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The Laplacian with Robin Boundary Conditions on Arbitrary Domains

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Abstract. Using a capacity approach, we prove in this article that it is always possible to define a realization Δ_μ of the Laplacian on $L^2(\Omega)$ with generalized Robin boundary conditions where Ω is an arbitrary open subset of $\mathbb{R}^n$ and μ is a Borel measure on the boundary $\partial\Omega$ of Ω. This operator Δ_μ generates a sub-Markovian C_0-semigroup on $L^2(\Omega)$. If $\mathrm{d}\mu = \beta\,\mathrm{d}\sigma$ where β is a strictly positive bounded Borel measurable function defined on the boundary $\partial\Omega$ and σ the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$, we show that the semigroup generated by the Laplacian with Robin boundary conditions Δ_β has always Gaussian estimates with modified exponents. We also obtain that the spectrum of the Laplacian with Robin boundary conditions in $L^p(\Omega)$ is independent of $p \in [1, \infty)$. Our approach constitutes an alternative way to Daners who considers the $(n-1)$-dimensional Hausdorff measure on the boundary. In particular, it allows us to construct a conterexample disproving Daners' closability conjecture.

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0. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. Selfadjoint realizations of the Laplacian can be defined on $L^2(\Omega)$ via forms incorporating various boundary conditions.

In particular, there is a natural way to define Dirichlet and Neumann boundary conditions for arbitrary Ω. This is not so obvious for Robin boundary conditions

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where β is a positive bounded Borel measurable function defined on the boundary $\partial\Omega$. If Ω has Lipschitz boundary, in virtue of results on the trace, it can be easily done. One obtains a sub-Markovian semigroup on $L^2(\Omega)$ which allows Gaussian estimates [6]. For the variational formulation one uses the surface measure σ in this

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case. If Ω is an arbitrary bounded open set, then Daners [12] replaced the surface measure by the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ and showed a way how to define a selfadjoint realization of the Laplacian which satisfies in a weak sense the boundary condition (1) on a Borel set $S \subset \partial\Omega$ and Dirichlet boundary conditions on $\partial\Omega \setminus S$. In a certain sense the set S can be chosen maximal. Daners conjectured that $S = \partial\Omega$ if the boundary has finite $(n-1)$-dimensional Hausdorff measure. In this article we disprove Daners' conjecture.

In fact, we develop another approach to define selfadjoint realizations of the Laplacian with general Robin boundary conditions; i.e. boundary conditions defined by arbitrary Borel measures on the boundary. For this we use as a systematic tool, the notion of *relative capacity* (with respect to $\bar{\Omega}$), by which we mean the capacity induced on $\bar{\Omega}$ by the usual Dirichlet form with domain $\widetilde{H}^1(\Omega)$; the completion of $H^1(\Omega) \cap C(\bar{\Omega})$. It is an efficient tool to study the boundary $\partial\Omega$ of Ω. If Ω has Lipschitz boundary, polars for the usual and the relative capacity coincide. But for non-regular boundary there exist relatively polar sets (i.e. sets of relative capacity zero) which are not polar.

Given a measure μ on the boundary $\partial\Omega$ we show that there exists a set $S \subset \partial\Omega$ such that $\partial\Omega \setminus S$ has relative capacity zero such that the Laplacian with generalized Robin boundary conditions defined by μ on S and with Dirichlet boundary conditions on $\partial\Omega \setminus S$ is well-posed.

The Hausdorff measure $\mathcal{H}^{n-1}$ of dimension $n-1$ is of special interest. The corresponding realization of the Laplacian has several interesting properties. In particular, it generates a C_0-semigroup allowing Gaussian estimates with modified exponents. These Gaussian estimates do not have the strong consequences as the classical ones (see, e.g., [1, 6]), but with the help of a recent result of Kunstmann and Vogt [25] we obtain an interesting application. The spectrum of the Laplacian with Robin boundary conditions in $L^p(\Omega)$ is independent of $p \in [1, \infty)$. This question of spectral p-independence has a long history now, see [32, 20, 2] and [23]. If Ω has finite measure, then the Laplacian with Robin boundary conditions has a compact resolvent. This, as well as spectral p-independence, fail for Neumann boundary conditions [24].

All these properties are valid if we consider the $(n-1)$-dimensional Hausdorff measure. If the Hausdorff dimension s of the boundary is bigger than $n-1$, then the s-dimensional Hausdorff measure is more natural to define Robin boundary conditions. An interesting example of this kind is the snowflake of von Koch.

1. Relative Capacity

The aim of this section is to investigate the notion of "relative capacity" with respect to an open subset Ω of $\mathbb{R}^n$. It will allow us to analyse phenomena occuring on the boundary of Ω. Throughout this paper the underlying field is $\mathbb{R}$.

Given an arbitrary subset A of $\mathbb{R}^n$ the *capacity* $\mathrm{Cap}(A)$ of A is defined as

$$\mathrm{Cap}(A) := \inf\big\{\|u\|^2_{H^1(\mathbb{R}^n)} : u \in H^1(\mathbb{R}^n),\ \exists O \subset \mathbb{R}^n \text{ open such that } A \subset O \text{ and } u(x) \geqslant 1 \text{ a.e. on } O\big\}. \tag{2}$$

Here, for an open non-empty subset Ω of $\mathbb{R}^n$, we consider the Sobolev space

$$H^1(\Omega) := \{u \in L^2(\Omega) : D_j u \in L^2(\Omega),\ j = 1, \dots, n\}$$

with norm

$$\|u\|^2_{H^1(\Omega)} := \|u\|^2_{L^2(\Omega)} + \sum_{j=1}^n \|D_j u\|^2_{L^2(\Omega)},$$

where $D_j u = \partial u/\partial x_j$ is the distributional derivative. Moreover, we let

$$\widetilde{H}^1(\Omega) = \overline{H^1(\Omega) \cap C(\bar{\Omega})}^{H^1(\Omega)},$$

where $C(\bar{\Omega})$ denotes the space of all continuous real-valued functions on $\bar{\Omega}$.

For example, if $\Omega = (0, 1) \cup (1, 2)$, then $H^1(\Omega) \neq \widetilde{H}^1(\Omega) = H^1(0, 2)$. For conditions implying that $\widetilde{H}^1(\Omega) = H^1(\Omega)$ see [27, Section 1.1.6, Theorem 2] or [14, Chap. V, Theorem 4.7]. In paricular, if Ω has a Lipschitz boundary, then $\widetilde{H}^1(\Omega) = H^1(\Omega)$. But the point in the present paper is to consider arbitrary open sets with possibly bad boundary.

Now we fix an open set Ω in $\mathbb{R}^n$. The *relative capacity* $\mathrm{Cap}_{\bar{\Omega}}(A)$ with respect to Ω is defined for an arbitrary subset A of $\bar{\Omega}$ by

$$\mathrm{Cap}_{\bar{\Omega}}(A) := \inf\big\{\|u\|^2_{H^1(\Omega)} : u \in \widetilde{H}^1(\Omega),\ \exists O \subset \mathbb{R}^n \text{ open such that } A \subset O \text{ and } u(x) \geqslant 1 \text{ a.e. on } \Omega \cap O\big\}. \tag{3}$$

Here and further on, the word "relative" means relative with respect to the fixed open set Ω.

Our notion of relative capacity is a special case of the capacity associated with a Dirichlet form. We consider the topological space $X = \bar{\Omega}$, the σ-algebra $\mathcal{B}(X)$ of all Borel sets in X, and the measure m on $\mathcal{B}(X)$ given by $m(A) = \lambda(A \cap \Omega)$ for all $A \in \mathcal{B}(X)$ with λ the Lebesgue measure. Denoting by $L^2(\Omega)$ the usual L^2-space with respect to the Lebesgue measure, we then have $L^2(\Omega) = L^2(X, \mathcal{B}(X), m)$. The introduction of m is needed to ensure this identity in the case where $\partial\Omega$ has positive Lebesgue measure. Now we consider the Dirichlet form $(\mathcal{E}, \mathbb{D})$ on $L^2(X, \mathcal{B}(X), m)$ given by $\mathbb{D} = \widetilde{H}^1(\Omega)$ and

$$\mathcal{E}(u, v) = \int_\Omega \nabla u \nabla v \,\mathrm{d}x.$$

It satisfies the assumption (T) of [9, p. 52] and (D) of [9, p. 54]. The relative capacity of a subset A of $\bar{\Omega}$ is exactly the capacity of A associated with the Dirichlet form

$(\mathcal{E}, \mathbb{D})$ in the sense of [9, 8.1.1, p. 52]. Open subsets of $X = \bar{\Omega}$ are understood with respect to the relative topology of $\bar{\Omega}$. Thus $\mathrm{Cap}_{\bar{\Omega}}$ (just as Cap) has the properties of a capacity as described in [9, I.8] (or [17, Chapter 2]). In particular, $\mathrm{Cap}_{\bar{\Omega}}$ is an *outer measure*; i.e.

$$\mathrm{Cap}_{\bar{\Omega}}(\emptyset) = 0,$$

and

$$\mathrm{Cap}_{\bar{\Omega}}\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mathrm{Cap}_{\bar{\Omega}}(A_n)$$

whenever $A_n \subset \bar{\Omega}$. Moreover, for $A \subset B \subset \bar{\Omega}$ one has

$$\mathrm{Cap}_{\bar{\Omega}}(A) \leqslant \mathrm{Cap}_{\bar{\Omega}}(B).$$

A subset A of $\mathbb{R}^n$ is called a *polar set* if $\mathrm{Cap}(A) = 0$. Similarly, a subset A of $\bar{\Omega}$ is called a *relatively polar set* if $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$.

This, as several subsequent notions are defined with respect to the fixed open set Ω.

We say that a property holds on $\bar{\Omega}$ *relatively quasi-everywhere* (r.q.e.), if it holds for all $x \in (\bar{\Omega} \setminus N)$ where $N \subset \bar{\Omega}$ is relatively polar.

Our main point in this section is to compare polar and relatively polar sets. First we note that

$$\mathrm{Cap}_{\bar{\Omega}}(A) \leqslant \mathrm{Cap}(A) \tag{4}$$

for all $A \subset \bar{\Omega}$. In fact, since $C(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ one has $u_{|\Omega} \in \widetilde{H}^1(\Omega)$ for all $u \in H^1(\mathbb{R}^n)$. Now (4) follows from the definition.

It is clear from (4) that each polar subset of $\bar{\Omega}$ is also relatively polar. The converse is true for subsets of Ω.

PROPOSITION 1.1. *Let* $A \subset \Omega$. *Then* $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$ *if and only if* $\mathrm{Cap}(A) = 0$.

Proof. First case. There exists an open bounded set ω such that $A \subset \bar{\omega} \subset \Omega$. Since $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$ there exist open sets $O_k \subset \mathbb{R}^n$, $u_k(x) \geqslant 1$ on $O_k \cap \Omega$ and $\|u_k\|^2_{H^1(\Omega)} \leqslant 1/k$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\mathrm{supp}[\varphi] \subset \Omega$ and $\varphi = 1$ on ω. Let $v_k = \varphi u_k$ on Ω and $v_k = 0$ on $\mathbb{R}^n \setminus \Omega$. Then $v_k \in H^1(\mathbb{R}^n)$, $v_k = 1$ on $O_k \cap \omega$ and $\|v_k\|_{H^1(\mathbb{R}^n)} \to 0$ $(k \to \infty)$. Thus $\mathrm{Cap}(A) = 0$.

Second case. Assume that $A \subset \Omega$ is arbitrary. Take open bounded sets ω_k such that $\bar{\omega}_k \subset \omega_{k+1} \subset \Omega$ and $\bigcup_{k \in \mathbb{N}} \omega_k = \Omega$. It follows from the first case that $\mathrm{Cap}(A \cap \omega_k) = 0$. Hence $\mathrm{Cap}(A) = \lim_{k\to\infty} \mathrm{Cap}(A \cap \omega_k) = 0$ by [9, I, Proposition 8.1.3 c)]. □

Thus, by the preceding proposition, merely subsets of $\partial\Omega$ are of interest for our question. And indeed, we will show below that there may exist relatively polar sets in $\partial\Omega$ which are not polar. However, for this the boundary has to be irregular. In fact, our next proposition shows that both notions of polar sets coincide if the boundary is Lipschitz continuous.

DEFINITION 1.2. We say that $\widetilde{H}^1(\Omega)$ (resp. $H^1(\Omega)$) has the *extension property* if there exists a bounded linear operator P from $\widetilde{H}^1(\Omega)$ (resp. $H^1(\Omega)$) into $H^1(\mathbb{R}^n)$ such that $(Pu)_{|\Omega} = u$ for all $u \in \widetilde{H}^1(\Omega)$ (resp. $u \in H^1(\Omega)$).

If P can be chosen such that $Pu \in C(\mathbb{R}^n)$ whenever $u = \varphi_{|\Omega}$ for some $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then we say that $\widetilde{H}^1(\Omega)$ (resp. $H^1(\Omega)$) has the *special extension property*.

If $H^1(\Omega)$ has the extension property, then $H^1(\Omega) = \widetilde{H}^1(\Omega)$. However, consider $\Omega = (0,1) \cup (1,2) \subset \mathbb{R}$. Then $\widetilde{H}^1(\Omega) = H^1(0,2)$ has the extension property, but $H^1(\Omega) \neq \widetilde{H}^1(\Omega)$.

Since $H^1(\mathbb{R}) \subset C(\mathbb{R})$, in dimension 1, the extension property implies the special extension property for $H^1(\Omega)$ and $\widetilde{H}^1(\Omega)$ for all open sets $\Omega \subset \mathbb{R}$.

PROPOSITION 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then $H^1(\Omega) = \widetilde{H}^1(\Omega)$ has the special extension property.*

Proof. There exist bounded linear operators $P_p : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that $P_p u = P_q u$ for all $u \in W^{1,p}(\Omega) \cap W^{1,q}(\Omega)$ and $P_p u_{|\Omega} = u$ for all $u \in W^{1,p}(\Omega)$ and $1 \leqslant p \leqslant \infty$. Since $W^{1,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ for $p > n$, it follows that $P_2 u = P_p u \in C(\mathbb{R}^n)$ whenever $u = \varphi_{|\Omega}$ for some $\varphi \in \mathcal{D}(\mathbb{R}^n)$. □

PROPOSITION 1.4. *Assume that $\widetilde{H}^1(\Omega)$ has the special extension property. Then there exists a constant $c > 0$ such that*

$$\mathrm{Cap}(A) \leqslant c\,\mathrm{Cap}_{\bar{\Omega}}(A) \quad \textit{for all } A \subset \bar{\Omega}. \tag{5}$$

In particular, $\mathrm{Cap}(A) = 0$ *if and only if* $\mathrm{Cap}_{\bar{\Omega}}(A) = 0$ *for all* $A \subset \bar{\Omega}$.

Proof. Denote by P the extension operator from Definition 1.2. We show (5) for $c = \|P\|^2$.

(1) Let $A \subset \bar{\Omega}$ be a compact set. Let $\varepsilon > 0$. The space $\mathcal{D} := \{u_{|\Omega} : u \in \mathcal{D}(\mathbb{R}^n)\}$ is a special core of $\widetilde{H}^1(\Omega)$ in the sense of [17, p. 6]. By [17, Lemma 2.2.7, p. 80] there exists $u \in \mathcal{D}$ such that $u(x) \geqslant 1$ for all $x \in A$ and $\|u\|^2_{H^1(\Omega)} \leqslant \mathrm{Cap}_{\bar{\Omega}}(A) + \varepsilon$. Let $v = Pu$. Then $v \in H^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $u = v$ on Ω. Hence $u = v$ on $\bar{\Omega}$ by continuity. Thus $v(x) \geqslant 1$ on A. Hence by [17, Theorem 2.1.5],

$$\mathrm{Cap}(A) \leqslant \|v\|^2_{H^1(\mathbb{R}^n)} \leqslant \|P\|^2 \|u\|^2_{H^1(\Omega)} \leqslant \|P\|^2 (\mathrm{Cap}_{\bar{\Omega}}(A) + \varepsilon).$$

Letting $\varepsilon \downarrow 0$ one obtains (5).

(2) Let $A \subset \bar{\Omega}$ be relatively open. There exists $O \subset \mathbb{R}^n$ open such that $A = O \cap \bar{\Omega}$. By Choquet's theorem [17, Theorem A.1.1] and (1) we have

$$\begin{aligned}\mathrm{Cap}(A) &= \sup\{\mathrm{Cap}(K) : K \subset O \cap \bar{\Omega} \text{ compact}\}\\ &\leqslant \|P\|^2 \sup\{\mathrm{Cap}_{\bar{\Omega}}(K) : K \subset O \cap \bar{\Omega} \text{ compact}\}\\ &\leqslant \|P\|^2\,\mathrm{Cap}_{\bar{\Omega}}(O \cap \bar{\Omega}).\end{aligned}$$

(3) Let $A \subset \bar{\Omega}$ be arbitrary. Since by definition,

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(O) : A \subset O \subset \mathbb{R}^n,\ O \text{ open}\},$$

it follows from (2) that

$$\begin{aligned}\mathrm{Cap}(A) &= \inf\{\mathrm{Cap}(O \cap \bar{\Omega}) : A \subset O \subset \mathbb{R}^n,\ O \text{ open}\}\\ &\leqslant \|P\|^2 \inf\{\mathrm{Cap}_{\bar{\Omega}}(O \cap \bar{\Omega}) : A \subset O \subset \mathbb{R}^n,\ O \text{ open}\}\\ &= \|P\|^2 \inf\{\mathrm{Cap}_{\bar{\Omega}}(U) : A \subset U \subset \bar{\Omega},\ U \text{ relatively open}\}\\ &= \|P\|^2\, \mathrm{Cap}_{\bar{\Omega}}(A). \qquad \square\end{aligned}$$

We first produce a one-dimensional example of a relatively polar set which is not polar. Note that a subset A of $\mathbb{R}$ is polar if and only if it is empty.

EXAMPLE 1.5. Let $0 < a_{n+1} < b_{n+1} < a_n < 1$ ($n \in \mathbb{N}$) such that $\lim_{n\to\infty} a_n = 0$, and $\Omega = (0,1) \setminus \bigcup_{n\in\mathbb{N}}[a_n, b_n]$. Then $0 \in \partial\Omega$ and $\mathrm{Cap}_{\bar{\Omega}}(\{0\}) = 0$ whereas $\mathrm{Cap}(\{0\}) > 0$. In fact, the characteristic function $u_n = 1_{[0,a_n]}$ of $[0, a_n]$ is in $\widetilde{H}^1(\Omega)$ and $u_n' = 0$. Since $u_n(x) \geqslant 1$ on $(0, a_n)$ one has

$$\mathrm{Cap}_{\bar{\Omega}}(\{0\}) \leqslant \|u_n\|^2_{H^1(\Omega)} = \|u_n\|^2_{L^2(\Omega)} = \|u_n\|^2_{L^2(0,a_n)} \to 0 \quad (n \to \infty).$$

Next we modify Example 1.5 in order to produce a connected bounded open set Ω in $\mathbb{R}^2$ and a closed subset of $\partial\Omega$ which is relatively polar but not polar. Note that if $A \subset \mathbb{R}^2$ is a polar set then it is totally disconnected; that is, every component of A is a singleton (see [8, Corollary 5.8.9, p. 155]).

EXAMPLE 1.6. *Let* $0 < a_{n+1} < b_{n+1} < a_n < 1$ ($n \in \mathbb{N}$) *such that* $\lim_{n\to\infty} a_n = 0$. *Let*

$$\Omega = \left\{(x, y) \in (0,1) \times (0,1) \setminus \bigcup_{n\in\mathbb{N}} [a_n, b_n] \times \left[\frac{1}{2}, 1\right]\right\}.$$

Then the segment $S = \{0\} \times [3/4, 1]$ *has relative capacity* $\mathrm{Cap}_{\bar{\Omega}}(S) = 0$ *but* $\mathrm{Cap}(S) > 0$.

Proof. Let $\varphi \in C^\infty[0,1]$ such that $\mathrm{supp}[\varphi] \subset [1/4, 1]$ and $\varphi(y) \geqslant 1$ for $y \in [1/2, 1]$. Let

$$u_n(x, y) := \begin{cases} \varphi(y) & \text{if } x < a_n,\\ 0 & \text{if } x \geqslant a_n.\end{cases}$$

Then $u_n \in C^\infty(\bar{\Omega})$ and $u_n \geqslant 1$ on $\Omega \cap O$ where $O = (-\infty, a_n) \times (\frac{1}{2}, \infty)$. Moreover,

$$\begin{aligned}\int_\Omega |\nabla u_n|^2\, \mathrm{d}x\, \mathrm{d}y &= \int_0^{a_n} \int_0^1 |\varphi'(y)|^2\, \mathrm{d}y\, \mathrm{d}x\\ &= a_n \int_0^1 |\varphi'(y)|^2\, \mathrm{d}y \to 0 \quad (n \to \infty)\end{aligned}$$

and

$$\int_{\Omega} |u_n|^2 \,\mathrm{d}x\,\mathrm{d}y \to 0 \quad (n \to \infty).$$

Since $S \subset O$, it follows that $\mathrm{Cap}_{\bar{\Omega}}(S) = 0$. □

2. $H_0^1(\Omega)$ in Terms of Relative Capacity

In the preceding section we produced an example of an entire segment S contained in the boundary $\partial\Omega$ of a bounded open connected set $\Omega \subset \mathbb{R}^2$ with relative capacity $\mathrm{Cap}_{\bar{\Omega}}(S) = 0$. We will show here that on the other hand $\mathrm{Cap}_{\bar{\Omega}}(\partial\Omega) > 0$ for all bounded open sets Ω in $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We start to describe the space

$$H_0^1(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$$

with the help of relative capacity. Here $\mathcal{D}(\Omega)$ denotes the space of all infinitely differentiable functions with compact support. We denote by $C_c(\bar{\Omega})$ those functions in $C(\bar{\Omega})$ which have a compact support in $\bar{\Omega}$. Thus $C_c(\bar{\Omega}) = C(\bar{\Omega})$ if Ω is bounded.

LEMMA 2.1. *The space* $\widetilde{H}^1(\Omega) \cap C_c(\bar{\Omega})$ *is dense in* $\widetilde{H}^1(\Omega)$.

Proof. Let $\xi \in \mathcal{D}(\mathbb{R}^n)$ such that $\xi(x) = 1$ for $|x| \leqslant 1$. Let $\xi_n(x) = \xi(x/n)$. If $u \in H^1(\Omega) \cap C(\bar{\Omega})$, then $\xi_n u \in H^1(\Omega) \cap C_c(\bar{\Omega})$ and $\xi_n u \to u$ $(n \to \infty)$ in $H^1(\Omega)$. Since $\widetilde{H}^1(\Omega)$ is the closure of $H^1(\Omega) \cap C(\bar{\Omega})$ the claim follows. □

Thus, the Dirichlet form $(\mathcal{E}, \mathbb{D})$ on $L^2(\bar{\Omega}, m)$ introduced in Section 1 is *regular*; i.e., $\mathbb{D} \cap C_c(\bar{\Omega})$ is dense in $(\mathbb{D}, \|\cdot\|_{\mathcal{E}})$, and also in $(C_c(\bar{\Omega}), \|\cdot\|_\infty)$ by the Stone–Weierstrass theorem. Hence we can apply the usual results on quasi-continuity.

DEFINITION 2.2. A scalar function u on $\bar{\Omega}$ is called *relatively quasi-continuous*, if for each $\varepsilon > 0$ there exists an open set $G \subset \mathbb{R}^n$ such that $\mathrm{Cap}_{\bar{\Omega}}(G \cap \bar{\Omega}) < \varepsilon$ and u is continuous on $\bar{\Omega} \setminus G$.

Recall that a scalar function u defined on $\mathbb{R}^n$ is called *quasi-continuous* if for each $\varepsilon > 0$ there exists an open set $G \subset \mathbb{R}^n$ such that $\mathrm{Cap}(G) < \varepsilon$ and u is continuous on $\mathbb{R}^n \setminus G$. It follows from (4) that then $u_{|\bar{\Omega}}$ is also relatively quasi-continuous.

Now, it follows from [9, I, Proposition 8.2.1] that for each $u \in \widetilde{H}^1(\Omega)$ there exists a relatively quasi-continuous function $\tilde{u} : \bar{\Omega} \to \mathbb{R}$ such that $\tilde{u}(x) = u(x)$ m-a.e. This function is unique relatively quasi-everywhere. We call $\tilde{u}$ the *relatively quasi-continuous representative* of u. It can be seen from the proof of [9, I, Proposition 8.2.1] that $\tilde{u}$ can be chosen Borel measurable. Now we can describe $H_0^1(\Omega)$ as a subspace of $\widetilde{H}^1(\Omega)$ in the following way.

THEOREM 2.3. *One has*

$$H_0^1(\Omega) = \{u \in \widetilde{H}^1(\Omega) : \tilde{u}(x) = 0 \text{ r.q.e. on } \partial\Omega\}.$$

Proof. $H_0^1(\Omega)$ is a closed ideal of $\widetilde{H}^1(\Omega)$ (see [7]). We apply a result of Stollmann [33, Theorem 1.1]. There exists a Borel set $M \subset \bar{\Omega}$ such that

$$H_0^1(\Omega) = \{u \in \widetilde{H}^1(\Omega) : \tilde{u}(x) = 0 \text{ r.q.e. on } M\}. \tag{6}$$

Since $\mathcal{D}(\Omega) \subset H_0^1(\Omega)$ it follows that $M \cap \Omega$ is relatively polar. In fact, let $K_n \subset \Omega$ be compact sets such that $K_n \subset K_{n+1}$ and $\Omega = \bigcup_{n\in\mathbb{N}} K_n$. For each $n \in \mathbb{N}$ there exists $\varphi_n \in \mathcal{D}(\Omega)$ such that $\varphi_n(x) = 1$ on K_n. Since $\varphi_n \in H_0^1(\Omega)$ it follows that $\varphi_n(x) = 0$ r.q.e. on M. Thus $K_n \cap M$ is relatively polar. Hence $\mathrm{Cap}_{\bar{\Omega}}(M \cap \Omega) = \lim_{n\to\infty} \mathrm{Cap}_{\bar{\Omega}}(M \cap K_n) = 0$.

Now let $u \in \widetilde{H}^1(\Omega)$ such that $\tilde{u}(x) = 0$ r.q.e. on $\partial\Omega$. Then $\tilde{u}(x) = 0$ r.q.e. on M and hence $\tilde{u} \in H_0^1(\Omega)$. To prove the converse, observe that the set $\{u \in \widetilde{H}^1(\Omega) : \tilde{u}(x) = 0 \text{ r.q.e. on } \partial\Omega\}$ is a closed ideal of $\widetilde{H}^1(\Omega)$ containing $\mathcal{D}(\Omega)$. Hence it also contains $H_0^1(\Omega)$. □

One should compare Theorem 2.3 with the following known result (which will be needed in the proof of Proposition 2.5). It allows one to identify $H_0^1(\Omega)$ with a subspace of $H^1(\mathbb{R}^n)$.

PROPOSITION 2.4. *One has*

$$H_0^1(\Omega) = \{u_{|\Omega} : u \in H^1(\mathbb{R}^n) : \tilde{u}(x) = 0 \text{ q.e. on } \mathbb{R}^n \setminus \Omega\}.$$

Proof. See [7, Theorem 1.1] or [17, Example 3.2.2, p. 81] or [18, Theorem 3.1, p. 241]. □

PROPOSITION 2.5. *Let $\Omega \subset \mathbb{R}^n$ be open. Then the following assertions are equivalent.*

(i) $\mathrm{Cap}_{\bar{\Omega}}(\partial\Omega) = 0$;
(ii) $\widetilde{H}^1(\Omega) = H_0^1(\Omega)$;
(iii) $\Omega = \mathbb{R}^n \setminus K$ *where* $K \subset \mathbb{R}^n$ *is closed and* $\mathrm{Cap}(K) = 0$;
(iv) $H_0^1(\Omega) = \{u_{|\Omega} : u \in H^1(\mathbb{R}^n)\}$.

Proof. (i) ⇔ (ii) This follows from Theorem 2.3.

(ii) ⇒ (iii) Assume that $\widetilde{H}^1(\Omega) = H_0^1(\Omega)$. For each $n \in \mathbb{N}$ there exists a function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\varphi(x) = 1$ on $K_n = \{z \in \partial\Omega : |z| \leqslant n\}$. Since $\widetilde{H}^1(\Omega) = H_0^1(\Omega)$, it follows from Theorem 2.3 that $\varphi(x) = 0$ q.e. on K_n. Thus $\mathrm{Cap}(\partial\Omega) = \lim_{n\to\infty} \mathrm{Cap}(K_n) = 0$. Now it follows from [3, Proposition 3.10] that $\mathbb{R}^n \setminus \partial\Omega$ is connected. Since $\mathbb{R}^n \setminus \partial\Omega = \Omega \cup (\mathbb{R}^n \setminus \bar{\Omega})$ this implies that $\bar{\Omega} = \mathbb{R}^n$. Taking $K = \partial\Omega$ we obtain (iii).

(iii) ⇒ (iv) This follows from Proposition 2.4 since $\mathrm{Cap}_{\bar{\Omega}}(\partial\Omega) = \mathrm{Cap}_{\bar{\Omega}}(K) \leqslant \mathrm{Cap}(K) = 0$.

(iv) $\Rightarrow$ (i) Assume that $\mathrm{Cap}_{\bar{\Omega}}(\partial\Omega) > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\mathrm{Cap}_{\bar{\Omega}}(\{x \in \partial\Omega : |x| \leqslant n_0\}) > 0$. Choose $u \in \mathcal{D}(\mathbb{R}^n)$ such that $u(x) = 1$ if $|x| \leqslant n_0$. Then $u \notin H_0^1(\Omega)$ by Theorem 2.3. Thus (iv) does not hold. □

COROLLARY 2.6. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set. Then* $\mathrm{Cap}_{\bar{\Omega}}(\partial\Omega) > 0$.

3. Robin Boundary Condition for Arbitrary Measures

In this section we define boundary conditions by Borel measures on the boundary of an open set. We will use the theory of quadratic forms (see [13, Chapter 1] and [29]).

Let H be a real Hilbert space. A *positive form* on H is a bilinear mapping $a : D(a) \times D(a) \to \mathbb{R}$ such that $a(u, v) = a(v, u)$ and $a(u, u) \geqslant 0$ for all $u, v \in D(a)$ where $D(a)$ (the *domain* of the form) is a dense subspace of H. The form is *closed* if $D(a)$ is complete for the norm $\|u\|_a = (a(u, u) + \|u\|_H^2)^{1/2}$. Then *the operator* A *on* H *associated with* a is defined by

$$\begin{cases} D(A) := \{u \in D(a) : \exists v \in H \ a(u, \varphi) = (v, \varphi)_H \ \forall \varphi \in D(a)\}, \\ Au = v. \end{cases} \tag{7}$$

The operator A is selfadjoint and $-A$ generates a contraction semigroup $(\mathrm{e}^{-tA})_{t\geqslant 0}$ of symmetric operators on H (and each symmetric contraction semigroup occurs in this way).

More generally, the form a is called *closable* if for each Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $(D(a), \|\cdot\|_a)$

$$\lim_{n\to\infty} u_n = 0 \quad \text{in } H \quad \text{implies} \quad \lim_{n\to\infty} a(u_n, u_n) = 0. \tag{8}$$

In that case the *closure* $\bar{a}$ is the unique positive closed form extending a such that $D(a)$ is dense in $D(\bar{a})$. We will give several examples of non-closable forms below.

Given two positive forms a and b we write here $a \leqslant b$ if $D(b) \subset D(a)$ and $a(u) \leqslant b(u)$ for all $u \in D(b)$ where $a(u) = a(u, u)$. Thus if a is closable, then $\bar{a} \leqslant a$.

Given a positive form a there always exists a closable positive form $a_r \leqslant a$ such that $b \leqslant a_r$ whenever b is a closable positive form such that $b \leqslant a$. Thus a_r is the largest closable form smaller or equal than a (see [29, Theorem S15, p. 373]). Clearly, a is closable if and only if $a = a_r$.

Note that these properties imply that, given an arbitrary positive form, one always has

$$D(a_r) = D(a). \tag{9}$$

In fact, $D(a) \subset D(a_r)$ by definition. Let b be the restriction of a_r to $D(a) \times D(a)$. Then b is closable and $b(u) = a_r(u) \leqslant a(u)$ for all $u \in D(b)$. Thus $b \leqslant a$.

Hence $b \leqslant a_r$ and in particular, $D(a_r) \subset D(b)$. However, it can happen though that $a_r(u) < a(u)$ whenever $u \neq 0$.

Let $\Omega \subset \mathbb{R}^n$ be an open set with boundary $\Gamma = \partial\Omega$. We consider positive forms on $L^2(\Omega)$ (formed with respect to Lebesgue measure). Denote by $\mathcal{B}(\Gamma)$ the Borel σ-algebra on Γ and let $\mu : \mathcal{B}(\Gamma) \to [0, \infty]$ be a measure. We consider the positive form a_μ on $L^2(\Omega)$ given by

$$a_\mu(u, v) = \int_\Omega \nabla u \nabla v \,\mathrm{d}x + \int_\Gamma uv \,\mathrm{d}\mu \tag{10}$$

with domain

$$D(a_\mu) = \left\{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \int_\Gamma |u|^2 \,\mathrm{d}\mu < \infty\right\}.$$

If X is a locally compact space, $C_c(X)$ denotes the space of all continuous real-valued functions on X with compact support.

We want to determine $(a_\mu)_r$ and in particular characterize when a_μ is closable.

Let

$$\Gamma_\mu := \{z \in \Gamma : \exists\, r > 0 \text{ such that } \mu(\Gamma \cap B(z, r)) < \infty\}$$

be the part of Γ on which μ is locally finite. Then $u = 0$ on $\Gamma\backslash\Gamma_\mu$ for all $u \in D(a_\mu)$.

We consider two special cases.

EXAMPLE 3.1 (Dirichlet Laplacian). Assume that $\Gamma_\mu = \emptyset$. Then $u = 0$ on Γ for all $u \in D(a_\mu)$. It follows from [10, Théorème IX.17, p. 171] that $D(a_\mu) \subset H_0^1(\Omega)$ and $a_\mu(u, v) = \int_\Omega \nabla u \nabla v \,\mathrm{d}x$. Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$ by definition, a_μ is closable and $D(\overline{a_\mu}) = H_0^1(\Omega)$, $\overline{a_\mu}(u, v) = \int_\Omega \nabla u \nabla v \,\mathrm{d}x$. Let A be the operator associated with $\overline{a_\mu}$. It is easy to see that $D(A) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$, $Au = -\Delta u$. We call $\Delta^D := -A$ the Dirichlet Laplacian.

EXAMPLE 3.2 (Neumann Laplacian). Let $\mu = 0$. Then $a_0 := a_\mu$ is closable, $D(\overline{a_0}) = \widetilde{H}^1(\Omega)$, $\overline{a_0}(u, u) = \int_\Omega \nabla u \nabla v \,\mathrm{d}x$. Let A_0 be the operator associated with $\overline{a_0}$. We call $\Delta^N := -A_0$ the Neumann Laplacian (cf. [13, Theorem 1.2.10]).

In the following we consider the case where $\Gamma_\mu \neq \emptyset$. Then Γ_μ is relatively open in Γ. Thus Γ_μ is a locally compact space and μ is a regular Borel measure on Γ_μ (by [30, 2.18, p. 48]). We let $L^2(\Gamma_\mu) = L^2(\Gamma_\mu, \mathcal{B}(\Gamma_\mu), \mu)$.

We say that the measure μ is *admissible* if for each Borel set $A \subset \Gamma_\mu$ one has

$$\mathrm{Cap}_{\bar{\Omega}}(A) = 0 \Longrightarrow \mu(A) = 0. \tag{11}$$

Then the following holds.

THEOREM 3.3. *The form a_μ is closable if and only if μ is admissible.*

The proof of Theorem 3.3 will be given later. In order to determine the closable part $(a_\mu)_r$, it will be natural to restrict μ to subsets of Γ_μ. For each Borel set S in Γ_μ we define the form a_S on $L^2(\Omega)$ with domain $D(a_S) = D(a_\mu)$ by

$$a_S(u, v) = \int_\Omega \nabla u \nabla v \,\mathrm{d}x + \int_S uv \,\mathrm{d}\mu. \tag{12}$$

First we establish a uniqueness result.

PROPOSITION 3.4. *Let $S_1, S_2 \subset \Gamma_\mu$ be two Borel sets. Then $a_{S_1} = a_{S_2}$ if and only if $\mu(S_1 \triangle S_2) = 0$.*

Proof. Assume that $a_{S_1} = a_{S_2}$. Then $\int_{S_1 \triangle S_2} |u|^2 \,\mathrm{d}\mu = 0$ for all $u \in D(a_\mu)$. Assume that $\mu(S_1 \triangle S_2) > 0$. Then there exists a compact set $K \subset S_1 \triangle S_2$ such that $\mu(K) > 0$. Let $O \subset \mathbb{R}^n$ such that $\Gamma_\mu = \Gamma \cap O$. Take $u \in \mathcal{D}(\mathbb{R}^n)$ such that $\mathrm{supp}[u] \subset O$ and $u = 1$ on K. Then $u_{|\Omega} \in D(a_\mu)$ and $\mu(K) \leqslant \int_{S_1 \triangle S_2} |u|^2 \,\mathrm{d}\mu = 0$, a contradiction. We have shown that $\mu(S_1 \triangle S_2) = 0$. The other implication is obvious. □

A set $S \subset \Gamma_\mu$ is called *μ-admissible* if S is a Borel set and

$$\mathrm{Cap}_{\bar{\Omega}}(A) = 0 \quad \text{implies} \quad \mu(A \cap S) = 0 \tag{13}$$

for each Borel set $A \subset \Gamma_\mu$. Thus μ is admissible if and only if Γ_μ is μ-admissible.

Note that the countable union of admissible sets is admissible. Moreover, if S_1 and S_2 are Borel sets and $\mu(S_1 \triangle S_2) = 0$, then S_1 is admissible if and only if S_2 is admissible. The following assertion gives one implication of Theorem 3.3.

PROPOSITION 3.5. *If $S \subset \Gamma_\mu$ is μ-admissible, then a_S is closable.*

Proof. Let (u_k) be a Cauchy sequence in $D(a_\mu)$ such that $u_k \to 0$ $(k \to \infty)$ in $L^2(\Omega)$. Then $\lim_{k\to\infty} u_k = 0$ in $H^1(\Omega)$ (since a_0 is closable). Taking a subsequence if necessary we can assume that u_k converges to 0 r.q.e. The assumption implies that $(u_k \chi_S)$ converges to a function f in $L^2(S, \mu)$. Hence $f = 0$ r.q.e. Since S is admissible, this implies that $f = 0$ μ-a.e. This shows that $\lim_{k\to\infty} u_k = 0$ in $D(a_\mu)$. □

The following proposition is similar to the Hahn decomposition theorem and similar to a result of Stollmann and Voigt [34, Proposition 1.1]. In Section 5 it will be used to describe the closable part of the surface measure for some special open set.

PROPOSITION 3.6. *There exists a μ-admissible set $S \subset \Gamma_\mu$ such that $\mathrm{Cap}_{\bar{\Omega}}(\Gamma_\mu \setminus S) = 0$.*

Proof. (a) Let $K \subset \Gamma_\mu$ be a compact set. Then there exists a μ-admissible set $S \subset K$ such that $\mathrm{Cap}_{\bar{\Omega}}(K \setminus S) = 0$. In fact, let

$$\alpha := \sup\{\mu(A) : A \subset K \text{ Borel set } \mathrm{Cap}_{\bar{\Omega}}(A) = 0\}.$$

Then $\alpha \leqslant \mu(K) < \infty$. Let $A_m \subset K$ be Borel sets such that $\mathrm{Cap}_{\bar\Omega}(A_m) = 0$ and $\mu(A_m) \to \alpha$ $(m \to \infty)$. Let $N = \bigcup_{m\in\mathbb{N}} A_m$. Then $\mathrm{Cap}_{\bar\Omega}(N) = 0$ and $\mu(N) = \alpha$. We show that $\mu(B \setminus N) = 0$ if $B \subset K$ is a Borel set such that $\mathrm{Cap}_{\bar\Omega}(B) = 0$. Assume that $\mu(B \setminus N) > 0$. Let $A = B \cup N$. Then $\mathrm{Cap}_{\bar\Omega}(A) = 0$ but $\mu(A) = \mu(B \setminus N) + \mu(N) > \alpha$ contradicting the definition of α. Now let $S = K \setminus N$. Then S is μ-admissible and $\mathrm{Cap}_{\bar\Omega}(K \setminus S) = 0$.

(b) Let $K_n \subset \Gamma_\mu$ be compact sets such that $K_n \subset K_{n+1}$ and $\bigcup_{n\in\mathbb{N}} K_n = \Gamma_\mu$ (one may take $K_n = \{z \in \Gamma : |z| \leqslant n\}$ if $\Gamma = \Gamma_\mu$ and $K_n = \{z \in \Gamma_\mu : \mathrm{dist}(z, \Gamma \setminus \Gamma_\mu) \geqslant 1/n,\ |z| \leqslant n\}$ if $\Gamma \neq \Gamma_\mu$). By (a) there exist μ-admissible sets $S_n \subset K_n$ such that $\mathrm{Cap}_{\bar\Omega}(K_n \setminus S_n) = 0$. Then $S := \bigcup_{n\in\mathbb{N}} S_n$ is μ-admissible and

$$\mathrm{Cap}_{\bar\Omega}(\Gamma_\mu \setminus S) = \lim_{n\to\infty} \mathrm{Cap}_{\bar\Omega}(K_n \setminus S) \leqslant \limsup_{n\to\infty} \mathrm{Cap}_{\bar\Omega}(K_n \setminus S_n) = 0. \qquad \square$$

We now show that the μ-admissible set S of Proposition 3.6 yields the closable part of a_μ. This is similar to Stollmann and Voigt [34, Proposition 1.1]. The proof we give here will also be needed in Section 5.

THEOREM 3.7. *Let $S \subset \Gamma_\mu$ be μ-admissible such that* $\mathrm{Cap}_{\bar\Omega}(\Gamma_\mu \setminus S) = 0$. *Then* $(a_\mu)_r = a_S$.

Proof. By Proposition 3.5, a_S is closable. Since $a_S \leqslant a_\mu$, it follows that $a_S \leqslant (a_\mu)_r$. We have to prove the converse inequality.

By inner regularity we find compact sets $K_n \subset K_{n+1} \subset \Gamma_\mu \setminus S$ such that $\mu\left((\Gamma_\mu \setminus S) \setminus \bigcup_{n\in\mathbb{N}} K_n\right) = 0$. Since $\mathrm{Cap}_{\bar\Omega}(K_n) = 0$, by [17, Lemma 2.2.7, p. 80], there exist $\psi_n \in H^1(\Omega) \cap C_c(\bar\Omega)$ such that $0 \leqslant \psi_n \leqslant 1$, $\psi_n = 1$ on K_n and $\psi_n \to 0$ in $H^1(\Omega)$. Taking a subsequence we may also assume that $\psi_n \to 0$ r.q.e. on $\bar\Omega$ and a.e. on Ω. In order to show that $(a_\mu)_r \leqslant a_S$, let $u \in D(a_\mu) = D(a_S)$. Let $\varphi_n = (1 - \psi_n)u$. Then $\varphi_n \in D(a_\mu)$, $|\varphi_n| \leqslant |u|$ and $\varphi_n \to 0$ μ-a.e. on $\Gamma_\mu \setminus S$. Since $\psi_n \to 0$ r.q.e. on $\bar\Omega$ and since S is μ-admissible, it follows that $\varphi_n \to u \cdot 1_S$ μ-a.e. on Γ_μ. It follows from the Dominated Convergence Theorem that $\varphi_n \to u \cdot 1_S$ in $L^2(\Gamma_\mu)$. Since

$$\int_\Omega |\varphi - u|^2 \,\mathrm{d}x = \int_\Omega |u\psi_n|^2 \,\mathrm{d}x \leqslant \|u\|_\infty^2 \int_\Omega |\psi_n|^2 \,\mathrm{d}x \to 0 \quad (n \to \infty)$$

one has $\varphi_n \to u$ in $L^2(\Omega)$. Moreover, $D_j\varphi_n = (1 - \psi_n)D_j u - (D_j\psi_n)u \to D_j u$ $(n \to \infty)$ in $L^2(\Omega)$ since $\psi_n \to 0$ a.e. in Ω and $D_j\psi_n \to 0$ in $L^2(\Omega)$. We have shown that $\varphi_n \to u$ in $H^1(\Omega)$ and that $(\varphi_{n|_{\Gamma_\mu}})$ converges to $u \cdot 1_S$ in $L^2(\Gamma_\mu)$. Thus $\varphi_n \to u$ in $D(a_\mu)$. Since $(a_\mu)_r$ is continuous on $D(a_\mu) \times D(a_\mu)$ and $\varphi_{n|_{\Gamma_\mu\setminus S}} \to 0$ in $L^2(\Gamma_\mu \setminus S)$, we conclude that

$$\begin{aligned}(a_\mu)_r(u) &= \lim_{n\to\infty} (a_\mu)_r(\varphi_n) \leqslant \liminf_{n\to\infty} a_\mu(\varphi_n) \\ &= \liminf_{n\to\infty} \left(a_S(\varphi_n) + \int_{\Gamma_\mu\setminus S} |\varphi_n|^2 \,\mathrm{d}\mu \right)\end{aligned}$$

$$= \lim_{n\to\infty} a_S(\varphi_n)$$
$$= a_S(u). \qquad \square$$

Proof of Theorem 3.3. If a_μ is closable, then $a_\mu = a_S$. By Proposition 3.4 it follows that $\mu(\Gamma_\mu \setminus S) = 0$. Since S is admissible, it follows that Γ_μ is admissible; i.e. μ is admissible. The converse follows from Proposition 3.5. $\square$

Next we show a uniqueness result.

PROPOSITION 3.8. *Let $S \subset \Gamma_\mu$ be a μ-admissible set such that* $\mathrm{Cap}_{\bar{\Omega}}(\Gamma_\mu \setminus S) = 0$ (*cf. Proposition* 3.6). *Let $S_1 \subset \Gamma_\mu$ be a Borel set. The following assertions are equivalent.*

(i) $a_{S_1} = (a_\mu)_r$;
(ii) $\mu(S_1 \triangle S) = 0$;
(iii) *S_1 is μ-admissible and $\mu(S_1 \setminus S_2) = 0$ for each μ-admissible set $S_2 \subset \Gamma_\mu$; i.e. S_1 is* maximal μ-admissible.

Proof. (i) $\Leftrightarrow$ (ii) follows from Theorem 3.7 and Proposition 3.4.

(ii) $\Rightarrow$ (iii) Since $\mu(S_1 \triangle S) = 0$, the set S_1 is μ-admissible and since S is so. Let $S_2 \subset \Gamma_\mu$ be μ-admissible. Since $\mathrm{Cap}_{\bar{\Omega}}(\Gamma_\mu \setminus S) = 0$, it follows that $\mu(S_1 \setminus S) = \mu(S_2 \cap (\Gamma_\mu \setminus S)) = 0$. Hence also $\mu(S_2 \setminus S_1) = 0$.

(iii) $\Rightarrow$ (ii) Since S is μ-admissible, it follows that $\mu(S \setminus S_1) = 0$. Since $\mathrm{Cap}_{\bar{\Omega}}(\Gamma_\mu \setminus S) = 0$ and S_1 is μ-admissible, it follows that $\mu(S_1 \setminus S) = \mu(S_1 \cap (\Gamma_\mu \setminus S)) = 0$. Thus $\mu(S_1 \triangle S) = 0$. $\square$

Let S be the set of Proposition 3.6. Then the trace

$$u \mapsto u_S : D(a_\mu) \to L^2(S, \mu) \tag{14}$$

has a continuous extension to $D(\overline{(a_\mu)_r})$. In Section 5 we will determine the set S for the surface measure of an irregular boundary. There we will see that $\mu(S) > 0$ (Proposition 5.5). In the next example, the set S is empty (or rather μ-equivalent to the empty set).

EXAMPLE 3.9. Let Ω be an arbitrary bounded open set in $\mathbb{R}^2$ and let $N := \{z_n : n \in \mathbb{N}\}$ be dense in $\Gamma = \partial\Omega$. Let $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{z_n}$. Then

$$a_\mu(u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \sum_{n=1}^{\infty} 2^{-n} |u(z_n)|^2$$

for all $u \in D(a_\mu) = H^1(\Omega) \cap C(\bar{\Omega})$. Let $S = \Gamma \setminus N$. Since $\mathrm{Cap}_{\bar{\Omega}}(\{z_n\}) \leqslant \mathrm{Cap}(\{z_n\}) = 0$ one has $\mathrm{Cap}_{\bar{\Omega}}(\Gamma \setminus S) = \mathrm{Cap}_{\bar{\Omega}}(N) = 0$. Since $\mu(S) = 0$, the set S is μ-admissible. Hence

$$(a_\mu)_r(u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

for all $u \in D(a_\mu)$.

Next we show some properties of the semigroup induced on $L^2(\Omega)$. Let μ be an admissible Borel measure on Γ. Denote by $-\Delta_\mu$ the operator associated with $\overline{a_\mu}$. Then it follows from the definiton (7) that Δ_μ is a realization of the Laplacian in $L^2(\Omega)$; i.e.

$$\Delta_\mu u = \Delta u$$

in the sense of distributions for all $u \in D(\Delta_\mu)$.

PROPOSITION 3.10. *The operator Δ_μ generates a symmetric sub-Markovian semigroup on $L^2(\Omega)$; i.e. $\mathrm{e}^{t\Delta_\mu} \geqslant 0$ for all $t \geqslant 0$ and*

$$\|\mathrm{e}^{t\Delta_\mu} f\|_\infty \leqslant \|f\|_\infty \quad (t \geqslant 0)$$

for all $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $u \in D(a_\mu)$. Then it is clear that $|u| \in D(a_\mu)$ and $a_\mu(|u|) = a_\mu(u)$. Since $D(a_\mu)$ is dense in $D(\overline{a_\mu})$, we obtain that if $u \in D(\overline{a_\mu})$ then $|u| \in D(\overline{a_\mu})$ and $\overline{a_\mu}(|u|) = \overline{a_\mu}(u)$ for all $u \in D(\overline{a_\mu})$. Now it follows from the first Beurling–Deny criterion [13, Theorem 1.3.2] that $\mathrm{e}^{t\Delta_\mu} \geqslant 0$ for all $t \geqslant 0$.

Let $0 \leqslant u \in D(a_\mu)$, then $u \wedge 1 \in D(a_\mu)$. Moreover,

$$\begin{aligned} a_\mu(u \wedge 1) &:= \int_\Omega |\nabla(u \wedge 1)|^2 \,\mathrm{d}x + \int_{\Gamma_\mu} (u \wedge 1)^2 \,\mathrm{d}\mu \\ &= \int_\Omega |\nabla u|^2 1_{\{u \leqslant 1\}} + \int_{\Gamma_\mu} (u \wedge 1)^2 \,\mathrm{d}\mu \\ &\leqslant a_\mu(u). \end{aligned}$$

By density of $D(a_\mu)$ of $D(\overline{a_\mu})$ we also obtain that if $0 \leqslant u \in D(\overline{a_\mu})$, then $u \wedge 1 \in D(\overline{a_\mu})$ and $\overline{a_\mu}(u \wedge 1) \leqslant \overline{a_\mu}(u)$ for all $0 \leqslant u \in D(\overline{a_\mu})$. It follows from the second Beurling–Deny criterion [13, Theorem 1.3.3] that $\|\mathrm{e}^{t\Delta_\mu} f\|_\infty \leqslant \|f\|_\infty \quad (t \geqslant 0)$ for all $f \in L^2(\Omega) \cap L^\infty(\Omega)$ which completes the proof. □

Notice that Davies [13] calls a symmetric Markov semigroup what we call a symmetric sub-Markovian semigroup on $L^2(\Omega)$.

Finally we describe Daners' approach [12] (given for the Hausdorff measure), making some modifications, and we compare it with ours. Let μ be a Borel measure on Γ. Consider the space

$$\begin{aligned} F := \{ w \in L^2(\Gamma_\mu) : \exists\, u_n \in D(a_\mu),\ u_n \to 0 \text{ in } H^1(\Omega) \text{ and} \\ u_n \to w \text{ in } L^2(\Gamma_\mu)\}. \end{aligned}$$

It is clear that $\varphi_{|\Gamma_\mu} w \in F$ for each $\varphi \in C(\bar{\Omega})$, $w \in F$. By the Stone–Weierstrass theorem it follows that $C_0(\Gamma_\mu)F \subset F$. Now Daners uses a clever argument involving Lusin's theorem to show that $L^\infty(\Gamma_\mu)F \subset F$. Thus F is a closed lattice ideal in $L^2(\Gamma_\mu)$. Since Γ_μ is σ-finite, by [31, Example 2, pp. 157–158] there exists a Borel set $S \subset \Gamma_\mu$ such that

$$F = \{w \in L^2(\Gamma_\mu) : w = 0 \ \mu\text{-a.e. on } S\}.$$

Consider $D(a_\mu)$ as a subspace of $\widetilde{H}^1(\Omega) \oplus L^2(\Gamma_\mu)$. The closure $\widetilde{V}$ in this space is isomorphic to the completion of $D(a_\mu)$ with respect to the norm $\|\cdot\|_{a_\mu}$. The imbedding of $D(a_\mu)$ into $L^2(\Omega)$ has the unique continuous extension $j : \widetilde{V} \to L^2(\Omega)$ given by $j(u, w) = u$. Thus a_μ is closable if and only if $\ker j = 0$. If a_μ is not closable, then by the proof of [29, Theorem S15, p. 373], the domain $D(\overline{(a_\mu)_r})$ of the closure of $(a_\mu)_r$ can be identified with $V = (\ker j)^\perp$. By the argument above we have $\ker j = 0 \oplus F$. Thus $V = \{(u, w) \in \widetilde{V} : w \in L^2(S, \mu)\}$ where we identified $L^2(S, \mu)$ with a subspace of $L^2(\Gamma_\mu)$ by extending functions by 0. Given $u \in \widetilde{H}^1(\Omega)$ there is at most one $w \in L^2(S, \mu)$ such that $(u, w) \in \widetilde{V}$. Thus we may identify $D(\overline{(a_\mu)_r})$ with the space of all $u \in \widetilde{H}^1(\Omega)$ such that there exists a function $u_{|S} \in L^2(S, \mu)$ such that $(u, u_{|S}) \in \widetilde{V}$ and

$$\overline{(a_\mu)_r}(u) = \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_S |u_{|S}|^2 \, \mathrm{d}\mu.$$

The notation $u_{|S}$ is purely symbolic here. Proposition 3.8 now shows that the set S is μ-equivalent to the set S constructed by the capacity approach.

4. Admissibility of Hausdorff Measures

Let Ω be an open set in $\mathbb{R}^n$ with boundary Γ. Let σ be the restriction of the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ to Γ (see [15, Chapter 2] for the definition). If Ω is a Lipschitz domain, then σ is just the surface measure. The measure σ follows the geometry of Γ. If $n = 2$ and $S \subset \Gamma$ is a segment, then $\sigma(S)$ is just the length of the segment. If $n = 3$ and S is a rectangle, then $\sigma(S)$ is just the surface of the rectangle. In any case, without any restriction on Ω, σ is a Borel measure on Γ. However, it may happen that σ is locally infinite, i.e., $\Gamma_\sigma = \emptyset$.

The question we want to answer here is whether a_σ is closable; i.e., whether σ is admissible. It is known that

$$\operatorname{Cap}(A) = 0 \quad \text{implies} \quad \sigma(A) = 0, \tag{15}$$

for each Borel set $A \subset \Gamma$ (see [15, Theorem 4, p. 156]). Thus, if $\widetilde{H}^1(\Omega)$ has the special extension propety, then σ is admissible by Proposition 1.4. In particular, we can state the following.

PROPOSITION 4.1. *If Ω is bounded with Lispschitz boundary, then σ is admissible.*

On the other hand, Example 1.6 shows that there exists a bounded, open, connected set $\Omega \subset \mathbb{R}^2$ such that a segment S is in the boundary Γ of Ω and $\operatorname{Cap}_{\bar{\Omega}}(S) = 0$.

Since $\sigma(S)$ is the length of S and thus positive, (11) is not satisfied. Still this example is not suitable for our purposes since $\sigma(\Gamma \cap B(z, r)) = \infty$ for each $z \in S$, $r > 0$; i.e., σ is locally infinite on S. A closer look at the example shows that σ is in fact admissible since (11) is only needed for subsets of Γ_σ.

We will construct a more complicated bounded, connected, open set Ω in $\mathbb{R}^3$ such that $\sigma(\Gamma) < \infty$ but σ is not admissible. First we give an example in $\mathbb{R}^2$ which has the disadvantage of not being connected. The three-dimensional example will be a modification of the two-dimensional one. In the following two examples σ_1 denotes the one-dimensional and σ_2 the two-dimensional Hausdorff measure.

EXAMPLE 4.2. Let $\mathbb{Q} \cap (0, 1) = \{q_1, q_2, \ldots\}$ where $\mathbb{Q}$ denotes the set of rational numbers. Consider the following Figure 1.

Let $\Omega := \bigcup_{n=1}^{\infty} \Omega_n$ where $\Omega_n = \bigcup_{i=1}^{n} \Omega_{n,i}$ is as in the Figure 1. We assume that the breadth of each rectangle $\Omega_{n,i}$ is 2^{-2n}. Then Ω is an open bounded subset of $\mathbb{R}^2$, but it is not connected. Since $\mathbb{Q} \cap (0, 1)$ is dense in [0, 1], we have $E := \{0\} \times [0, 1] \subset \partial\Omega$ and

$$\partial\Omega \subset \bigcup_{n=1}^{\infty} \partial\Omega_n \cup E.$$

Thus $\sigma_1(\partial\Omega) \leqslant 1 + \sum_{n=1}^{\infty} \sigma_1(\partial\Omega_n)$. Since the one-dimensional Hausdorff measure of a segment is its length, we have

$$\sigma_1(\partial\Omega_n) \leqslant n(2^{-(n-1)} + 2^{-(2n-1)}).$$

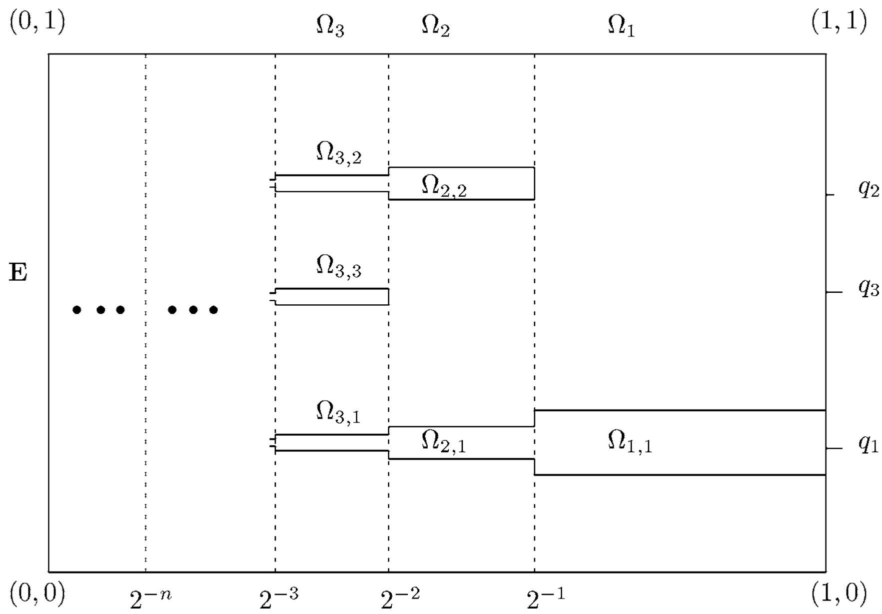

Figure 1.

Hence

$$\sigma_1(\partial\Omega) \leqslant 1 + \sum_{n=1}^{\infty} n(2^{-(n-1)} + 2^{-(2n-1)}) < \infty.$$

Let $\rho \in C^\infty[0, \infty)$ such that

$$\begin{cases} 0 \leqslant \rho(x) \leqslant 1, & \\ \rho(x) = 1 & \text{if } 0 \leqslant x \leqslant 1/2, \\ \rho(x) = 0 & \text{if } x > 3/4. \end{cases}$$

We define the sequence of functions u_n on Ω by letting $u_n(x, y) := \rho(2^n x)$. Then $u_n \in H^1(\Omega) \cap C(\bar{\Omega})$ and $0 \leqslant u_n(x, y) \leqslant 1$. Since $u_n(x, y) = 1$ for $0 \leqslant x \leqslant 2^{-(n+1)}$, one has that $u_n = 1$ on a relative neighborhood of E. Moreover,

$$\lim_{n\to\infty} u_n(x, y) = \lim_{n\to\infty} \rho(2^n x) = 0.$$

Since $|u_n(x, y)| = |\rho(2^n x)| \leqslant 1$, the Dominated Convergence Theorem implies that the sequence u_n converges to 0 in $L^2(\Omega)$. Furthermore, $\operatorname{supp}[\nabla u_n] \subset \{2^{-n-1} \leqslant x \leqslant 2^{-n}\}$ and

$$\begin{aligned} \int_\Omega |\nabla u_n|^2 \,\mathrm{d}x\,\mathrm{d}y &= 2^{2n} \int_\Omega |\rho'(2^n x)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant 2^{2n}(n+1)2^{-2(n+1)} \int_{2^{-(n+1)}}^{2^{-n}} |\rho'(2^n x)|^2 \,\mathrm{d}x \\ &= (n+1)2^{-(n+2)} \int_{1/2}^{1} |\rho'(r)|^2 \,\mathrm{d}r \\ &\longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned} \tag{16}$$

By definition of the relative capacity, this implies that $\mathrm{Cap}_{\bar{\Omega}}(E) = 0$, but clearly, $\sigma_1(E) = 1$. Thus σ_1 is not admissible.

We now add one dimension, interpreting the lines in Ω as walls. In order to make Ω connected we join the walls by tiny tubes.

EXAMPLE 4.3. Let $D = \Omega \times (0, 1) \cup \bigcup_{n=1}^{\infty} P_n$ where Ω is the set of Example 4.2 and P_n is a tube of radius $r_n = 2^{-2n}$ connecting the walls $(2^{-(n+1)}, 2^{-n}) \times \{0\} \times (0, 1)$ and $(2^{-(n+1)}, 2^{-n}) \times \{1\} \times (0, 1)$. Thus D is an open, connected, bounded subset of $\mathbb{R}^3$. Since the two-dimensional Hausdorff measure of a rectangle in $\mathbb{R}^3$ is its surface, we have

$$\begin{aligned} \sigma_2(\partial D) &\leqslant \sigma_2(\partial\Omega \times [0, 1]) + \sum_{n=1}^{\infty} \sigma_2(\partial P_n) \\ &\leqslant \sigma_1(\partial\Omega) + \sum_{n=1}^{\infty} 2\pi 2^{-2n} < \infty. \end{aligned}$$

Let $u_n(x, y, z) = \rho(2^n x)$. Then u_n converges to 0 in $L^2(\Omega)$ as $n \to \infty$. Moreover, since $\mathrm{supp}[\nabla u_n] \subset \{2^{-n-1} \leqslant x \leqslant 2^{-n}\}$, by (16) we have,

$$\begin{aligned}\int_{\tilde\Omega} |\nabla u_n|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &\leqslant c_1(n+1)2^{-(n+2)} + 2^{2n}\int_{P_{n+1}} |\rho'(2^n x)|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\leqslant c_1(n+1)2^{-(n+2)} + \|\rho'\|_\infty^2 2^{2n}\pi r_n^2\\ &\longrightarrow 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus $\lim_{n\to\infty} u_n = 0$ in $H^1(D)$. Since $u_n = 1$ on a relative neighborhood of $\widetilde{E} := \{0\} \times [0, 1] \times [0, 1] \subset \partial D$, it follows that $\widetilde{E}$ has relative capacity 0 but $\sigma_2(\widetilde{E}) = 1$. Thus σ_2 is not admissible.

The point in the above examples is that the $(n-1)$-dimensional Hausdorff measure of Γ is finite. If the boundary of Ω is irregular, then $\mathcal{H}^{n-1}$ may be locally infinite on the boundary. A more natural measure in that case would be $\mathcal{H}^s$ where s is the Hausdorff dimension of the boundary. We give an example.

EXAMPLE 4.4 (von Koch curve). *Let $\Omega \subset \mathbb{R}^2$ be the interior of the von Koch curve* Γ [16, p. XV]. *Then Ω is bounded and connected with boundary $\Gamma = \partial\Omega$. The Hausdorff dimension of Γ is $\log 4/\log 3 \in (1, 2)$, see* [16, Example 2.7]. *Moreover, $\mathcal{H}^s$ is a finite Borel measure on Γ. We claim that $\mathcal{H}^s$ is admissible.*

Proof. By [27, Section 1.5.1, Example 1], the boundary Γ is a quasicircle and then by [21, Theorem B and Theorem 4], there exist compatible extension operators $P_p : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$. Hence $H^1(\Omega) = \widetilde{H}^1(\Omega)$, and it follows as in the proof of Proposition 1.3 that $H^1(\Omega)$ has the special extension property. It follows from Proposition 1.4 that each relatively polar subset of Γ is polar. On the other hand, by [15, Theorem 4, p. 156], one has $\mathcal{H}^s(A) = 0$ whenever $A \subset \Gamma$ is a polar Borel set. Thus $\mathcal{H}^s$ is admissible. □

5. Robin Boundary Condition for the Hausdorff Measure

In this section we investigate properties of the semigroup generated by the Laplacian with Robin boundary conditions in the restricted sense; i.e. with respect to a measure which is absolutely continuous with respect to the Hausdorff measure of dimension $n-1$. But our point is to allow arbitrary open sets.

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set with boundary Γ. Consider the $(n-1)$-dimensional Hausdorff measure σ on Γ. As before, $\Gamma_\sigma = \{z \in \Gamma : \exists\, r > 0 \text{ such that } \sigma(\Gamma \cap B(z, r)) < \infty\}$ and we assume that $\Gamma_\sigma \neq \emptyset$. Denote by $S \subset \Gamma$ the maximal σ-admissible set; i.e., S is σ-admissible and $\mathrm{Cap}_{\bar\Omega}(\Gamma_\sigma \setminus S) = 0$.

Let $0 \leqslant \beta \in L^\infty(S, \sigma)$. Define the form a_β on $L^2(\Omega)$ by

$$a_\beta(u, v) = \int_\Omega \nabla u \nabla v \,\mathrm{d}x + \int_S uv\beta \,\mathrm{d}\sigma, \tag{17}$$

where $D(a_\beta) = \{u \in H^1(\Omega) \cap C_c(\bar{\Omega}) : \int_S |u|^2 \beta \, d\sigma < \infty\}$. Since $\beta \, d\sigma$ is admissible, the form a_β is closable. Denote by $-\Delta_\beta$ the operator associated with $\overline{a_\beta}$. Then Δ_β is selfadjoint and by Proposition 3.10 it generates a sub-Markovian C_0-semigroup $(e^{t\Delta_\beta})_{t\geqslant 0}$ on $L^2(\Omega)$. If $\beta = 0$, then $\Delta_0 = \Delta^N$.

It is well-known that the Neumann Laplace operator may have all kind of strange properties if Ω is not regular (see, e.g., [19]). For instance, it is easy to see that the injection $\widetilde{H}^1(\Omega) \hookrightarrow L^2(\Omega)$ is not compact if Ω is the set of Example 1.6. Thus $e^{t\Delta^N}$ is not compact in that case. It is surprising that $e^{t\Delta_\beta}$ has much better behaviour if $\inf_S \beta > 0$. This phenomenon has been discovered by Daners [12] and is a consequence of a remarkable inequality due to Maz'ya. Whereas Daners studies bounded open sets we consider here arbitrary open sets. Moreover, we use our different approach based on capacity and now know by the examples in Section 4 that a restriction to an admissible subset of Γ is needed.

We assume that the dimension n is larger than 1. Maz'ya's result [27, Theorem 3.6.3] says that there exists a constant (merely depending on the dimension and not on the set Ω) such that

$$\|u\|_{L^{n/(n-1)}(\Omega)} \leqslant c\left(\int_\Omega |\nabla u| \, dx + \int_\Gamma |u| \, d\sigma \right) \tag{18}$$

for all $u \in W^{1,1}(\Omega) \cap C_c(\bar{\Omega})$. From this we deduce that

$$\|u\|^2_{L^{2n/(n-1)}(\Omega)} \leqslant c\left(\|u\|^2_{H^1(\Omega)} + \int_\Gamma |u|^2 \, d\sigma \right) \tag{19}$$

for all $u \in H^1(\Omega) \cap C_c(\bar{\Omega})$.

Proof. Let $u \in H^1(\Omega) \cap C_c(\bar{\Omega})$. Applying (18) to u^2 gives

$$\begin{aligned}
\|u\|^2_{L^{2n/(n-1)}(\Omega)} &= \|u^2\|_{L^{n/(n-1)}(\Omega)} \\
&\leqslant c\left(\int_\Omega |\nabla u^2| \, dx + \int_\Gamma |u|^2 \, d\sigma \right) \\
&= c\left(\int_\Omega 2|u \nabla u| \, dx + \int_\Gamma |u|^2 \, d\sigma \right) \\
&\leqslant c\left(2\left(\int_\Omega |u|^2 \, dx \right)^{1/2} \cdot \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2} + \int_\Gamma |u|^2 \, d\sigma \right) \\
&\leqslant c\left(\int_\Omega |u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx + \int_\Gamma |u|^2 \, d\sigma \right). \qquad \square
\end{aligned}$$

We now show that we can replace Γ by S in (19):

$$\|u\|^2_{L^{2n/(n-1)}(\Omega)} \leqslant c\left(\|u\|^2_{H^1(\Omega)} + \int_S |u|^2 \, d\sigma \right) \tag{20}$$

for all $u \in H^1(\Omega) \cap C_c(\bar{\Omega})$.

Proof. Let $u \in H^1(\Omega) \cap C_c(\bar{\Omega})$ such that $\int_\Gamma |u|^2 \, d\sigma < \infty$. The proof of Theorem 3.7 yields a sequence $\varphi_n \in H^1(\Omega) \cap C_c(\bar{\Omega})$ such that $\int_\Gamma |\varphi_n|^2 \, d\sigma < \infty$, $\varphi_n \to u 1_S$ in $L^2(\Gamma, \sigma)$, and $\varphi_n \to u$ in $H^1(\Omega)$ and a.e. and such that $|\varphi_n| \leqslant |u|$. Applying (19) to φ_n instead of u and passing to the limit as $n \to \infty$ gives (20). □

Now assume that $\beta(z) \geqslant \delta > 0$ for all $z \in S$. Then by (20), $D(a_\beta) \subset L^{2n/(n-1)}(\Omega)$ and

$$\|u\|^2_{L^{2n/(n-1)}(\Omega)} \leqslant c \max(1, 1/\delta)\big(a_\beta(u) + \|u\|^2_{L^2(\Omega)}\big) \tag{21}$$

for all $u \in D(a_\beta)$. It follows that also $D(\overline{a_\beta}) \subset L^{2n/(n-1)}(\Omega)$ and that (21) remains true for all $u \in D(\overline{a_\beta})$.

Inequality (21) implies ultracontractivity of the semigroup $e^{t\Delta_\beta}$.

THEOREM 5.1 *Let $\beta \in L^\infty(S, \sigma)$ such that $\beta(z) \geqslant \delta > 0$ for all $z \in S$. Then there exists a constant $c_2 > 0$ such that*

$$\|e^{t\Delta_\beta} f\|_{L^\infty(\Omega)} \leqslant c_2 t^{-n} e^t \|f\|_{L^1(\Omega)} \tag{22}$$

for all $f \in L^1(\Omega) \cap L^2(\Omega)$.

Proof. This follows from [13, Theorem 2.4.2] applied to $(e^{-t} e^{t\Delta_\beta})_{t \geqslant 0}$. □

COROLLARY 5.2 *Assume that Ω has finite measure. Then $e^{t\Delta_\beta}$ is compact for all $t > 0$.*

Proof. Since $e^{t\Delta_\beta} L^2(\Omega) \subset L^\infty(\Omega)$, it follows that $e^{t\Delta_\beta}$ is a Hilbert–Schmidt operator. □

By the Dunford–Pettis criterion (see, e.g., [5, 6]), it follows that there exists a kernel $K_t \in L^\infty(\Omega \times \Omega)$ such that $0 \leqslant K_t(x, y) \leqslant c_2 t^{-n} e^t$ a.e. on $\Omega \times \Omega$ for all $t > 0$ and

$$(e^{t\Delta_\beta} f)(x) = \int_\Omega K_t(x, y) f(y) \, dy.$$

Now one can also prove Gaussian estimates following the lines of the proof of [6, Theorem 4.4] (see also [12]).

THEOREM 5.3 *Let $\beta(z) \geqslant \delta > 0$ for all $z \in S$. There exist constants c_3, $\omega \geqslant 0$, $b > 0$ such that*

$$0 \leqslant K_t(x, y) \leqslant c_3 t^{-n} e^{-\frac{|x-y|^2}{bt}} e^{\omega t}. \tag{23}$$

The estimate (23) is not a usual Gaussian estimate. If Ω is Lipschitz, we know that we may replace t^{-n} by the natural factor $t^{-n/2}$ (see [6]). Thus several of the interesting consequences of Gaussian estimates (holomorphy on L^1, $\mathcal{H}^\infty$-calculus in L^p, see, e.g., [6, 1]) cannot be deduced from (23). However, it is remarkable that we can deduce p-independence of the spectrum in virtue of recent results of Kunstmann and Vogt [25] and Karrmann [22].

Note that $(e^{t\Delta_\beta})_{t\geqslant 0}$ is a symmetric sub-Markovian semigroup on $L^2(\Omega)$ (see Proposition 3.10). In particular, there exist consistent positive contraction semigroups $(e^{t\Delta_{\beta p}})_{t\geqslant 0}$ on $L^p(\Omega)$, $1 \leqslant p < \infty$, such that $\Delta_{\beta 2} = \Delta_\beta$ ([13, Theorem 1.4.1]).

THEOREM 5.4 *Let $\beta \in L^\infty(S, \sigma)$ such that $\beta(z) \geqslant \delta > 0$ for all $z \in S$. Then*

$$\sigma(\Delta_{\beta p}) = \sigma(\Delta_\beta)$$

for all $p \in [1, \infty)$.

Proof. Consider $B = \Delta_\beta - \omega$ (where ω is given by Theorem 5.3). Let $B_p = \Delta_{\beta p} - \omega$. It follows from [25, Proposition 5] that the spectrum of $R(\lambda, B_p)^k$ is p-indepedent for λ large and $k \geqslant 1 + n/2$. By [22, Lemma 6.3] we obtain that $\sigma(B_p) = \sigma(B_2)$. □

We remark that Theorem 5.4 is not true if $\beta = 0$; i.e. for the Neumann Laplace operator. A counterexample of an unbounded domain with finite measure is given by Kunstmann [24]. In fact, let

$$\Omega := \{(x, y) \in \mathbb{R}^2 : x > 0,\ |y| < e^{-x}\}.$$

Then Ω has finite measure and $\widetilde{H}^1(\Omega) = H^1(\Omega)$ by [14, Chap. V, Theorem 4.7]. Kunstmann [24] shows that the spectrum $\sigma(\Delta_{p,\beta})$ for $\beta = 0$ (i.e., the spectrum of the Neumann Laplacian in $L^p(\Omega)$) does depend on $p \in [1, \infty]$.

Finally, we consider the case where Ω has finite Lebesgue measure. Then $L^{2n/(n-1)}(\Omega) \hookrightarrow L^2(\Omega)$; i.e., $\|u\|^2_{L^2(\Omega)} \leqslant c_2\|u\|^2_{L^{2n/(n-1)}(\Omega)}$ for all $u \in L^{2n/(n-1)}(\Omega)$. Now by the proof of (19) and Young's inequality $2ab \leqslant \varepsilon a^2 + b^2/\varepsilon$ for all $a, b \geqslant 0$ and for every $\varepsilon > 0$ we obtain

$$\begin{aligned}\|u\|^2_{L^{2n/(n-1)}(\Omega)} &\leqslant c\left(\varepsilon\|u\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\int_\Omega |\nabla u|^2\,dx + \int_\Gamma |u|^2\,d\sigma\right)\\ &\leqslant c\left(c_2\varepsilon\|u\|^2_{L^{2n/(n-1)}(\Omega)} + \frac{1}{\varepsilon}\int_\Omega |\nabla u|^2\,dx + \int_\Gamma |u|^2\,d\sigma\right).\end{aligned}$$

Hence

$$\|u\|^2_{L^{2n/(n-1)}(\Omega)} \leqslant c_3\left(\int_\Omega |\nabla u|^2\,dx + \int_\Gamma |u|^2\,d\sigma\right). \tag{24}$$

Reasoning as for (20) we deduce that

$$\|u\|^2_{L^{2n/(n-1)}(\Omega)} \leqslant c_3\left(\int_\Omega |\nabla u|^2\,dx + \int_S |u|^2\,d\sigma\right) \tag{25}$$

for all $u \in H^1(\Omega) \cap C_c(\bar{\Omega})$.

This inequality has several interesting consequences.

PROPOSITION 5.5 *Assume that Ω is bounded and $\sigma(\Gamma) < \infty$. Then $\sigma(S) > 0$.*

Proof. If $\sigma(S) = 0$, then applying (25) to the constant 1 function we obtain a contradiction. □

The next consequence concerns the asymptotic behaviour.

PROPOSITION 5.6 *Assume that Ω has finite Lebesgue measure. Assume that $\beta(z) \geqslant \delta > 0$ for all $z \in S$. Let $1 \leqslant p < \infty$. Then $e^{t\Delta_{\beta p}} \in \mathcal{L}(L^p(\Omega))$ is compact for all $t > 0$. Moreover, the semigroup $(e^{t\Delta_{\beta p}})_{t \geqslant 0}$ is exponentially stable.*

Proof. Let $1 \leqslant p \leqslant 2$. By ultracontractivity we may factorize $e^{t\Delta_{\beta p}} = j \circ e^{t/2\Delta_{\beta p}} e^{t/2\Delta_{\beta p}} : L^p(\Omega) \to L^2(\Omega) \to L^2(\Omega) \hookrightarrow L^p(\Omega)$. Since $e^{t/2\Delta_{\beta 2}}$ is compact also $e^{t\Delta_{\beta p}}$ is compact if $t > 0$. Since $(e^{t\Delta_{\beta p}})^* = e^{t\Delta_{\beta p^*}}$ where $1/p + 1/p^* = 1$, the first claim is proved. Instead of Theorem 5.4 we can here use the simpler [13, Corollary 1.6.2] to show that the spectrum of $\Delta_{\beta p}$ is p-independent.

Inequality (25) implies that $D(\overline{a_\beta}) \subset L^{2n/(n-1)}(\Omega)$ and $\|u\|_{L^{2n/(n-1)}(\Omega)} \leqslant c_3 \max(1, 1/\delta)\overline{a_\beta}(u)$ for all $u \in D(\overline{a_\beta})$. Since Δ_β has compact resolvent, this implies that the first eigenvalue λ_1 of $-\Delta_\beta$ is strictly positive. Thus $\|e^{t\Delta_\beta}\|_{\mathcal{L}(L^2(\Omega))} \leqslant e^{-t\lambda_1}$ $(t \geqslant 0)$. Since $\sigma(\Delta_{\beta p}) \subset (-\infty, -\lambda_1)$, it follows from [4, Proposition 5.3.7] that $(e^{t\Delta_{\beta 1}})_{t \geqslant 0}$ is exponentially bounded. Now the same is valid for the semigroup on $L^p(\Omega)$ by the Riesz–Thorin theorem (or by [4, Theorem 5.3.6]). □

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