} R_q\{|s|^\theta R(is, A) : s \in [\lambda_n, \lambda_{n+1}]\}$$
$$\cdot\ 2 \sup_{s\in[\lambda_n \lambda_{n+1}]} |s|^{\beta-\theta} \leq 16c \sum_{n\geq 0} \frac{1}{\lambda_n^{\theta-\beta}} < \infty$$

by Lemma 4.2 since $\theta - \beta > 1 - \theta$. The estimate for $s \leq -\lambda_0$ is similar. □

Proof of Theorem 4.1. a) Using Taylor's formula (4.2) one sees that $s \in \varrho(A)$ and $\|R(is, A)\| \leq C_1$ whenever $|s| \geq k_0$ where $k_0 \in \mathbb{N}$, $C_1 \geq 0$ are suitable. Let $s \geq k_0$. Choose $s \in [k, k+1]$. Then

$$\begin{aligned} s^\theta R(is, A) - k^\theta R(ik, A) =& (s^\theta - k^\theta)R(is, A) + k^\theta(R(is, A) - R(ik, A)) \\ =& (s^\theta - k^\theta)R(is, A) \\ &+ k^\theta R(ik, A)R(is, A)i(k - s)\,. \end{aligned}$$

Similar for $s \leq -k_0$. This shows that

$$C := \sup_{|s|\geq k_0} |s|^\theta \|R(is, A)\| < \infty\,.$$

b) Let $0 \leq \beta < 4\theta - 3$. It follows from Proposition 4.3 that the set

$$\{|s|^{\frac{\beta+1}{2}} R(is, A) : |s| \geq k_0\} \tag{4.3}$$

is R-bounded. It follows from Lemma 1.8 that also $\{|s|^{\beta+1}R(is, A)^2 : |s| \geq k_0\}$ is R-bounded. Since $\beta < 2\theta - 1$, also $\{|s|^\beta R(is, A) : |s| \geq k_0\}$ is R-bounded by Proposition 4.3. Let $M(s) = s^\beta R(is, A)$ $(|s| > k_0)$. Then $sM'(s) = \beta M(s) - is^{\beta+1}R(is, A)^2$. Hence $\{M(s) : |s| \geq k_0\}$ and $\{sM'(s) : |s| \geq k_0\}$ are R-bounded. Since $M_{k+1} - M_k = \int_k^{k+1} M'(s)ds$,

the set $\{k(M_{k+1} - M_k) : k \in \mathbb{Z};\, |k| \geq k_0\}$ is contained in $\overline{co}\{sM'(s) : |s| \geq k_0\}$ and so R-bounded by Lemma 1.9. Theorem 1.3 implies that $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier.
c) Let $f \in L^p(0, 2\pi; X)$. Applying b) to $\beta_0 = 0$ and $0 < \beta < 4\theta - 3$ we find unique functions $u, v \in L^p(0, 2\pi; X)$ such that

$$\hat{u}(k) = R(ik, A)\hat{f}(k)\ ,\ \hat{v}(k) = (ik)^{\beta} R(ik, A)\hat{f}(k)$$

for all $k \in \mathbb{Z}$. Thus $u \in H^{\beta,p}_{per} := \{w \in L^p(0, 2\pi; X)$: there exists $g \in L^p(0, 2\pi; X)$ satisfying $\hat{g}(k) = (ik)^{\beta}\hat{w}(k)$ for all $k \in \mathbb{Z}\}$. Now we choose $1/p < \beta < 4\theta - 3$. Then by [Zy, Theorem 9.1, p. 138] one has

$$H^{\beta,p}_{per} \subset \{w \in C^{\beta - \frac{1}{p}}([0, 2\pi]; X) : w(0) = w(2\pi)\}\ .$$

□

5. Maximal regularity

In this section we compare the periodic problem P_{per} with the first order problem with Dirichlet boundary condition

$$P_0(\tau) \begin{cases} u'(t) = Au(t) + f(t) & (t \in [0, \tau]) \\ u(0) = 0\ , \end{cases}$$

where A is the generator of a C_0-semigroup T and $f \in L^1(0, \tau; X)$, $\tau > 0$. There exists a unique mild solution $u = T * f$ (see Lemma 3.5).
We say that $P_0(\tau)$ is *strongly L^p-well-posed* if for every $f \in L^p(0, \tau; X)$ one has $T * f \in H^{1,p}(0, \tau; X)$.
It is easy to see that strong L^p-well-posedness of $P_0(\tau)$ implies the same property if A is replaced by $A - \lambda$ for all $\lambda \in \mathbb{C}$. Moreover, it is well-known that L^p-well-posedness of $P_0(\tau)$ for some $\tau > 0$ implies the same property for $P_0(\tau')$ for all $\tau' > 0$ (see Dore [Do]).

Theorem 5.1. *Let A be the generator of a C_0-semigroup T on a Banach space X. Let $1 < p < \infty$. The following assertions are equivalent:*

(i) $P_0(2\pi)$ *is strongly L^p-well-posed and* $1 \in \varrho(T(2\pi))$*;*
(ii) P_{per} *is strongly L^p-well-posed.*

Proof. If $P_0(2\pi)$ is L^p-well-posed, then T is holomorphic (see Dore [Do]). Conversely, it is not difficult to see from the necessity of condition (iii) in Theorem 2.3 (for which the UMD-property is not needed) that L^p-well-posedness of P_{per} implies that T is holomorphic. Thus, for the proof of equivalence of (i) and (ii) we can assume that T is holomorphic. By the trace theorem we have

$$\begin{aligned}(X, D(A))_{1-\frac{1}{p},p} &:= \{x \in X : AT(\cdot)x \in L^p(0, 2\pi; X)\} \\ &= \{u(0) : u \in L^p(0, 2\pi; D(A)) \cap H^{1,p}(0, 2\pi; X)\} ,\end{aligned}$$

see Lunardi [Lu, 1.2.2 and 2.2.1].
$(i) \Rightarrow (ii)$. Let $f \in L^p(0, 2\pi; X)$. Then by assumption $v = T * f \in H^{1,p}(0, 2\pi; X)$. It follows from the trace theorem above that $v(2\pi) \in (X, D(A))_{1-1/p,p}$. Hence also

$$x := (I - T(2\pi))^{-1} v(2\pi) \in (X, D(A))_{1-1/p,p} .$$

Since $\frac{d}{dt}T(t)x = AT(t)x$ on $(0, \infty)$, it follows that $T(\cdot)x \in H^{1,p}(0, 2\pi; X)$. Let $u(t) = T(t)x + v(t)$. Then $u \in H^{1,p}(0, 2\pi; X)$ and $u(0) = x = T(2\pi)x + v(2\pi) = u(2\pi)$. Thus u is a strong solution of P_{per}. Since $e^{2\pi\sigma(A)} \subset \sigma(T(2\pi))$ and $1 \in \varrho(T(2\pi))$, it follows that $i\mathbb{Z} \subset \varrho(A)$, and uniqueness of the solution of P_{per} follows from Corollary 3.3.
$(ii) \Rightarrow (i)$. Let $f \in L^p(0, 2\pi; X)$. By assumption, there exists $v \in H^{1,p}_{per}$ solution of P_{per}. It follows from the trace theorem again that $x := v(0) \in (X, D(A))_{1-1/p,p}$; hence $T(\cdot)x \in H^{1,p}(0, 2\pi; X)$. Let $u(t) = v(t) - T(t)x$. Then u is a strong L^p-solution of $P_0(2\pi)$. □

With the help of Theorem 1.3 we now obtain the following characterization of strong L^p-well-posedness of $P_0(\tau)$.

Corollary 5.2. *Let A be the generator of a C_0-semigroup on a UMD-space X and let $1 < p < \infty$. The following assertions are equivalent:*

(i) *$P_0(\tau)$ is strongly L^p-well-posed for all $\tau > 0$;*
(ii) *there exists $w > \omega(T)$ such that $\{kR(w + ik, A) : k \in \mathbb{Z}\}$ is R-bounded.*

Proof. Replacing A by $A - \omega$ this follows directly from Theorem 5.1 and Theorem 2.3 □

Corollary 5.2 shows in particular that strong L^p-well-posedness of $P_0(\tau)$ is independent of $p \in (1, \infty)$ (which is well-known). It became customary to say that a closed operator A *has the property* (MR) (for *maximal regularity*) if $P_0(\tau)$ is strongly L^p-well-posed for one and hence all $p \in (1, \infty)$, $\tau > 0$. Thus condition (ii) is a characterization of (MR).

We obtain this characterization as a consequence of the discrete multiplier theorem (Theorem 1.3). It is also possible to use Weis' multiplier theorem [W1, Theorem 3.4]) and the criterion [W2, Section 1e)(i)]. For that one has to show that condition (ii) of Corollary 5.2 implies that the set $\{sR(is+\omega,A) : s\in\mathbb{R}\}$ is R-bounded. This is not difficult to do.

Finally, we should mention that in contrast to the periodic problem, in the context of problem $P_0(\tau)$ it is natural to assume that A generates a holomorphic C_0-semigroup. In fact, for densely defined closed operators this is a necessary assumption (see Dore [Do]). By a spectacular result of Kalton and Lancien [KL] it is not sufficient if X is a Banach space with unconditional basis, which is not isomorphic to a Hilbert space.

6. The second order problem

Let A be a closed operator on a UMD-space X and let $1<p<\infty$. In this section we characterize strong L^p-well-posedness of the problem

$$u''(t)+Au(t)=f(t)$$

on bounded intervall with periodic, Dirichlet and Neumann boundary conditions. For $a<b$ we denote by

$$H^{2,p}(a,b;X) := \{u\in H^{1,p}(a,b;X) : u'\in H^{1,p}(a,b;X)\}$$

the second Sobolev space. Note that $H^{2,p}(a,b;X)\subset C^1([a,b];X)$. Using the notion of Sect. 2 we let

$$H^{2,p}_{per} := \{u\in H^{1,p}_{per} : u'\in H^{1,p}_{per}\}\ .$$

Let $u\in L^p(0,2\pi;X)$. It is easy to see that $u\in H^{2,p}_{per}$ if and only if there exists $v\in L^p(0,2\pi;X)$ such that $\hat{v}(k)=-k^2\hat{u}(k)$ for all $k\in\mathbb{Z}$. In that case $v=(u')'=:u''$.

Theorem 6.1. *The following are equivalent:*

(i) *For all $f\in L^p(0,2\pi;X)$ there exists a unique*

$$u\in L^p(0,2\pi;D(A))\cap H^{2,p}(0,2\pi;X)$$

such that

$$P_2(2\pi)\begin{cases} u''(t)+Au(t)=f(t) \quad a.e. \\ u(0) \ =u(2\pi)\ ,\ u'(0)=u'(2\pi)\ ; \end{cases}$$

(ii) *one has $k^2\in\varrho(A)$ for all $k\in\mathbb{Z}$ and $\{k^2R(k^2,A) : k\in\mathbb{Z}\}$ is R-bounded.*

Proof. $(i) \Rightarrow (ii)$. One shows as in Theorem 2.3 that $k^2 \in \varrho(A)$ for all $k \in \mathbb{Z}$. Let $f \in L^p(0, 2\pi; X)$. Let u be the solution of (i). Then $\hat{u}(k) \in D(A)$ and $-k^2\hat{u}(k) + A\hat{u}(k) = \hat{f}(k)$. Hence $\hat{u}(k) = -R(k^2, A)\hat{f}(k)$ and $(u'')\hat{}(k) = -k^2\hat{u}(k) = k^2R(k^2, A)\hat{f}(k) \quad (k \in \mathbb{Z})$. Since $u'' \in L^p(0, 2\pi; X)$ this proves (ii).
$(ii) \Rightarrow (i)$. Let $M_k = k^2R(k^2, A) \quad (k \in \mathbb{Z})$. Then

$$\begin{aligned} k(M_{k+1} - M_k) &= kR((k+1)^2, A)\{(k+1)^2(k^2 - A) \\ &\qquad -k^2((k+1)^2 - A)\}R(k^2, A) \\ &= -k(2k+1)R((k+1)^2, A)(k^2R(k^2, A) - I)\,. \end{aligned}$$

It follows that the set $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ is R-bounded. Now Theorem 1.3 implies that $(k^2R(k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. Let $f \in L^p(0, 2\pi; X)$. Then there exists $u'' \in L^p(0, 2\pi; X)$ such that $(u'')\hat{}(k) = k^2R(k^2, A)\hat{f}(k) \quad (k \in \mathbb{Z})$. A simple computation shows that there exist $y, z \in X$ such that if we let $u(t) = \int_0^t (t-s)u''(s)ds + ty + z$ for $t \in [0, 2\pi]$, then $\hat{u}(k) = -R(k^2, A)\hat{f}(k)$ for all $k \in \mathbb{Z}$.
Since $AR(k^2, A) = k^2R(k^2, A) - I$, it follows that $(R(k^2, A))_{k\in\mathbb{Z}}$ is an $\mathcal{L}(X, D(A))$-multiplier. Thus $u \in L^p(0, 2\pi; D(A))$. Since

$$\begin{aligned} (u'' + Au)\hat{}(k) &= k^2R(k^2, A)\hat{f}(k) - AR(k^2, A)\hat{f}(k) \\ &= \hat{f}(k) \quad (k \in \mathbb{Z}) \end{aligned}$$

it follows from the Uniqueness Theorem that u is a solution of $P_2(2\pi)$. Since $(u'')\hat{}(0) = (u')\hat{}(0) = 0$ it follows that $u'(0) = u'(2\pi)$ and $u(0) = u(2\pi)$. Uniqueness is proved as in Sect. 2. □

In order to treat Dirichlet boundary conditions we will consider odd functions f on $(-\pi, \pi)$; i.e. functions satisfying $\hat{f}(k) = -\hat{f}(-k)$ for all $k \in \mathbb{Z}$. We need the following lemma.

Lemma 6.2. *Let* $M_k \in \mathcal{L}(X)$ *such that* $M_k = M_{-k} \quad (k \in \mathbb{Z})$. *Assume that for each odd* $f \in L^p(-\pi, \pi; X)$ *there exists* $u \in L^p(-\pi, \pi; X)$ *such that* $\hat{u}(k) = M_k\hat{f}(k) \quad (k \in \mathbb{Z})$. *Then* $(M_k)_{k\in\mathbb{Z}}$ *is an* L^p*-multiplier.*

Proof. Let $f \in L^p(-\pi, \pi; X)$. We have to show that there exists $g \in L^p(-\pi, \pi; X)$ such that $M_k\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$. We can assume that $\hat{f}(0) = 0$. Consider $f_1 \in L^p(-\pi, \pi; X)$ such that

$$\hat{f}_1(k) = \begin{cases} \hat{f}(k) & \text{if } k > 0 \\ -\hat{f}(-k) & \text{if } k < 0 \\ 0 & \text{if } k = 0\,. \end{cases}$$

Notice that $f_1 \in L^p(-\pi,\pi;X)$ exists as X is a UMD-space [Bur]. There exists $h_1 \in L^p(-\pi,\pi;X)$ such that $M_k \hat{f}_1(k) = \hat{h}_1(k) \quad (k \in \mathbb{Z})$. Since the Riesz projection is bounded we find $g_1 \in L^p(-\pi,\pi;X)$ such that $\hat{g}_1(k) = \hat{h}_1(k)$ for $k \geq 0$ and $\hat{g}_1(k) = 0$ for $k < 0$. Thus $\hat{g}_1(k) = M_k \hat{f}(k)$ for $k \geq 0$. Similarly, we find $g_2 \in L^p(-\pi,\pi;X)$ such that $\hat{g}_2(k) = M_k \hat{f}(k)$ for $k < 0$ and $\hat{g}_2(k) = 0$ for $k \geq 0$. Choose $g = g_1 + g_2$. $\square$

Now we obtain the following characterization of strong L^p-well-posedness in the case of Dirichlet boundary conditions. Here 0 may be in the spectrum of A.

Theorem 6.3. *The following are equivalent:*

(i) For all $f \in L^p(0,\pi;X)$ *there exists a unique* $u \in L^p(0,\pi;D(A)) \cap H^{2,p}(0,\pi;X)$ *satisfying*

$$\begin{cases} u''(t) + Au(t) \;=\; f(t) \quad a.e. \\ u(0) \;=\; u(\pi) = 0 \end{cases}$$

(ii) $k^2 \in \varrho(A)$ *for all* $k \in \mathbb{N}$ *and* $\{k^2 R(k^2,A) : k \in \mathbb{N}\}$ *is* R*-bounded.*

Proof. $(i) \Rightarrow (ii)$. Let $k \in \mathbb{N}$. We show that $k^2 \in \varrho(A)$. If $x \in D(A)$ such that $(-k^2 + A)x = 0$, then $u(t) = (\sin kt)x$ defines a solution of $u'' + Au = 0$. Hence $u = 0$, and so $x = 0$. Let $y \in X$. There exists a strong solution u of $u'' + Au = (\sin kt)y$. Extend u to an odd function. Comparing Fourier coefficients we see that $u(t) = (\sin kt) \cdot x$ for some $x \in D(A)$ satisfying $(-k^2 + A)x = y$. We have shown that $(-k^2 + A)$ is bijective, thus $k^2 \in \varrho(A)$.
Let $f \in L^p(0,\pi;X)$. There exists a unique function u satisfying (i). Extending u and f to odd functions we see that $-k^2\hat{u}(k) + A\hat{u}(k) = \hat{f}(k)$, hence $\hat{u}(k) = -R(k^2,A)\hat{f}(k)$ for all $k \in \mathbb{Z}$. Moreover, $(u'')\hat{}(k) = -k^2 R(k^2,A)\hat{f}(k) \quad (k \in \mathbb{Z})$. Now (ii) follows from Lemma 6.2.
$(ii) \Rightarrow (i)$. Let $M_0 = 0$, $M_k = k^2 R(k^2,A)$ for $k \in \mathbb{Z} \setminus \{0\}$. One sees as in the proof of Theorem 6.1 that $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier. Let $f \in L^p(0,\pi;X)$. Extend f to an odd function. Then there exists $u'' \in L^p(-\pi,\pi;X)$ such that $(u'')\hat{}(k) = k^2 R(k^2,A)\widehat{f}(k)$ for $k \neq 0$ and $(u'')\hat{}(0) = 0$. A simple computation shows that there exists $x \in X$ such that if we let $u(t) = \int_0^t (t-s)u''(s)ds + tx$ for $t \in [0,\pi]$ and extend u to an odd function on $[-\pi,\pi]$, then $\hat{u}(k) = -R(k^2,A)\hat{f}(k)$ for $k \neq 0$. So $u_{|[0,\pi]}$ solves the problem in (i). $\square$

Finally we consider Neumann boundary conditions.

Theorem 6.4. *The following assertions are equivalent:*

(i) For all $f \in L^p(0,\pi;X)$ *there exists a unique* $u \in L^p(0,\pi;D(A)) \cap H^{2,p}(0,\pi;X)$ *satisfying*

$$\begin{cases} u''(t) + Au(t) = f(t) \ a.e. \\ u'(0) = u'(\pi) \ = 0\,; \end{cases}$$

(ii) one has $k^2 \in \varrho(A)$ *for all* $k \in \mathbb{N}_0$ *and* $\{k^2R(k^2,A) : k \in \mathbb{N}\}$ *is* R*-bounded.*

The proof may be given similarly to the one of Theorem 6.3 replacing odd by even functions there and in Lemma 6.2.

Finally we mention that Clément and Guerre-Delabrière [CG] studied the relation of first and second order problems. To be more precise, let B be a closed operator and consider the periodic problem

$$P_{per} \ \begin{cases} u' + Bu = f \\ u(0) = u(2\pi) \end{cases}$$

of Sect. 2. Let $A = -B^2$. Let $1 < p < \infty$. Assume that P_{per} is strongly L^p-well-posed. Then by Theorem 2.3 we have $i\mathbb{Z} \subset \varrho(-B)$ and $\{k(ik+B)^{-1} : k \in \mathbb{Z}\}$ is R-bounded. Then $k^2 \in \varrho(A)$ and $R(k^2,A) = (k^2 + B^2)^{-1} = (ik+B)^{-1}(-ik+B)^{-1}$ for all $k \in \mathbb{Z}$. It follows that $\{k^2R(k^2,A) : k \in \mathbb{Z}\}$ is R-bounded and Theorem 6.1, 6.3 and 6.4 give strong L^p-well-posedness of the second order problems defined by A. This is shown in [CG] by different methods in the case when $-B$ generates an exponentially stable holomorphic C_0-semigroup T. In that case they also show the other implication. From our results this other implication can be seen as follows. One may represent the resolvent of B by the resolvent of B^2 via a contour integral [Ta, (2.29) page 36]. If the equivalent conditions appearing in Theorem 6.1, 6.3 or 6.4 are satisfied, then it is not difficult with help of this formula to prove R-boundedness of $\{k(ik - B)^{-1} : k \in \mathbb{Z}\}$ which implies strong L^p-well-posedness of P_{per} by Theorem 2.3 again.

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