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Dirichlet and Neumann boundary conditions: What is in between?

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Abstract. Given an admissible measure μ on $\partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is an open set, we define a realization Δ_μ of the Laplacian in $L^2(\Omega)$ with general Robin boundary conditions and we show that Δ_μ generates a holomorphic C_0-semigroup on $L^2(\Omega)$ which is sandwiched by the Dirichlet Laplacian and the Neumann Laplacian semigroups. Moreover, under a locality and a regularity assumption, the generator of each sandwiched semigroup is of the form Δ_μ. We also show that if $D(\Delta_\mu)$ contains smooth functions, then μ is of the form $d\mu = \beta d\sigma$ (where σ is the $(n-1)$-dimensional Hausdorff measure and β a positive measurable bounded function on $\partial\Omega$); i.e. we have the classical Robin boundary conditions.

0. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. Then it is standard to define selfadjoint realizations Δ^D and Δ^N of the Laplacian on $L^2(\Omega)$ with **Dirichlet boundary conditions**

$$u_{|\partial\Omega} = 0 \qquad \text{on} \quad \partial\Omega \tag{1}$$

or **Neumann boundary conditions**

$$\frac{\partial u}{\partial \nu}_{|\partial\Omega} = 0 \qquad \text{on} \quad \partial\Omega. \tag{2}$$

The exterior normal derivative $\frac{\partial u}{\partial \nu}$ may only exist in a weak form and actually the boundary of Ω may be so bad that no exterior normal can be defined.

Here we consider boundary conditions of the third kind

$$u d\mu + \frac{\partial u}{\partial \nu} d\sigma = 0 \qquad \text{on} \quad \partial\Omega \tag{3}$$

where μ is an (admissible) Borel measure on $\partial\Omega$ and σ is the surface measure if Ω is Lipschitz, or more generally the $(n-1)$-dimensional Hausdorff measure if Ω is arbitrary.

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If $d\mu = \beta d\sigma$ for some $0 \leq \beta \in L^\infty(\partial\Omega, \sigma)$, then (3) reduces to the usual **Robin boundary conditions**

$$\beta u + \frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \quad \partial\Omega. \tag{4}$$

We show by the method of quadratic forms that a selfadjoint realization Δ_μ of the Laplacian on $L^2(\Omega)$ can be associated to these kind of boundary conditions.

The semigroup $(e^{t\Delta_\mu})_{t\geq 0}$ generated by Δ_μ satisfies the following sandwich property

$$e^{t\Delta^D} \leq e^{t\Delta_\mu} \leq e^{t\Delta^N} \qquad (t \geq 0). \tag{5}$$

We show in this paper that conversely each symmetric semigroup T on $L^2(\Omega)$ satisfying $e^{t\Delta^D} \leq T(t) \leq e^{t\Delta^N}$ is of the form $T(t) = e^{t\Delta_\mu}$ provided a locality and a regularity assumption are satisfied.

This paper is based on some properties of admissible measures and relative capacity introduced in [AW]. But here we define the closed form directly with precise form domain. This is most convenient in order to establish the desired domination properties. On the way we prove a regularity result for such forms (Theorem 2.4) which is of independent interest. We also show that the measure μ is of the form $\beta d\sigma$ as soon as the domain of Δ_μ contains smooth functions.

Finally, we establish some asymptotic properties of the semigroup $e^{t\Delta_\mu}$ as $t \to \infty$ which are similar to those studied for Schrödinger semigroups in [ABB] and [Bat].

1. Preliminaries.

Let H be a Hilbert space over $\mathbb{R}$. A **positive form** on H is a bilinear mapping $a : D(a) \times D(a) \to \mathbb{R}$ such that $a(u, v) = a(v, u)$ and $a(u) := a(u, u) \geq 0$. Here $D(a)$ is a dense subspace of H, the **domain** of the form. The form a is called **closed** if $D(a)$ is complete for the norm $\|u\|_a = (a(u) + \|u\|_H^2)^{1/2}$. In that case we define the **operator A on H associated with a** by

$$\begin{cases} D(A) &= \{u \in D(a) : \exists\, v \in H,\ a(u, \varphi) = (v, \varphi)_H\ \forall\, \varphi \in D(a)\} \\ Au &= v. \end{cases}$$

Then A is selfadjoint and $-A$ generates a contraction C_0-semigroup $S = (S(t))_{t\geq 0}$ of symmetric operators on H. We also write $e^{-tA} = S(t)$ and call S **the semigroup associated with** a.

Now assume that $H = L^2(\Omega)$ where $(\Omega, \Sigma, \lambda)$ is a σ-finite measure space. We let $L^2(\Omega)_+ = \{f \in L^2(\Omega) : f \geq 0 \text{ a.e.}\}$ and $F_+ = L^2(\Omega)_+ \cap F$ if F is a subspace of $L^2(\Omega)$.

Let S be the semigroup associated with a closed, positive form a on $L^2(\Omega)$. The **first Beurling-Deny criterion** [Dav, Theorem 1.3.2] asserts that S is positive (i.e., $S(t)L^2(\Omega)_+ \subset L^2(\Omega)_+$ for all $t \geq 0$) if and only if

$$u \in D(a) \text{ implies } |u| \in D(a) \text{ and } a(|u|) \leq a(u). \tag{6}$$

In that case, $u, v \in D(a)$ implies that $u \wedge v = \inf\{v, u\},\ u \vee v = \sup\{v, u\} \in D(a)$ and the lattice operations are continuous in $(D(a), \|\cdot\|_a)$.

Assume that S is positive. Then the **second Beurling-Deny criterion** [Dav, Theorem 1.3.3] asserts that S is L^∞-contractive (i.e., if $f \in L^2(\Omega)$ satisfy $0 \leq f \leq 1$ then $0 \leq S(t)f \leq 1$ for all $t \geq 0$) if and only if

$$0 \leq u \in D(a) \text{ implies } u \wedge 1 \in D(a) \text{ and } a(u \wedge 1) \leq a(u). \tag{7}$$

In that case, the mapping $u \mapsto u \wedge 1$ is continuous from $D(a)_+$ into $D(a)_+$.

We say that a C_0-semigroup T on $L^2(\Omega)$ is **submarkovian** if it is positive and L^∞-contractive. For further information on this property we refer to [ArBe] and to [BC] in the nonlinear case. Notice that several authors ([Dav], [BH], [FOT]) call a symmetric Markov semigroup what we call a symmetric submarkovian semigroup on $L^2(\Omega)$. A **Dirichlet form** is a closed positive form satisfying the two Beurling-Deny criteria.

Now let b be a second closed, positive form on $L^2(\Omega)$ such that the associated semigroup T is positive. We say that $D(a)$ is an **ideal** of $D(b)$ if

a) $u \in D(a)$ implies $|u| \in D(a)$ and,
b) $0 \leq u \leq v,\ v \in D(a),\ u \in D(b)$ implies $u \in D(a)$.

Ouhabaz's domination criterion [Ouh] says that

$$0 \leq S(t) \leq T(t) \qquad (t \geq 0) \tag{8}$$

if and only if $D(a)$ is an ideal of $D(b)$ and

$$a(u, v) \geq b(u, v) \quad \text{for all} \quad u, v \in D(a)_+. \tag{9}$$

2. Relative capacity and Robin boundary conditions

Let $\Omega \subset \mathbb{R}^n$ be an open set with boundary Γ. Let $H^1(\Omega) := \{u \in L^2(\Omega) : D_j u \in L^2(\Omega),\ j = 1, \ldots n\}$ be the first order Sobolev space and let $\widetilde{H}^1(\Omega)$ be the closure of $H^1(\Omega) \cap C_c(\bar{\Omega})$ in $H^1(\Omega)$. Here

$$C_c(\bar{\Omega}) = \{f : \bar{\Omega} \to \mathbb{R} \text{ continuous with compact support}\}.$$

If Ω has Lipschitz boundary, then $H^1(\Omega) = \widetilde{H}^1(\Omega)$. But e.g., $\widetilde{H}^1((0, 1) \cup (1, 2)) = H^1(0, 2) \neq H^1((0, 1) \cup (1, 2))$.

Using [GT, Lemma 7.6 p.152] one easily sees that $u \in \widetilde{H}^1(\Omega)$ implies that $|u| \in \widetilde{H}^1(\Omega)$ and $D_j|u| = (\operatorname{sign} u)D_j u \quad (j = 1, \dots n)$. Hence $\| \, |u| \, \|_{\widetilde{H}^1(\Omega)} = \|u\|_{\widetilde{H}^1(\Omega)}$. This implies in particular that the mapping $u \mapsto |u|$ is continuous on $\widetilde{H}^1(\Omega)$. Also for $v \in \widetilde{H}^1(\Omega)$, the mappings $u \mapsto v \wedge u$ and $u \mapsto v \vee u$ are continuous.

We define the **relative capacity** $\mathrm{Cap}_{\bar{\Omega}}(A)$ of a subset A of $\bar{\Omega}$ by

$$\mathrm{Cap}_{\bar{\Omega}}(A) := \inf \{\|u\|^2_{H^1(\Omega)} : \ u \in \widetilde{H}^1(\Omega), \ \exists \, O \subset \mathbb{R}^n \ \text{open such that} \ A \subset O \ \text{and} \ u(x) \geq 1 \ \text{a.e. on} \ \Omega \cap O\}. \tag{10}$$

Here and elsewhere the word relative stands for relative with respect to $\bar{\Omega}$.

Then $\mathrm{Cap}_{\bar{\Omega}}$ is an outer measure on $\bar{\Omega}$. This notion of capacity is induced by the regular Dirichlet form $\mathcal{E}$ on $L^2(\Omega)$ given by

$$\mathcal{E}(u, v) := \int_\Omega \nabla u \nabla v \, dx$$

with domain $D(\mathcal{E}) = \widetilde{H}^1(\Omega)$, in the sense of [BH, I 8.1.1 p.52]. Here the underlying locally compact space is $X = \bar{\Omega}$, with Borel σ-algebra $\mathcal{B}(\bar{\Omega})$ and the measure $m(A) = \lambda(A \cap \Omega)$ $(A \in \mathcal{B}(\bar{\Omega}))$ (to make sure that $L^2(X, \mathcal{B}(\bar{\Omega}), m) = L^2(\Omega)$, the usual space with Lebesgue measure).

But now we may consider functions in $\widetilde{H}^1(\Omega)$ as defined on $\bar{\Omega}$.

We say that $A \subset \bar{\Omega}$ is **relatively polar** if $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$. A property is said to hold **relatively quasi-everywhere (r.q.e.)** if it holds on $\bar{\Omega} \setminus N$ where $N \subset \bar{\Omega}$ is relatively polar.

A function $u : \bar{\Omega} \to \mathbb{R}$ is called **relatively quasi-continuous**, if for each $\varepsilon > 0$ there exists a relatively open set $G \subset \bar{\Omega}$ such that $\mathrm{Cap}_{\bar{\Omega}}(G) < \varepsilon$ and u is continuous on $\bar{\Omega} \setminus G$. Then by [BH, I, Proposition 8.2.1] for each $u \in \widetilde{H}^1(\Omega)$ there exists a relatively quasi-continuous function $\tilde{u} : \bar{\Omega} \to \mathbb{R}$ such that $u = \tilde{u}$ a.e. The function $\tilde{u}$ is relatively quasi-everywhere unique and we call it the **relatively quasi-continuous representative** of u. Moreover, $\tilde{u}$ may be chosen Borel measurable.

Finally we recall the following result which we shall use frequently (see [FOT, Theorem 2.1.4 p.69] or [BH, I, Proposition 8.2.5]).

PROPOSITION 2.1. *Let* $\lim_{m\to\infty} u_m = u$ *in* $\widetilde{H}^1(\Omega)$. *Then there exists a subsequence* $(\tilde{u}_{m_k})$ *such that* $\lim_{k\to\infty} \tilde{u}_{m_k}(x) = \tilde{u}(x)$ *r.q.e.*

REMARK 2.2. The notion of relative capacity was introduced in [AW]. It is clear that polar subsets of $\bar{\Omega}$ are relatively polar. The converse is true for subsets of Ω, and it is also true for subsets of $\bar{\Omega}$ if the boundary is Lipschitz. But if Ω is not regular, then there may exist relatively polar sets in the boundary which are not polar.

With the help of the relatively quasi-continuous representative the space $H_0^1(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$ may now be described as follows [AW, Theorem 2.3]

$$H_0^1(\Omega) = \{u \in \widetilde{H}^1(\Omega) : \ \tilde{u} = 0 \ \text{ r.q.e. on } \ \Gamma\}. \tag{11}$$

Now we define the class of measures by which we define general Robin boundary conditions.

DEFINITION 2.3. An **admissible measure** is a measure $\mu : \ \mathcal{B}(\Gamma_\mu) \to [0, \infty)$ where $\Gamma_\mu \subset \Gamma$ is relatively open, such that

a) $\mu(K) < \infty$ for each compact set $K \subset \Gamma_\mu$; and
b) $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$ implies $\mu(A) = 0$ for each Borel set $A \subset \Gamma_\mu$.

The set Γ_μ is called **the domain of** μ. Here $\mathcal{B}(\Gamma_\mu)$ denotes the Borel σ-algebra of Γ_μ.

Let μ be an admissible measure with domain Γ_μ. Let $L^2(\Gamma_\mu) = L^2(\Gamma_\mu, \mathcal{B}(\Gamma_\mu), \mu)$. We define a form a_μ on $L^2(\Omega)$ by

$$D(a_\mu) = \left\{ u \in \widetilde{H}^1(\Omega) : \ \tilde{u} = 0 \ \text{ r.q.e. on } \Gamma \setminus \Gamma_\mu, \ \int_{\Gamma_\mu} |\tilde{u}|^2 \, d\mu < \infty \right\},$$

$$a_\mu(u, v) = \int_\Omega \nabla u \nabla v \, dx + \int_{\Gamma_\mu} \tilde{u}\tilde{v} \, d\mu.$$

Here $\tilde{u}$ is the relatively quasi-continuous representative of u which we always choose Borel measurable. Since μ is admissible, a_μ is well-defined. In fact, let $u, u_1, v, v_1 \in D(a_\mu)$ such that $u = u_1$, $v = v_1$ a.e. on Ω. Then it follows from [BH, I, Proposition 8.1.6] that $\tilde{u} = \tilde{u_1}$ and $\tilde{v} = \tilde{v_1}$ r.q.e. on $\bar{\Omega}$. Since μ is admissible, it follows that $\tilde{u} = \tilde{u_1}$ and $\tilde{v} = \tilde{v_1}$ μ-a.e. on Γ_μ and thus $a_\mu(u, v) = a_\mu(u_1, v_1)$.

THEOREM 2.4. *Let μ be an admissible measure. Then a_μ is a Dirichlet form on $L^2(\Omega)$. The space $H^1(\Omega) \cap C_c(\Omega \cup \Gamma_\mu)$ is a form core of a_μ.*

Proof. a) The mapping $D(a_\mu) \to \widetilde{H}^1(\Omega) \oplus L^2(\Gamma_\mu)$, $u \mapsto (u, \tilde{u}_{|\Gamma_\mu})$ is isometric. In order to show that a_μ is closed, it suffices to show that the image of the mapping is closed. Let $u_m \in D(a_\mu)$ such that $u_m \to u$ in $\widetilde{H}^1(\Omega)$ and $\tilde{u}_m \to f$ in $L^2(\Gamma_\mu)$. Taking a subsequence, we can assume that $\tilde{u}_m \to \tilde{u}$ r.q.e. and $\tilde{u}_m \to f$ μ-a.e. on Γ_μ. Since μ is admissible it follows that $f = \tilde{u}$ in $L^2(\Gamma_\mu)$.

b) Since $|u|^\sim = |\tilde{u}|$ and $(u \wedge 1)^\sim = \tilde{u} \wedge 1$ it follows that the two criteria of Beurling-Deny are satisfied.

c) In order to show that $H^1(\Omega) \cap C_c(\Omega \cup \Gamma_\mu)$ is a form core, let $u \in D(a_\mu)$. We can assume that $u \geq 0$ a.e.

FIRST CASE. We assume that u is bounded; $u \le c$, say. There exists a sequence $(u_m)_{m\in\mathbb{N}}$ in $H^1(\Omega) \cap C_c(\Omega \cup \Gamma_\mu)$ such that $u_m \to u$ in $\widetilde{H}^1(\Omega)$. We may assume that $u_m \to \tilde{u}$ r.q.e. Let $v_m = 0 \vee (u_m \wedge \tilde{u})$. Then $v_m \to u$ in $\widetilde{H}^1(\Omega)$ and, by the Dominated Convergence Theorem, $v_{m_{|\Gamma_\mu}} \to \tilde{u}_{|\Gamma_\mu}$ in $L^2(\Gamma_\mu, \mu)$. Fix $m \in \mathbb{N}$. Let $O \subset \Gamma_\mu$ be relatively open such that $\bar{O}$ is compact, $\bar{O} \subset \Gamma_\mu$ and $\operatorname{supp}[v_m] \subset O \cup \Omega$. Since v_m is bounded and $\mu(O) < \infty$, it follows that $v_m \in D(a_\mu)$. By [FOT, Corollary 2.3.1] there exists a sequence $(w_k)_{k\in\mathbb{N}}$ in $H^1(\Omega)\cap C_c(\Omega\cup\Gamma_\mu)$ such that $w_k \to v_m$ $(k \to \infty)$ in $\widetilde{H}^1(\Omega)$ and r.q.e. Let $f_k = (0 \vee w_k) \wedge c$. Then $f_k \in H^1(\Omega) \cap C_c(\Omega \cup \Gamma_\mu)$ and $f_k \to v_m$ in $\widetilde{H}^1(\Omega)$ and $f_{k_{|\Gamma_\mu}} \to v_{m_{|\Gamma_\mu}}$ in $L^2(\Gamma_\mu)$ as $k \to \infty$. Hence u is in the closure of $H^1(\Omega) \cap C_c(\Omega \cup \Gamma_\mu)$ in $D(a_\mu)$.

SECOND CASE. The function u is not bounded. By the first case $u \wedge k$ can be approximated by functions in $H^1(\Omega) \cap C_c(\Omega \cup \Gamma_\mu)$. But $u_k \to u$ in $\widetilde{H}^1(\Omega)$ and $u_{k_{|\Gamma_\mu}} \to u_{|\Gamma_\mu}$ in $L^2(\Gamma_\mu)$ as $k \to \infty$. This proves the claim of the theorem. □

3. Monotonicity properties

Let μ be an admissible measure. We denote by A_μ the operator associated with a_μ. Then

$$A_\mu u = -\Delta u \qquad \text{in } \mathcal{D}(\Omega)' \tag{12}$$

for all $u \in D(A_\mu)$. In fact, let $A_\mu u = v$. Then it follows from the definition of the associated operator that

$$\int_\Omega \nabla u \nabla \varphi \, dx = a_\mu(u, \varphi) = \int_\Omega v\varphi \, dx$$

for all $\varphi \in \mathcal{D}(\Omega)$. This implies (12). We let $\Delta_\mu := -A_\mu$. Thus Δ_μ is a symmetric realization of the Laplacian in $L^2(\Omega)$.

If $\Gamma_\mu = \emptyset$, then $D(a_\mu) = H_0^1(\Omega)$ and $-A_\mu$ is just the **Dirichlet Laplacian** Δ^D given by

$$\begin{cases} D(\Delta^D) = \{u \in H_0^1(\Omega) : \ \Delta u \in L^2(\Omega)\} \\ \Delta^D u = \Delta u \qquad (\text{in } \mathcal{D}(\Omega)'). \end{cases}$$

If $\Gamma_\mu = \Gamma$ and $\mu = 0$, then $-A_\mu$ is the **Neumann Laplacian** Δ^N whose domain consists of all $u \in \widetilde{H}^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$ and

$$\int_\Omega \Delta u \varphi \, dx = -\int_\Omega \nabla u \nabla \varphi \, dx \tag{13}$$

for all $\varphi \in \widetilde{H}^1(\Omega)$. In view of Green's formula, (13) may be seen as a weak formulation of Neumann boundary conditions

$$\frac{\partial u}{\partial \nu}_{|\Gamma} = 0 \qquad \text{on} \quad \Gamma.$$

Next, we show the following domination property.

THEOREM 3.1. *For each admissible measure μ, the semigroup $(e^{t\Delta_\mu})_{t\geq 0}$ satisfies*

$$e^{t\Delta^D} \leq e^{t\Delta_\mu} \leq e^{t\Delta^N} \tag{14}$$

for all $t \geq 0$ in the sense of positive operators.

Proof. 1) We show that $e^{t\Delta^D} \leq e^{t\Delta_\mu}$. By Ouhabaz's domination criterion, it suffices to prove that $H_0^1(\Omega)$ is an ideal of $D(a_\mu)$ and $a_\mu(u, v) \leq \int_\Omega \nabla u \nabla v \, dx$ for all $u, v \in H_0^1(\Omega)_+$. We may assume that functions in $\widetilde{H}^1(\Omega)$ are r.q.c.

a) We claim that $H_0^1(\Omega)$ is an ideal of $D(a_\mu)$. In fact, let $u \in H_0^1(\Omega)$ and $v \in D(a_\mu)$ such that $0 \leq v \leq u$. Since $\bar{\Omega}$ is relatively open, it follows from [FOT, Lemma 2.1.4] that $0 \leq v \leq u$ r.q.e. on $\bar{\Omega}$. Using the characterization of $H_0^1(\Omega)$ given by (11), we have that $u = 0$ r.q.e. on Γ and thus $v = 0$ r.q.e. on Γ. Therefore $v \in H_0^1(\Omega)$ which proves the claim.

b) Let $u, v \in H_0^1(\Omega)_+$. By the characterization of $H_0^1(\Omega)$, we have that $u = v = 0$ r.q.e. on Γ. Since μ is admissible, it follows that $u = v = 0$ μ a.e. on Γ_μ. We finally obtain that

$$\begin{aligned} a_\mu(u, v) &:= \int_\Omega \nabla u \nabla v \, dx + \int_{\Gamma_\mu} uv \, d\mu \\ &= \int_\Omega \nabla u \nabla v \, dx \end{aligned}$$

and the proof of this part is complete.

2) The proof of the inequality $e^{t\Delta_\mu} \leq e^{t\Delta_N}$ is a simple modification of the first part. □

We will see in the next section that (14) characterizes the semigroups $(e^{t\Delta_\mu})_{t\geq 0}$. Before that we prove a monotonicity and uniqueness result.

THEOREM 3.2. *Let μ, ν be two admissible measures. The following assertions are equivalent.*

(i) $e^{t\Delta_\mu} \leq e^{t\Delta_\nu} \qquad (t \geq 0)$.

(ii) (a) $\mathrm{Cap}_{\bar{\Omega}}(\Gamma_\mu \setminus \Gamma_\nu) = 0$ *and*

(b) $\mu(A) \geq \nu(A)$ *for each Borel set $A \subset \Gamma_\mu \cap \Gamma_\nu$.*

Proof. (i) $\Rightarrow$ (ii). (a) Let $K_m \subset \Gamma_\mu$ be compact sets such that $K_m \subset K_{m+1}$ and $\cup_{m\in\mathbb{N}} K_m = \Gamma_\mu$. Let $O \subset \mathbb{R}^n$ be open such that $\Gamma_\mu = O \cap \Gamma$. Let $u \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq u \leq 1$, $u = 1$ on K_m and $\mathrm{supp}[u] \subset O$. Then $u_{|\bar{\Omega}} \in D(a_\mu)$. Since $D(a_\mu) \subset D(a_\nu)$ by the domination criterion, it follows that $u(z) = 0$ r.q.e. on $\Gamma \setminus \Gamma_\nu$. In particular, $\mathrm{Cap}_{\bar{\Omega}}(K_m \setminus \Gamma_\nu) = 0$. Thus $\mathrm{Cap}_{\bar{\Omega}}(\Gamma_\mu \setminus \Gamma_\nu) = \lim_{n\to\infty} \mathrm{Cap}_{\bar{\Omega}}(K_m \setminus \Gamma_\nu) = 0$.

(b) Let $A \subset \Gamma_\mu \cap \Gamma_\nu$ be a Borel set. Since by [Rud, 2.18 p.48] μ and ν are regular, it suffices to show that $\nu(K) \le \mu(O)$ where $K \subset A$ is a compact set and O is a relatively open subset in $\Gamma_\mu \cap \Gamma_\nu$ containing A. Let $V \subset \mathbb{R}^n$ be open such that $V \cap \Gamma = \emptyset$. Let $u \in \mathcal{D}(\mathbb{R}^n)$ such that $\text{supp}[u] \subset V$, $0 \le u \le 1$ and $u = 1$ on K. Then $u_{|\bar{\Omega}} \in D(a_\mu)$. Hence

$$\begin{aligned}
&\int_\Omega |\nabla u|^2\, dx + \int_{\Gamma_\mu} |u|^2\, d\mu \\
&\quad = a_\mu(u) \ge a_\nu(u) \\
&\quad = \int_\Omega |\nabla u|^2\, dx + \int_{\Gamma_\nu} |u|^2\, d\nu
\end{aligned}$$

by the domination criterion. Consequently,

$$\nu(K) \le \int_{\Gamma_\nu} |u|^2\, d\nu \le \int_{\Gamma_\mu} |u|^2\, d\mu \le \mu(O).$$

(ii) $\Rightarrow$ (i). Let $u \in D(a_\mu)$. Then (a) implies that $\tilde{u} = 0$ r.q.e. on $\Gamma \setminus \Gamma_\nu$. Since μ is admissible, $\mu(\Gamma_\mu \setminus \Gamma_\nu) = 0$. Hence

$$\begin{aligned}
\int_{\Gamma_\nu} |\tilde{u}|^2\, d\nu &= \int_{\Gamma_\mu \cap \Gamma_\nu} |\tilde{u}|^2\, d\nu \\
&= \int_0^\infty \nu(\{z \in \Gamma_\mu \cap \Gamma_\nu :\ |\tilde{u}(z)|^2 > t\})\, dt \\
&\le \int_0^\infty \mu(\{z \in \Gamma_\mu \cap \Gamma_\nu :\ |\tilde{u}(z)|^2 > t\})\, dt \\
&= \int_{\Gamma_\mu} |\tilde{u}(z)|^2\, d\mu(z) < \infty.
\end{aligned}$$

Thus $u \in D(a_\nu)$. We have shown that $D(a_\mu) \subset D(a_\nu)$. Since $D(a_\mu)$ is an ideal of $\widetilde{H}^1(\Omega)$, it is also an ideal of $D(a_\nu)$. Let $u, v \in D(a_\nu)_+$. One proves similarly as in Theorem 3.1. that $a_\mu(u, v) \ge a_\nu(u, v)$. Now the domination criterion implies (i). □

As corollary we note the following uniqueness theorem.

COROLLARY 3.3. *Let μ and ν be two admissible measures. The following assertions are equivalent.*

(i) $e^{t\Delta_\mu} = e^{t\Delta_\nu} \qquad (t \ge 0)$.

(ii) $\text{Cap}_{\bar{\Omega}}(\Gamma_\mu \Delta \Gamma_\nu) = 0$ *and* $\mu(A) = \nu(A)$ *for each Borel set* $A \subset \Gamma_\mu \cap \Gamma_\nu$.

4. Sandwiched semigroups.

In this section we show that the sandwich property (14) characterizes the semigroups $(e^{t\Delta_\mu})_{t\ge 0}$ under suitable conditions. Let Ω be an open subset of $\mathbb{R}^n$ with boundary Γ.

THEOREM 4.1. *Let T be a symmetric C_0-semigroup on $L^2(\Omega)$ associated with a positive closed form $(a, D(a))$. Then the following assertions are equivalent.*

(i) *There exists an admissible measure μ such that $a = a_\mu$.*
(ii) (a) *One has $e^{t\Delta^D} \le T(t) \le e^{t\Delta^N}$ $(t \ge 0)$;*
(b) $\mathrm{supp}[u] \cap \mathrm{supp}[v] = \emptyset$ *implies* $a(u, v) = 0$ *for all* $u, v \in D(a) \cap C_c(\bar{\Omega})$.
(c) *$D(a) \cap C_c(\bar{\Omega})$ is dense in $(D(a), \|\cdot\|_a)$.*

Proof. We know that the conditions in (ii) are necessary. In order to prove the converse assume that (ii) is satisfied. Then by the domination criterion, $D(a)$ is an ideal of $\widetilde{H}^1(\Omega)$ containing $H_0^1(\Omega)$, and

$$b(u, v) := a(u, v) - \int_\Omega \nabla u \nabla v \, dx \tag{15}$$

is positive whenever $0 \le u, v \in D(a)$. Let $\Gamma_0 := \{z \in \Gamma : \exists\, u \in D(a) \cap C_c(\bar{\Omega}), u(z) \neq 0\}$ and let $Y = \Omega \cup \Gamma_0$. Notice that Y is a locally compact space. Since $D(a)$ is an ideal of $\widetilde{H}^1(\Omega)$ one has

$$\widetilde{H}^1(\Omega) \cap C_c(Y) = D(a) \cap C_c(Y) =: E_c. \tag{16}$$

The space E_c is a subalgebra of $C_c(Y)$ by [BH, I, Corollary 3.3.2] (or [FOT, Theorem 1.4.2 (ii)]). It follows from the Stone-Weierstrass Theorem that E_c is uniformly dense in $C_c(Y)$. From this follows that E_c is also dense in $C_c(Y)$ for the inductive topology. In fact, we observe first that a is a Dirichlet form since $T(t) \le e^{t\Delta^N}$ and $(e^{t\Delta^N})_{t\ge0}$ is submarkovian.

Let $0 \le u \in C_c(Y)$ and $\varepsilon > 0$. There exists $0 \le v \in E_c$ such that $\|u - v\|_\infty \le \varepsilon$. Then $(v - \varepsilon)^+ \in E_c$, $\mathrm{supp}[(v - \varepsilon)^+] \subset \mathrm{supp}[u]$ and

$$\|u - (v - \varepsilon)^+\|_\infty \le \|u - v\|_\infty + \|v - (v - \varepsilon)^+\|_\infty \le 2\varepsilon.$$

This shows that u can be approximated in the inductive topology by functions in E_c.

Now b is a positve bilinear form on E_c (i.e., $b(u, v) \ge 0$ whenever $0 \le u, v \in E_c$). Thus b is continuous for the inductive topology. Hence there exists a unique positive bilinear form $\tilde{b}$ on $C_c(Y)$ extending b. Consequently, there exists a unique positive functional Φ on $C_c(Y \times Y)$ such that $\Phi(u \otimes v) = \tilde{b}(u, v)$ for all $u, v \in C_c(Y)$ (cf. [Bou, Chap. III., Section 4] or [Sch, p.297] and the proof of [FOT, Lemma 1.4.1]). Hence there exists a unique regular Borel measure ν on $Y \times Y$ such that

$$b(u, v) = \int_{Y \times Y} u(x) v(y) \, d\nu$$

for all $u, v \in E_c$. Observe that $\int_{Y\times Y} u(x)v(y)\, d\nu = 0$ for $u, v \in C_c(Y)$ such that $\mathrm{supp}[u] \cap \mathrm{supp}[v] = \emptyset$. In fact, if $u, v \in E_c$ this follows from the assumption. But in general, by [FOT, Lemma 1.4.2 (ii)] there exist $u_n, v_n \in E_c$ with $\mathrm{supp}[u_n] \subset \{y \in Y : u(y) \neq 0\}$

and $\text{supp}[v_n] \subset \{y \in Y : \ v(y) \neq 0\}$ such that u_n, v_n converge uniformly to u and v, respectively. Hence $\int_{Y\times Y} u(x)v(y)\ d\nu = \lim_{n\to\infty} \int_{Y\times Y} u_n(x)v_n(y)\ d\nu = 0$. Thus $\text{supp}[\nu] \subset \{(y,y) : \ y \in Y\} \subset Y \times Y$. Hence there exists a regular Borel measure μ on Y such that

$$b(u,v) = \int_Y u(x)v(x)\ d\mu$$

for all $u, v \in E_c$.

By the domination property (9), one has $b = 0$ on $H_0^1(\Omega) \times H_0^1(\Omega)$. Thus it follows that $\text{supp}[\mu] \subset \Gamma_0$. We have shown that

$$a(u,v) = \int_\Omega \nabla u \nabla v\ dx + \int_{\Gamma_0} uv\, d\mu \tag{17}$$

for all $u \in E_c$. Next we show that

$$D(a) \cap C_c(\bar{\Omega}) = \left\{ u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \ u_{|\Gamma\setminus\Gamma_0} = 0,\ \int_{\Gamma_0} |u|^2\, d\mu < \infty \right\} =: F_\mu \tag{18}$$

and that (17) remains true for all $u, v \in F_\mu$.

In order to prove (18) it suffices to consider positive functions. Let $0 \leq u \in F_\mu$. Then $(u-\varepsilon)^+ \in H^1(\Omega) \cap C_c(\Omega \cup \Gamma_0) = E_c$ (by (16)) for all $\varepsilon > 0$. Moreover, $(u-\varepsilon)^+ \to u$ in $H^1(\Omega)$ and $(u-\varepsilon)^+_{|\Gamma_0} \to u_{|\Gamma_0}$ in $L^2(\Gamma_0)$ as $\varepsilon \downarrow 0$. Hence $(u-\varepsilon)^+$ is a Cauchy net in $D(a)$. Thus $u \in D(a)$ and

$$\begin{aligned} a(u) &= \lim_{\varepsilon\downarrow 0} a((u-\varepsilon)^+) = \lim_{\varepsilon\downarrow 0} \left(\int_\Omega |\nabla(u-\varepsilon)^+|^2\, dx + \int_{\Gamma_0} ((u-\varepsilon)^+)^2\, d\mu \right) \\ &= \int_\Omega |\nabla u|^2\, dx + \int_{\Gamma_0} |u|^2\, d\mu. \end{aligned}$$

Conversely, let $0 \leq u \in D(a) \cap C_c(\bar{\Omega})$. Since a is a Dirichlet form $(u-\varepsilon)^+$ converges to u in $D(a)$ as $\varepsilon \downarrow 0$. Moreover, $(u-\varepsilon)^+ \in F_\mu$. Hence

$$a(u) = \lim_{\varepsilon\downarrow 0} a((u-\varepsilon)^+) = \int_\Omega |\nabla u|^2\, dx + \int_{\Gamma_0} |u|^2\, d\mu.$$

We have proved (18) and (17) for $u = v$. The polarization identity shows that (17) holds for all $u, v \in F_\mu$. Since a is closed it follows from [AW, Theorem 2.3] that μ is admissible. Let $\Gamma_\mu = \Gamma_0$. Now Theorem 2.4 implies that $a = a_\mu$. □

Next we characterize those sandwiched semigroups which come from a bounded measure.

COROLLARY 4.2. *Let Ω be bounded. Let T be a symmetric C_0-semigroup on $L^2(\Omega)$ associated with a positive closed form $(a, D(a))$. Then the following assertions are equivalent.*

(i) *There exists a bounded admissible measure μ on Γ such that $a = a_\mu$.*
(ii) (a) *One has $e^{t\Delta^D} \leq T(t) \leq e^{t\Delta^N}$ $(t \geq 0)$;*
(b) $\text{supp}[u] \cap \text{supp}[v] = \emptyset$ *implies* $a(u, v) = 0$ *for all* $u, v \in D(a) \cap C(\bar{\Omega})$.
(c) $1 \in D(a)$.

Proof. Assume that (ii) holds. We keep the notations of the proof of Theorem 4.1. Since $1 \in D(a)$, it follows from (18) that $\Gamma_\mu = \Gamma_0 = \Gamma$, that μ is a bounded admissible measure and that $D(a_\mu) \subset D(a)$ and $a(u, v) = a_\mu(u, v)$ for all $u, v \in D(a_\mu)$. Let $0 \leq u \in D(a)$. Then for $k \in \mathbb{N}$, $u \wedge k \in D(a_\mu)$ and by (7),

$$a_\mu(u \wedge k) = a(u \wedge k) = k^2 a\left(\frac{u}{k} \wedge 1\right) \leq k^2 a\left(\frac{u}{k}\right) = a(u).$$

Thus $(u \wedge k)$ is bounded in $(D(a_\mu), \|\cdot\|_{a_\mu})$ and converges to u in $L^2(\Omega)$. It follows that $(u \wedge k)$ converges weakly to u in $D(a)$. Thus $u \in D(a_\mu)$. We have shown that $D(a) = D(a_\mu)$. This proves (i). The other implication is clear. □

We give several comments concerning Theorem 4.1 and Corollary 4.2. First of all, it is remarkable that in the situation of Corollary 4.2; i.e. assuming that $D(a)$ contains a stricty positive continuous function, the form a is automatically regular (i.e., $D(a) \cap C(\bar{\Omega})$ is dense in $D(a)$). In general, the situation is more complicated. Choosing Γ_μ open in Definition 2.3 we could prove in Theorem 2.4 that the form a_μ is regular. This shows in particular that condition (c) in Theorem 4.1 is satisfied for $a = a_\mu$. But we might consider the more general case where Γ_μ is merely a Borel set. In the following we do this for the special case where the measure μ is 0.

Let $\Omega \subset \mathbb{R}^n$ be an open set with boundary Γ.

EXAMPLE 4.3. (Dirichlet-Neumann boundary conditions) Let $\Gamma_0 \subset \Gamma$ be a Borel set. We define

$$J(\Gamma_0) := \{u \in \widetilde{H}^1(\Omega) : \tilde{u} = 0 \text{ r.q.e. on } \Gamma \setminus \Gamma_0\}.$$

Then $J(\Gamma_0)$ is a closed ideal of $\widetilde{H}^1(\Omega)$. Let $D(a) = J(\Gamma_0)$, $a(u, v) = \int_\Omega \nabla u \nabla v \, dx$. Then a is a Dirichlet form on $L^2(\Omega)$ and the associated semigroup T satisfies

$$e^{t\Delta^D} \leq T(t) \leq e^{t\Delta^N} \qquad (t \geq 0). \tag{19}$$

This follows from the domination criterion (9).

Now we describe under which conditions Γ_0 may be chosen relatively open in Γ. If $\Gamma_0 \subset \Gamma$ is relatively open, then it follows from [FOT, Corollary 2.3.1] that the space

$H^1(\Omega) \cap C_c(\Omega \cup \Gamma_0)$ is dense in $J(\Gamma_0)$. Conversely, assume that J is a closed ideal of $\widetilde{H}^1(\Omega)$ containing $H_0^1(\Omega)$. Assume that $J \cap C_c(\bar{\Omega})$ is a dense subspace of J. Let $\Gamma_0 = \{z \in \Gamma : \exists u \in J \cap C_c(\bar{\Omega})$ such that $u(z) \neq 0\}$. Then $J = J(\Gamma_0)$. In fact, since J is an ideal, and $H_0^1(\Omega) \subset J$ it follows that $H^1(\Omega) \cap C_c(\Omega \cup \Gamma_0) \subset J \subset J(\Gamma_0)$. Now the claim follows from the preceding.

REMARK 4.4. By a result of Stollmann [Sto] each closed ideal J of $\widetilde{H}^1(\Omega)$ containing $H_0^1(\Omega)$ is of the form $J = J(\Gamma_0)$ for some Borel set $\Gamma_0 \subset \Gamma$.

Next we comment on the locality condition. It cannot be omitted as the following simple example shows.

EXAMPLE 4.5. (Non-local boundary conditions) Let $\Omega = (0, 1)$. Define the form a by $D(a) = H^1(0, 1)$,

$$a(u, v) = \int_0^1 u'v'\,dx + u(0)v(0) + u(1)v(0) + u(0)v(1) + u(1)v(1).$$

Then a is a closed positive form which is not local. Let T be the associated semigroup on $L^2(0, 1)$. Then condition (a) of Corollary 4.2 is satisfied by the domination criterion. However condition (b) is not satisfied.

For further properties of local forms we refer to [BH], [FOT] and [MR]. For locality properties of the Laplacian we refer to Bénilan-Pierre [BP].

5. The surface measure

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary Γ. By $\sigma = \mathcal{H}^{n-1}$ we denote the surface measure on Γ. Then σ is admissible [AW, Proposition 4.1]. Recall that $H^1(\Omega) \cap C(\bar{\Omega})$ is dense in $H^1(\Omega)$ (i.e. $\widetilde{H}^1(\Omega) = H^1(\Omega)$) and the trace $u \mapsto u_{|\Gamma}$ defined for $u \in H^1(\Omega) \cap C(\bar{\Omega})$ has a continuous extension from $H^1(\Omega)$ into $L^2(\Gamma)$. In order words, one has $\tilde{u} \in L^2(\Gamma)$ for all $u \in H^1(\Omega)$.

Let $u \in C^2(\bar{\Omega})$. Then

$$\int_\Omega \Delta u \varphi\,dx = -\int_\Omega \nabla u \nabla \varphi\,dx + \int_\Gamma \frac{\partial u}{\partial \nu} \varphi\,d\sigma \tag{20}$$

for all $\varphi \in H^1(\Omega)$, where $\frac{\partial u}{\partial \nu} = \langle \nabla u, \nu \rangle \in L^\infty(\Omega)$, $\nu(z)$ being the exterior normal at $z \in \Gamma$. We want to define the weak normal derivative $\frac{\partial u}{\partial \nu}$ of u. Let

$$D(\Delta_{\max}) := \{u \in H^1(\Omega) : \ \Delta u \in L^2(\Omega)\}.$$

For $u \in D(\Delta_{\max})$ we say that $\frac{\partial u}{\partial \nu}$ **exists weakly**, if there exists a function $b \in L^2(\Gamma)$ such that

$$\int_\Omega \Delta u \varphi \, dx = -\int_\Omega \nabla u \nabla \varphi \, dx + \int_\Gamma b\varphi \, d\sigma \tag{21}$$

for all $\varphi \in H^1(\Omega)$. In that case $b \in L^2(\Gamma)$ is unique and we write $\frac{\partial u}{\partial \nu} := b$.

Now let $0 \le \beta \in L^\infty(\Gamma) := L^\infty(\Gamma, \sigma)$. Then the measure μ given by $d\mu = \beta d\sigma$ is admissible [AW]. The form $a_\beta := a_\mu$ is given by $D(a_\beta) = H^1(\Omega)$,

$$a_\beta(u, v) = \int_\Omega \nabla u \nabla v \, dx + \int_\Gamma \tilde{u}\tilde{v}\beta \, d\sigma.$$

Denote by $-\Delta_\beta$ the operator associated with a_β. We now can describe Δ_β as follows.

PROPOSITION 5.1. *One has*

$$\begin{cases} D(\Delta_\beta) &= \{u \in D(\Delta_{\max}) : \ \frac{\partial u}{\partial \nu} \ \text{exists weakly in } L^2(\Gamma) \ \text{and} \ \frac{\partial u}{\partial \nu} + \beta u_{|\Gamma} = 0\}, \\ \Delta_\beta u &= \Delta u \quad \text{in} \quad \mathcal{D}(\Omega)'. \end{cases} \tag{22}$$

Proof. Denote by A the operator associated with a_β. Let $u \in D(A)$ and $Au = v$. Then

$$\int_\Omega v\varphi \, dx = a_\beta(u, \varphi) = \int_\Omega \nabla u \nabla \varphi \, dx + \int_\Gamma u\varphi\beta \, d\sigma$$

for all $\varphi \in H^1(\Omega)$. Choosing $\varphi \in \mathcal{D}(\Omega)$ this implies that $v = -\Delta u$. This shows that $D(\Delta_\beta)$ is included in the right-hand-side of (22).

Conversely, let $u \in D(\Delta_{\max})$ such that $\frac{\partial u}{\partial \nu}$ exists weakly and $\frac{\partial u}{\partial \nu} + \beta u_{|\Gamma} = 0$. Then one has for all $\varphi \in H^1(\Omega)$,

$$\begin{aligned} -\int_\Omega \Delta u \varphi \, dx &= \int_\Omega \nabla u \nabla \varphi \, dx + \int_\Gamma u\varphi\beta \, d\sigma \\ &= a_\beta(u, \varphi). \end{aligned}$$

Hence $u \in D(A)$ and $Au = -\Delta u$. □

In particular, in this case of *classical Robin boundary conditions* one has $\{u \in C^2(\bar{\Omega}) : \frac{\partial u}{\partial \nu} + \beta u_{|\Gamma} = 0\} \subset D(\Delta_\beta)$.

Next we show that an admissible measure μ is necessarily of the form $\beta d\sigma$ whenever $D(\Delta_\mu)$ contains smooth functions. More generally, we have the following.

PROPOSITION 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1 with boundary Γ. Let T be a symmetric C_0-semigroup associated with a closed form a. Denote by A the generator of T. Assume that*

a) $e^{t\Delta^D} \leq T(t) \leq e^{t\Delta^N}$ $(t \geq 0)$, *that*
b) *a is* **local***; i.e. $a(u, v) = 0$ whenever $u, v \in D(a) \cap C(\bar{\Omega})$ have disjoint support, and that*
c) *there exists $u \in D(A) \cap C^2(\bar{\Omega})$ such that $u(z) > 0$ for all $z \in \Gamma$.*

Then there exists a function $\beta \in C(\Gamma)_+$ such that $A = -\Delta_\beta$.

Proof. It follows from Theorem 4.1 that there exists an admissible measure μ on Γ such that $a = a_\mu$ and $A = -\Delta_\mu$. One considers the function u in c). Then for all $\varphi \in C^1(\bar{\Omega})$ one has

$$\begin{aligned}\int_\Omega \nabla u \nabla \varphi \, dx + \int_\Gamma u\varphi \, d\mu &= a(u, \varphi) \\ &= -\int_\Omega \Delta u \varphi \, dx \\ &= \int_\Omega \nabla u \nabla \varphi \, dx - \int_\Gamma \frac{\partial u}{\partial \nu} \varphi \, d\sigma.\end{aligned}$$

It follows from the Stone-Weierstrass Theorem that

$$\int_\Gamma u\varphi \, d\mu + \int_\Gamma \frac{\partial u}{\partial \nu} \varphi \, d\sigma = 0$$

for all $\varphi \in C(\Gamma)$. This implies that $d\mu = -\frac{1}{u}\frac{\partial u}{\partial \nu} \, d\sigma$. Thus the claim is proved with $\beta = -\frac{1}{u}\frac{\partial u}{\partial \nu}$. □

Also a converse version of Proposition 5.2 holds.

PROPOSITION 5.3. *Assume that Ω is a bounded open set of class $C^{2,\alpha}$ where $0 < \alpha < 1$. Let $\beta \in C^{1,\alpha}(\Gamma)$ with $0 < \beta(z)$ $(z \in \Gamma)$. Then there exists $u \in D(\Delta_\beta) \cap C^{2,\alpha}(\bar{\Omega})$ such that* $\inf_{x \in \bar{\Omega}} u(x) > 0$.

Proof. By [GT, Theorem 6.31] there exists $u \in C^{2,\alpha}(\bar{\Omega})$ such that $-\Delta u = 1$ on $\bar{\Omega}$ and $\beta u + \frac{\partial u}{\partial \nu} = 0$ on Γ. Then $u \in D(\Delta_\beta)$ and $\Delta_\beta u = 1$. By Proposition 6.3 below, one has $0 \in \rho(\Delta_\beta)$. Thus, $u = R(0, \Delta_\beta)1$. It follows from the domination property (Theorem 3.2) that $u = R(0, \Delta_\beta)1 \geq R(0, \Delta^D)1$. Now it follows from the maximum principle (see e.g. [Are, Theorem 1.5]) that $u(x) > 0$ for all $x \in \Omega$. Assume that there exists $z_0 \in \Gamma$ such that $u(z_0) = 0$. Then by [RR, Lemma 4.7], it follows that $\frac{\partial u}{\partial \nu}(z_0) < 0$ which is impossible since u satisfies the boundary condition. Thus $u(x) > 0$ for all $x \in \bar{\Omega}$. □

6. Asymptotics

Let $\Omega \subset \mathbb{R}^n$ be open and let μ be an admissible measure on Γ with domain Γ_μ. The semigroup $(e^{t\Delta_\mu})_{t\geq 0}$ on $L^2(\Omega)$ is submarkovian. Thus there exist consistent C_0-semigroups $(e^{t\Delta_{\mu,p}})_{t\geq 0}$ on $L^p(\Omega)$, $1 \leq p < \infty$, such that $\Delta_{\mu,2} = \Delta_\mu$ (cf. [Dav, Theorem 1.4.1]).

PROPOSITION 6.1. *Assume that Ω is connected. Assume that $\Gamma_\mu \neq \emptyset$ and $\mu \neq 0$. Then*

$$\lim_{t\to\infty} \|e^{t\Delta_{\mu,p}} f\|_{L^p(\Omega)} = 0 \tag{23}$$

for all $f \in L^p(\Omega)$ and $1 < p < \infty$.

Proof. a) We show that $a_\mu(u) = 0$ implies that $u = 0$ for all $u \in D(a_\mu)$. In fact, if $a_\mu(u) = 0$, then $\nabla u = 0$, hence u is a constant c since Ω is connected. It follows that $0 = a_\mu(u) = \int_{\Gamma_\mu} |u|^2 \, d\mu = \mu(\Gamma_\mu)c^2$. Thus $c = 0$.

b) Property (23) is true for $p = 2$. This follows from the spectral theorem. In fact the semigroup $(e^{t\Delta_\mu})_{t\geq 0}$ is unitarily equivalent to a semigroup T on $H = L^2(Y, \nu)$ given by $T(t)f = e^{tm} f$ where $m : Y \to [0, \infty)$ is measurable and (Y, ν) is a σ-finite measure space. Via the unitary equivalence the form a_μ becomes the form a on H given by $a(u) = \int_Y |u|^2 m \, d\nu$ with $D(a) = \{u \in H : \int_Y |u|^2 m \, d\nu < \infty\}$, see e.g. [ABHN, Section 7.1]. By a) we have $a(u) = 0$ only if $u = 0$. Thus $m(y) > 0$ ν-a.e. Now it follows from the Dominated Convergence Theorem that $\lim_{t\to\infty} T(t)f = 0$ in $H = L^2(Y, \nu)$.

c) Now the claim (23) follows from the interpolation inequality for arbitrary $1 < p < \infty$ as in [ABB, Proposition 3.1].

□

COROLLARY 6.2. *Let $\Omega \subset \mathbb{R}^n$ be open, and connected of finite Lebesgue measure. Assume that $\Gamma_\mu \neq \emptyset$ and $\mu \neq 0$. Then*

$$\lim_{t\to\infty} \|e^{t\Delta_{\mu,1}} f\|_{L^1(\Omega)} = 0 \tag{24}$$

for all $f \in L^1(\Omega)$.

Proof. Since $L^2(\Omega) \hookrightarrow L^1(\Omega)$, (24) follows from (23) if $f \in L^2(\Omega)$. Since the semigroup $(e^{t\Delta_{\mu,1}})_{t\geq 0}$ is contractive on $L^1(\Omega)$ the claim follows from a density argument. □

If Ω is a bounded, regular open set, then we obtain even exponential stability.

PROPOSITION 6.3. *Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary. Let μ be an admissible measure on Γ. Then $\Delta_{\mu,p}$ has compact resolvent for $1 \leq p < \infty$ and* *the spectrum $\sigma(\Delta_{\mu,p})$ is independent of $p \in [1, \infty)$. Moreover, there exist $c > 0, \omega > 0$ such that*

$$\|e^{t\Delta_{\mu,p}}\|_{\mathcal{L}(L^p(\Omega))} \leq ce^{-\omega t} \qquad (t \geq 0)$$

for all $1 \leq p < \infty$.

Proof. Since Ω has Lipschitz boundary, one has $H^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$ if $n > 2$ and $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$ if $n = 1, 2$. It follows from [Dav, Section 2.4] that $e^{t\Delta_{\mu,1}} L^1(\Omega) \subset L^\infty(\Omega)$ and

$$\|e^{t\Delta_{\mu,1}} f\|_\infty \leq ct^{-n/2}\|f\|_1$$

for all $0 < t \leq 1$, $f \in L^1(\Omega)$. In particular, $e^{t\Delta_{\mu,2}}$ is a Hilbert-Schmidt operator and hence compact. Writing $e^{t\Delta_{\mu,1}} = e^{t/2\Delta_\mu} e^{t/2\Delta_\mu}$ one sees that $e^{t\Delta_{\mu,1}}$ is a compact operator on $L^1(\Omega)$ for $t > 0$. Now spectral p-independence follows from [Dav, Theorem 1.6.4]. It follows from [ArBa, Theorem 1.3] that $0 \notin \sigma(\Delta_{\mu,p})$. Thus $\Delta_{\mu,p}$ has negative spectral bound, which coincides with the growth bound of the semigroup. □

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