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Tools for maximal regularity†

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Abstract

Let A be the generator of an analytic C_0-semigroup on a Banach space X. We associate a closed operator $\mathscr{A}_1$ with A defined on $\mathrm{Rad}(X)$ and show that when X is a UMD-space, the Cauchy problem associated with A has maximal regularity if and only if the operator $\mathscr{A}_1$ generates an analytic C_0-semigroup on $\mathrm{Rad}(X)$. This allows us to exploit known results on analytic C_0-semigroups to study maximal regularity. Our results show that $\mathscr{R}$-boundedness is a local property for semigroups: an analytic C_0-semigroup T of negative type is $\mathscr{R}$-bounded if and only if it is $\mathscr{R}$-bounded at $z = 0$. As applications, we give a perturbation result for positive semigroups. Finally, we show the following: when X is a UMD-space, T is an analytic C_0-semigroup of negative type, then for every $f \in L^p(\mathbb{R}_+; X)$, the mild solution of the corresponding inhomogeneous Cauchy problem with initial value 0 belongs to $W^{\theta,p}(\mathbb{R}_+; X)$ for every $0 < \theta < 1$.

1. *Introduction*

Let A be the generator of a C_0-semigroup T on a Banach space X and let $0 < \tau < \infty$. If $f \in L^1([0, \tau); X)$ then $u(t) = T * f(t) = \int_0^t T_{t-s} f(s)\, ds$ defines the unique mild solution of the problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0, \tau) \\ u(0) = 0. \end{cases} \tag{1}$$

We say that A or T satisfies *property* $\mathscr{M}R$ (for *maximal regularity*) if $u \in L^p([0, \tau); D(A)) \bigcap W^{1,p}([0, \tau); X)$ whenever $f \in L^p([0, \tau); X)$ for some (equivalently for all) $1 < p < \infty$. This property of maximal regularity is important for non-linear problems and has been studied extensively in the last years, see [**AB**, **CPSW**, **CL**, **DPG**, **Do**, **DoV**,

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HP, **KL**, **KW**, **LLM**, **LeM**, **W1**, **W2**]. It is shown by Dore that property $\mathcal{MR}$ implies that T is analytic [**Do**]. Moreover on Hilbert space, each analytic semigroup has $\mathcal{MR}$ [**DS**]. However, by a remarkable recent result by [**KL**] this property characterizes Hilbert spaces among a large class of Banach spaces. This result makes it desirable to have a characterization of $\mathcal{MR}$ for an individual operator A. And indeed, this was done recently by [**W2**]: if X is a UMD-space (see [**Bo1**]), then the operator A has $\mathcal{MR}$ if and only if the set $\{sR(\omega + is, A)\colon s \in \mathbb{R}\}$ is $\mathcal{R}$-bounded for some ω larger than the growth bound $\omega(A)$ of A. The notion of $\mathcal{R}$-boundedness for sets of operators is due to [**Bo2**] and is stronger than boundedness in operator norm (unless X is a Hilbert space [**AB**]).

In the present article we find a reformulation of the characterization given by Weis. We associate two closed operators $\mathcal{A}_1$ and $\mathcal{A}_2$ with A which are defined on a Banach space $\mathrm{Rad}(X)$ (the closed space of all Rademacher functions in $L^1([0,1];X)$). Assuming that T has negative growth bound, we show that A has $\mathcal{MR}$ if and only if $\mathcal{A}_j$ generates an analytic C_0-semigroup, where j may be 1 or 2. Both operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are useful to exploit known results on analytic semigroups to study maximal regularity. For example, we obtain as an immediate corollary that $A + B$ has $\mathcal{MR}$ if A has $\mathcal{MR}$ and B is a small perturbation of A whenever X is a UMD-space (a perturbation result due to [**W1**]). For positive C_0-semigroups on $X = L^p(\Omega)$, $1 < p < \infty$, we obtain the following new perturbation result using the operator $\mathcal{A}_2$: if A generates a positive C_0-semigroup on X satisfying $\mathcal{MR}$ and if $B\colon D(A) \to X$ is a positive operator such that $A + B$ is resolvent positive, then $A + B$ satisfies $\mathcal{MR}$.

We also find a sufficient condition for positive interpolating semigroups on $L^p(\Omega)$ to have $\mathcal{MR}$. Finally, we refind a result due to [**HP**] showing that Gaussian estimates implies $\mathcal{MR}$. Gaussian estimates can be established for elliptic operators of second order for diverse boundary conditions (see [**AE**]).

In view of Weis' characterization and other results in this direction (e.g. [**AB**, **CP**]) it seems important to find criteria to verify $\mathcal{R}$-boundedness. Our result shows that maximal regularity is a local property of the semigroup T: if X is a UMD-space, then the semigroup T has $\mathcal{MR}$ if and only if the set $\{T_z : |\arg(z)| < \theta, |z| < \epsilon\}$ is $\mathcal{R}$-bounded for some $\theta > 0$ and some $\epsilon > 0$.

It is not easy to verify $\mathcal{R}$-boundedness in concrete cases. However, if A generates an analytic C_0-semigroup of negative type, then we show that $\{|s|^\beta R(is, A)\colon s \in \mathbb{R}\}$ is $\mathcal{R}$-bounded for each $0 < \beta < 1$. This is used to show that when X is a UMD-space, the mild solution u of (1) is actually in $W^{\theta,p}(\mathbb{R}_+;X)$ for all $0 < \theta < 1$ whenever $f \in L^p(\mathbb{R}_+;X)$, $1 < p < \infty$.

We will also consider the problem (1) on $\mathbb{R}_+$. We say that A or T satisfies $\mathcal{MR}_\infty$ if the mild solution u of (1) belongs to $L^p(\mathbb{R}_+;D(A)) \bigcap W^{1,p}(\mathbb{R}_+;X)$ for $f \in L^p(\mathbb{R}_+;X)$. It is known that A has $\mathcal{MR}_\infty$ if and only if A has $\mathcal{MR}$ and $\omega(A) < 0$ [**Do**], where $\omega(A)$ denotes the growth bound of A. This result is most useful in the study of $\mathcal{MR}$. In fact, to study the property $\mathcal{MR}$ for A, it suffices to study the property $\mathcal{MR}_\infty$ for $A - \beta$ for some $\beta > 0$ satisfying $\omega(A-\beta) < 0$. Notice that for a C_0-semigroup T with generator A, one has $\omega(A) < 0$ if and only if T is exponentially stable (or equivalently, of negative type), i.e. there exist $M \geqslant 1$ and $\epsilon > 0$ such that $\|T_t\| \leqslant Me^{-\epsilon t}$ whenever $t \geqslant 0$. When T is analytic, we have $s(A) = \omega(A)$, where $s(A) = \sup\{\mathrm{Re}(z)\colon z \in \sigma(A)\}$ is the spectral bound of A.

Let X, Y be Banach spaces, we will denote by $B(X,Y)$ the space of all bounded linear operators from X to Y. When $X = Y$, we will denote $B(X, X)$ simply by $B(X)$. For $0 < \theta < \pi$, $\sum_\theta$ will be the sector $\{z \in \mathbb{C} : |\arg(z)| < \theta\}$. For a closed operator A on X, we denote by $\rho(A)$ the resolvent set of A, and $R(\lambda, A) = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$.

2. $\mathscr{R}$-*boundedness*

Let X, Y be Banach spaces. A set $M \subset B(X; Y)$ is called $\mathscr{R}$-*bounded* if there exists a constant $C > 0$, such that for all $T_1, T_2, \ldots, T_n \in M$, $x_1, x_2, \ldots, x_n \in X$, $n \in \mathbb{N}$,

$$\int_0^1 \left\| \sum_{j=1}^n \gamma_j(t) T_j x_j \right\|_X dt \leqslant C \int_0^1 \left\| \sum_{j=1}^n \gamma_j(t) x_j \right\|_X dt, \tag{2}$$

where $(\gamma_j)_{j \geqslant 1}$ is a fixed sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$, e.g. the Rademacher functions $\gamma_j(t) = \operatorname{sign}(\sin(2^j \pi t))$. We will denote by $\mathscr{R}(M)$ the smallest constant in (2). This concept was already used in [**BG**] and [**Bo2**] in connection with multiplier theorems and more recently in [**AB**, **CPSW**, **KW**, **W1**, **W2**]. Using Kahane's inequality [**LT**, theorem 1·*e*·13], it is easy to see that we can replace the L^1-norm by any L^p-norm in (2). We should also notice that if we require (2) only for distinct $T_1, T_2, \ldots, T_n \in M$, we will obtain the same notion of $\mathscr{R}$-boundedness with the same constant $\mathscr{R}(M)$ [**CPSW**].

The notion of $\mathscr{R}$-boundedness plays an important role in the study of $\mathscr{M}R$ and $\mathscr{M}R_\infty$. For instance we recall the following characterization of $\mathscr{M}R_\infty$ in term of $\mathscr{R}$-boundedness due to [**W2**]. Its original statement is more general, but we will only use the following more simple version.

THEOREM 2·1. *Let A be the generator of an analytic C_0-semigroup T on X. Assume that X is a* UMD-*space and $\omega(A) < 0$. Then the operator A has $\mathscr{M}R_\infty$ if and only if the set $\{sR(is, A) : s \in \mathbb{R}\}$ is $\mathscr{R}$-bounded.*

The following proposition summarizes the most useful properties concerning $\mathscr{R}$-boundedness (see [**CPSW**, lemma 3·2] and [**W2**, 2·4, 2·6, 2·8]).

PROPOSITION 2·2. (i) *Let Ω be an open subset of $\mathbb{C}$, and let $T : \Omega \to B(X)$ be an analytic mapping. Then for every compact subset $K \subset \Omega$, $\mathscr{R}\{T(z) : z \in K\} < \infty$.*

(ii) *Let $\Omega \subset \mathbb{C}$ be a simply connected Jordan region such that $\mathbb{C} \backslash \Omega$ has interior points. Let $T : \overline{\Omega} \to B(X)$ be a bounded, strongly measurable function, analytic in Ω. If $\mathscr{R}\{T(z) : z \in \partial\Omega\} < \infty$, then $\mathscr{R}\{T(z) : z \in \Omega\} < \infty$.*

(iii) *Let $M_1, M_2, \ldots, M_n$ be subsets of $B(X)$, then*

$$\mathscr{R}\left(\bigcup_{i=1}^n M_i\right) \leqslant \sum_{i=1}^n \mathscr{R}(M_i).$$

(iv) *Let M be an $\mathscr{R}$-bounded set, then $\mathscr{R}(co(M)) \leqslant 2\mathscr{R}(M)$, where $co(M) = \{\sum_{j=1}^m \lambda_j S_j : \lambda_j \in \mathbb{C}, S_j \in M, \sum_{j=1}^m |\lambda_j| \leqslant 1, m \in \mathbb{N}\}$.*

(v) *Let $T \in B(X)$ be fixed, $\Omega \subset \mathbb{C}$ be a bounded subset. Then $\mathscr{R}\{\lambda T : \lambda \in \Omega\}$ is $\mathscr{R}$-bounded and*

$$\mathscr{R}\{\lambda T : \lambda \in \Omega\} \leqslant 2\|T\| \sup_{\lambda \in \Omega} |\lambda|.$$

The following lemma will be very useful in the study of $\mathscr{R}$-boundedness.

Lemma 2·3. *Let S be a set, I be an interval of $\mathbb{R}$, $f\colon S\times I\to B(X)$. Assume that for each $s\in S$, $f(s,.)\in L^1(I,B(X))$ and that there exists a measurable function g on I, such that $\mathscr{R}\{f(s,t)\colon s\in S\}\leqslant g(t)$ for each $t\in I$. Then*

$$\mathscr{R}\left\{\int_I f(s,t)\,dt\colon s\in S\right\}\leqslant\int_I g(t)\,dt.$$

Proof. Let $s_1,s_2,\dots,s_n\in S$, $x_1,x_2,\dots,x_n\in X$. Then

$$\begin{aligned}\int_0^1\left\|\sum_{j=1}^n\gamma_j(\omega)\int_I f(s_j,t)\,dt\,x_j\right\|d\omega&\leqslant\int_I\int_0^1\left\|\sum_{j=1}^n\gamma_j(\omega)f(s_j,t)x_j\right\|d\omega\,dt\\&\leqslant\int_I g(t)\,dt\int_0^1\left\|\sum_{j=1}^n\gamma_j(\omega)x_j\right\|d\omega.\end{aligned}$$

Thus

$$\mathscr{R}\left\{\int_I f(s,t)\,dt : s\in S\right\}\leqslant\int_I g(t)\,dt.$$

Corollary 2·4. *Let T be a C_0-semigroup on X with generator A. Assume that there exist constants $M\geqslant 1$ and $\omega>0$ such that $\|T_t\|\leqslant Me^{-\omega t}$ for all $t\geqslant 0$. Then*

$$\mathscr{R}\{R(z,A)\colon Re(z)\geqslant 0\}\leqslant 2M/\omega.$$

Proof. This is a simple consequence of Lemma 2·3 and the equality

$$R(z,A)=\int_0^\infty e^{-zt}T_t\,dt,\quad Re(z)\geqslant 0.$$

For $0<\theta<\pi/2$, $M\geqslant 1$ and $\omega>0$, we denote by $\mathscr{E}(\theta,M,\omega)$ the class of all analytic C_0-semigroups T defined on $\sum_\theta$ satisfying

$$\|T_z\|\leqslant Me^{-\omega|z|},\quad z\in\Sigma_\theta.$$

We will use the following lemma.

Lemma 2·5. *Let A be the generator of an analytic C_0-semigroup T on X. Assume that $\omega(A)<0$. Then there exist $0<\theta<\pi/2$, $\omega>0$ and $M\geqslant 1$ such that $T\in\mathscr{E}(\theta,M,\omega)$.*

Proof. As the semigroup T is analytic, we have $s(A)=\omega(A)<0$ and there exist $\omega>0$ and $\alpha>0$ such that $\{z\in\mathbb{C} : |\arg(z-\omega)|<\alpha+\pi/2\}\subset\rho(A)$. This implies that $\sum(A)\subset\{z\in\mathbb{C}\colon \mathrm{Re}(z)\leqslant s(A)\}\bigcap\{z\in\mathbb{C} : |(z-\omega)|\geqslant\alpha+\pi/2\}$. From this we can find $\beta>0$ such that $e^{\pm i\beta}A$ generate analytic C_0-semigroup and $\omega(e^{\pm i\beta}A)=s(e^{\pm i\beta}A)<0$. In particular, the semigroup $(T_{te^{\pm i\beta}})_{t\geqslant 0}$ generated by $e^{\pm i\beta}A$ is exponentially stable. There exist $\omega>0$ and $M\geqslant 1$ satisfying

$$\|T_{te^{\pm i\beta}}\|\leqslant Me^{-\omega t},\quad t>0.$$

[**AMH**, proposition 4·5] implies that

$$\|e^{zA}\|\leqslant Me^{-\omega Re(z)},\quad z\in\Sigma_\beta.$$

As $\mathrm{Re}(z)\geqslant|z|/\cos\beta$ for $z\in\Sigma_\beta$, the claim is proved.

Notice that when a C_0-semigroup T has $\mathscr{M}R_\infty$, then T is analytic and has negative growth bound [**Do**]. So by Lemma 2·5, $T\in\mathscr{E}(\theta,M,\omega)$ for some $\theta>0$, $M\geqslant 1$ and $\omega>0$. Lemma 2·3 has the following useful corollary.

Corollary 2·6. *Let A be the generator of an analytic C_0-semigroup T. Assume that $\omega(A) < 0$. Then for some $\theta > 0$ and each $r > 0$, we have $\mathscr{R}\{T_z : |z| \geqslant r, z \in \Sigma_\theta\} < \infty$.*

Proof. By Lemma 2·5, there exist $\theta_0 > 0$, $M \geqslant 1$ and $\omega > 0$ such that $T \in \mathscr{E}(\theta_0, M, \omega)$. For each $|\alpha| < \theta_0$, $e^{i\alpha}A$ generates an analytic C_0-semigroup in the class $\mathscr{E}(\theta_0 - |\alpha|, M, \omega)$. By Proposition 2·2, it suffices to show that $\mathscr{R}\{T(t): t \geqslant t_0\} < \infty$ for each $t_0 > 0$.

Let $t_0 > 0$ be fixed. Denote by $\overline{B}(s, r)$ the closed ball centered on s with radius r in the complex plane. We have $\overline{B}(t_0, t_0 \sin \theta_0/2) \subset \Sigma_{\theta_0}$. For $z \in \overline{B}(t_0, t_0 \sin \theta_0/4)$, $z = t_0 + t_0 \sin \theta_0 r e^{i\alpha}/2$, $0 \leqslant r < 1/2$, $\alpha \in [0, 2\pi]$, since T is analytic in Σ_{θ_0}, we have

$$T_z = \int_0^{2\pi} T_{t_0 + t_0 \sin \theta_0 e^{i\beta}/2} P_r(\alpha - \beta) \frac{d\beta}{2\pi},$$

where

$$P_r(\beta) = \frac{1 - r^2}{1 + r^2 - 2r \cos \beta}$$

is the Poisson kernel. By Lemma 2·3,

$$\begin{aligned}\mathscr{R}\{T_z : z \in \overline{B}(t_0, t_0 \sin \theta_0/4)\} &\leqslant \sup_{0 \leqslant r \leqslant 1/2, \alpha \in [0,2\pi]} 2P_r(\alpha) \int_0^{2\pi} \|T_{t_0 + t_0 \sin \theta_0 e^{i\beta}/2}\| \frac{d\beta}{2\pi} \\ &\leqslant 6Me^{-\omega t_0 (1 - \sin \theta_0/2)}.\end{aligned}$$

In particular

$$\mathscr{R}\{T_t : t \in [(1 - \sin \theta_0/4)t_0, (1 + \sin \theta_0/4)t_0]\} \leqslant 6Me^{-\omega t_0(1 - \sin \theta_0/2)}.$$

Let $\alpha = (4 + \sin \theta_0/4 - \sin \theta_0)$, then $[t_0, \infty) \subset \bigcup_{n \geqslant 0} [(1 - \sin \theta_0/4)\alpha^n t_0, (1 + \sin \theta_0/4)\alpha^n t_0]$ and thus by Proposition 2·2

$$\begin{aligned}\mathscr{R}\{T_t : t \geqslant t_0\} &\leqslant \sum_{n \geqslant 0} \mathscr{R}\{T_t : t \in [(1 - \sin \theta_0/4)\alpha^n t_0, (1 + \sin \theta_0/4)\alpha^n t_0]\} \\ &\leqslant \sum_{n \geqslant 0} 6Me^{-\omega t_0 \alpha^n (1 - \sin \theta_0/2)} < \infty\end{aligned}$$

since $\alpha > 1$. The claim is proved.

3. *Associated semigroups on* Rad(X)

Let X be a Banach space and let A be the generator of an analytic C_0-semigroup T on X. Let $(\gamma_j)_{i \geqslant 1}$ be the sequence of Rademacher functions on $[0, 1]$. Define

$$\mathscr{R}(X) = \left\{ \sum_{j=1}^{n} \gamma_j x_j : x_j \in X, n \in \mathbb{N} \right\}$$

and Rad(X) the closure of $\mathscr{R}(X)$ in $L^1([0, 1]; X)$. We obtain the same space Rad(X) if we replace the L^1-norm by any other L^p-norm by Kahane's inequality, see [**LT**, theorem 1·*e*·13]. Notice that

$$\text{Rad}(X) = \left\{ \sum_{j=1}^{\infty} \gamma_j x_j : \text{the series } \sum_{j=1}^{\infty} \gamma_j x_j \text{ converges in } L^1([0,1]; X) \right\}.$$

In fact, the subset in the right hand side is closed in $L^1([0,1];X)$. Indeed, let $f_n = \sum_{j=1}^{\infty}\gamma_j x_j^{(n)} \in L^1([0,1];X)$ and let $f = \lim_{n\to\infty} f_n$ in $L^1([0,1];X)$. For $j\in\mathbb{N}$, let $\mathscr{F}_j$ be the σ-algebra of Borel subsets of $[0,1]$ generated by the functions $\gamma_1,\gamma_2,\ldots,\gamma_j$. Then the σ-algebra generated by $\bigcup_{j=1}^{\infty}\mathscr{F}_j$ is exactly the σ-algebra of all Borel subsets of $[0,1]$. Therefore if $g_j = \mathbf{E}(f|\mathscr{F}_j)$, then $(g_j)_{j\geqslant 1}$ is a martingale with respect to the filtration $\mathscr{F}_j$ and $\lim_{j\to\infty} g_j = f$ in $L^1([0,1];X)$, where $\mathbf{E}(f|\mathscr{F}_j)$ denotes the expectation of f with respect to $\mathscr{F}_j$. Each g_j is of the form

$$g_j = \sum_{k=1}^{j} h_k(\gamma_1,\ldots,\gamma_{k-1})\gamma_k,$$

where h_k is a function defined on $\{-1,1\}^{k-1}$ with values in X. It is easy to see that for $m,n,j\in\mathbb{N}$

$$\|x_j^{(n)} - x_j^{(n+m)}\|_X \leqslant \|f_n - f_{n+m}\|_{L^1([0,1];X)}.$$

This implies that $\lim_{n\to\infty} x_j^{(n)} := x_j$ exists in X. On the other hand, for $j,n\in\mathbb{N}$

$$\|h_j(\gamma_1,\ldots,\gamma_{j-1}) - x_j^{(n)}\|_{L^1([0,1];X)} \leqslant \|f - f_n\|_{L^1([0,1];X)}.$$

Let n tend to ∞, we obtain that h_j is a constant function and $h_j \equiv x_j$. We deduce from this that the series $\sum_{j=1}^{\infty}\gamma_j x_j$ converges in $L^1([0,1];X)$ to f.

Let $(q_j)_{j\geqslant 1}$ be a fixed dense sequence in $(0,1)$ and let $(p_j)_{j\geqslant 1}$ be a fixed dense sequence in $(1,\infty)$. We introduce two operators $\mathscr{A}_1$ and $\mathscr{A}_2$ on $\mathrm{Rad}(X)$ in the following way:

$$\begin{cases} D(\mathscr{A}_1) = \left\{\sum_{j=1}^{\infty}\gamma_j x_j \in \mathrm{Rad}(X)\colon x_j\in D(A), \quad \sum_{j=1}^{\infty} q_j\gamma_j A x_j \in \mathrm{Rad}(X)\right\} \\ \mathscr{A}_1\left(\sum_{j=1}^{\infty}\gamma_j x_j\right) = \sum_{j=1}^{\infty} q_j\gamma_j A x_j. \end{cases} \tag{3}$$

$$\begin{cases} D(\mathscr{A}_2) = \left\{\sum_{j=1}^{\infty}\gamma_j x_j \in \mathrm{Rad}(X)\colon x_j\in D(A), \quad \sum_{j=1}^{\infty} p_j\gamma_j A x_j \in \mathrm{Rad}(X)\right\} \\ \mathscr{A}_2\left(\sum_{j=1}^{\infty}\gamma_j x_j\right) = \sum_{j=1}^{\infty} p_j\gamma_j A x_j. \end{cases} \tag{4}$$

It is easy to verify that $\mathscr{A}_1$ and $\mathscr{A}_2$ are densely defined closed operators. We will use the following lemma.

LEMMA 3·1. *Let $\lambda\in\mathbb{C}$, $\lambda = re^{i\theta}$, then $\lambda\in\rho(\mathscr{A}_1)$ if and only if $\mathscr{R}\{se^{i\theta}R(se^{i\theta},A)\colon s\geqslant r\} < \infty$ and $\lambda\in\rho(\mathscr{A}_2)$ if and only if $\mathscr{R}\{se^{i\theta}R(se^{i\theta},A)\colon 0 < s\leqslant r\} < \infty$. In that case we have*

$$\|\lambda R(\lambda,\mathscr{A}_1)\| = \mathscr{R}\{se^{i\theta}R(se^{i\theta},A)\colon s\geqslant r\}$$

$$\|\lambda R(\lambda,\mathscr{A}_2)\| = \mathscr{R}\{se^{i\theta}R(se^{i\theta},A)\colon 0 < s\leqslant r\}.$$

Proof. We will only give the proof for $\mathscr{A}_1$, the proof for $\mathscr{A}_2$ is similar. First assume that $\lambda = re^{i\theta}\in\rho(\mathscr{A}_1)$, let $s_1,s_2,\ldots,s_m$ be distinct such that $s_i > r$. We have $0 < r/s_i < 1$, so there exists $q_{i,n}$ subsequence of q_n satisfying $\lim_{n\to\infty} q_{i,n} = r/s_i$. We may assume furthermore that for $n\geqslant 1$, $q_{1,n}, q_{2,n},\ldots,q_{m,n}$ are distinct. For

$x_1, x_2, \ldots, x_n \in X$, we have

$$\begin{aligned}
&\int_0^1 \left\| \sum_{j=1}^m \gamma_j(t) s_j e^{i\theta} R(s_j e^{i\theta}, A) x_j \right\|_X dt \\
&\quad = \lim_{n\to\infty} \int_0^1 \left\| \sum_{j=1}^m \gamma_j(t) \frac{r}{q_{j,n}} e^{i\theta} R\left(\frac{r}{q_{j,n}} e^{i,\theta}, A\right) x_j \right\|_X dt \\
&\quad = \lim_{n\to\infty} \int_0^1 \left\| \sum_{j=1}^m \gamma_j(t) r e^{i\theta} R(r e^{i\theta}, q_{j,n} A) x_j \right\|_X dt \\
&\quad \leqslant \|\lambda R(\lambda, \mathscr{A}_1)\| \left\| \sum_{j=1}^m \gamma_j x_j \right\|_{\mathrm{Rad}(X)}.
\end{aligned}$$

Thus we deduce that

$$\mathscr{R}\{s e^{i\theta} R(s e^{i\theta}, A) : s \geqslant r\} = \mathscr{R}\{s e^{i\theta} R(s e^{i\theta}, A) : s > r\} \leqslant \|\lambda R(\lambda, \mathscr{A}_1\|.$$

Conversely let $\sum_{j=1}^\infty \gamma_j x_j \in \mathrm{Rad}(X)$. Then

$$\begin{aligned}
\left\| \lambda R(\lambda, \mathscr{A}_1) \left(\sum_{j=1}^\infty \gamma_j x_j \right) \right\|_{\mathrm{Rad}(X)} &= \left\| \sum_{j=1}^\infty \gamma_j \lambda R(\lambda, q_j A) x_j \right\|_{\mathrm{Rad}(X)} \\
&= \left\| \sum_{j=1}^\infty \gamma_j \frac{\lambda}{q_j} R\left(\frac{\lambda}{q_j}, A\right) x_j \right\|_{\mathrm{Rad}(X)} \\
&\leqslant \mathscr{R}\{s e^{i\theta} R(s e^{i\theta}, A) : s \geqslant r\} \left\| \sum_{j=1}^\infty \gamma_j x_j \right\|_{\mathrm{Rad}(X)}.
\end{aligned}$$

Thus $\|\lambda R(\lambda, \mathscr{A}_1)\| \leqslant \mathscr{R}\{s e^{i\theta} R(s e^{i\theta}, A)\colon s \geqslant r\}$. The claim is proved.

The following is the main result of this section.

THEOREM 3·2. *Let A be the generator of an analytic C_0-semigroup on X. Assume that X is a* UMD*-space and $\omega(A) < 0$. Then the following are equivalent*:

(i) *A has $\mathscr{M}R_\infty$.*
(ii) *$\mathscr{A}_1$ generates a bounded analytic C_0-semigroup on* $\mathrm{Rad}(X)$.
(iii) *$\mathscr{A}_2$ generates a bounded analytic C_0-semigroup on* $\mathrm{Rad}(X)$.

Proof. First notice that there exist $\theta > 0$, $M \geqslant 1$ and $\omega > 0$ such that $T \in \mathscr{E}(\theta, M, \omega)$ by Lemma 2·5. Assume that A has $\mathscr{M}R_\infty$. Then by Theorem 2·1 the set $\{\lambda R(\lambda, A)\colon \lambda \in i\mathbb{R}\}$ is $\mathscr{R}$-bounded. By an argument used in [**CP**], there exists $\pi/2 < \theta' < \pi$, such that the set $\{\lambda R(\lambda, A)\colon \lambda \in \Sigma_{\theta'} \cup \{0\}\}$ is $\mathscr{R}$-bounded. By Lemma 3·1 we have $\Sigma_{\theta'} \subset \rho(\mathscr{A}_1) \cap \rho(\mathscr{A}_2)$ and

$$\sup_{\lambda \in \Sigma_{\theta'}} \|\lambda R(\lambda, \mathscr{A}_1)\| \leqslant \mathscr{R}\{\lambda R(\lambda, A)\colon \lambda \in \Sigma_{\theta'}\} < \infty$$

$$\sup_{\lambda \in \Sigma_{\theta'}} \|\lambda R(\lambda, \mathscr{A}_2)\| \leqslant \mathscr{R}\{\lambda R(\lambda, A)\colon \lambda \in \Sigma_{\theta'}\} < \infty.$$

So $\mathscr{A}_1$ and $\mathscr{A}_2$ generate bounded analytic C_0-semigroups on $\mathrm{Rad}(X)$ (see [**N**, theorem 4·6]). This shows the implications (i) ⇒ (ii) and (i) ⇒ (iii).

If $\mathscr{A}_1$ generates a bounded analytic C_0-semigroup on $\mathrm{Rad}(X)$, we have $\{is\colon s \in \mathbb{R}, s \neq 0\} \subset \rho(\mathscr{A}_1)$ and

$$\sup_{s\in\mathbb{R}, s\neq 0} \|isR(is, \mathscr{A}_1)\| < \infty.$$

By Lemma 3·1 for each $r > 0$

$$\mathscr{R}\{itR(it, A)\colon t \geqslant r\} \leqslant \sup_{s\in\mathbb{R}, s\neq 0} \|isR(is, \mathscr{A}_1)\|.$$

So

$$\mathscr{R}\{itR(it, A)\colon t \geqslant 0\} \leqslant \sup_{s\in\mathbb{R}, s\neq 0} \|isR(is, \mathscr{A}_1)\|.$$

Therefore

$$\mathscr{R}\{itR(it, A)\colon t \in \mathbb{R}\} \leqslant 2 \sup_{s\in\mathbb{R}, s\neq 0} \|isR(is, \mathscr{A}_1)\| < \infty.$$

It follows that A has $\mathscr{M}R_\infty$ by Theorem 2·1. This shows the implication (ii) ⇒ (i). The implication (iii) ⇒ (i) is similar.

Remark 3·3. (i) When A has $\mathscr{M}R_\infty$, the bounded analytic C_0-semigroups generated by $\mathscr{A}_1$ and $\mathscr{A}_2$ are given by

$$\mathscr{T}_z\left(\sum_{j=1}^{\infty} \gamma_j x_j\right) = \sum_{j=1}^{\infty} \gamma_j T_{q_j z} x_j$$

$$\mathscr{S}_z\left(\sum_{j=1}^{\infty} \gamma_j x_j\right) = \sum_{j=1}^{\infty} \gamma_j T_{p_j z} x_j,$$

respectively.

(ii) Let A be the generator of an analytic C_0-semigroup T. Assume that $\omega(A) < 0$. The semigroup $\mathscr{T}$ does not exist in general since $\{T_t\colon 0 \leqslant t \leqslant 1\}$ is not necessarily $\mathscr{R}$-bounded. However by Corollary 2·6, $\mathscr{S}$ is well defined and $\mathscr{S}$ defines a semigroup on $\mathrm{Rad}(X)$, but it is not always strongly continuous (or equivalently, $\mathscr{S}$ is not always bounded on $\Sigma_\theta \cap B(0, 1)$ for $\theta > 0$).

As application of Theorem 3·2, we deduce the following result due to [**W1**].

Theorem 3·4. *Let T be an analytic C_0-semigroup with generator A. Assume that X is a* UMD*-space and $\omega(A) < 0$. Then A has $\mathscr{M}R_\infty$ if and only if $\{T_z\colon z \in \Sigma_\theta\}$ is $\mathscr{R}$-bounded for some* $0 < \theta < \pi/2$.

Proof. When A has $\mathscr{M}R_\infty$, by Theorem 3·2 $\mathscr{A}_1$ generates a bounded analytic C_0-semigroup $\mathscr{T}$ on Σ_θ for some $0 < \theta < \pi/2$. By Remark 3·3 the semigroup generated by $\mathscr{A}_1$ is given by

$$\mathscr{T}_z\left(\sum_{j=1}^{\infty} \gamma_j x_j\right) = \sum_{j=1}^{\infty} \gamma_j T_{q_j z} x_j$$

for $z \in \Sigma_\theta$. A similar argument as in the proof of Lemma 3·1 shows that

$$\mathscr{R}\{T_{se^{\pm i\theta'}}\colon s \geqslant 0\} < \infty$$

for all $0 < \theta' < \theta$. It follows from Proposition 2·2 that $\mathscr{R}\{T_z\colon z \in \Sigma_{\theta'}\} < \infty$.

Conversely, suppose that $\mathscr{R}\{T_z\colon z \in \Sigma_\theta\} < \infty$ for some $0 < \theta < \pi/2$. Then the C_0-semigroup

$$\mathscr{T}_z\left(\sum_{j=1}^{\infty} \gamma_j x_j\right) = \sum_{j=1}^{\infty} \gamma_j T_{q_j z} x_j$$

is bounded analytic on Σ_θ.

Furthermore, $\mathscr{T}$ is strongly continuous. Indeed, it is clear that for $\sum_{j=1}^{N} \gamma_j x_j \in \mathrm{Rad}(X)$, we have $\lim_{z\to 0, z\in\sum_\theta} \mathscr{T}_z(\sum_{j=1}^{N}\gamma_j x_j) = \sum_{j=1}^{N}\gamma_j \lim_{z\to 0, z\in\sum_\theta} T_{q_j z} x_j = \sum_{j=1}^{N}\gamma_j x_j$. It follows from the uniform boundedness of $\mathscr{T}$ and the density of $\{\sum_{j=1}^{n}\gamma_j x_j\colon x_j \in X, n \in \mathbb{N}\}$ in $\mathrm{Rad}(X)$ that for every $\sum_{j=1}^{\infty}\gamma_j x_j \in \mathrm{Rad}(X)$, we have $\lim_{z\to 0, z\in\sum_\theta} \mathscr{T}_z(\sum_{j=1}^{\infty}\gamma_j x_j) = \sum_{j=1}^{\infty}\gamma_j x_j$.

The generator of $\mathscr{T}$ is given by $\mathscr{A}_1$. Therefore $\mathscr{A}_1$ generates a bounded analytic C_0-semigroup. It follows from Theorem 3·2 that A has $\mathscr{M}R_\infty$.

As an immediate consequence of Corollary 2·6 and Theorem 3·4, we obtain the following.

COROLLARY 3·5. *Let T be an analytic C_0-semigroup on X and let A be the generator of T. Assume that X is a* UMD*-space and $\omega(A) < 0$. Then T has $\mathscr{M}R_\infty$ if and only if there exist $r > 0$ and $\theta > 0$ such that $\mathscr{R}\{T(z)\colon |z| \leqslant r, z \in \Sigma_\theta\} < \infty$.*

For $\mathscr{M}R$, it is more natural to work with analytic C_0-semigroups which are not necessarily of negative type. In that case, the semigroup generated by $\mathscr{A}_1$ is not necessarily bounded and there is no analogous characterization involving the operator $\mathscr{A}_2$. The following characterization of $\mathscr{M}R$ is analogous to the equivalence between (i) and (ii) in Theorem 3·2.

THEOREM 3·6. *Let A be the generator of an analytic C_0-semigroup on X. Assume that X is a* UMD*-space. Then A has $\mathscr{M}R$ if and only if $\mathscr{A}_1$ generates an analytic C_0-semigroup on* $\mathrm{Rad}(X)$.

Proof. First recall that A generates an analytic C_0-semigroup if and only if there exists $r > 0$, such that $\{z\colon \mathrm{Re}(z) \geqslant r\} \subset \rho(A)$ and $\sup_{Re(z)\geqslant r} \|zR(z, A)\| < \infty$, see [**N**, A-II, theorem 1·14].

Assume now that A has $\mathscr{M}R$. Let T be the analytic C_0-semigroup generated by A. There exists $\omega > 0$ such that $(e^{-\omega t}T_t)_{t\geqslant 0}$ is exponentially stable, so $A - \omega$ has $\mathscr{M}R_\infty$. We have by Theorem 2·1 and Corollary 2·4 that

$$\mathscr{R}\{isR(is, A-\omega)\colon s \in \mathbb{R}\} < \infty$$

$$\mathscr{R}\{R(is, A-\omega)\colon s \in \mathbb{R}\} < \infty.$$

Thus

$$\mathscr{R}\{\lambda R(\lambda, A)\colon \mathrm{Re}(\lambda) = \omega\} < \infty.$$

By Proposition 2·2

$$\mathscr{R}\{\lambda R(\lambda, A)\colon \mathrm{Re}(z) \geqslant \omega\} < \infty.$$

For $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geqslant w$, $\lambda = a + ib$, $a \geqslant \omega$,

$$\left\| \lambda R(\lambda, \mathscr{A}_1) \left(\sum_{j=1}^{\infty} \gamma_j x_j \right) \right\|_{\mathrm{Rad}(X)} = \left\| \sum_{j=1}^{\infty} \gamma_j \left(\frac{a}{q_j} + i\frac{b}{q_j} \right) R\left(\frac{a}{q_j} + i\frac{b}{q_j}, A \right) x_j \right\|_{\mathrm{Rad}(X)}$$

$$\leqslant \mathscr{R}\{\lambda R(\lambda, A) \colon Re(\lambda) \geqslant \omega\} \left\| \sum_{j=1}^{\infty} \gamma_j x_j \right\|_{\mathrm{Rad}(X)}$$

as $a/q_j \geqslant a \geqslant \omega$. So $\sup_{Re(\lambda)\geqslant\omega} \|\lambda R(\lambda, \mathscr{A}_1)\| < \infty$, thus $\mathscr{A}_1$ generates an analytic C_0-semigroup on $\mathrm{Rad}(X)$.

Conversely assume that $\mathscr{A}_1$ generates an analytic C_0-semigroup on $\mathrm{Rad}(X)$, there exists $r > 0$ satisfying $\{\lambda \colon \mathrm{Re}(\lambda) \geqslant 0, |\lambda| \geqslant r\} \subset \rho(\mathscr{A}_1)$ and there exists $C > 0$ such that

$$\sup_{\mathrm{Re}(\lambda)\geqslant 0, |\lambda|\geqslant r} \|\lambda R(\lambda, \mathscr{A}_1)\| \leqslant C.$$

As A also generates an analytic C_0-semigroup on X, we can suppose that $\{\lambda \colon \mathrm{Re}(\lambda) \geqslant 0, |\lambda| \geqslant r\} \subset \rho(A)$ and

$$\sup_{\mathrm{Re}(\lambda)\geqslant 0, |\lambda|\geqslant r} \|\lambda R(\lambda, A)\| \leqslant C.$$

From $\|irR(ir, \mathscr{A}_1)\| \leqslant C$, we get $\mathscr{R}\{isR(is, A) \colon s \geqslant r\} \leqslant C$ by Lemma 3·1. Similarly $\mathscr{R}\{isR(is, A)(:) \; s \leqslant -r\} \leqslant C$. Define

$$\Lambda = \{\lambda \in \mathbb{C} : |\mathrm{Im}(\lambda)| = r, 0 \leqslant \mathrm{Re}(\lambda) \leqslant r\} \cup \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) = r, |\mathrm{Im}(\lambda)| \leqslant r\}.$$

We have $\Lambda \subset \rho(A)$ and Λ is a compact subset. By Proposition 2·2, $\mathscr{R}\{\lambda R(\lambda, A) \colon \lambda \in \Lambda\} < \infty$, hence $\mathscr{R}\{\lambda R(\lambda, A) \colon \lambda \in \Lambda \text{ or } \lambda \in i\mathbb{R}, |\lambda| \geqslant r\} < \infty$. Again by Proposition 2·2

$$\mathscr{R}\{\lambda R(\lambda, A) \colon \mathrm{Re}(\lambda) \geqslant r\} < \infty.$$

Without loss of generality, we can assume that the semigroup $(e^{-rt}T_t)_{t\geqslant 0}$ is exponentially stable. By Corollary 2·4 this implies that

$$\mathscr{R}\{R(is + r, A) \colon s \in \mathbb{R}\} < \infty.$$

Therefore

$$\mathscr{R}\{isR(is + r, A) \colon s \in \mathbb{R}\} < \infty.$$

By Theorem 2·1 this is equivalent to say that $A - r$ has $\mathscr{M}R_\infty$ which implies that A has $\mathscr{M}R$.

We have also the following characterization of $\mathscr{M}R$ in term of $\mathscr{R}$-boundedness of the semigroup T.

PROPOSITION 3·7. *Let A be the generator of an analytic C_0-semigroup T. Assume that X is a* UMD-*space. Then A has $\mathscr{M}R$ if and only if for some $r > 0$ and $\theta > 0$, the set $\{T_z \colon z \in \Sigma_\theta, |z| \leqslant r\}$ is $\mathscr{R}$-bounded.*

Proof. First assume that A has $\mathscr{M}R$, there exists $\epsilon > 0$ such that the analytic C_0-semigroup $(e^{-\epsilon z}T_z)_{z\in\Sigma_\theta}$ belongs to the class $\mathscr{E}(\theta, M, \omega)$ for some $\theta > 0$, $M \geqslant 1$ and $\omega > 0$. As $(e^{-\epsilon t}T_t)_{t\geqslant 0}$ is exponentially stable, $A - \epsilon$ has $\mathscr{M}R_\infty$ and by Theorem 3·4 the set $\{e^{-\epsilon z}T_z \colon z \in \Sigma_{\theta'}\}$ is $\mathscr{R}$-bounded for some $0 < \theta' < \theta$. On the other hand by

Corollary 2·6 there exists $r > 0$ such that the set $\{e^{-\epsilon z}T_z\colon z \in \Sigma_{\theta'}, |z| \geqslant r\}$ is $\mathscr{R}$-bounded, and so

$$\mathscr{R}\{e^{-\epsilon z}T_z\colon z \in \Sigma_{\theta'}, |z| \leqslant r\} < \infty,$$

by Kahane's contraction principle

$$\mathscr{R}\{T_z\colon z \in \Sigma_{\theta'}, |z| \leqslant r\} < \infty.$$

For the converse, assume that there exist $r > 0$ and $\theta > 0$ such that $\mathscr{R}\{T_z\colon z \in \Sigma_\theta, |z| \leqslant r\} < \infty$. There exists $\epsilon > 0$ such that the C_0-semigroup $(e^{-\epsilon t}T_t)_{t\geqslant 0}$ belongs to the class $\mathscr{E}(\theta', M, \omega)$ for some $0 < \theta' < \theta$, $M \geqslant 1$ and $\omega > 0$. We deduce from Corollary 2·6 that $\mathscr{R}\{e^{-\epsilon z}T_z\colon z \in \Sigma_{\theta'}, |z| \geqslant r\} < \infty$. By hypothesis we have $\mathscr{R}\{T_z\colon z \in \Sigma_{\theta'}, |z| \leqslant r\} < \infty$, so $\mathscr{R}\{e^{-\epsilon z}T_z\colon z \in \Sigma_{\theta'}, |z| \leqslant r\} < \infty$ by Kahane's contraction principle. Finally we get $\mathscr{R}\{e^{-\epsilon z}T_z\colon z \in \Sigma_{\theta'}\} < \infty$, this implies by Theorem 3·4 that $A - \epsilon$ has $\mathscr{M}R_\infty$; and so A has $\mathscr{M}R$. The claim is proved.

Let A be the generator of an analytic C_0-semigroup T. Assume that $\omega(A) < 0$. By Lemma 2·5 the semigroup T belongs to the class $\mathscr{E}(\theta, M, \omega)$ for some $0 < \theta < \pi/2$, $M \geqslant 1$ and $\omega > 0$. By Proposition 2·2 and Corollary 2·6 there exists $0 < \theta_0 < \pi/2$ such that for all $r > 0$

$$\sup_{z\in\Sigma_{\theta_0}, |z|\geqslant r} \|\mathscr{S}_z\| < \infty.$$

One can easily check that when X is a UMD-space, A has $\mathscr{M}R_\infty$ if and only if the semigroup $\mathscr{S}_z$ is strongly continuous at $z = 0$. One can also show that when X is a UMD-space, A has $\mathscr{M}R_\infty$ if and only if the semigroup $\mathscr{S}_z$ is bounded on $\{z \in \Sigma_{\theta_0}, |z| \leqslant 1\}$.

In the following we study the behaviour of $\|\lambda R(\lambda, \mathscr{A}_2)\|$ when $|\lambda| \to \infty$, and the behaviour of $\|\mathscr{S}_z\|$ when $|z| \to 0$. We will need the following lemma.

LEMMA 3·8. *Let A be the generator of an analytic C_0-semigroup T. Assume that $\omega(A) < 0$. Then there exists $\pi/2 < \theta < \pi$ such that for each $0 < \alpha < 1$*

$$\mathscr{R}\left\{|\lambda|^\alpha R(\lambda, A)\colon \lambda \in \Sigma_\theta\right\} < \infty.$$

Proof. Under the assumption of the lemma, the semigroup T belongs to $\mathscr{E}(\theta, M, \omega)$ for some $\theta > 0$, $M \geqslant 1$ and $\omega > 0$ by Lemma 2·5. As for each $|\beta| < \theta$, $e^{i\beta}A$ also generates an analytic C_0-semigroup on X in the class $\mathscr{E}(\theta - |\beta|, M, \omega)$, to show the lemma it suffices to show that for each $0 < \theta < \pi/2$, $0 < \alpha < 1$, we have

$$\mathscr{R}\{|\lambda|^\alpha R(\lambda, A)\colon \lambda \in \Sigma_\theta\} < \infty.$$

We have

$$|\lambda|^\alpha R(\lambda, A) = \int_0^\infty |\lambda|^\alpha e^{-\lambda t} T_t \, dt.$$

By Lemma 2·3

$$\begin{aligned}
\mathscr{R}\{|\lambda|^\alpha R(\lambda, A)\colon \lambda \in \Sigma_\theta\} &\leqslant \int_0^\infty 2\|T_t\| \sup_{\lambda\in\Sigma_\theta} |\lambda|^\alpha e^{-Re(\lambda)^t} \, dt \\
&\leqslant \frac{2M}{\cos^\alpha \theta} \int_0^\infty e^{-\omega t} \sup_{\lambda>0} \lambda^\alpha e^{-\lambda t} \, dt \\
&= \frac{2M\alpha^\alpha}{e\cos^\alpha \theta} \int_0^\infty e^{-\omega t} \frac{dt}{t^\alpha} < \infty.
\end{aligned}$$

Under the hypothesis of the previous lemma, there exists $\pi/2 < \theta < \pi$ such that for all $0 < \alpha < 1$, the subset $\{|\lambda|^\alpha R(\lambda, A) \colon \lambda \in \Sigma_\theta\}$ is $\mathscr{R}$-bounded. For $\lambda \in \Sigma_\theta$

$$|\lambda|^\alpha R(\lambda, \mathscr{A}_2)\left(\sum_{j=1}^{\infty} \gamma_j x_j\right) = \sum_{j=1}^{\infty} \gamma_j \frac{|\lambda|^\alpha}{p_j} R\left(\frac{\lambda}{p_j}, A\right) x_j$$

as $|\lambda|^\alpha/p_j \leqslant |\lambda|^\alpha/p_j^\alpha$ for $p_j \geqslant 1$ and $0 < \alpha < 1$, by Kahane's contraction principle $\||\lambda|^\alpha R(\lambda, \mathscr{A}_2)\| \leqslant C_{\alpha,\theta}$, where $C_{\alpha,\theta}$ is a constant depending only on $0 < \alpha < 1$ and θ. So

$$\sup_{\lambda \in \Sigma_\theta} \||\lambda|^\alpha R(\lambda, \mathscr{A}_2)\| \leqslant C_{\alpha,\theta}.$$

It is clear by Remark 3·3 that the semigroup $\mathscr{S}_z$ is well defined and

$$\mathscr{S}_z\left(\sum_{j=1}^{\infty} \gamma_j x_j\right) = \sum_{j=1}^{\infty} \gamma_j T_{p_j z} x_j.$$

One can represent S_z in term of $R(\lambda, \mathscr{A}_2)$ in a standard way:

$$\mathscr{S}_t = \int_\Gamma e^{\lambda t} R(\lambda, \mathscr{A}_2)\, d\lambda,$$

where for $t > 0$, the path Γ is composed by Γ_1, Γ_2 and Γ_3, where $\Gamma_1 = \{re^{-i\theta'} : t^{-1} \leqslant r < \infty\}$, $\Gamma_2 = \{t^{-1}e^{i\phi} : -\theta' \leqslant \phi \leqslant \theta'\}$ and $\Gamma_3 = \{re^{i\theta'} : t^{-1} \leqslant r < \infty\}$ for a fixed $\pi/2 < \theta' < \theta$. Γ is orientated in such a way that $\mathrm{Im}(z)$ increases. Using the known estimate of $R(\lambda, \mathscr{A}_2)$ when $|\lambda|$ is big, we easily obtain that

$$\|\mathscr{S}_t\| \leqslant C_\alpha/t^\alpha \qquad 0 < t \leqslant 1$$

for some constant C_α depending only on $0 < \alpha < 1$. So there exists $0 < \beta < \pi/2$ and a constant $C_{\alpha,\beta}$ depending only on $0 < \alpha < 1$ and β, such that

$$\|\mathscr{S}_z\| \leqslant C_{\alpha,\beta}/|z|^\alpha$$

for all $z \in \Sigma_\beta, 0 < |z| \leqslant 1$. So we have shown the following.

PROPOSITION 3·9. *Let A be the generator of an analytic C_0-semigroup T. Assume that $\omega(A) < 0$. Then there exists $0 < \beta < \pi/2$ such that for each $0 < \alpha < 1$, there exists $C_{\alpha,\beta} < \infty$ depending only on α and β such that*

$$\sup_{\lambda \in \Sigma_{\beta+\pi/2}} \||\lambda|^\alpha R(\lambda, \mathscr{A}_2)\| \leqslant C_{\alpha,\beta}$$

$$\sup_{z \in \Sigma_\beta} \||z|^{1-\alpha} S_z\| \leqslant C_{\alpha,\beta}.$$

Remark 3·10. By the counterexample of Kalton and Lancien, when X is a Banach space with an unconditional basis and if X is not isomorphic to l^2, there exists an analytic C_0-semigroup T with generator A satisfying $\omega(A) < 0$ such that the corresponding Cauchy problem does not have $\mathscr{M}R_\infty$. This implies that $\mathscr{R}\{sR(is, A) \colon s \in \mathbb{R}\} = \infty$, or equivalently $\sup_{\lambda \in \Sigma_{\beta+\pi/2}} \||\lambda| R(\lambda, \mathscr{A}_2)\| = \infty$ for some $0 < \beta < \pi/2$. This means that we cannot expect to extend the conclusion of Lemma 3·8 or Proposition 3·9 to the case $\alpha = 1$.

4. *Applications*

Let (Ω, Σ, μ) be a σ-finite measure space. Let $1 \leqslant p_0 < p_1 < \infty$ and let T_p, $p_0 \leqslant p \leqslant p_1$ be a family of interpolating C_0-semigroups on $L^p(\Omega)$ (i.e., $T_p(t)f = T_q(t)f$ for all $f \in L^p(\Omega) \bigcap L^q(\Omega)$ whenever $t \geqslant 0$, $p_0 \leqslant p, q \leqslant p_1$). If each T_p is bounded (on $\mathbb{R}_+$) and if T_2 is bounded analytic in a sector Σ_θ for some $0 < \theta < \pi/2$, then by Stein's interpolation theorem (see [**RS**, theorem IX·21]), each T_p is bounded analytic on some sector (depending on p), $p_0 < p < p_1$.

Next we suppose that $p_1 = 2$, and let T_p is exponentially stable, $p_0 \leqslant p \leqslant 2$. By Corollary 2·6 the semigroup

$$\mathscr{S}_p(t)\left(\sum_{j=1}^{\infty} \gamma_j f_j\right) = \sum_{j=1}^{\infty} \gamma_j T_p(tp_j) f_j$$

is well defined on $\mathrm{Rad}(L^p(\Omega))$ and in general it is not a C_0-semigroup. As T_2 is bounded analytic and exponentially stable and $L^2(\Omega)$ is a Hilbert space, T_2 has $\mathscr{MR}_\infty$ [**DS**]. By Theorem 3·2 this implies that the C_0-semigroup $\mathscr{S}_2$ is bounded analytic. Again by Theorem 3·2 T_p has $\mathscr{MR}_\infty$ if and only if $\mathscr{S}_p$ is bounded analytic on $\mathrm{Rad}(L^p(\Omega))$.

Recall the well-known Khintchine's inequality: for $1 < q < \infty$, there exists $C_q > 0$ such that for $f_j \in L^q(\Omega)$

$$\frac{1}{C_q}\left\|\left(\sum_{j=1}^{\infty} |f_j|^2\right)^{\frac{1}{2}}\right\|_{L^q(\Omega)} \leqslant \left\|\sum_{j=1}^{\infty} \gamma_j f_j\right\|_{\mathrm{Rad}(L^q(\Omega))} \leqslant C_q \left\|\left(\sum_{j=1}^{\infty} |f_j|^2\right)^{\frac{1}{2}}\right\|_{L^q(\Omega)}.$$

This shows that $\mathrm{Rad}(L^q(\Omega))$ is isomorphic to the space $L^q(\Omega; l^2)$. In order to show that $\mathscr{S}_p$ is bounded analytic, it suffices to show that the semigroup

$$\overline{\mathscr{S}}_p(t)(f_1, f_2, \ldots) = (T_p(p_1 t) f_1,\ T_p(p_2 t) f_2, \ldots)$$

is bounded analytic on $L^p(\Omega, l^2)$. It is easy to verify that for $p_0 \leqslant p, q \leqslant 2$

$$\overline{\mathscr{S}}_p(t)(f_1, f_2, \ldots) = \overline{\mathscr{S}}_q(t)(f_1, f_2, \ldots)$$

for $(f_1, f_2, \ldots) \in L^p(\Omega; l^2) \bigcap L^q(\Omega; l^2)$ and $t > 0$. So $\overline{\mathscr{S}}_p$ are again 'vector-valued' interpolating semigroups, $p_0 \leqslant p \leqslant 2$. Now we are in the position to state a result which gives a sufficient condition for an interpolating C_0-semigroup of negative type to have $\mathscr{MR}_\infty$.

THEOREM 4·1. *Let (Ω, Σ, μ) be a σ-finite measure space and let T_p be interpolating C_0-semigroups on $L^p(\Omega)$, $p_0 \leqslant p \leqslant 2$. Assume that for all $p_0 \leqslant p \leqslant 2$, T_p is exponentially stable, that T_2 is analytic and $\mathscr{R}\{T_{p_0}(t) \colon t > 0\} < \infty$. Then T_p has $\mathscr{MR}_\infty$ for all $p_0 < p \leqslant 2$.*

This theorem is an immediate consequence of the above discussion and the following vector-valued Stein's interpolation theorem. Its proof is similar to the scalar case and can be omitted (see [**RS**, theorem IX·21] for the scalar case).

THEOREM 4·2. *Let (Ω, Σ, μ) be a σ-finite measure space, X be a Banach space, $1 < p_1$, $p_2 < \infty$, $0 < \theta < \pi/2$. Let $z \rightarrow T(z)$ be an application defined on $\overline{\Sigma}_\theta$ with values in $B(L^{p_2}(\Omega; X))$ which is bounded continuous on $\overline{\Sigma}_\theta$ and analytic on Σ_θ. Assume that there* *exist* $M_1, M_2 < \infty$ *such that*

$$\sup_{r>0} \|T(re^{\pm i\theta})\|_{L_{p_2}\to L_{p_2}} \leqslant M_2$$

and

$$\|T(r)f\|_{L_{p_1}} \leqslant M_1\|f\|_{L_{p_1}}$$

for all $r > 0$, $f \in L^{p_1}(\Omega; X) \cap L^{p_2}(\Omega; X)$. *Then for all* $0 < \alpha < 1$,

$$\|T(re^{i\alpha\theta})f\|_{L^p} \leqslant M_1^{1-\alpha}M_2^{\alpha}\|f\|_{L^p}$$

for all $f \in L^p(\Omega; X) \cap L^{p_2}(\Omega; X)$, *where*

$$\frac{1}{p} = \frac{\alpha}{p_2} + \frac{1-\alpha}{p_1}.$$

Now we can prove the following interpolation result for maximal regularity.

THEOREM 4·3. *Let* (Ω, Σ, μ) *be a* σ*-finite measure space*, $1 < p_0 < 2$. *For* $p_0 < p \leqslant 2$, *let* T_p *be a* C_0*-semigroups on* $L^p(\Omega)$ *verifying* $T_p(t)f = T_q(t)f$ *for* $f \in L^p(\Omega) \cap L^q(\Omega)$ *and* $t > 0$. *Assume that* T_2 *is analytic and there exists* $\delta > 0$ *such that* $\mathscr{R}\{T_{p_0}(t)\colon 0 < t < \delta\} < \infty$. *Then* T_p *has* $\mathscr{M}R$ *for all* $p_0 < p \leqslant 2$.

Proof. Define for $\epsilon > 0$

$$W_p(t) = e^{-\epsilon t}T_p(t).$$

Fix one $\epsilon > 0$ big enough to ensure that each W_p is exponentially stable. In this case W_2 is bounded analytic and by Corollary 2·6, $\mathscr{R}\{W_{p_0}(t)\colon t \geqslant \delta\} < \infty$. As $\mathscr{R}\{T_{p_0}(t)\colon 0 < t \leqslant \delta\} < \infty$, we deduce that $\mathscr{R}\{W_{p_0}(t)\colon t > 0\} < \infty$. By Theorem 4·2 W_p has $\mathscr{M}R_\infty$ for $p_0 < p \leqslant 2$, and so T_p has $\mathscr{M}R$.

Remark 4·4. Of course, if we have $2 < p_0 < \infty$, we can give similar results for $2 \leqslant p < p_0$ as in Theorem 4·1 and Theorem 4·3.

As a direct application of Theorem 4·3, we deduce the following result due to [**HP**] (see also [**CP**] and [**W1**]).

COROLLARY 4·5. *Let* $\Omega \subset \mathbb{R}^n$ *be a measurable subset. For* $1 < p < \infty$, *let* T_p *be a* C_0*-semigroup on* $L^p(\Omega)$ *such that for* $t > 0$, $f \in L^p(\Omega) \cap L^q(\Omega)$, *we have* $T_p(t)f = T_q(t)f$. *Assume that* T_2 *is analytic and has Gaussian estimates. Then* T_p *has* $\mathscr{M}R$ *for* $1 < p < \infty$.

Proof. As T_2 has Gaussian estimates, there exist constants $C > 0$ and $a > 0$ such that for $f \in L^2(\Omega)$, one has $|T_2(t)f|(\omega) \leqslant C[G_2(at)|f|](\omega)$ for almost all $\omega \in \Omega$, and all $0 < t \leqslant 1$, where G_p denotes the Gaussian semigroup on $L^p(\mathbb{R}^n)$. We have $\mathscr{R}\{G_p(t) : 0 < t \leqslant 1\} < \infty$ by [**St**, theorem 1, p. 51]. This implies that $\mathscr{R}\{T_p(t) : 0 < t \leqslant 1\} < \infty$ by Khintchine's inequality. The result follows from Theorem 4·3.

Remark 4·6. Let $(\Omega, \sum, \mu)$ be a σ-finite measure space, $1 < p < \infty$, and let T_p be interpolating C_0-semigroups on $L^p(\Omega; X)$, i.e. for all $1 < p, q < \infty$ and $f \in L^p(\Omega; X) \cap L^q(\Omega; X)$, we have $T_p(t)f = T_q(t)f$ for all $t > 0$. Using Kahane's inequality it is easy to see that the linear mapping

$$\mathrm{Rad}(L^p(\Omega; X)) \to L^p(\Omega; \mathrm{Rad}(X))$$

$$\sum_{j=1}^{\infty} \gamma_j f_j \to \left\{ \omega \to \sum_{j=1}^{\infty} \gamma_j f_j(\omega) \right\}$$

is an isomorphism between Banach spaces. So we can use Theorem 4·2 to obtain a vector-valued version of Theorem 4·1, Theorem 4·3 and Corollary 4·5 (for this we have to introduce the notion of vector-valued Gaussian estimates). Since the adaptation is standard, we omit the detail.

For $1 < p < \infty$, if T_p are positive interpolating C_0-semigroups on $L^p(\Omega)$, we can establish the following result which gives a sufficient condition for $\mathscr{M}R$. Notice that by Proposition 2·2 and Lemma 2·3, the hypothesis here is weaker than that of Theorem 4·3.

Theorem 4·7. *Let $(\Omega, \sum, \mu)$ be a σ-finite measure space and let $1 < p_0 < 2$. For $p_0 \leqslant p \leqslant 2$, let T_p be a positive C_0-semigroup on $L^p(\Omega)$ with generator A_p. Assume that T_2 is analytic and for $p_0 \leqslant p,\ q \leqslant 2$ and $t > 0$, we have $T_p(t)f = T_q(t)f$ whenever $f \in L^p(\Omega) \bigcap L^q(\Omega)$. Assume that $\mathscr{R}\{tR(t+\omega, A_{p_0})\colon t \geqslant 0\} < \infty$ for some $\omega > \omega(A_{p_0})$. Then T_p satisfies $\mathscr{M}R$ for $p_0 < p \leqslant 2$.*

Proof. First notice that it suffices to show the same conclusion for the semigroups $(e^{-\epsilon t}T_p(t))_{t\geqslant 0}$ for some $\epsilon > 0$. So without loss of generality, we can assume that each T_p is exponentially stable and $\mathscr{R}\{tR(t, A_{p0})\colon t \geqslant 0\} < \infty$. $L^2(\Omega)$ is a Hilbert space, T_2 is analytic and exponentially stable, so T_2 has $\mathscr{M}R_\infty$ [**DS**]. By Theorem 2·1 and an argument used in [**CP**], there exist $\alpha > 0$ such that $\mathscr{R}\{\lambda R(\lambda, A_2)\colon \lambda \in \Sigma_{\pi/2+\alpha}\} < \infty$.

Let $0 < \theta < \pi/2$ be fixed. For $\lambda \in \Sigma_\theta$ and $f \in L^{p_0}(\Omega)$, one has

$$\begin{aligned}|\lambda R(\lambda, A_{p_0})f| &\leqslant |\lambda| \int_0^\infty e^{-\mathrm{Re}(\lambda)t} T_{p_0}(t)|f|\, dt = |\lambda| R(\mathrm{Re}(\lambda), A_{p_0})|f| \\ &\leqslant \frac{1}{\cos\theta} \mathrm{Re}(\lambda) R(\mathrm{Re}(\lambda), A_{p_0})|f|.\end{aligned}$$

By Khintchine's inequality this implies that $\mathscr{R}\{\lambda R(\lambda, A_{p_0}) : \lambda \in \Sigma_\theta\} < \infty$.

Notice that $0 < \theta < \pi/2$ is arbitrary, so a similar argument as in [**La**, section II] shows that for each $p_0 < p \leqslant 2$, the set $\{sR(is, A_p)\colon s \in \mathbb{R}\}$ is $\mathscr{R}$-bounded. By Theorem 2·1, T_p has $\mathscr{M}R$.

Using Theorem 3·6, we can also give an easy new proof of the following perturbation result due to [**W1**].

Theorem 4·8. *Let A be the generator of an analytic C_0-semigroup on a* UMD*-space X, B a closed operator in X such that $D(A) \subset D(B)$. Assume that for each $a > 0$ there exists $b > 0$ satisfying*

$$\|Bx\| \leqslant a\|Ax\| + b\|x\|, \quad x \in D(A).$$

Then if A has $\mathscr{M}R$, $A + B$ also has $\mathscr{M}R$.

Proof. Let $\mathscr{A}_1$ and $\mathscr{B}_1$ be the corresponding closed operators associated to A and B respectively, defined by (3·1). We have for $x_j \in D(A)$

$$\left\| \mathscr{B}_1 \left(\sum_{j=1}^{\infty} \gamma_j x_j \right) \right\|_{\mathrm{Rad}(X)} \leqslant a \left\| \mathscr{A}_1 \left(\sum_{j=1}^{\infty} \gamma_j x_j \right) \right\|_{\mathrm{Rad}(X)} + b \left\| \sum_{j=1}^{\infty} \gamma_j x_j \right\|_{\mathrm{Rad}(X)}.$$

This follows from Kahane's contraction principle and the estimate

$$\left\| \sum_{j=1}^{\infty} \gamma_j B x_j \right\|_{\mathrm{Rad}(X)} \leqslant a \left\| \sum_{j=1}^{\infty} \gamma_j A x_j \right\|_{\mathrm{Rad}(X)} + b \left\| \sum_{j=1}^{\infty} \gamma_j x_j \right\|_{\mathrm{Rad}(X)}.$$

The semigroup generated by $\mathscr{A}_1$ is strongly continuous and analytic since A has $\mathscr{M}R$. By [**ABHN**, theorem 3·7·23], $\mathscr{A}_1 + \mathscr{B}_1$ generates an analytic C_0-semigroup. This implies that $A + B$ has $\mathscr{M}R$ by Theorem 3·6.

Theorem 3·2 combined with [**AR**, theorem 1·1] gives the following perturbation result for positive C_0-semigroups. Recall that an operator A on $L^p(\Omega)$ is called *resolvent positive* if, for some $\lambda_0 \in \mathbb{R}$ one has $[\lambda_0, \infty) \subset \rho(A)$ and $R(\lambda, A) \geqslant 0$ for all $\lambda \geqslant \lambda_0$.

THEOREM 4·9. *Let A be the generator of a positive C_0-semigroup on $L^p(\Omega)$ for some measure space (Ω, Σ, μ) and $1 < p < \infty$. Let $B : D(A) \to L^p(\Omega)$ be linear and positive. Assume that $A+B$ is resolvent positive and that A has $\mathscr{M}R$. Then $A + C$ has $\mathscr{M}R$ whenever C: $D(A) \to L^p(\Omega)$ is a linear mapping satisfying $|Cu| \leqslant Bu$ for all $u \in D(A)_+$.*

Proof. By [**AR**, theorem 1·1], $A + B$ and $A + C$ generate analytic C_0-semigroups. Since $A + B$ is resolvent positive, there exist $\lambda > \omega(A)$ and $k \in \mathbb{N}$ satisfying $\|(BR(\lambda, A))^k\| < 1$, see [**V**, theorem 1·1]. Without loss of generality we can assume that $\omega(A) < 0$, $\omega(A + B) < 0$, $\omega(A + C) < 0$ and $\lambda = 0$. Let $\mathscr{A}_2$, $\mathscr{B}_2$ and $\mathscr{C}_2$ be the corresponding closed operators associated with A, B and C respectively, defined by (3·2). By Khintchine's inequality, there exists a constant $C > 0$ such that for $f_j \in L^p(\Omega)$

$$\frac{1}{C} \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^p(\Omega)} \leqslant \left\| \sum_{j=1}^{\infty} \gamma_j f_j \right\|_{\mathrm{Rad}(L^p(\Omega))} \leqslant C \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^p(\Omega)}.$$

So the application J

$$\mathrm{Rad}(L^p(\Omega)) \longrightarrow L^p(\Omega; l^2)$$

$$\sum_{j=1}^{\infty} \gamma_j f_j \longrightarrow \{\omega \longrightarrow (f_j(\omega))_{j \geqslant 1}\}$$

is an isomorphism between Banach spaces. We will consider $L^p(\Omega; l^2)$ as a Banach lattice in the natural way. Let $\mathscr{A} = J\mathscr{A}_2 J^{-1}$, $\mathscr{B} = J\mathscr{B}_2 J^{-1}$ and $\mathscr{C} = J\mathscr{C}_2 J^{-1}$ be the corresponding operators on $L^p(\Omega; l^2)$.

The operator $\mathscr{B}$: $\mathscr{D}(\mathscr{A}) \to L^p(\Omega; l^2)$ is well defined and positive. Indeed, for $(f_j)_{j\geqslant 1} \in \mathscr{D}(\mathscr{A})$, we have $(p_j A f_j)_{j \geqslant 1} \in L^p(\Omega; l^2)$. As BA^{-1} is bounded, we deduce that $(p_j B f_j)_{j\geqslant 1} \in L^p(\Omega; l^2)$ and so $\mathscr{D}(\mathscr{A}) \subset \mathscr{D}(\mathscr{B})$. It is also clear that $\mathscr{C}$: $\mathscr{D}(\mathscr{A}) \to L^p(\Omega; l^2)$ is well defined and satisfies $|\mathscr{C}u| \leqslant \mathscr{B}u$ for each $u \in \mathscr{D}(\mathscr{A})_+$ and $\mathscr{A}$ generates an analytic C_0-semigroup by Theorem 3·2.

Notice that A is injective with dense range since $\mathscr{A}$ is injective with dense range. Furthermore for $(f_j)_{j\geqslant 1} \in L^p(\Omega; l^2)$, one has $\|\mathscr{A}^{-1}((f_j)_{j\geqslant 1})\| \leqslant \|(\frac{1}{p_j}A^{-1}f_j)_{j\geqslant 1}\| \leqslant \|A^{-1}\|\|((f_j)_{j\geqslant 1})\|$ since $p_j \geqslant 1$. This means that $0 \in \rho(\mathscr{A})$. Next for $(f_j)_{j\geqslant 1} \in L^p(\Omega; l^2)$ and $n \in \mathbb{N}$, one has

$$\begin{aligned}\|(\mathscr{B}R(0,\mathscr{A}))^{nk}(f_j)_{j\geqslant 1}\|_{L^p(\Omega;l^2)} &= \|((BR(0,A))^{nk}f_j)_{j\geqslant 1}\|_{L^p(\Omega,l^2)}\\ &\leqslant C\left\|\sum_{j=1}^{\infty}\gamma_j(BR(0,A))^{nk}f_j\right\|_{\mathrm{Rad}(L^p(\Omega))}\\ &\leqslant C\|(BR(0,A))^k\|^n\left\|\sum_{j=1}^{\infty}\gamma_j f_j\right\|_{\mathrm{Rad}(L^p(\Omega))}\\ &\leqslant C^2\|(BR(0,A))^k\|^n\|(f_j)_{j\geqslant 1}\|_{L^p(\Omega;l^2)}.\end{aligned}$$

We deduce that $\|(\mathscr{B}R(0,\mathscr{A}))^{nk}\| \leqslant C^2\|B(R(0,A))^k\|^n$ and so $\|(\mathscr{B}R(0,\mathscr{A}))^{nk}\| < 1$ for large $n \in \mathbb{N}$. This implies by [**V**, theorem 1·1] that $\mathscr{A} + \mathscr{B}$ is resolvent positive. By [**AR**, theorem 1·1], $\mathscr{A} + \mathscr{C}$ generates an analytic C_0-semigroup on $L^p(\Omega; l^2)$. We will show that $\omega(\mathscr{A} + \mathscr{C}) < 0$, this will finish the proof by Theorem 3·2 and Lemma 2·5.

We have $\mathscr{A} + \mathscr{B} = \mathscr{A}(I - R(0,\mathscr{A})\mathscr{B})$, so $(\mathscr{A} + \mathscr{B})^{-1} = (I - \mathscr{B}R(0,\mathscr{A}))^{-1}\mathscr{A}^{-1} = \sum_{j=0}^{\infty}(\mathscr{B}R(0,\mathscr{A}))^j\mathscr{A}^{-1}$. As $\mathscr{B}R(0,\mathscr{A})$ and $R(0,\mathscr{A})$ are positive, we deduce that $R(0,\mathscr{A} + \mathscr{B})$ is positive. This implies that $s(\mathscr{A} + \mathscr{B}) = \omega(\mathscr{A} + \mathscr{B}) < 0$, see [**ABHN**, proposition 3·11·2]. By [**AR**, theorem 1·2], the semigroup generated by $\mathscr{A} + \mathscr{C}$ is exponentially stable and so $\omega(\mathscr{A} + \mathscr{C}) < 0$. The claim is proved.

When T is a positive contractive analytic C_0-semigroup on $L^p(\Omega)(1 < p < \infty)$, T has $\mathscr{M}R$ [**W1**]. The following corollary is an immediate consequence of the previous theorem, and enlarges the class of semigroups to which Weis' theorem is applicable.

COROLLARY 4·10. *Let A be the generator of a positive contractive analytic C_0-semigroup on $L^p(\Omega)$ for some measure space (Ω, Σ, μ) and $1 < p < \infty$. Let B: $D(A) \to L^p(\Omega)$ be linear and positive. Assume that $\mathscr{A} + \mathscr{B}$ is resolvent positive. Then $\mathscr{A} + \mathscr{C}$ has $\mathscr{M}\mathscr{R}$ whenever C: $D(A) \to L^p(\Omega)$ is a linear mapping satisfying $|\mathscr{C}u| \leqslant Bu$ for all $u \in D(A)_+$.*

We give a concrete example of a Schrödinger operator.

Example 4·11. Let $X = L^p(\mathbb{R}^n)$, $1 < p < \infty$, $Af := \Delta f$, $D(A) = W^{2,p}(\mathbb{R}^n)$. Let $0 \leqslant V \in L^r(\mathbb{R}^n)$, where $r \geqslant \max\{p, n/2\}$ if $p \neq n/2$ and $r > n/2$ if $p = n/2$. Then $A + V$ with domain $D(A + V) = D(A)$ generates an analytic C_0-semigroup on $L^p(\mathbb{R}^n)$ which satisfies $\mathscr{M}R$. We refer to [**AR**, section 3] for more details and further examples.

Finally we prove that the mild solutions of the inhomogeneous Cauchy problem are always in some fractional Sobolev space without any assumption of $\mathscr{R}$-boundedness. Let X be a Banach space, $1 < p < \infty$, let $\mathscr{S}(\mathbb{R}; X)$ be the space of all rapidly decreasing smooth X-valued functions and denote by $\mathscr{S}'(\mathbb{R}; X) := B(\mathscr{S}(\mathbb{R}); X)$ the X-valued Schwartz space. As usual, we identify $L^p(\mathbb{R}; X)$ with a subspace of $\mathscr{S}'(\mathbb{R}; X)$. For

$f \in \mathscr{S}(\mathbb{R}; X)$, the Fourier transform of f is defined by

$$(\mathscr{F} f)(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ixy} f(x)\, dx, \quad y \in \mathbb{R}.$$

It is known that for $f \in S(\mathbb{R}; X)$, $\mathscr{F} f \in \mathscr{S}(\mathbb{R}; X)$. So for $T \in \mathscr{S}'(\mathbb{R}; X)$, we can define $\mathscr{F} T \in \mathscr{S}'(\mathbb{R}; X)$ in a natural way: $\langle \mathscr{F} T, \phi \rangle = -\langle T, \mathscr{F} \phi \rangle$ for $\phi \in \mathscr{S}(\mathbb{R}; \mathbb{C})$. It is known that $\mathscr{F}$ is an isomorphism from $\mathscr{S}(\mathbb{R}; X)$ onto $\mathscr{S}(\mathbb{R}; X)$, and it is also an isomorphism from $\mathscr{S}'(\mathbb{R}; X)$ onto $\mathscr{S}'(\mathbb{R}; X)$.

For $\beta > 0$, we can define the *fractional Sobolev space* $W^{\beta,p}(\mathbb{R}; X)$ by

$$W^{\beta,p}(\mathbb{R}; X) = \{f \in \mathscr{S}'(\mathbb{R}; X) \colon \mathscr{F}^{-1}((1 + y^2)^{\beta/2}(\mathscr{F} f)(y)) \in L^p(\mathbb{R}; X)\}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform on $\mathscr{S}'(\mathbb{R}; X)$. Define $W^{\beta,p}(\mathbb{R}_+; X) = \{f \in W^{\beta,p}(\mathbb{R}; X) \colon f(x) = 0 \text{ for } x < 0\}$.

Let T be an analytic C_0-semigroup with generator A. Assume that $\omega(A) < 0$. When X is a UMD-space and $\{isR(is, A) \colon s \in \mathbb{R}\}$ is $\mathscr{R}$-bounded, then the Cauchy problem (1) has maximal regularity in $W^{\beta,p}(\mathbb{R}_+; X)$ for $1 < p < \infty$ and $0 < \beta < 1$: for each $f \in W^{\beta,p}(\mathbb{R}_+; X)$, there exists a unique u, solution of (1) such that $Au \in W^{\beta,p}(\mathbb{R}_+; X)$. This follows easily from the operator-valued Fourier multiplier theorem due to [**W2**].

Using Lemma 3·8 and the same operator-valued Fourier multiplier theorem, we can establish the following result.

THEOREM 4·12. *Let A be the generator of an analytic C_0-semigroup T on a* UMD-*space X and let A be the generator of an analytic C_0-semigroup on X satisfying $\omega(A) < 0$. Then the following holds.*

(i) *For all $f \in W^{\beta,p}(\mathbb{R}_+, X)$, $0 < \beta < 1$, there exists a unique solution u of the problem* (1) *such that $Au \in W^{\beta',p}(\mathbb{R}_+; X)$ for all $0 < \beta' < \beta$.*
(ii) *For all $f \in L^p(\mathbb{R}_+, X)$, the mild solution u of the problem* (1) *belongs to $W^{\beta,p}$ $(\mathbb{R}_+, X)$ for every $0 < \beta < 1$.*

Proof. First we give the proof for the second conclusion. For $f \in L^p(\mathbb{R}_+; X)$, the mild solution of the problem (1) is given by $u(t) = T * f(t) = \int_0^t T_{t-s} f(s)\, ds$. Since T is exponentially stable, one has $u \in L^p(\mathbb{R}^n; X)$ and the Fourier transform of u is given by $\hat{u}(y) = R(iy, A)\hat{f}(y)$, $y \in \mathbb{R}$. Let $0 < \beta < 1$ be fixed, by Lemma 3·8, the set $\{|y|^\beta R(iy, A) \colon y \in \mathbb{R}\}$ is $\mathscr{R}$-bounded. In order to show that $u \in W^{\beta,p}(\mathbb{R}_+, X)$, it suffices to show that $M_\beta \colon y \to (1 + y^2)^{\beta/2} R(iy, A)$ is a Fourier multiplier, see [**W2**] for a definition. By the operator-valued Fourier multiplier theorem of [**W2**], it will suffice to show that both $\{M_\beta(y) \colon y \in \mathbb{R}\}$ and $\{yM'_\beta(y) \colon y \in \mathbb{R}\}$ are $\mathscr{R}$-bounded. Notice that $yM'_\beta(y) = \beta y^2(1 + y^2)^{\beta/2-1} R(iy, A) - iy(1 + y^2)^{\beta/2} R(iy, A)^2$. So the $\mathscr{R}$-boundedness of $\{M_\beta(y) \colon y \in \mathbb{R}\}$ and $\{yM'_\beta(y) \colon y \in \mathbb{R}\}$ follows from Kahane's contraction principle, the $\mathscr{R}$-boundedness of $\{|y|^\beta R(iy, A) \colon y \in \mathbb{R}\}$ and Proposition 2·2.

Now let $f \in W^{\beta,p}(\mathbb{R}_+, X)$, $0 < \beta < 1$ and let u be the mild solution of the problem (1). We have to show that $\mathscr{F}^{-1}((1 + y^2)^{\beta'/2} A(\mathscr{F} u)(y)) = \mathscr{F}^{-1}((1 + y^2)^{\beta'/2} AR$ $(iy, A)(\mathscr{F} f)(y)) \in L^p(\mathbb{R}; X)$. Since $\mathscr{F}^{-1}((1 + y^2)^{\beta/2}(\mathscr{F} f)(y)) \in L^p(\mathbb{R}; X)$, it suffices to show that

$$y \longrightarrow (1 + y^2)^{\frac{\beta' - \beta}{2}} AR(iy, A)$$

is a Fourier multiplier. This follows from Kahane's contraction principle, the $\mathscr{R}$-

boundedness of $\{|y|^{\alpha} R(iy, A) \colon y \in \mathbb{R}\}$ for $0 < \alpha < 1$ and Proposition 2·2. The claim is proved.

Remark 4·13. By [**KL**], for each Banach space X with an unconditional basis, if X is not isomorphic to l^2, there exists an analytic C_0-semigroup T on X of negative type, such that the corresponding Cauchy problem does not have $\mathscr{M}R_\infty$. This implies that we cannot expect to extend the first conclusion of Theorem 4·12 to the case $\beta' = \beta$, or the second conclusion of Theorem 4·12 to the case $\beta = 1$.

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