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RESEARCH ARTICLE

# The Laplacian with Wentzell-Robin Boundary Conditions on Spaces of Continuous Functions*

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*Dedicated to Jerry Goldstein on the occasion of his 60$^{th}$ birthday*

**Abstract**

We investigate the Laplacian $\Delta$ on a smooth bounded open set $\Omega \subset \mathbf{R}^n$ with Wentzell-Robin boundary condition $\beta u + \frac{\partial u}{\partial \nu} + \Delta u = 0$ on the boundary $\Gamma$. Under the assumption $\beta \in C(\Gamma)$ with $\beta \geq 0$, we prove that $\Delta$ generates a differentiable positive contraction semigroup on $C(\bar{\Omega})$ and study some monotonicity properties and the asymptotic behaviour.

**Key words:** Wentzell-Robin boundary conditions, positive contraction semigroups

**Mathematics subject classification (2000):** 47D06, 35J20, 35J25

## Introduction

The aim of this article is to show that the Laplacian $\Delta$ with Wentzell-Robin boundary condition

$$\beta u + \frac{\partial u}{\partial \nu} + \Delta u = 0 \text{ on } \Gamma \tag{1}$$

generates a positive contraction semigroup $T$ on $C(\bar{\Omega})$. Here $\Omega$ is a bounded open subset of $\mathbf{R}^n$ with smooth boundary $\Gamma$ and $0 \leq \beta \in C(\Gamma)$. Note that (1) is a dynamic boundary condition. In fact, let $f$ be an element of $C(\bar{\Omega})$ and $u(t) = T(t)f$. Then $u'(t) = \Delta u(t)$. Introducing this in (1) we obtain

$$\frac{d}{dt} u(t) = -\beta u(t) - \frac{\partial}{\partial \nu} u(t) \qquad \text{on} \quad \Gamma.$$

We also establish monotonicity properties of this semigroup with respect to $\beta$. Also the asymptotic behaviour for $t \to \infty$ is studied. The boundary condition (1) was first studied in [9] in the space $C([0,1])$ and then in [10] in the space

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$L^p(\Omega) \oplus L^p(\Gamma)$ for $1 \leq p < \infty$ and in $C(\bar\Omega)$ by a direct energy method and the Lumer-Phillips theorem. The semigroup is shown to be holomorphic on $L^p(\Omega) \oplus L^p(\Gamma)$ for $1 < p < \infty$ and also in $H^1(\Omega)$, as has been shown in a subsequent paper [11].

Here we follow another path: the semigroup is constructed on $L^2(\Omega) \oplus L^2(\Gamma)$ by form methods and extended to $L^p(\Omega) \oplus L^p(\Gamma)$ by the Beurling-Deny criterion. Finally, using Schauder estimates, it is shown that $C(\bar\Omega) \oplus C(\Gamma)$ is an invariant subspace, and this leads to a Feller semigroup on $C(\bar\Omega)$, maybe the most natural space for such boundary conditions. The semigroup in $C(\bar\Omega)$ is even more regular. In fact, K. J. Engel (see [7]) has proved very recently, using a completely different approach, that it is analytic.

The idea to incorporate boundary conditions into a product space goes back to Greiner [16] and has also been used by Amann-Escher [1] and in [2, Chapter 6]. Robin boundary conditions

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on } \Gamma \tag{2}$$

have already been treated by form methods (see [3] and [5]), whereas for the Wentzell-Robin conditions (1) this seems to be new. Concerning generation theorems for elliptic operators (possibly degenerate) with pure Wentzell boundary conditions (i.e., $\Delta u_{|\Gamma} = 0$) in spaces of continous functions we refer to pioneer work of Feller [13] (in dimension one) and subsequent results by Clément-Timmermans [4], Goldstein-Lin [15] and Taira, Favini and Romanelli [24] among others. Concerning regularity properties and holomorphy of the generated semigroup in the case of pure Wentzell conditions see Vespri [25], Favini and Romanelli [12], Metafune [20], Engel-Nagel [8, Chapter VI, Section 4], and most recently Warma [26] for Wentzell-Robin boundary conditions in $C([0,1])$.

## 1. Beurling-Deny criteria and ultracontractivity

In this section we recall some results on positive forms which we will use in the sequel refering essentially to Davies [6] for the proofs.

Let $(H, (\ |\ )_H)$ be a real Hilbert space. By a **positive form** on $H$ we mean a bilinear mapping

$$Q\colon\ D(Q) \times D(Q) \to \mathbf{R}$$

such that

$$\begin{aligned} Q(u,v) &= Q(v,u) \quad \text{for all} \quad u,v \in D(Q), \\ Q(u,u) &\geq 0 \quad \text{for all} \quad u \in D(Q), \end{aligned}$$

where $D(Q)$ is a dense subspace of $H$, the **domain** of the form $Q$. We set

$$Q(u) = Q(u,u) \quad \text{for all} \quad u \in D(Q).$$

The form $Q$ is called **closed** if the space $D(Q)$ is complete for the norm

$$\|u\|_Q = (Q(u) + \|u\|_H^2)^{1/2}.$$

If $Q$ is closed, then the operator $A$ associated with $Q$ is defined in the following way:

$$\begin{aligned} D(A) &= \{u \in D(Q) : \exists f \in H \quad \text{such that} \\ &\qquad Q(u,\varphi) = (f \mid \varphi)_H \quad \text{for all} \quad \varphi \in D(Q)\}, \\ Au &= f. \end{aligned}$$

The operator $-A$ is selfadjoint and generates a $C_0$-semigroup $T$ on $H$ satisfying $T(t) = T^*(t)$ and $\|T(t)\| \leq 1$ for all $t \geq 0$. We call $T$ **the semigroup associated with the form** $Q$. Let us recall the following compactness criterion. The following are equivalent:

(i) $T(t)$ is compact for each $t > 0$;

(ii) the injection of $(D(Q), \| \ \|_Q)$ into $H$ is compact;

(iii) the operator $(I + A)^{-1} \in \mathcal{L}(H)$ is compact.

We now suppose that $H = L^2(Y)$ where $(Y, \Sigma, \mu)$ is a $\sigma$-finite measure space. One says that $T = (T(t))_{t \geq 0}$ is a **symmetric Markov semigroup** if the following conditions are satisfied:

$$T(t) = T(t)^* \quad \text{for all} \quad t \geq 0; \tag{1.1}$$

$$T(t) \geq 0 \quad \text{for all} \quad t \geq 0 \tag{1.2}$$

$$\|T(t)f\|_\infty \leq \|f\|_\infty \quad \text{for all} \quad f \in L^2(Y) \cap L^\infty(Y) \text{ and all } t \geq 0. \tag{1.3}$$

A **Dirichlet form** on $L^2(Y)$ is a closed positive form satisfying the following **two conditions of Beurling-Deny**

$$u \in D(Q) \ \text{ implies } \ |u| \in D(Q) \text{ and } Q(|u|) \leq Q(u) \tag{1.4}$$

$$0 \leq u \in D(Q) \ \text{ implies } \ u \wedge 1 \in D(Q) \text{ and } Q(u \wedge 1) \leq Q(u). \tag{1.5}$$

**Theorem 1.1** ([6, Theorem 1.3.3]). *Let $A$ be an operator on $L^2(Y)$. The following assertions are equivalent:*

(i) *$-A$ generates a symmetric Markov semigroup;*

(ii) *$A$ is associated with a Dirichlet form.*

Next we recall a notion of ultracontractivity.

**Theorem 1.2** ([6, Corollary 2.4.3]). *Let $Q$ be a Dirichlet form and $T = (T(t))_{t \geq 0}$ the associated semigroup. Let $\mu > 2$. The following assertions are* *equivalent:*

(i) $D(Q) \subset L^{2\mu/(\mu-2)}(Y)$*;*

(ii) *there exists* $c > 0$ *such that*

$$\|T(t)f\|_\infty \le ct^{-\mu/4}\|f\|_2 \qquad (0 < t < 1)$$

*for all* $f \in L^2(Y)$*.*

If a $\mu > 2$ exists such that these equivalent conditions are satisfied, we call the semigroup $T$ **ultracontractive**.

## 2. The semigroup on $L^2(\Omega) \oplus L^2(\Gamma)$

Let $\Omega$ be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary $\Gamma = \partial\Omega$. We denote by

$$u \mapsto u_{|\Gamma}$$

the **trace** function, which is a bounded operator from the Sobolev space $H^1(\Omega)$ into $L^2(\Gamma, \sigma)$, where $\sigma$ is the surface measure on $\Gamma$. To simplify the notation, we frequently write $u$ instead of $u_{|\Gamma}$. Denote by $\Delta_{\max}$ the Laplacian in $L^2(\Omega)$ with maximal domain, i.e.,

$$\begin{aligned} D(\Delta_{\max}) &:= \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\} \\ \Delta_{\max} u &:= \Delta u \quad \text{(in the sense of distributions)}, \end{aligned}$$

and denote by $\nu(z)$ the exterior normal in $z \in \Gamma$. Let us introduce the notion of **weak normal derivative**.

**Definition 2.1.** Let $u \in D(\Delta_{\max})$. We say that $u$ has a **weak normal derivative** if there exists a function $b \in L^2(\Gamma)$ such that

$$\int_\Omega \nabla u \nabla \varphi \, dx + \int_\Omega \Delta u \varphi \, dx = \int_\Gamma b\varphi d\sigma \tag{2.1}$$

for all $\varphi \in H^1(\Omega)$. In that case the function $b \in L^2(\Gamma)$ verifying (2.1) is unique and we denote it by $\frac{\partial u}{\partial \nu}$.

We now consider the space $H = L^2(\Omega) \oplus L^2(\Gamma)$. Note that $H$ can be identified with a space $L^2(Y)$ for a suitable finite measure space $(Y, \Sigma, \mu)$ such that $L^\infty(Y)$ can be identified with $L^\infty(\Omega) \oplus L^\infty(\Gamma)$ with the norm

$$\|(u,b)\|_\infty := \max\{\|u\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Gamma)}\}$$

for each $(u,b) \in L^\infty(\Omega) \oplus L^\infty(\Gamma)$. Let $\beta \in L^\infty(\Gamma)$ be such that $\beta(z) \ge 0$ for $\sigma$-a.a. $z \in \Gamma$ and define the operator $A_\beta$ on $H$ by

$$\begin{aligned} D(A_\beta) &:= \left\{(u, u_{|\Gamma}) : u \in D(\Delta_{\max}), \frac{\partial u}{\partial \nu} \text{ exists in } L^2(\Gamma)\right\}, \\ A_\beta(u, u_{|\Gamma}) &:= \left(\Delta u, -\beta u_{|\Gamma} - \frac{\partial u}{\partial \nu}\right) \end{aligned}$$

**Remark 2.2.** It is possible to characterise the domain $D(A_\beta)$ in terms of fractional Sobolev spaces and traces. We have:

$$D(A_\beta) = \{u \in H^{3/2}(\Omega) : \Delta u \in L^2(\Omega)\}.$$

In fact, for every $u \in H^{3/2}(\Omega)$ with $\Delta u \in L^2(\Omega)$ a weak normal derivative exists, so one inclusion follows. Conversely, if $u$ belongs to $D(A_\beta)$, setting $f = \Delta u$ and $b = \frac{\partial u}{\partial \nu}$, $u$ is a variational solution of the boundary value problem $\Delta u = f$ in $\Omega$, $\frac{\partial u}{\partial \nu} = b$ on $\Gamma$. Moreover, there is a unique (up to constants) $v \in H^{3/2}(\Omega)$ solving the same problem, hence $u \in H^{3/2}(\Omega)$, as claimed.

If $\Gamma$ is $C^\infty$ these results are classical (see e.g. [19, Theorem 7.3, 7.4 p. 186–7]), whereas if $\Gamma$ is only Lipschitz continuous, the proof is much more delicate, and we refer to [17], [18].

**Theorem 2.3.** *The operator $A_\beta$ generates a symmetric Markov semigroup on the space $L^2(\Omega) \oplus L^2(\Gamma)$.*

**Proof.** We define the positive form $Q$ on $H$ by

$$\begin{aligned} D(Q) &:= \{(u, u_{|\Gamma}) : u \in H^1(\Omega)\} \\ Q((u, u_{|\Gamma}), (v, v_{|\Gamma})) &:= \int_\Omega \nabla u \nabla v \, dx + \int_\Gamma uv\beta \, d\sigma. \end{aligned}$$

The proof is now given in several steps.

a) $D(Q)$ *is dense* in $H$. Let $b \in \mathcal{D}(\mathbf{R}^n)$ (i.e., $b$ is a test function). Then there exists a sequence $(u_k)_{k\in\mathbf{N}}$ in $\mathcal{D}(\mathbf{R}^n)$ such that $u_{k_{|\Gamma}} = b$ and $u_k \to 0$ in $L^2(\Omega)$ as $k \to \infty$. Thus $(0, b) \in \overline{D(Q)}$. It follows that

$$\{0\} \oplus L^2(\Gamma) \subset \overline{D(Q)}.$$

Moreover,

$$(u, 0) = (u, u_{|\Gamma}) - (0, u_{|\Gamma}) \in \overline{D(Q)}$$

for all $u \in H^1(\Omega)$. Hence $L^2(\Omega) \oplus \{0\} \subset \overline{D(Q)}$.

b) *The form $Q$ is closed.* Since the trace is a continuous operator from $H^1(\Omega)$ into $L^2(\Gamma)$, there exists a constant $c > 0$ such that

$$\|u_{|\Gamma}\|_{L^2(\Gamma)} \leq c \|u\|_{H^1(\Omega)}$$

for all $u \in H^1(\Omega)$. It follows that the form norm

$$\|(u, u_{|\Gamma})\|_Q = (Q(u, u_{|\Gamma}) + \|(u, u_{|\Gamma})\|_H^2)^{1/2}$$

is equivalent to the norm

$$\|(u, u_{|\Gamma})\| := \|u\|_{H^1(\Omega)}, u \in D(Q).$$

Since $H^1(\Omega)$ is complete, also $D(Q)$ is complete.

c) *The first Beurling-Deny condition (1.4) is satisfied.* Let $u \in H^1(\Omega)$. Then $|u| \in H^1(\Omega)$ and $\nabla |u| = (\operatorname{sign} u)\nabla u$ (see [14, § 7.6]). In particular, $|\nabla |u||^2 = |\nabla u|^2$. Moreover, the trace of $|u|$ coincides with $|u_{|\Gamma}|$. Hence

$$Q(|u|, |u|_{|\Gamma}) = \int_\Omega |\nabla u|^2 \, dx + \int_\Gamma |u|^2 \beta d\sigma = Q(u, u_{|\Gamma}).$$

d) *The second Beurling-Deny condition (1.5) holds.* Let $0 \le u \in H^1(\Omega)$. Then $u \wedge 1 \in H^1(\Omega)$ and $\nabla(u \wedge 1) = 1_{\{u<1\}} \nabla u$ (see [14, § 7.6]). Hence

$$(u, u_{|\Gamma}) \wedge (1_\Omega, 1_\Gamma) = (u \wedge 1_\Omega, (u \wedge 1_\Omega)_{|\Gamma}) \in D(Q)$$

and

$$\begin{aligned} Q((u, u_{|\Gamma}) \wedge (1_\Omega, 1_\Gamma)) &= \int_\Omega |\nabla (u \wedge 1_\Omega)|^2 \, dx + \int_\Gamma (u \wedge 1_\Gamma)^2 \beta \, d\sigma \\ &\le \int_\Omega |\nabla u|^2 \, dx + \int_\Gamma |u|^2 \beta \, d\sigma \\ &= Q(u, u_{|\Gamma}). \end{aligned}$$

Hence $Q$ is a Dirichlet form.

e) *$-A_\beta$ is the operator associated with $Q$.* Denote by $B$ the operator associated with $Q$. Let $(u, u_{|\Gamma}) \in D(B)$, and let

$$B(u, u_{|\Gamma}) = (f, b) \in L^2(\Omega) \oplus L^2(\Gamma).$$

Then

$$\begin{aligned} \int_\Omega \nabla u \nabla \varphi \, dx + \int_\Gamma u \varphi \beta \, d\sigma &= Q((u, u_{|\Gamma}), (\varphi, \varphi_{|\Gamma})) \\ &= ((f, b) \mid (\varphi, \varphi_{|\Gamma}))_H \\ &= \int_\Omega f \varphi \, dx + \int_\Gamma b \varphi \, d\sigma, \end{aligned}$$

for all $\varphi \in H^1(\Omega)$. Choosing $\varphi \in \mathcal{D}(\Omega)$ we deduce that $f = -\Delta u$. Hence

$$\int_\Omega \nabla u \nabla \varphi \, dx + \int_\Omega \Delta u \varphi \, dx = \int_\Gamma (b - \beta u) \varphi \, d\sigma$$

for all $\varphi \in H^1(\Omega)$, i.e.,

$$\frac{\partial u}{\partial \nu} \text{ exists and } \frac{\partial u}{\partial \nu} = b - \beta u_{|\Gamma}.$$

Thus we have proved that

$$(u, u_{|\Gamma}) \in D(A_\beta) \text{ and } A_\beta(u, u_{|\Gamma}) = -B(u, u_{|\Gamma}).$$

In order to prove the converse, let $(u, u_{|\Gamma}) \in D(A_\beta)$. Then

$$\begin{aligned} \int_\Omega \nabla u \nabla \varphi \, dx + \int_\Omega \Delta u \varphi \, dx &= \int_\Gamma \frac{\partial u}{\partial \nu} \varphi \, d\sigma \\ &= \int_\Gamma (b - \beta u_{|\Gamma}) \varphi \, d\sigma \end{aligned}$$

where $b = \frac{\partial u}{\partial \nu} + \beta u_{|\Gamma}$ for all $\varphi \in H^1(\Omega)$. Hence

$$Q((u, u_{|\Gamma}), (\varphi, \varphi_{|\Gamma})) = -\int_\Omega \Delta u \varphi \, dx + \int_\Gamma b \varphi \, d\sigma$$

for all $\varphi \in H^1(\Omega)$. By the definition of the operator associated with the form $Q$ we deduce that $(u, u_{|\Gamma}) \in D(B)$ and

$$B(u, u_{|_\Gamma}) = (-\Delta u, b) = -A_\beta(u, u_{|_\Gamma}).$$ ■

**Corollary 2.4.** *Let $\lambda > 0$ and let $u \in D(\Delta_{\max})$ such that $\frac{\partial u}{\partial \nu}$ exists. Let*

$$\begin{aligned} f &= \lambda u - \Delta u \\ b &= \lambda u_{|\Gamma} + \beta u_{|\Gamma} + \frac{\partial u}{\partial \nu}. \end{aligned}$$

*Then*

$$\begin{aligned} \lambda \|(u, u_{|\Gamma})\|_{L^\infty(\Omega) \oplus L^\infty(\Gamma)} &\leq \|(f, b)\|_{L^\infty(\Omega) \oplus L^\infty(\Gamma)} \\ &= \max\{\|f\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Gamma)}\}. \end{aligned}$$

**Proof.** It suffices to observe that

$$\|\lambda(\lambda - A_\beta)^{-1}\| \leq 1$$

where the norm is considered in the space of all linear operators on $L^\infty(\Omega) \oplus L^\infty(\Gamma)$. ■

**Remark 2.5.** The $C_0$-semigroup given by the previous theorem extends to a positive contraction $C_0$-semigroup on $L^p(\Omega) \oplus L^p(\Gamma)$ for $1 \leq p < \infty$ which is holomorphic for $1 < p < \infty$. This follows directly from [5, Theorem 1.4.1 and 1.4.2] and can be also obtained as a special case of [10, Theorem 3.1].

Next we show that the $C_0$-semigroup $(T_\beta(t))_{t \geq 0}$ generated by $A_\beta$ is ultracontractive.

**Proposition 2.6.** *Let $n \geq 3$. Then there exists a constant $c > 0$ such that*

$$\|T_\beta(t)(f,b)\|_\infty \leq ct^{-\frac{n-1}{2}}\|(f,b)\|_2 \quad (0 < t < 1)$$

*for all $(f,b) \in L^2(\Omega) \oplus L^2(\Gamma)$. If $n \leq 2$, then for all $\mu > \frac{1}{2}$ there exists $c > 0$ such that*

$$\|T_\beta(t)(f,b)\|_\infty \leq ct^{-\mu}\|(f,b)\|_2 \qquad (0 < t < 1)$$

*for all $(f,b) \in L^2(\Omega) \oplus L^2(\Gamma)$.*

**Proof.** By the embedding theorem [22, Chapter 2, Theorem 4.2], the trace operator $u \mapsto u_{|\Gamma}$ is continuous from $H^1(\Omega)$ to $L^q(\Gamma)$ where $q = \frac{2n-2}{n-2}$ for $n > 2$ and where $2 \leq q < \infty$ is arbitrary if $n = 2$. On the other hand, one has the following inclusions:

$$H^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega) \quad \text{if } n > 2$$

and

$$H^1(\Omega) \subset L^q(\Omega) \text{ for } 2 \leq q < \infty \text{ arbitrary if } n = 2.$$

Hence

$$D(Q) \subset L^q(\Omega) \oplus L^q(\Gamma) \text{ for } q = \frac{2n-2}{n-2} \text{ if } n > 2.$$

Hence, letting $\mu = 2n - 2$, one has $q = \frac{2\mu}{\mu - 2}$ and the claim follows from Theorem 1.2 if $n > 2$. If $n \leq 2$, then $D(Q) \subset L^q(\Omega) \oplus L^q(\Gamma)$ for all $2 \leq q < \infty$, and the claim follows from Theorem 1.2 again. ∎

**Corollary 2.7.** *For all $t > 0$ the operator $T_\beta(t)$ is compact and the resolvent of $A_\beta$ is compact.*

**Proof.** From Proposition 2.6 it follows that the operator $T_\beta(t)$ is Hilbert-Schmidt for all $t > 0$. ∎

Next we investigate how the semigroups depend on $\beta$. We denote by $A_\infty$ the operator on $L^2(\Omega) \oplus L^2(\Gamma)$ with domain $D(A_\infty)$ given by

$$\begin{aligned} D(A_\infty) &:= \{(u,0) : u \in H_0^1(\Omega), \Delta u \in L^2(\Omega)\} \\ A_\infty(u,0) &:= (\Delta u, 0). \end{aligned}$$

Then $A_\infty$ generates a semigroup $T_\infty = (T_\infty(t))_{t \geq 0}$ on $L^2(\Omega) \oplus L^2(\Gamma)$ given by

$$T_\infty(t)(f,b) = (e^{t\Delta^D} f, b),$$

where $\Delta^D$ is the Dirichlet Laplacian on $L^2(\Omega)$.

**Proposition 2.8.** *Let $\beta_1, \beta_2 \in L^\infty(\Gamma)$ such that $0 \le \beta_1 \le \beta_2$. Then*

$$T_\infty(t) \le T_{\beta_2}(t) \le T_{\beta_1}(t) \le T_0(t)$$

*in the sense of positive semigroups, where $T_0 = (T_0(t))_{t\ge 0}$ denotes the semigroup $T_\beta$ for $\beta = 0$.*

**Proof.** Let $Q_\beta$ be the form associated with $0 \le \beta \in L^\infty(\Gamma)$. Then $D(Q_\beta)$ is independent of $\beta$. Moreover,

$$Q_0((u, u_{|\Gamma}), (v, v_{|\Gamma})) \le Q_{\beta_1}((u, u_{|\Gamma}), (v, v_{|\Gamma})) \le Q_{\beta_2}((u, u_{|\Gamma}), (v, v_{|\Gamma}))$$

if $u, v \ge 0$. From this, the second and third inequality in the statement follow from the domination criterion [23, Theorem 3.7] of Ouhabaz. Moreover, the form domain $D(Q_0) = \{(u, 0) : u \in H_0^1(\Omega)\}$ is an ideal in $D(Q_{\beta_1})$ and the two forms $Q_0$ and $Q_{\beta_1}$ coincide on $D(Q_0)$. Hence also the first inequality follows from Ouhabaz' criterion. ■

**Remark 2.9** (Semigroup on $H^1(\Omega)$). The operator $B_\beta$ on $H^1(\Omega)$ given by $D(B_\beta) = \{u \in H^1(\Omega) : \Delta u \in H^1(\Omega), \frac{\partial u}{\partial \nu}$ exists in $L^2(\Gamma), (\Delta u)_{|\Gamma} + \beta u_{|\Gamma} + \frac{\partial u}{\partial \nu} = 0\}, B_\beta u = \Delta u$ generates a holomorphic $C_0$-semigroup on $H^1(\Omega)$. This follows directly from the proof of Theorem 2.3. In fact, the part $\tilde{B}_\beta$ of $A_\beta$ in $D(Q)$ generates a holomorphic $C_0$-semigroup. This is just a property of forms which can easily be seen from the spectral theorem (cf. [2, § 7.1]). Now the mapping $u \in H^1(\Omega) \mapsto (u, u_{|\Gamma}) \in D(Q)$ is an isomorphism. With this identification, the operator $\tilde{B}_\beta$ induces the operator $B_\beta$ on $H^1(\Omega)$. This result is valid on bounded open sets with Lipschitz boundary. For another approach on smooth domains allowing also degenerate elliptic operators we refer to [11].

## 3. The semigroup in the space $C(\bar{\Omega})$

In order to use Schauder estimates, we suppose in the sequel that $\Omega$ is a bounded open set in $\mathbf{R}^n$ of class $C^{2,\alpha}$ where $0 < \alpha < 1$.

We first consider the space $C(\bar{\Omega}) \oplus C(\Gamma)$ endowed with the norm

$$\|(f, b)\|_\infty := \max\{\|f\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Gamma)}\}.$$

Define the operator $B_1$ on the space $C(\bar{\Omega}) \oplus C(\Gamma)$ by

$$\begin{aligned} B_1(u, u_{|\Gamma}) &:= \left(\Delta u, -u_{|\Gamma} - \frac{\partial u}{\partial \nu}\right) \\ D(B_1) &:= \left\{(u, u_{|\Gamma}) : u \in C(\bar{\Omega}) \cap H^1(\Omega), \Delta u \in C(\bar{\Omega}), \frac{\partial u}{\partial \nu} \text{ exists in } C(\Gamma)\right\} \end{aligned}$$

**Proposition 3.1.** *The operator $B_1$ is $m$-dissipative and resolvent positive.*

**Proof.** At first we recall that the operator $A_1$ (i.e., $A_\beta$ with $\beta \equiv 1$) is defined on $L^2(\Omega) \oplus L^2(\Gamma)$ by

$$A_1(u, u_{|\Gamma}) = \left(\Delta u, -u_{|\Gamma} - \frac{\partial u}{\partial \nu}\right)$$

with domain

$$D(A_1) = \left\{(u, u_{|\Gamma}) : u \in D(\Delta_{\max}) \text{ such that } \frac{\partial u}{\partial \nu} \text{ exists}\right\}.$$

Observe that $B_1$ is the part of $A_1$ in $C(\bar{\Omega}) \oplus C(\Gamma)$, i.e.,

$$D(B_1) = \{w \in D(A_1) \cap (C(\bar{\Omega}) \oplus C(\Gamma)) : A_1 w \in C(\bar{\Omega}) \oplus C(\Gamma)\}$$

and

$$B_1 w = A_1 w, w \in D(B_1).$$

Hence $B_1$ is dissipative by Corollary 2.4. Since $A_1$ is closed in $L^2(\Omega) \oplus L^2(\Gamma)$, also $B_1$ is closed in $C(\bar{\Omega}) \oplus C(\Gamma)$. Let $f \in C^\alpha(\bar{\Omega})$ and $b \in C^{1,\alpha}(\Gamma)$. By [14, Theorem 6.31] there exists $u \in C^{2,\alpha}(\bar{\Omega})$ such that

$$\Delta u = f \text{ and } -u_{|\Gamma} - \frac{\partial u}{\partial \nu} = b.$$

Hence $(u, u_{|\Gamma}) \in D(B_1)$ and $B_1(u, u_{|\Gamma}) = (f, b)$. By the Stone-Weierstrass Theorem, the space $C^\alpha(\bar{\Omega}) \oplus C^{1,\alpha}(\Gamma)$ is dense in $C(\bar{\Omega}) \oplus C(\Gamma)$, and then $B_1$ is $m$-dissipative. Since $B_1$ is the part of $A_1$ in the space $C(\bar{\Omega}) \oplus C(\Gamma)$, the resolvent $R(\lambda, B_1)$ of $B_1$ in $\lambda > 0$ is the restriction of $R(\lambda, A_1)$ to $C(\bar{\Omega}) \oplus C(\Gamma)$. Since the latter operator is positive, the same is true for $R(\lambda, B_1)$. ■

Now we consider a perturbation of $B_1$. Let $0 \leq \beta \in C(\Gamma)$ and let $B_\beta$ be the operator on the space $C(\bar{\Omega}) \oplus C(\Gamma)$ defined in the following way:

$$\begin{aligned} B_\beta(u, u_{|\Gamma}) &:= \left(\Delta u, -\frac{\partial u}{\partial \nu} - \beta u\right) \\ D(B_\beta) &= D(B_1). \end{aligned}$$

Recall that $B$ is called a Hille-Yosida operator if there exist $\omega \in \mathbf{R}, M \geq 0$ such that $(\omega, \infty) \subset \varrho(B)$ and

$$\|(\lambda - \omega)^{n+1} R(\lambda, B)^n\| \leq M$$

for all $\lambda > \omega, n \in \mathbf{N}, n \geq 1$ (see [2, § 3.5.]). We now show the following.

**Proposition 3.2.** *The operator $B_\beta$ is a Hille-Yosida operator on $C(\bar{\Omega}) \oplus C(\Gamma)$ which is resolvent positive.*

**Proof.** Consider the bounded operator $C$ on $L^2(\Omega)\oplus L^2(\Gamma)$ given by $C(f,b) = (0,(-\beta+1)b)$ and its restriction $C_0$ to $C(\bar\Omega)\oplus C(\Gamma)$. Then from [2, Theorem 3.5.5] it follows that $A_1+C$ and $B_1+C_0$ are both Hille-Yosida operators. The semigroup $(e^{tC})_{t\geq 0}$ generated by $C$ on $L^2(\Omega)\oplus L^2(\Gamma)$ is positive (in fact, $e^{tC}(f,b) = (f, e^{t(-\beta+1)}b))$. Hence $A_1+C$ is resolvent positive. Consequently, also its part $B_1+C_0$ in $C(\bar\Omega)\oplus C(\Gamma)$ is resolvent positive. ∎

Note that the operator $B_\beta$ is not the generator of a $C_0$-semigroup since its domain is not dense. But its part in the closure of its domain generates a $C_0$-semigroup. This observation will finally lead to the principal result of the article.

Let $0\leq\beta\in C(\Gamma)$. Define the Laplacian with Wentzell-Robin boundary conditions on $C(\bar\Omega)$ as the operator $G_\beta$ given by

$$\begin{aligned} G_\beta u &:= \Delta u \\ D(G_\beta) &:= \Big\{ u\in C(\bar\Omega)\cap H^1(\Omega) : \Delta u\in C(\bar\Omega), \\ &\qquad \frac{\partial u}{\partial\nu} \text{ exists in } C(\Gamma) \text{ and} \\ &\qquad (\Delta u)_{|\Gamma} + \frac{\partial u}{\partial\nu} + \beta u_{|\Gamma} = 0 \Big\}. \end{aligned}$$

**Theorem 3.3.** *The operator $G_\beta$ generates a compact, positive $C_0$-semigroup $S_\beta$ on $C(\bar\Omega)$.*

**Proof.** Consider the closed subspace $F$ of $C(\bar\Omega)\oplus C(\Gamma)$ given by

$$F := \{(u, u_{|\Gamma}) : u\in C(\bar\Omega)\},$$

which we will identify with $C(\bar\Omega)$ in the sequel. Observe the following properties:

a) **$F$ is the closure of $D(B_\beta)$ in $C(\bar\Omega)\oplus C(\Gamma)$.**

In fact, the domain $D(B_\beta)$ is contained in $F$ and contains the set $\{(u,u_{|\Gamma}) : u\in C^\infty(\bar\Omega)\}$, which is dense in $F$ by the Stone-Weierstrass theorem.

b) By [2, Lemma 3.3.12] the part $\tilde G_\beta$ of $B_\beta$ in $F$ generates a $C_0$-semigroup $\tilde S_\beta$. This semigroup is positive since $B_\beta$ is resolvent positive. Identifying $F$ and $C(\bar\Omega)$ the operator $\tilde G_\beta$ becomes $G_\beta$. Thus $G_\beta$ generates the $C_0$-semigroup $S_\beta$ which can be identified with $\tilde S_\beta$.

c) **The semigroup $S_\beta$ is compact.**

It is sufficient to prove that $\tilde S_\beta(t)$ is compact for $t>0$. Recall that $\tilde S_\beta(t)$ is the restriction of $T_\beta(t)$ to $F$. This follows from the exponential formula $\tilde S_\beta(t) = \lim_{n\to\infty}(I-\frac{t}{n}\tilde G_\beta)^{-n}$ strongly. Recall that the operator $T_\beta(t)$ is compact. Since the semigroup $T_\beta$ is ultracontractive, one has

$$T_\beta(t)(L^2(\Omega)\oplus L^2(\Gamma)) \subset L^\infty(\Omega)\oplus L^\infty(\Gamma).$$

Factorising $T_\beta(2t)_{|L^\infty(\Omega)\oplus L^\infty(\Gamma)}$ as

$$L^\infty(\Omega)\oplus L^\infty(\Gamma)\hookrightarrow L^2(\Omega)\oplus L^2(\Gamma)\xrightarrow{T_\beta(t)} L^2(\Omega)\oplus L^2(\Gamma)\xrightarrow{T_\beta(t)} L^\infty(\Omega)\oplus L^\infty(\Gamma)$$

we deduce that $T_\beta(2t)_{|L^\infty(\Omega)\oplus L^\infty(\Gamma)}$ is compact. Hence also the restriction $S_\beta(2t)$ to $F$ is compact. ■

As a consequence of the previous results, we show that the semigroup is differentiable in $C(\bar\Omega)$.

**Corollary 3.4.** *The semigroup $(S_\beta(t))_{t\geq 0}$ is differentiable on $C(\bar\Omega)$.*

**Proof.** Let us show that for every $t>0$ the operator $G_\beta S_\beta(t)$ is bounded on $C(\bar\Omega)$. First, recall that $(S_\beta(t))_{t\geq 0}$ concides in $C(\bar\Omega)$ with the semigroup $(T_\beta(t))_{t\geq 0}$ acting on $L^2(\Omega)\oplus L^2(\Gamma)$, which is holomorphic and ultracontractive. Therefore, we may write

$$G_\beta S_\beta(t)=G_\beta S_\beta(t/2)S_\beta(t/2)=S_\beta(t/2)G_\beta S_\beta(t/2)=T_\beta(t/2)A_\beta T_\beta(t/2).$$

But $A_\beta T_\beta(t/2)$ is a bounded operator from $L^2(\Omega)\oplus L^2(\Gamma)$ (hence, from $C(\bar\Omega)$) in $L^2(\Omega)\oplus L^2(\Gamma)$ and $T_\beta(t/2)$ is continuous from $L^2(\Omega)\oplus L^2(\Gamma)$ in $C(\bar\Omega)$ and the thesis follows. ■

We next treat a monotonicity property. Denote by $G_\infty$ the Dirichlet Laplacian on $C(\bar\Omega)$, i.e.,

$$\begin{aligned} D(G_\infty) &:= \{u\in C(\bar\Omega): u_{|\Gamma}=0, \Delta u\in C(\bar\Omega)\}\\ G_\infty u &:= \Delta u. \end{aligned}$$

Then $G_\infty$ generates a positive holomorphic semigroup $S_\infty$ on $C(\bar\Omega)$ such that $\|S_\infty(t)\|\leq 1$ for $t>0$, which is **not** strongly continuous in $0$ (see [2, Example 3.7.8, p. 156]).

**Theorem 3.5.** *Let $\beta_1,\beta_2\in C(\Gamma)$ such that $0\leq\beta_1\leq\beta_2$. Then*

$$S_\infty(t)\leq S_{\beta_2}(t)\leq S_{\beta_1}(t)\leq S_0(t)\quad (t\geq 0).$$

**Proof.** We identify $C(\bar\Omega)$ with the subspace $\{(u,u_{|\Gamma}): u\in C(\bar\Omega)\}$ of $L^2(\Omega)\oplus L^2(\Gamma)$, so that the semigroups $S_\infty, S_{\beta_2}, S_{\beta_1}, S_0$ are restrictions of the semigroups $T_\infty, T_{\beta_2}, T_{\beta_1}$ and $T_0$ considered in Proposition 2.8. Thus the corresponding generators are obtained as parts of the corresponding generators on $L^2(\Omega)\oplus L^2(\Gamma)$ and the theorem is a consequence of Proposition 2.8. ■

Finally, we consider the asymptotic behaviour of $(S_\beta(t))_{t\geq 0}$ as $t\to\infty$. If $\beta\geq 0$, then the operator $G_\beta$ is dissipative and hence $\|S_\beta(t)\|\leq 1$ for $t\geq 0$.

If $\beta\equiv 0$, then $1_{\bar\Omega}\in D(G_0)$ and $G_0 1_{\bar\Omega}=0$. Hence $S_\beta(t)1_{\bar\Omega}=1_{\bar\Omega}$ for all $t\geq 0$ and the norm $\|S_\beta(t)\|$ does not converge to $0$ as $t\to\infty$. But this case is exceptional. In fact, the following result holds:

**Theorem 3.6.** *Suppose that* $\Omega$ *is connected and* $0 \le \beta \in C(\Gamma), \beta \not\equiv 0$. *Then there exist* $\varepsilon > 0, M \ge 0$ *such that*

$$\|S_\beta(t)\| \le Me^{-\varepsilon t} \quad (t \ge 0).$$

**Proof.** It suffices to consider the $C_0$-semigroup $\tilde{S}_\beta$ on $F := \{(u, u_{|\Gamma}) : u \in C(\bar{\Omega})\} \subset C(\bar{\Omega}) \oplus C(\Gamma)$, with generator $\tilde{G}_\beta$ (as in the proof of Theorem 3.3). Since $\tilde{S}_\beta(t)$ is compact, the resolvent of $\tilde{G}_\beta$ is compact (see [21, A-II Theorem 1.25]). We show that $\tilde{G}_\beta$ is injective. In order to do so, recall that $\tilde{G}_\beta$ is the part of $A_\beta$ defined on $L^2(\Omega) \oplus L^2(\Gamma)$. Let $(u, u_{|\Gamma}) \in D(A_\beta)$ such that $A_\beta u = 0$, then

$$0 = Q(u, u_{|\Gamma}) = \int_\Omega |\nabla u|^2 \, dx + \int_\Gamma \beta |u|^2 \, d\sigma.$$

Hence, since $\nabla u = 0$ and since $\Omega$ is connected, $u$ is constant on $\Omega$, and since $\beta \not\equiv 0$, it follows that $u \equiv 0$. Thus $\tilde{G}_\beta$ is injective and hence invertible. Observing that $\tilde{S}_\beta$ is bounded, it follows that

$$\sigma(\tilde{G}_\beta) \cap i\mathbf{R} \subset \{0\}.$$

Moreover, the spectrum of $\tilde{G}_\beta$ is either finite or a sequence going to infinity. Thus, since $\tilde{S}_\beta$ is norm continuous, we deduce that the set

$$\{\lambda \in \sigma(\tilde{G}_\beta) : \operatorname{Re}\lambda \ge -1\}$$

is bounded (see [21, A-II Theorem 1.20]). This implies that the spectral bound $s(\tilde{G}_\beta)$ is negative. Applying [8, Theorem 1.10. p. 302] the result follows. ∎

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