J. Aust. Math. Soc. **77** (2004), 175–184

OPERATOR-VALUED MULTIPLIER THEOREMS CHARACTERIZING HILBERT SPACES

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(Received 22 May 2002; revised 4 June 2003)

Communicated by G. Willis

Abstract

We show that the operator-valued Marcinkiewicz and Mikhlin Fourier multiplier theorem are valid if and only if the underlying Banach space is isomorphic to a Hilbert space.

2000 *Mathematics subject classification*: primary 42A45, 42C99, 46C15, 47A56.
Keywords and phrases: Operator-valued Fourier multiplier, Hilbert space and Rademacher boundedness.

1. Introduction

Mikhlin's multiplier theorem is of great importance in analysis. It says that a bounded function $m \in C^1(\mathbb{R} \setminus \{0\})$ such that $tm'(t)$ is bounded, defines an $L^p(\mathbb{R})$-multiplier for $1 < p < \infty$. In the context of partial differential equations vector-valued spaces $L^p(\mathbb{R}; X)$ occur in a natural way, where X is a Banach space. Thus the function m should take its values in $\mathscr{L}(X)$. Our aim is to show that Mikhlin's multiplier theorem does hold for such operator-valued functions if and only if X is isomorphic to a Hilbert space.

The phenomenon that operator-valued versions of certain classical multiplier theorems are only valid in Hilbert spaces was first observed by Pisier (unpublished) as a consequence of Kwapien's deep characterization of Hilbert spaces. More recently, new versions of operator-valued multiplier theorems turned out to be most useful in the theory of evolution equations (see the references and comments below) and it

This research is part of the DFG-project: "*Regularität und Asymptotik für elliptische und parabolische Probleme*". The second author is supported by the Alexander-von-Humboldt Foundation, the NSF of China and the Excellent Young Teacher Program of MOE, P.R.C.

seems to us that it is important to elaborate in some detail why the classical result merely holds on Hilbert spaces.

In another context, it helps to impose a Mikhlin's condition of order k

$$m \in C^k(\mathbb{R} \setminus \{0\}; \mathscr{L}(X)), \qquad \sup_{t \in \mathbb{R}\setminus\{0\}, 0 \le l \le k} \|t^l m^{(l)}(t)\| < \infty. \tag{1}$$

In fact, Amann [1] discovered that if m satisfies (1) with $k = 2$, then m is a multiplier for Besov spaces and in particular for the space $C^\theta(\mathbb{R}; X)$, $0 < \theta < 1$ (see also [2] and [11]). We show here that imposing higher order Mikhlin's conditions does not help in the context of operator-valued L^p-multipliers.

We also consider the groups $\mathbb{T}$ and $\mathbb{Z}$ instead of $\mathbb{R}$. In fact, the case $\mathbb{T}$ corresponds to Marcinkiewicz's classical theorem and its operator-valued version is already treated in [3] for the order-1-case.

Now we would like to comment on the new vector-valued multiplier theorems which were found recently. It were Berkson-Gillespie [4] who introduced the notion of R-boundedness (after implicit use of Bourgain [6]). They use R as an abbreviation for Riesz, but in many subsequent papers people seem think rather of Rademacher or 'Randomized' because the definition involves Rademacher functions. A multiplier theorem of Marcinkiewicz type was established by Clément-de Pagter-Sukochev-Witvliet [8] for multipliers of the form $m(t)I$ (I is the identity operator) clarifying the role of R-boundedness. Then Weis [18] established Mikhlin's theorem for operator-valued functions (without restriction) replacing boundedness by the stronger condition of R-boundedness. Then in [3] the corresponding periodic theorem (that is, Marcinkiewicz's theorem) was proved on the basis of results in [8]. Štrkalj and Weis [17] gave an R-version of the variational version of the Marcinkiewicz theorem. Further important contributions were given by Clément-Prüss [9], Denk-Hieber-Prüss [10], and Girardi-Weis [11].

2. Periodic multipliers

Let us first recall some notions. Let X be a Banach space. Denote by r_j the j-th Rademacher function on $[0, 1]$. For $x \in X$, we denote by $r_j \otimes x$ the vector-valued function $t \mapsto r_j(t)x$. Let Y be another Banach space. We denote by $\mathscr{L}(X, Y)$ the set of all bounded linear operators from X to Y. If $X = Y$ we will denote $\mathscr{L}(X, Y)$ simply by $\mathscr{L}(X)$. A family $\mathbf{T} \subset \mathscr{L}(X, Y)$ is called *R-bounded* if for some $q \in [1, \infty)$ there exists a constant $c_q \ge 0$ such that

$$\left\| \sum_{j=1}^n r_j \otimes T_j x_j \right\|_{L^q(0,1;Y)} \le c_q \left\| \sum_{j=1}^n r_j \otimes x_j \right\|_{L^q(0,1;X)} \tag{2}$$

for all $T_1, \dots, T_n \in \mathbf{T}$, $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$. By Kahane's inequality [14, Theorem 1.e.13] if such constant c_q exists for some $q \in [1, \infty)$, then it also exists for each $q \in [1, \infty)$.

It is known that R-boundedness is strictly stronger than boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space. More precisely, each bounded subset in $\mathscr{L}(X, Y)$ is R-bounded if and only if X is of cotype 2 and Y is of type 2 (see [3, Proposition 1.13]). In particular, by a result of Kwapien [14, pages 73–74], each bounded subset in $\mathscr{L}(X)$ is R-bounded if and only if X is isomorphic to a Hilbert space.

For $1 \le p < \infty$, consider the Banach space $L^p(0, 2\pi; X)$ with norm $\|f\|_p := \left(\int_0^{2\pi} \|f(t)\|^p \, dt\right)^{1/p}$. For $f \in L^p(0, 2\pi; X)$ we denote by

$$\hat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) \, dt,$$

the k-th *Fourier coefficient* of f, where $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, $x \in X$ we let $e_k(t) = e^{ikt}$ and $(e_k \otimes x)(t) = e_k(t)x \quad (t \in \mathbb{R})$. A function $f \in L^p(0, 2\pi; X)$ is called a *trigonometric polynomial* if f is given by $f = \sum_{k \in \mathbb{Z}} e_k \otimes x_k$, where $x_k \in X$ is 0 for all but finitely many $k \in \mathbb{Z}$.

Let $M = (M_k)_{k \in \mathbb{Z}} \subset \mathscr{L}(X, Y)$ be a sequence and let $1 \le p, q < \infty$. We say that $(M_k)_{k \in \mathbb{Z}}$ is a *periodic L^p-L^q-Fourier multiplier* if there exists a constant $C > 0$ such that

$$\left\| \sum_{k \in \mathbb{Z}} e_k \otimes M_k x_k \right\|_{L^q(0,2\pi;Y)} \le C \left\| \sum_{k \in \mathbb{Z}} e_k \otimes x_k \right\|_{L^p(0,2\pi;X)}$$

for all X-valued trigonometric polynomials $\sum_{k \in \mathbb{Z}} e_k \otimes x_k$. In this case, there exists a unique operator $M \in \mathscr{L}(L^p(0, 2\pi; X), L^q(0, 2\pi; Y))$ such that $(Mf)\hat{}(k) = M_k \hat{f}(k)$ for $k \in \mathbb{Z}$ [3]. When $p = q$, we say simply that $(M_k)_{k \in \mathbb{Z}}$ is a *periodic L^p-Fourier multiplier*. For $k \in \mathbb{Z}$, we let $(\Delta^1 M)(k) = M_{k+1} - M_k$ and $(\Delta^m M)(k) = (\Delta^1(\Delta^{m-1} M))(k)$ for $m \ge 2$. Notice that $\Delta^m M$ is a discrete analogue of the m-th derivative of M.

The classical Marcinkiewicz Fourier multiplier theorem has been extended to the operator-valued case in the following way: let X and Y be UMD spaces and let $(M_k)_{k \in \mathbb{Z}} \subset \mathscr{L}(X, Y)$; if both $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k \in \mathbb{Z}}$ defines a periodic L^p-Fourier multiplier for each $1 < p < \infty$ [3]. Indeed, $(M_k)_{k \in \mathbb{Z}}$ is a periodic L^p-L^q-Fourier multiplier whenever $1 \le q \le p < \infty$.

We will need the following inequality of Pisier [15]. Let $1 \le p < \infty$ and let $\Lambda = \{n_k : k \in N\} \subset \mathbb{Z}$ be a Sidon subset [16, page 120]. Then there exists $C > 0$ such that for any Banach space X and for any finite sequence $(y_k)_{1 \le k \le N}$ of X, we have

$$C^{-1} \left\| \sum_k r_k \otimes y_k \right\|_2 \le \left\| \sum_k e_{n_k} \otimes y_k \right\|_p \le C \left\| \sum_k r_k \otimes y_k \right\|_2 . \tag{3}$$

Note that if $\lambda > 1$, then any subset $\{n_k : k \in \mathbb{N}\}$ satisfying $n_{k+1}/n_k \geq \lambda$ $(k \in \mathbb{N})$ is a Sidon subset of $\mathbb{Z}$ [16, page 127].

The following result shows that one cannot replace R-boundedness in the operator-valued Marcinkiewicz theorem above by boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space.

THEOREM 1. *Let X be a Banach space. Then the following assertions are equivalent:*

(i) *X is isomorphic to a Hilbert space.*

(ii) *For some $1 \leq q < p < \infty$, each sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ satisfying*

(a) $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$,

(b) $\sup_{k\in\mathbb{Z}} \|k^l(\Delta^l M)(k)\| < \infty$ *for* $l \in \mathbb{N}$,

(c) $M_k = 0$ *for* $k \leq 0$,

is a periodic L^p-L^q-Fourier multiplier.

(iii) *For all $1 < p < \infty$, each sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ satisfying*

(a) $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$,

(b) $\sup_{k\in\mathbb{Z}} \|k(\Delta^1 M)(k)\| < \infty$,

is a periodic L^p-Fourier multiplier.

REMARK 2. For $l = 1$, the condition formulated in (iii) is the classical condition considered by Marcinkiewicz in the scalar case. For arbitrary $l \in \mathbb{N}$ we therefore speak of the *Marcinkiewicz condition of order l*. For $p = q$ and $l = 1$, Theorem 1 has been proved in [3, Proposition 1.17.]. However, a more refined choice of test functions is needed here for the general case. The motivation to consider $l > 1$ stems from the results on Fourier multipliers for spaces of Hölder continuous functions where, indeed, the Marcinkiewicz condition of order 2 suffices (see [2] and also the Concluding Remarks at the end of this article). Theorem 1 shows that this is not the case in the L^p-context even if we consider weaker multipliers by allowing $q < p$. This has also been done by Kalton-Lancien in the context of maximal regularity for Cauchy problems [13] (see also the Concluding Remarks 5 (b) below).

PROOF. (i) $\Rightarrow$ (iii). Assume that X is isomorphic to a Hilbert space, then considering an orthonormal basis one easily verifies that each bounded subset in $\mathcal{L}(X)$ is actually R-bounded, so the result follows from the operator-valued Marcinkiewicz Fourier multiplier theorem in [3].

(iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). Assume that for some $1 \leq q < p < \infty$, each sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ satisfying $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^l(\Delta^l M)(k)\| < \infty$ for $l \in \mathbb{N}$ and $M_k = 0$ for $k \leq 0$, is a periodic L^p-L^q-Fourier multiplier. Let $N = (N_k)_{k\in\mathbb{N}} \subset \mathcal{L}(X)$ be a bounded sequence.

Let $\mathscr{S}(\mathbb{R})$ be the Schwartz space. Let $\phi_1 \in \mathscr{S}(\mathbb{R})$ be such that $\operatorname{supp}(\phi_1) \subset [2,4]$ and $\phi_1(3) = 1$. For $n \geq 1$, we let $h_n = 2^{2n-2}$. Define $\phi_n = \phi_1(\cdot/h_n)$. Then $\operatorname{supp}(\phi_n) \subset [2h_n, 4h_n]$ and $\phi_n(3h_n) = 1$. Let $\phi : \mathbb{R} \to \mathscr{L}(X)$ be defined by

$$\phi(t) = \begin{cases} \phi_n(t)N_n & \text{if } 2h_n \leq t \leq 4h_n \text{ for some } n \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Let $M = (\phi(k))_{k\in\mathbb{Z}}$. We claim that

$$\sup_{k\in\mathbb{Z}} \|\phi(k)\| < \infty, \tag{4}$$

$$\sup_{k\in\mathbb{Z}} \|k^l(\Delta^l M)(k)\| < \infty, \tag{5}$$

for $l \in \mathbb{N}$. Indeed (4) is clearly true. We will only give the proof for (5) when $l = 2$, the proof for the general case is similar.

First notice that when $4h_n \leq k \leq 8h_n - 2$ for some $n \in \mathbb{N}$, or $k \leq 0$, then $(\Delta^2 M)(k) = 0$. While when $2h_n - 2 < k < 4h_n$ for some $n \in \mathbb{N}$

$$\begin{aligned} (\Delta^2 M)(k) &= (\phi_n(k+2) - 2\phi_n(k+1) + \phi_n(k))N_n \\ &= \left(\phi_1\left(\frac{k+1}{h_n} + \frac{1}{h_n}\right) - 2\phi_1\left(\frac{k+1}{h_n}\right) + \phi_1\left(\frac{k+1}{h_n} - \frac{1}{h_n}\right)\right) N_n \\ &= \frac{1}{2h_n^2}(\phi_1''(\eta_1) + \phi_1''(\eta_2))N_n \end{aligned}$$

for some $\eta_1, \eta_2 \in \mathbb{R}$. We deduce that

$$\sup_{k\in\mathbb{Z}} \|k^2(\Delta^2 M)(k)\| \leq \sup_{n\in\mathbb{N}} \frac{16h_n^2}{4h_n^2}\|N_n\| \sup_{x\in\mathbb{R}} |\phi_1''(x)| \leq 4 \sup_{n\in\mathbb{N}} \|N_n\| \sup_{x\in\mathbb{R}} |\phi_1''(x)|.$$

Thus $M = (\phi(k))_{k\in\mathbb{Z}}$ is a periodic L^p-L^q-Fourier multiplier by assumption. Hence there exists $C > 0$ such that for $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$, we have

$$\left\| \sum_k e_k \otimes \phi(k)x_k \right\|_q \leq C \left\| \sum_k e_k \otimes x_k \right\|_p,$$

and, in particular,

$$\left\| \sum_{n\geq 1} e_{3h_n} \otimes M_k x_{3h_n} \right\|_q \leq C \left\| \sum_{n\geq 1} e_{3h_n} \otimes x_{3h_n} \right\|_p.$$

By (3), this implies that the sequence $(M_k)_{k\geq 1}$ is R-bounded. It is easy to check that if each countable subset of T is R-bounded then so is T. We deduce from this that each bounded subset in $\mathscr{L}(X)$ is actually R-bounded. By [3, Proposition 1.13], this implies that X is isomorphic to a Hilbert space. □

3. Multipliers on the line

Let X be a Banach space and consider the Banach space $L^p(\mathbb{R};X)$ for $1 < p < \infty$. We denote by $\mathscr{D}(\mathbb{R};X)$ the space of all X-valued C^∞-functions with compact support. $\mathscr{S}(\mathbb{R};X)$ will be the X-valued Schwartz space and we let $\mathscr{S}'(\mathbb{R};X) := \mathscr{L}(\mathscr{S}(\mathbb{R});X)$, where $\mathscr{S}(\mathbb{R})$ denotes the $\mathbb{C}$-valued Schwartz space. Let Y be another Banach space. Then given $M \in L^1_{\mathrm{loc}}(\mathbb{R};\mathscr{L}(X,Y))$, we may define an operator $T : \mathscr{F}^{-1}\mathscr{D}(\mathbb{R};X) \to \mathscr{S}'(\mathbb{R};Y)$ by means of

$$T\phi := \mathscr{F}^{-1}M\mathscr{F}\phi \quad \text{for all } \mathscr{F}\phi \in \mathscr{D}(\mathbb{R};X),$$

where $\mathscr{F}$ denotes the Fourier transform. Since $\mathscr{F}^{-1}\mathscr{D}(\mathbb{R};X)$ is dense in $L^p(\mathbb{R};X)$, we see that T is well defined on a dense subset of $L^p(\mathbb{R};X)$. We say that M is an *L^p-Fourier multiplier* on $L^p(\mathbb{R};X)$ if T can be extended to a bounded linear operator from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};Y)$.

The classical Mikhlin Fourier multiplier theorem has been extended to the operator-valued case by Weis. Let X and Y be UMD spaces, $1 < p < \infty$ and let $M \in C^1(\mathbb{R}\setminus\{0\}, \mathscr{L}(X,Y))$. If both $\{M(x) : x \neq 0\}$ and $\{xM'(x) : x \neq 0\}$ are R-bounded, then M defines a L^p-Fourier multiplier on $L^p(\mathbb{R};X)$ [18].

The following result shows that one cannot replace R-boundedness in the operator-valued Mikhlin theorem above by boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space.

THEOREM 3. *Let X be a Banach space. Then the following assertions are equivalent*:

(i) *X is isomorphic to a Hilbert space.*

(ii) *For some $1 < p < \infty$, each function $M \in C^\infty(\mathbb{R};\mathscr{L}(X))$ satisfying*

(a) $M(x) = 0$ *for* $x \leq 0$,

(b) $\sup_{x\in\mathbb{R}} \|M(x)\| < \infty$,

(c) $\sup_{x\in\mathbb{R}} (1+|x|)^l \|M^{(l)}(x)\| < \infty$ *for* $l \in \mathbb{N}$,

defines an L^p-Fourier multiplier on $L^p(\mathbb{R};X)$.

(iii) *For all $1 < p < \infty$, each function $M \in C^1(\mathbb{R}\setminus\{0\};\mathscr{L}(X))$ satisfying the conditions*

(a) $\sup_{x\neq 0} \|M(x)\| < \infty$,

(b) $\sup_{x\neq 0} \|xM'(x)\| < \infty$,

defines an L^p-Fourier multiplier on $L^p(\mathbb{R};X)$.

PROOF. (i) $\Rightarrow$ (iii). Assume that X is isomorphic to a Hilbert space. Then considering an orthonormal basis one easily verifies that each bounded subset in $\mathscr{L}(X)$ is actually R-bounded, so the result follows from the operator-valued Mikhlin Fourier multiplier theorem of Weis [18].

(iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). Assume (ii) holds. Let $(M_k)_{k\geq 0} \subset \mathscr{L}(X)$ be a bounded sequence and let $\phi \in \mathscr{D}(\mathbb{R})$ satisfying $\operatorname{supp}(\phi) \subset [1,2]$, $\sup_{x\in\mathbb{R}} |\phi(x)| = 1$ and $\phi(3/2) = 1$. Define $M \in C^\infty(\mathbb{R}; \mathscr{L}(X))$ by

$$M(x) = \begin{cases} 0 & \text{if } x \leq 1; \\ \phi(2^{-k}x)M_k & \text{if } 2^k \leq x < 2^{k+1} \text{ for some } k \geq 0. \end{cases}$$

Then $\sup_{x\in\mathbb{R}} \|M(x)\| = \sup_{k\geq 0} \|M_k\| < \infty$ and for $l \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}} (1+|x|)^l \|M^{(l)}(x)\| \leq 2^m \Big(\sup_{x\in\mathbb{R}} |x^l \phi^{(l)}(x)| \sup_{k\geq 0} \|M_k\| + \sup_{x\in\mathbb{R}} |\phi^{(l)}(x)| \sup_{k\geq 0} 2^{-lk} \|M_k\| \Big) < \infty.$$

So M is an L^p-Fourier multiplier on $L^p(\mathbb{R}; X)$ by assumption. By [9, Proposition 1] this implies that the set $\{M(x) : x \in \mathbb{R}\}$ is R-bounded. In particular, the sequence $(M_k)_{k\geq 0}$ is R-bounded. We deduce from this that each bounded subset in $\mathscr{L}(X)$ is R-bounded, by [3, Proposition 1.13] X is isomorphic to a Hilbert space. □

4. Multipliers on $\mathbb{Z}$

Let X, Y be Banach spaces and consider the Banach space $\ell^p(\mathbb{Z}; X)$ for $1 < p < \infty$. Let $\mathbb{T} = \{e^{it} : 0 \leq t < 2\pi\}$ be the torus. We consider the dense subspace P of $\ell^p(\mathbb{Z}; X)$ consisting of all elements having a finite support. Then for $f = (f_n)_{n\in\mathbb{Z}} \in P$, the Fourier transform of f is a function on $[-\pi, \pi]$ defined by $(\mathscr{F} f)(t) = \sum_{n\in\mathbb{Z}} f_n e^{int}$. Let $M \in L^\infty(-\pi, \pi; \mathscr{L}(X, Y))$. Then the function $M\mathscr{F} f$ is in $L^\infty(-\pi, \pi; Y)$, where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. We deduce that that $Tf := \mathscr{F}^{-1}(M\mathscr{F} f) \in c_0(\mathbb{Z}; Y)$ makes sense. We say that M is an *L^p-Fourier multiplier* on $\ell^p(\mathbb{Z}; X)$ if the mapping T can be extended to a bounded linear operator from $\ell^p(\mathbb{Z}; X)$ to $\ell^p(\mathbb{Z}; Y)$.

The classical Mikhlin Fourier multiplier theorem on $\ell^p(\mathbb{Z})$ has been extended to the operator-valued case by Blunck. Let $1 < p < \infty$, X be a UMD space, let $M \in C^1((-\pi, 0) \cup (0, \pi); \mathscr{L}(X))$ be such that both $\{M(t) : t \in (-\pi, 0) \cup (0, \pi)\}$ and $\{(e^{it} - 1)(e^{it} + 1)M'(t) : t \in (-\pi, 0) \cup (0, \pi)\}$ are R-bounded. Then M is an L^p-Fourier multiplier on $\ell^p(\mathbb{Z}; X)$ [5]. In particular, each $M \in C^1([-\pi, 0) \cup (0, \pi]; \mathscr{L}(X))$ such that both $\{M(t) : t \neq 0\}$ and $\{tM'(t) : t \neq 0\}$ are R-bounded, defines an L^p-Fourier multiplier on $\ell^p(\mathbb{Z}; X)$. Blunck has also established the R-boundedness of

L^p-Fourier multipliers on $\ell^p(\mathbb{Z};X)$: when M is an L^p-Fourier multiplier on $\ell^p(\mathbb{Z};X)$, then $\{M(t) : t$ is a Lebesgue point of $M\}$ is R-bounded.

The following result shows that one cannot replace the R-boundedness in Blunck's result by the boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space. As the proof is similar to that of Theorem 3, we omit it.

THEOREM 4. *Let X be a Banach space. Then the following assertions are equivalent*:

(i) *X is isomorphic to a Hilbert space.*

(ii) *For some $1 < p < \infty$, each function $M \in C^\infty([-\pi,\pi];\mathscr{L}(X))$ satisfying*

(a) $\sup_{x\in[-\pi,\pi]} \|M(x)\| < \infty$,

(b) $\sup_{x\in[-\pi,\pi]} |x|^l \|M^{(l)}(x)\| < \infty$ *for* $l \in \mathbb{N}$,

(c) $M(x) = 0$ *for* $x \leq 0$,

defines an L^p-Fourier multiplier on $\ell^p(\mathbb{Z};X)$.

(iii) *For all $1 < p < \infty$, each function $M \in C^1([-\pi,0)\cup(0,\pi];\mathscr{L}(X))$ satisfying*

(a) $\sup_{x\neq 0} \|M(x)\| < \infty$,

(b) $\sup_{x\neq 0} \|xM'(x)\| < \infty$,

defines an L^p-Fourier multiplier on $\ell^p(\mathbb{Z};X)$.

5. Concluding remarks

(a) One can actually show by using [3, Theorem 1.3] and the same argument as in the proof of Theorem 1, that when X and Y are UMD-spaces, then the assertions (ii) and (iii) in Theorem 1 are still equivalent for sequences in $\mathscr{L}(X,Y)$. Similarly, using [18, Theorem 3.4] (respectively, [5, Theorem 1.3]) one can show that when X and Y are UMD-spaces, the assertions (ii) and (iii) in Theorem 3 (respectively, Theorem 4) are still equivalent for functions with values in $\mathscr{L}(X,Y)$. Furthermore, these assertions are equivalent to X having cotype 2 and Y having type 2. This contains our Theorem 1, Theorem 3 and Theorem 4 by a result of Kwapien [14, pages 73–74], saying that a Banach space X is isomorphic to a Hilbert space if and only if X is of cotype 2 and of type 2.

(b) A restricted version of our results follows from the recent work of Kalton and Lancien on the maximal regularity problem [12]. In particular, the counterexample constructed in [12] can be used to show that the equivalences in Theorem 1 and Theorem 3 are true within the class of UMD Banach spaces which have an unconditional basis.

(c) In contrast to the L^p-spaces case, the situation for Hölder continuous function spaces is quite different. It has been shown that the operator-valued Marcinkiewicz (respectively, Mikhlin) Fourier multiplier theorem holds true on $C^\alpha_{\text{per}}([0, 2\pi]; X)$ (respectively, $C^\alpha(\mathbb{R}; X)$) for every Banach space X and $0 < \alpha < 1$ and for each sequence $M = (M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X)$ satisfying a second order condition:

$$\sup_{k\in\mathbb{Z}} \|M_k\| + \sup_{k\in\mathbb{Z}} \|k(\Delta^1 M)(k)\| + \sup_{k\in\mathbb{Z}} \|k^2(\Delta^2 M)(k)\| < \infty$$

(respectively, each function $M \in C^2(\mathbb{R} \setminus \{0\}; \mathscr{L}(X))$ satisfying a second order condition: $\sup_{x\neq 0} \|M(x)\| + \sup_{x\neq 0} \|xM'(x)\| + \sup_{x\neq 0} \|x^2 M''(x)\| < \infty$) (see Amann [1] and [2]). Here $C^\alpha_{\text{per}}([0, 2\pi]; X)$ denotes the space of all functions in $C^\alpha(\mathbb{R}, X)$ which are 2π-periodic. If the Banach space has a non-trivial type, then even the Marcinkiewicz condition of order 1 suffices (see [2] and [11]).

(d) Periodic L^p-Fourier multipliers (respectively, L^p-Fourier multipliers on $L^p(\mathbb{R}; X)$) of the form $M = (m_k I)_{k\in\mathbb{Z}}$, where $m_k \in \mathbb{C}$ for $k \in \mathbb{Z}$ (respectively, $M = f I$, where $f \in C^1(\mathbb{R} \setminus \{0\})$) on $L^p(0, 2\pi; X)$ (respectively, on $L^p(\mathbb{R}; X)$) have been studied by Zimmermann [19], where I denotes the identity operator on X. Actually Zimmermann's results follow from the operator-valued Marcinkiewicz (respectively, Mikhlin) Fourier multiplier theorem established in [3] (respectively, in [18]) as each subset $M \subset \mathscr{L}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R-bounded whenever $\Omega \subset \mathbb{C}$ is bounded. Zimmermann's results together with a result of Burkholder [7] show that the scalar-valued Marcinkiewicz (respectively, Mikhlin) Fourier multiplier theorem holds true for $L^p(0, 2\pi; X)$ (respectively, $L^p(\mathbb{R}; X)$) for some $1 < p < \infty$ if and only if X is a UMD space. A similar result characterizing UMD spaces via a scalar-valued Fourier multiplier theorem on $\ell^p(\mathbb{Z}; X)$ can be established based on results in [4].

(e) It is remarkable that in all three cases we consider here (Theorem 1, Theorem 3 and Theorem 4), the sequence $(M_k)_{k\in\mathbb{Z}}$ (or the function M) satisfying the Marcinkiewicz condition (of order l) without being a Fourier multiplier consists of operators of rank 1 (see [3, Proposition 1.13.]).

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