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Integral Equations and Operator Theory

Sums of Bisectorial Operators and Applications

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Abstract. We study sums of bisectorial operators on a Banach space X and show that interpolation spaces between X and $D(A)$ (resp. $D(B)$) are maximal regularity spaces for the problem $Ay + By = x$ in X. This is applied to the study of regularity properties of the evolution equation $u' + Au = f$ on $\mathbb{R}$ for $f \in L^p(\mathbb{R}; X)$ or $BUC(\mathbb{R}; X)$, and the evolution equation $u' + Au = f$ on $[0, 2\pi]$ with periodic boundary condition $u(0) = u(2\pi)$ in $L^p_{2\pi}(\mathbb{R}; X)$ or $C_{2\pi}(\mathbb{R}; X)$.

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1. Introduction

The method of sums of operators has been first used by Da Prato and Grisvard [13] for sectorial operators (see also [6], [9]). It gives conditions under which the equation $Ay + By = x$ can be solved. Here A and B are closed linear operators on a Banach space X with domain $D(A)$ and $D(B)$, respectively. It is known that in general for arbitrary $x \in X$, only the existence of a mild solution can be guaranteed. However, when x is in an interpolation space between X and $D(A)$ (resp. $D(B)$), then the solution y is in $D(A) \cap D(B)$. Moreover, one has Ay and By belong to the same interpolation space, *i.e.*, interpolation spaces between X and $D(A)$ (resp. $D(B)$) are maximal regularity spaces for the equation $Ay + By = x$.

In this paper, we are interested in the method of sums for bisectorial operators and we establish similar results as in the case of sectorial operators. More precisely let A and B be linear operators on X, assume that both A and B are sectorial in two sectors (see the next section for the definition), that A and B commute in the sense of resolvents and that $\sigma(A)$ and $\sigma(-B)$ are disjoint. Then we can

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find a curve Γ inside $\rho(A) \cap \rho(-B)$ which separates $\sigma(A)$ and $\sigma(-B)$, and this is not at all obvious, see Appendix. As in the sectorial operator case, we then define a bounded linear operator S on X by a contour integral over Γ using the resolvent of A and $-B$. For $x \in X$, the element Sx is a solution of the equation $Ay + By = x$ in a weak sense. In particular, when $D(A) + D(B)$ is dense in X, there exist $y_n \in D(A) \cap D(B)$ such that $y_n \to Sx$ and $Ay_n + By_n \to x$ as $n \to \infty$. We should notice that it is known that when $x \in X$, the equation $Ay + By = x$ does not necessarily have a solution $y \in D(A) \cap D(B)$. However, when x is in an interpolation space $D_A(\theta, p)$ (resp. $D_B(\theta, p)$) between X and $D(A)$ (resp. $D(B)$), then $Sx \in D(A) \cap D(B)$, $ASx \in D_A(\theta, p) \cap D_B(\theta, p)$ and $BSx \in D_A(\theta, p)$ (resp. $BSx \in D_A(\theta, p) \cap D_B(\theta, p)$ and $ASx \in D_B(\theta, p)$), this means that $D_A(\theta, p)$ and $D_B(\theta, p)$ are maximal regularity spaces for the equation $Ay + By = x$. In our treatment of interpolation spaces we are also inspired by Clément-Gripenberg-Högnäs [9] who proved "cross regularity" extending the Da Prato-Grisvard's result for sectorial operators (see also [10]). A few words should be said concerning our more complicated spectral conditions we consider and which demand sophisticated contours. In the case of sectorial operators A and B, one may always reduce the situation to the case where the spectra of A and $-B$ are situated in disjoint sectors by replacing A and B by $A + \lambda$ and $B + \lambda$, and this is actually done in [13]. However, for bisectorial operators this is no longer possible. On the other hand, the more complicated spectra occur naturally in the context of periodic problems, see section 5. In addition, our method also allows us to prove the spectral inclusion

$$\sigma(\overline{A + B}) \subset \sigma(A) + \sigma(B)$$

for bisectorial operators. This relation was proved independently in [21, 8.3] and [5] in the sectorial operators case.

In section 5, our results are applied to study regularity properties of the evolution equation

$$u'(t) + Au(t) = f(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where $f \in L^p(\mathbb{R}; X)$ for some $1 \leq p < \infty$ (resp. $BUC(\mathbb{R}; X)$), A is an invertible linear operator on X, sectorial in two symmetric sectors $\Sigma_\theta = \{z \in \mathbb{C} : |arg(z) - \frac{\pi}{2}| \leq \theta$ or $|arg(z) - \frac{3\pi}{2}| \leq \theta\}$ for some $0 < \theta < \frac{\pi}{2}$. Since the operator $\frac{d}{dt}$ generates the bounded translation group on $L^p(\mathbb{R}; X)$ (resp. $BUC(\mathbb{R}; X)$), for $0 < \theta < \frac{\pi}{2}$, $\frac{d}{dt}$ is sectorial in two symmetric sectors $\Sigma'_\theta = \{z \in \mathbb{C} : |arg(z) - \frac{\pi}{2}| \geq \theta$ or $|arg(z) - \frac{3\pi}{2}| \geq \theta\}$. Thus our abstract results can be applied to the operators $\frac{d}{dt}$ and $\mathcal{A}$, where $(\mathcal{A}f)(t) := A(f(t))$. As an immediate consequence of our maximal regularity results applied to the operator $\frac{d}{dt}$, we deduce that the Besov space $B^\alpha_{p,q}(\mathbb{R}; X)$ and the space $C^\alpha_b(\mathbb{R}; X)$ of all X-valued bounded α-Hölder continuous functions are maximal regularity spaces for the problem (1.1), where $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $0 < \alpha < 1$. When we apply our maximal regularity results to the operator $\mathcal{A}$, we obtain that $L^p(\mathbb{R}; D_A(\theta, p))$ and $BUC(\mathbb{R}; D_A(\theta, \infty_0))$ are maximal regularity spaces for the problem (1.1), where $1 \leq p < \infty$ and $0 < \theta < 1$. Our abstract results can be also applied to the equation $u' + Au = f$ on $[0, 2\pi]$ with periodic boundary condition $u(0) = u(2\pi)$ and we get similar results on maximal regularity spaces.

The sum method presented here is not the only one to study regularity of (1.1) and the analogous periodic problem. Another method is based on operator-valued multiplier theorems (see Weis [24], Denk-Hieber-Prüss [14], Schweiker [23], [2] for the L^p-case, and Amann [1], [4] for the C^α and Besov-case). Some of our results have been obtained by this other method. However, in many cases, the multiplier method needs a geometrical condition on the underlying Banach space (namely, the UMD-property).

2. Preliminaries

Let X be a Banach space and $A : D(A) \to X$ be a closed operator on X. We denote by $\rho(A)$ the resolvent set of A and $\sigma(A)$ will be the spectrum of A. For $\lambda \in \rho(A)$, we denote the resolvent $(\lambda - A)^{-1}$ by $R(\lambda, A)$.

Let $\omega > 0$ and $0 \leq \theta < 2\pi$. Assume that

$$(H_{\theta,\omega}) \qquad \begin{cases} \{re^{i\theta} : r > \omega\} \subset \rho(A) \\ C_{\theta,\omega} = \sup_{r>\omega} \|rR(re^{i\theta}, A)\| < \infty. \end{cases}$$

Then there exists $\alpha > 0$ such that $\{re^{i\phi} : r > \omega, \theta - \alpha \leq \phi \leq \theta + \alpha\} \subset \rho(A)$ and $\sup_{r>\omega,\theta-\alpha\leq\phi\leq\theta+\alpha} \|rR(re^{i\phi}, A)\| < \infty$. We say in this case that A is *sectorial* in the sector $\{re^{i\phi} : r > \omega, \theta - \alpha \leq \phi \leq \theta + \alpha\}$. When A is sectorial in two symmetric sectors with respect to the origin, we say that it is *bisectorial*.

Let $0 < s < 1$ and $1 \leq p \leq \infty$. Define

$$\begin{aligned} D_A(s,p) &= \{x \in X : \|t^s AR(te^{i\theta}, A)x\| \in L^p(\omega, \infty; \frac{dt}{t})\} \\ \|x\|_{D_A(s,p)} &= \|x\| + \|t^s AR(te^{i\theta}, A)x\|_{L^p(\omega,\infty;\frac{dt}{t})}, \end{aligned}$$

and

$$\begin{aligned} D_A(s,\infty_0) &= \{x \in D_A(s,\infty) : \lim_{t\to+\infty} \|t^s AR(te^{i\theta}, A)x\| = 0\} \\ \|x\|_{D_A(s,\infty_0)} &= \|x\|_{D_A(s,\infty)}. \end{aligned}$$

For $1 \leq p \leq \infty$, we define

$$\begin{aligned} D_A(1,p) &= \{x \in X : \|tA^2 R(te^{i\theta}, A)^2 x\| \in L^p(\omega, \infty; \frac{dt}{t})\} \\ \|x\|_{D_A(1,p)} &= \|x\| + \|tA^2 R(te^{i\theta}, A)^2 x\|_{L^p(\omega,\infty;\frac{dt}{t})}. \end{aligned}$$

Let $0 < s < 1,\ p \in [1, \infty] \cup \{\infty_0\}$ or $s = 1,\ p \in [1, \infty]$, then it is easy to verify that $D_A(s,p)$ equipped with the norm $\|\cdot\|_{D_A(s,p)}$ is a Banach space. For different $\omega > 0$, the different norms on $D_A(s,p)$ are equivalent. On the other hand, if $0 \leq \beta < 2\pi$

is such that the operator A satisfies the assumption $(H_{\beta,\omega})$, then for $r > \omega$,

$$\begin{aligned} &\|AR(re^{i\theta},A)x\| = \|(re^{i\beta}-A)R(re^{i\theta},A)AR(re^{i\beta},A)x\| \\ &\leq (\|AR(re^{i\theta},A)\| + \|rR(re^{i\theta},A)\|)\|AR(re^{i\beta},A)x\| \\ &\leq (1+2C_{\theta,\omega})\|AR(re^{i\beta},A)x\|. \end{aligned} \tag{2.1}$$

This implies that for different θ such that A satisfies the assumption $(H_{\theta,\omega})$, we define the same space $D_A(s,p)$ and equivalent norms on it whenever $0 < s < 1$ and $p \in [1,\infty] \cup \{\infty_0\}$. Applying (2.1) twice we show that this is also true when $s = 1$ and $1 \leq p \leq \infty$.

When $\omega = 0$, the spaces $D_A(s,p)$ were first introduced by Grisvard [16], who showed in particular that, when $0 < s < 1$ and $1 \leq p \leq \infty$, then

$$D_A(s,p) = (X, D(A))_{s,p} \tag{2.2}$$

(where $D(A)$ is equipped with the graph norm so that it becomes a Banach space), and when $1 > s' > s$, or $s' = s$ and $1 \leq q \leq p \leq \infty$,

$$D_A(s',q) \subset D_A(s,p). \tag{2.3}$$

It follows that when $s' > s,\ 1 \leq p \leq \infty$, we have

$$D_A(s',p) \subset D_A(s,\infty_0). \tag{2.4}$$

Our first task is to show that the relation (2.2) remains true when $\omega > 0$, and (2.3) is valid when $\omega > 0$ except for some special case. Without loss of generality, in the sequel we assume that $\theta = 0$. Let A be a closed linear operator satisfying the assumption $(H_{0,\omega})$ for some $\omega > 0$. Let $A_\omega := A - \omega$. Then A_ω satisfies the assumption $(H_{0,0})$. Indeed, when $r > 0$, we have $\|rR(r,A_\omega)\| = \|(r+\omega)R(r+\omega,A) - \omega R(r+\omega,A)\| \leq 2C_{0,\omega}$.

First for $0 < s < 1$ and $p \in [1,\infty] \cup \{\infty_0\}$, we compare the spaces $D_A(s,p)$ and $D_{A_\omega}(s,p)$.

Lemma 2.1. *For $0 < s < 1$ and $p \in [1,\infty] \cup \{\infty_0\}$, we have $D_A(s,p) = D_{A_\omega}(s,p)$ with equivalent norms.*

Proof. We will only give the proof for $1 \leq p \leq \infty$, the proof for the case $p = \infty_0$ is similar. Let $x \in D_A(s,p)$, *i.e.*, $\|t^s AR(t,A)x\| \in L^p(\omega,\infty;\frac{dt}{t})$. We have to show that $\|t^s A_\omega R(t+\omega,A)x\| \in L^p(0,\infty;\frac{dt}{t})$. We have

$$t^s A_\omega R(\omega+t,A)x = t^s AR(\omega+t,A)x - t^s \omega R(\omega+t,A)x.$$

Since the function $\|t^s\omega R(\omega+t,A)x\| \leq \frac{\omega C_{0,\omega} t^s \|x\|}{\omega+t}$ belongs to $L^p(0,\infty;\frac{dt}{t})$ and the function $\|t^s AR(\omega+t,A)x\| \leq (1+C_{0,\omega})t^s\|x\|$ belongs to $L^p(0,\omega;\frac{dt}{t})$, it will suffice to show that $\|t^s AR(\omega+t,A)x\| \in L^p(\omega,\infty;\frac{dt}{t})$. We have for $t > \omega$

$$\begin{aligned} &\|t^s AR(t,A)x - t^s AR(\omega+t,A)x\| \\ \leq\ &\|\omega t^s AR(t,A)R(\omega+t,A)x\| \leq C_{0,\omega}(1+C_{0,\omega})\omega\frac{t^s\|x\|}{\omega+t} \end{aligned} \tag{2.5}$$

which belongs to $L^p(\omega, \infty; \frac{dt}{t})$. This together with the assumption $\|t^s AR(t, A)x\| \in L^p(\omega, \infty; \frac{dt}{t})$ shows that $\|t^s AR(\omega + t, A)x\| \in L^p(\omega, \infty; \frac{dt}{t})$.

Conversely, Let $x \in D_{A_\omega}(s, p)$, *i.e.*, $\|t^s A_\omega R(t, A_\omega)x\| \in L^p(0, \infty; \frac{dt}{t})$. Then $\|t^s AR(\omega + t, A)x\| \in L^p(0, \infty; \frac{dt}{t})$ as $\|t^s \omega R(t, A_\omega)x\| \leq \frac{t^s \omega C_{0,\omega}\|x\|}{t+\omega} \in L^p(0, \infty; \frac{dt}{t})$. In particular, $\|t^s AR(\omega + t, A)x\| \in L^p(\omega, \infty; \frac{dt}{t})$. This together with the estimate (2.5) implies that $\|t^s AR(t, A)x\| \in L^p(\omega, \infty; \frac{dt}{t})$. Thus $x \in D_A(s, p)$ and the proof is finished. □

Lemma 2.2. *We have* $D_A(1, \infty) = D_{A_\omega}(1, \infty)$ *with equivalent norms.*

Proof. For $t > 0$ and $x \in X$, we have

$$\begin{aligned} tA_\omega^2 R(t, A_\omega)^2 x &= tA^2 R(\omega + t, A)^2 x \\ &\quad -2\omega t AR(\omega + t, A)^2 x + \omega^2 t R(\omega + t, A)^2 x. \end{aligned}$$

The last two terms on the right hand side are bounded on $(0, \infty)$ by the assumption $(H_{0,\omega})$. Hence $\|tA_\omega^2 R(t, A_\omega)^2 x\|$ is bounded on $(0, \infty)$ if and only if $\|tA^2 R(\omega + t, A)^2 x\|$ is bounded on $(0, \infty)$. On the other hand, we have

$$\begin{aligned} & \|tA^2 R(\omega + t, A)^2 x - tA^2 R(t, A)^2 x\| \\ \leq \quad & \|t\omega A^2 R(\omega + t, A)^2 R(t, A)x\| + \|t\omega A^2 R(\omega + t, A)R(t, A)^2 x\| \end{aligned}$$

is bounded on (ω, ∞) by the assumption $(H_{0,\omega})$. We deduce that $\|tA_\omega^2 R(t, A_\omega)^2 x\|$ is bounded on $(0, \infty)$ if and only if $\|tA^2 R(t, A)^2 x\|$ is bounded on (ω, ∞). The claimed result follows. □

Remarks 2.3. (a) For $0 < s < 1$ and $1 \leq p \leq \infty$, we have $D_A(s, p) = (X, D(A))_{s,p}$ with equivalent norms. This follows from Lemma 2.1 and the relation (2.2).

(b) When $0 < s < s' < 1$, or $0 < s = s' < 1$ and $q \leq p$, we have $D_A(s', q) \subset D_A(s, p)$. When $1 \leq p \leq \infty$ and $0 < s < 1$, we have $D_A(1, \infty) \subset D_A(s, p) \cap D_A(s, \infty_0)$. This follows from Lemma 2.1, Lemma 2.2 and the relations (2.3) and (2.4)

3. Sums of Bisectorial Operators

We need some preliminary results on separating curves. If $\Omega \subset \mathbb{C}$ is open and $K \subset \Omega$ is compact, then there exists a piecewise affine closed oriented path in $\Omega \setminus K$ surrounding K counterclockwise, see [11] or [8]. In addition to this, we need the following more complicated lemma on the existence of a separating curve, which is new. Its proof will be given in the appendix.

Lemma 3.1. *Let* $a, b > 0$ *and* $R = [-a, a] + i[-b, b]$. *Let* $S, T \subset \mathbb{C}$ *be open such that* $R \subset S \cup T$ *and* $S^c \cap T^c = \emptyset$, $\pm a + i[-b, b] \subset S$ *and* $[-a, a] \pm ib \subset T$. *Then at least one of the following two cases occurs:*

(a) *there exists a piecewise affine curve* Γ_1 *inside* $S \cap T \cap R$ *from* $-a - ib$ *to* $-a + ib$, *and another piecewise affine curve* Γ_2 *inside* $S \cap T \cap R$ *from* $a + ib$ *to* $a - ib$. *Moreover,* $\Gamma_1 \cap \Gamma_2 = \emptyset$.

(b) *there exists a piecewise affine curve* Γ_1' *inside* $S \cap T \cap R$ *from* $-a-ib$ *to* $a-ib$, *and another piecewise affine curve* Γ_2' *inside* $S \cap T \cap R$ *from* $a+ib$ *to* $-a+ib$. *Moreover,* $\Gamma_1' \cap \Gamma_2' = \emptyset$.

Let A, B be closed operators on X. Assume that there exist $0 < \theta_A, \theta_B < \frac{\pi}{2}$ and $\omega > 0$ such that

(H_1): $\theta_A + \theta_B > \frac{\pi}{2}$.

(H_2): A and B commute in the sense of resolvent, *i.e.* for $\lambda \in \rho(A)$, $\mu \in \rho(B)$, we have

$$R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A).$$

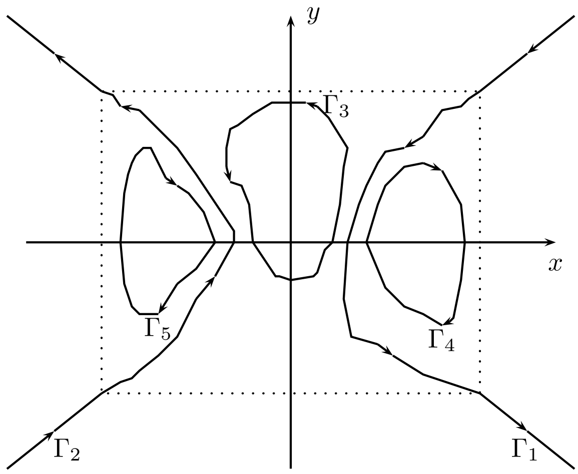
$$(H_3): \begin{cases} \Omega_B &= \{\, |arg(z)| < \theta_B \text{ or } |\pi - arg(z)| < \theta_B\} \\ & \quad \cap\{\, |Re(z)| \geq \omega\} \subset \rho(-B) \\ C_B &= \sup_{z \in \Omega_B} \|zR(z, -B)\| < \infty. \end{cases}$$

$$(H_4): \begin{cases} \Omega_A &= \{\, |\frac{\pi}{2} - arg(z)| < \theta_A \text{ or } |\frac{3\pi}{2} - arg(z)| < \theta_A\} \\ & \quad \cap\{\, |Im(z)| \geq \omega\} \subset \rho(A) \\ C_A &= \sup_{z \in \Omega_A} \|zR(z, A)\| < \infty. \end{cases}$$

(H_5): $\sigma(A) \cap \sigma(-B) = \emptyset$.

Then there exist $\frac{\pi}{2} - \theta_B < \theta < \theta_A$, $a, b > 0$ such that $arctan(\theta) = \frac{b}{a}$, $\{z \in \mathbb{C} : |arg(z)| \leq \theta \text{ or } |\pi - arg(z)| \leq \theta\} \cap \{z \in \mathbb{C} : \ |Re(z)| \geq a\} \subset \rho(-B)$, $\{z \in \mathbb{C} : |\frac{3\pi}{2} - arg(z)| \leq \frac{\pi}{2} - \theta \text{ or } |\frac{\pi}{2} - arg(z)| \leq \frac{\pi}{2} - \theta\} \cap \{z \in \mathbb{C} : \ |Im(z)| \geq b\} \subset \rho(A)$ and $\sigma(A) \cap \sigma(-B) \cap R = \emptyset$, where $R := \{z \in \mathbb{C} : \ |Re(z)| \leq a, |Im(z)| \leq b\}$. By Lemma 3.1, there exist a piecewise affine curve Γ_1 inside $R \cap \rho(A) \cap \rho(-B)$ from $a + ib$ to $a - ib$, another piecewise affine curve Γ_2 inside $R \cap \rho(A) \cap \rho(-B)$ from $-a - ib$ to $-a + ib$ satisfying $\Gamma_1 \cap \Gamma_2 = \emptyset$ or, there exist a piecewise affine curve Γ_1' inside $R \cap \rho(A) \cap \rho(-B)$ from $a + ib$ to $-a + ib$, another piecewise affine curve Γ_2' inside $R \cap \rho(A) \cap \rho(-B)$ from $-a - ib$ to $a - ib$ satisfying $\Gamma_1' \cap \Gamma_2' = \emptyset$. Assume in the rest of this paper that we are in the first case (the argument for the second case is similar).

Let Γ_3 be a closed piecewise affine curve inside $R \cap \rho(A) \cap \rho(-B)$ and the region limited by Γ_1, Γ_2, $\{t + ib : -a \leq t \leq a\}$ and $\{t - ib : -a \leq t \leq a\}$ such that the part of $\sigma(A)$ inside the region limited by Γ_1, Γ_2, $\{t + ib : -a \leq t \leq a\}$ and $\{t - ib : -a \leq t \leq a\}$ is contained in the region limited by Γ_3 and the region limited by Γ_3 is contained in $\rho(-B)$. Let Γ_4 be a piecewise affine closed curve inside $R \cap \rho(A) \cap \rho(-B)$ and the region limited by Γ_1 and $\{a + it : -b \leq t \leq b\}$, such that the part of $\sigma(-B)$ inside the region limited by Γ_1 and $\{a + it : -b \leq t \leq b\}$ is contained in the region limited by Γ_4 and the region limited by Γ_4 is contained in $\rho(A)$. Let Γ_5 be a piecewise affine closed curve inside $R \cap \rho(A) \cap \rho(-B)$ and the region limited by Γ_2 and $\{-a + it : -b \leq t \leq b\}$, such that the part of $\sigma(-B)$ inside the region limited by Γ_2 and $\{-a + it : -b \leq t \leq b\}$ is contained in the region limited by Γ_5 and the region limited by Γ_5 is contained in $\rho(A)$. We extend Γ_1 from $\infty e^{i\theta}$ to $a + ib$, and from $a - ib$ to $\infty e^{-i\theta}$, extend Γ_2 from $-a + ib$ to $\infty e^{i(\pi-\theta)}$, and from $\infty e^{i(\theta-\pi)}$ to $-a - ib$. So we have the following figure:



Let $\Gamma = \cup_{k=1}^{5}\Gamma_k$ be the closed Jordan curve oriented as in the figure. We can assume that $0 \notin \Gamma$ (if this is not the case, we may take a small perturbation of Γ, this is possible as $\rho(A) \cap \rho(-B)$ is open in $\mathbb{C}$). Then $\Gamma \subset \rho(A) \cap \rho(-B)$ and by the assumption (H_3) and (H_4), we have

$$C'_A = \sup_{z\in\Gamma} \|zR(z,A)\| < \infty \tag{3.1}$$

$$C'_B = \sup_{z\in\Gamma} \|zR(z,-B)\| < \infty. \tag{3.2}$$

We define

$$S = \frac{1}{2\pi i}\int_\Gamma R(z,A)R(z,-B)dz. \tag{3.3}$$

By (3.1), (3.2) and the fact that $\Gamma \subset \rho(A) \cap \rho(-B)$, S is linear and bounded on X. We will see that for $x \in X$, Sx is the solution of the equation $Ay + By = x$ in a weak sense.

Remark 3.2. Let A and B be two operators satisfying (H_1)-(H_5). For $x \in X$, it is clear that the integrals $\int_\Gamma R(z,A)x\frac{dz}{z}$ and $\int_\Gamma R(z,-B)x\frac{dz}{z}$ converge. An easy application of the Residue Theorem shows that when 0 is in the region limited by Γ, we have

$$\frac{1}{2\pi i}\int_\Gamma R(z,A)x\frac{dz}{z} = 0, \quad \frac{1}{2\pi i}\int_\Gamma R(z,-B)x\frac{dz}{z} = B^{-1}x.$$

When 0 is not in the region limited by Γ, we have

$$\frac{1}{2\pi i}\int_\Gamma R(z,A)x\frac{dz}{z} = A^{-1}x, \quad \frac{1}{2\pi i}\int_\Gamma R(z,-B)x\frac{dz}{z} = 0,$$

where the curve Γ is completed at infinity by identifying the points $\infty e^{i\theta}, \infty e^{i(\pi-\theta)}$, and the points $\infty e^{-i\theta}, \infty e^{i(\theta-\pi)}$.

Proposition 3.3. *Let A and B be two operators satisfying (H_1)-(H_5). Then S is linear and bounded from X to $D_A(1,\infty)$ (and to $D_B(1,\infty)$).*

Proof. Let $y \in X,\ \ x = Sy$. Then for large t,

$$B^2R(t,B)^2x = \frac{1}{2\pi i}\int_\Gamma \frac{z^2}{(t+z)^2}R(z,A)R(z,-B)ydz. \tag{3.4}$$

Indeed,

$$\begin{aligned} B^2R(t,B)^2x &= (BR(t,B))^2x \\ &= x - 2tR(t,B)x + t^2R(t,B)^2x. \end{aligned} \tag{3.5}$$

We have by the resolvent identity and Cauchy's theorem

$$\begin{aligned} R(t,B)x &= \frac{1}{2\pi i}\int_\Gamma R(z,A)R(z,-B)R(t,B)ydz \\ &= \frac{1}{2\pi i}\int_\Gamma R(z,A)R(z,-B)y\frac{dz}{t+z} \\ &\quad +\frac{1}{2\pi i}\int_\Gamma R(z,A)R(t,B)y\frac{dz}{t+z} \\ &= \frac{1}{2\pi i}\int_\Gamma R(z,A)R(z,-B)y\frac{dz}{t+z} \end{aligned} \tag{3.6}$$

for large $t>0$, as $t+z \neq 0$ when t is large enough and so the function $\frac{R(z,A)R(t,B)y}{t+z}$ is analytic in the region limited by Γ, where the path Γ is completed at infinity by identifying the points $\infty e^{i\theta},\ \infty e^{i(\pi-\theta)}$, and the points $\infty e^{-i\theta},\ \infty e^{i(\theta-\pi)}$. A similar computation shows that when t is large enough we have

$$R(t,B)^2x = \frac{1}{2\pi i}\int_\Gamma R(z,A)R(z,-B)y\frac{dz}{(t+z)^2}.$$

We deduce from (3.5) that

$$B^2R(t,B)^2x = \frac{1}{2\pi i}\int_\Gamma \frac{z^2}{(t+z)^2}R(z,A)R(z,-B)ydz,$$

when t is large enough. By (3.1) and (3.2),

$$\begin{aligned} \|tB^2R(t,B)^2x\| &\le \frac{1}{2\pi}\int_\Gamma \frac{C'_AC'_Bt\|y\|}{|t+z|^2}|dz| \\ &= \frac{C'_AC'_B\|y\|}{2\pi}\int_{\Gamma_t}\frac{|dz|}{|1+z|^2} \le C\|y\| \end{aligned}$$

is bounded for large $t>0$, where $\Gamma_t = \{z/t : z \in \Gamma\}$ and C is a constant. This shows that S is linear and bounded from X to $D_B(1,\infty)$. A similar argument shows that S is linear and bounded from X to $D_A(1,\infty)$ and finishes the proof. □

By Remarks 2.3 and Proposition 3.3, if A and B satisfy the assumptions (H_1)-(H_5), then for $0<\theta<1$ and $p \in [1,\infty]\cup\{\infty_0\}$, the operator S is linear and bounded from X to $D_A(\theta,p)$ and to $D_B(\theta,p)$.

In the following lemma, we will see that when $0<\theta<1, 1\le p\le\infty$ and $x \in D_A(\theta,p)+D_B(\theta,p)$, the equation $Ay+By=x$ is solvable with solution $y \in D(A)\cap D(B)$.

Lemma 3.4. *Let A and B be two operators satisfying (H_1)-(H_5). Let $0 < \theta < 1$, $1 \le p \le \infty$ and $y \in D_A(\theta, p) + D_B(\theta, p)$. Then $x = Sy \in D(A) \cap D(B)$ and $(A + B)Sy = y$.*

Proof. We will only give the proof for $y \in D_B(\theta, p)$, the proof for $y \in D_A(\theta, p)$ is similar. Let $y \in D_B(\theta, p)$ be fixed. Since $D_B(\theta, p) \subset D_B(\theta, \infty)$, the function $|z|^\theta BR(z, -B)y$ is bounded when $z \in \Gamma$ is far enough from 0. By (3.1) and (3.2) this implies that $x = Sy \in D(B)$ and

$$Bx = \frac{1}{2\pi i} \int_\Gamma R(z, A)BR(z, -B)y dz. \tag{3.7}$$

To show that $x = Sy \in D(A)$, we use the equality

$$R(z, -B)y = (y - BR(z, -B)y)/z.$$

Thus

$$x = \frac{1}{2\pi i} \int_\Gamma R(z, A)y \frac{dz}{z} - \frac{1}{2\pi i} \int_\Gamma R(z, A)BR(z, -B)y \frac{dz}{z}.$$

The first term is 0 (when 0 is not in the region limited by Γ) or $A^{-1}y$ (when 0 is in the region limited by Γ) by Residue Theorem. Thus it belongs to $D(A)$. Here the path Γ is completed by identifying the points $\infty e^{i\theta}$, $\infty e^{i(\pi-\theta)}$, and the points $\infty e^{-i\theta}$, $\infty e^{i(\theta-\pi)}$. The second term also belongs to $D(A)$ as

$$\|AR(z, A)BR(z, -B)y/z\| \le \frac{C}{|z|^{1+\theta}}$$

for some constant $C > 0$ independent from z. Hence $x \in D(A)$ and by Remark 3.2 and (3.7)

$$\begin{aligned} Ax &= \frac{1}{2\pi i} A \int_\Gamma R(z, A)y \frac{dz}{z} + \frac{1}{2\pi i} \int_\Gamma BR(z, -B)y \frac{dz}{z} \\ &- \frac{1}{2\pi i} \int_\Gamma R(z, A)BR(z, -B)y dz = y - Bx. \end{aligned}$$

Therefore $(A + B)Sx = x$. □

It is clear from the definition that $D(A) \subset D_A(\theta, p)$ and $D(B) \subset D_B(\theta, p)$ whenever $0 < \theta < 1$ and $1 \le p \le \infty$. Thus we have following corollary

Corollary 3.5. *Let A and B be two operators satisfying (H_1)-(H_5). For $x \in D(A) + D(B)$, we have $Sx \in D(A) \cap D(B)$ and $(A + B)Sx = x$.*

Even though for $x \in X$, the equation $Ay + By = x$ does not necessarily have a solution $y \in D(A) \cap D(B)$, the following result shows that when $D(A) + D(B)$ is dense in X, then Sx is a solution of $Ay + By = x$ in a weak sense: there exist $y_n \in D(A) \cap D(B)$ such that $y_n \to Sx$ and $Ay_n + By_n \to x$ as $n \to \infty$.

Theorem 3.6. *Let A and B be two operators satisfying (H_1)-(H_5). Then $A + B$ is closable. Furthermore when $D(A) + D(B)$ is dense in X, if we denote the closure of $A + B$ by L, then $0 \in \rho(L)$ and $L^{-1} = S$.*

Proof. Let $x_n \in D(A) \cap D(B)$, $y \in X$ be such that $x_n \to 0$ and $Ax_n + Bx_n \to y$ as $n \to \infty$. By Corollary 3.5, we have $S(A+B)x_n = x_n$. We deduce that $Sy = 0$. Now let $\mu \in \rho(A)$ which is not empty by assumption, then by the assumption (H$_2$) and Corollary 3.5

$$\begin{aligned} R(\mu, A)y &= (A+B)SR(\mu, A)y \\ &= (A+B)R(\mu, A)Sy = 0. \end{aligned}$$

Here we use the fact that $R(\mu, A)y \in D(A)$ so that we can apply Corollary 3.5. This implies that $y = 0$. Therefore $A+B$ is closable. When $D(A)+D(B)$ is dense in X, the facts that $0 \in \rho(L)$ and $L^{-1} = S$ follow immediately from Corollary 3.5. □

Theorem 3.6 can be transformed into a result on spectral inclusion. For sectorial operators this had been done before independently by [21, 8.2] and [5, Appendix]. In the following we omit the assumption (H_5) that $\sigma(A) \cap \sigma(-B) = \emptyset$.

Corollary 3.7. *Suppose that A and B are operators satisfying assumptions (H_1)-(H_4). Assume furthermore that $D(A) + D(B)$ is dense in X. If $\sigma(A) + \sigma(B) \neq \mathbb{C}$, then $A + B$ is closable and $\sigma(\overline{A+B}) \subset \sigma(A) + \sigma(B)$.*

Proof. Let $\lambda \in \mathbb{C} \setminus (\sigma(A) + \sigma(B))$. Then $A - \lambda$ and B satisfy assumptions (H_1)-(H_5). Thus $A + B - \lambda$ is closable and its closure is invertible by Theorem 3.6. Hence $A + B$ is closable and $\overline{A+B} - \lambda = \overline{A+B-\lambda}$ is invertible. □

4. Strict Solutions in Interpolation Spaces

Our next aim is to show that for $0 < \theta < 1$, $p \in [1,\infty] \cup \{\infty_0\}$ and $x \in D_A(\theta, p)$ (resp. $D_B(\theta, p)$), we have $Sx \in D(A) \cap D(B)$, $ASx, BSx \in D_A(\theta, p)$ and $ASx \in D_B(\theta, p)$ (resp. $ASx, BSx \in D_B(\theta, p)$ and $BSx \in D_A(\theta, p)$). Thus the spaces $D_A(\theta, p)$ and $D_B(\theta, p)$ are maximal regularity spaces for the equation $Ay + By = x$. The proof of this result is similar to that for sectorial operators given by Da Prato and Grisvard [13] (see also [6] [9]).

Theorem 4.1. *Let A and B be two operators satisfying (H$_1$)-(H$_5$). Let $0 < \theta < 1$, $p \in [1,\infty] \cup \{\infty_0\}$ and $y \in D_B(\theta, p)$ (resp. $D_A(\theta, p)$). Then $BSy \in D_A(\theta, p) \cap D_B(\theta, p)$, $ASy \in D_B(\theta, p)$ (resp. $ASy \in D_A(\theta, p) \cap D_B(\theta, p)$, $BSy \in D_A(\theta, p)$).*

Proof. We will only give the proof for $y \in D_B(\theta, p)$ and $1 \leq p \leq \infty$, the proof for the other case is similar. Let $y \in D_B(\theta, p)$ and $x = Sy$, by Lemma 3.4, $x \in D(A) \cap D(B)$. Then by (3.6)

$$BR(t, B)x = -\frac{1}{2\pi i} \int_\Gamma \frac{z}{t+z} R(z, A)R(z, -B)y dz$$

Thus

$$BR(t, B)Bx = -\frac{1}{2\pi i} \int_\Gamma \frac{z}{t+z} R(z, A)BR(z, -B)y dz. \tag{4.1}$$

An elementary computation shows that there exists $C > 0$ such that when $z \in \Gamma$ and when t is big enough, we have $|t+z| \geq C(t+r)$, where $z = re^{i\theta}$. By (3.1) and (3.2), we have

$$\begin{aligned}\|t^\theta BR(t,B)Bx\| &\leq \frac{C'_A}{2\pi}\int_\Gamma \frac{t^\theta \|BR(z,-B)y\|}{|t+z|}|dz| \\ &\leq \frac{C'_A}{2\pi C}\int_\Gamma \frac{t^\theta}{t+|z|}\phi(z)|dz|,\end{aligned}$$

where $\phi(z) := \|BR(z,-B)y\|$. Let Γ' be the part of Γ inside $R_{a,b}$ and Γ'' be the part of Γ outside $R_{a,b}$. Then by (3.1)

$$\int_{\Gamma'} \frac{t^\theta}{t+|z|}\phi(z)|dz| \leq C'' t^{\theta-1}$$

which is a function in $L^p(\sqrt{a^2+b^2}, +\infty; \frac{dt}{t})$, where C'' is a constant independent from t. We divide Γ'' into four parts $\Gamma'' = \cup_{k=1}^4 \Gamma''_k$, where $\Gamma''_1 = \{re^{i\theta} : r \geq \sqrt{a^2+b^2}\}$, $\Gamma''_2 = \{re^{-i\theta} : r \geq \sqrt{a^2+b^2}\}$, $\Gamma''_3 = \{re^{i(\theta-\pi)} : r \geq \sqrt{a^2+b^2}\}$ and $\Gamma''_4 = \{re^{i(\pi-\theta)} : r \geq \sqrt{a^2+b^2}\}$. Let $\varphi(r) := r^\theta \|BR(re^{i\theta}, -B)y\|$ if $r \geq \sqrt{a^2+b^2}$ and $\varphi(r) := 0$ otherwise. Then

$$\int_{\Gamma''_1} \frac{t^\theta}{t+|z|}\phi(z)|dz| = \int_0^\infty \frac{t^\theta r^{-\theta}}{t+r}\varphi(r)dr = h * \varphi(t)$$

where $h(t) = \frac{t^\theta}{1+t}$ and the convolution is for functions defined on the group $\mathbb{R}_+$ of multiplication equipped with the Haar measure $\frac{dt}{t}$. By Young's theorem

$$\|h*\varphi\|_{L^p(0,+\infty;\frac{dt}{t})} \leq \|h\|_{L^1(0,+\infty;\frac{dt}{t})}\|\varphi\|_{L^p(0,+\infty;\frac{dt}{t})} \leq \|h\|_{L^1(0,\infty,\frac{dt}{t})}\|y\|_{D_B(\theta,p)}.$$

Similar computations can be also done for the paths Γ''_2, Γ''_2 and Γ''_4 and thus we have shown that $Bx \in D_B(\theta,p)$. Since $Ax = y - Bx$, we also have $Ax \in D_B(\theta,p)$.

To show that $Bx \in D_A(\theta,p)$, we use

$$AR(it,A)R(z,A) = \frac{it}{z-it}R(it,A) - \frac{z}{z-it}R(z,A)$$

and (3.7), then we get

$$\begin{aligned}AR(it,A)Bx &= \frac{R(it,A)}{2\pi i}\int_\Gamma \frac{it}{z-it}BR(z,-B)ydz \\ &- \frac{1}{2\pi i}\int_\Gamma \frac{z}{z-it}R(z,A)BR(z,-B)ydz.\end{aligned}$$

The first integral is 0 for large $t > 0$, as $z - it \neq 0$ when $t > 0$ is large enough and so the function $\frac{BR(z,-B)}{z-it}$ is analytic in the region limited by Γ, where the path Γ is completed by identifying $\infty e^{i\theta}$, $\infty e^{-i\theta}$, and the points $\infty e^{i(\pi-\theta)}$, $\infty e^{i(\theta-\pi)}$. We conclude that

$$AR(it,A)Bx = -\frac{1}{2\pi i}\int_\Gamma \frac{z}{z-it}R(z,A)BR(z,-B)ydz.$$

We see that the right-hand side of this equality only differs from the right-hand side of (4.1) by $-it$ instead of t, so the same argument shows that $Bx \in D_A(\theta, p)$. □

When $y \in D_B(\theta, p)$, the conclusion $BSy \in D_A(\theta, p)$ is the so called "cross-regularity". For sectorial operators, the corresponding cross-regularity has been established in [9] (see also [10] [12]).

Theorem 4.1 can be reformulated by saying that A and B induce operators on interpolation spaces which have a closed sum. Recall, if C is an operator on X and Y is a Banach space continuously imbedded into X, then the part C_Y of C in Y is defined by $D(C_Y) = \{y \in D(C) \cap Y : \; Cy \in Y\}$ and $C_Y y = Cy$.

Corollary 4.2. *Let A and B be two operators satisfying (H_1)-(H_5). Let $0 < \theta < 1$, $p \in [1, \infty] \cup \{\infty_0\}$ and $Y = D_A(\theta, p)$ or $Y = D_B(\theta, p)$. Denote by A_Y and B_Y the parts of A and B in Y. Then $A_Y + B_Y$ is invertible.*

A similar proof as in [13, Theorem 3.14] shows the following result which gives a sufficient condition for the equation $Ay + By = x$ to be solvable with solution $y \in D(A) \cap D(B)$ when X is a Hilbert space.

Theorem 4.3. *Let H be a Hilbert space, A and B be two closed operators in H satisfying the assumptions (H_1)-(H_5). Assume that there exists $0 < \theta < 1$ such that $D_A(\theta, 2) = D_{A^*}(\theta, 2)$ or $D_B(\theta, 2) = D_{B^*}(\theta, 2)$, and $D(A)$ and $D(B)$ are dense in X. Then the sum $A + B$ is closed. Moreover $0 \in \rho(A + B)$ and $(A + B)^{-1} = S$.*

By the Remarks 2.3, we have $D_A(\theta, 2) = (H, D(A))_{\theta,2}$ and $D_{A^*}(\theta, 2) = (H, D(A^*))_{\theta,2}$. Thus Theorem 4.3 implies that when $D(A) = D(A^*)$, we have $D_A(\theta, 2) = D_{A^*}(\theta, 2)$, therefore the sum $A + B$ is closed and not only closable.

5. Applications

Let X be a Banach space and let $A : D(A) \to X$ be a closed operator. Assume that A generates a bounded strongly continuous semigroup T_t on X. Then by [17], [18], for $0 < \theta < 1$ and $1 \le p \le \infty$, we have

$$D_A(\theta, p) = \{x \in X : \; \|t^{-\theta}(T_t - I)x\| \in L^p(0, +\infty; \frac{dt}{t})\}, \tag{5.1}$$

and

$$\|x\| + \|t^{-\theta}(T_t - I)x\|_{L^p(0,+\infty;\frac{dt}{t})} \tag{5.2}$$

is an equivalent norm on $D_A(\theta, p)$. This is in particular the case when A generates a bounded strongly continuous group on X.

Now consider the Banach space $Y := L^p(\mathbb{R}; X)$ for $1 \le p < \infty$ (resp. $BUC(\mathbb{R}; X)$ the space of X-valued bounded and uniformly continuous functions defined on $\mathbb{R}$ equipped with the supremum norm $\|\cdot\|_\infty$), and let B the operator on Y defined by

$$\begin{aligned} Bf &= f' \\ D(B) &= \{f \in Y : \ f' \in Y\}. \end{aligned}$$

Then B is the generator of the translation group on Y defined by $(T_t f)(s) = f(t+s)$ for $t, s \in \mathbb{R}$. Since $\|T_t\| = 1$ for $t \in \mathbb{R}$, we have for $0 < \alpha < \pi/2$

$$\Omega_\alpha = \{z \in \mathbb{C} \setminus \{0\} : \ |\frac{\pi}{2} - arg(z)| \geq \alpha \text{ and } |\frac{3\pi}{2} - arg(z)| \geq \alpha\} \subset \rho(-B), \quad (5.3)$$

and

$$\sup_{z \in \Omega_\alpha} \|zR(z, -B)\| < \infty. \quad (5.4)$$

When $Y = L^p(\mathbb{R}; X)$, by (5.1) for $0 < \theta < 1$ and $1 \leq q \leq \infty$ we have

$$D_B(\theta, q) = \{f \in L^p(\mathbb{R}; X) : \ \int_0^{+\infty} t^{-\theta q} \|f(t + \cdot) - f(\cdot)\|_p^q \frac{dt}{t} < \infty\}. \quad (5.5)$$

with usual convention when $q = \infty$. By (5.2), an equivalent norm on $D_B(\theta, q)$ is defined by

$$\|f\|_p + (\int_0^{+\infty} t^{-\theta q} \|f(t + \cdot) - f(\cdot)\|_p^q \frac{dt}{t})^{1/q}. \quad (5.6)$$

This shows that $D_B(\theta, q)$ is precisely the X-valued Besov space $B^\theta_{p,q}(\mathbb{R}; X)$ (see [1] and [15]).

When $p = 2$ and X is a Hilbert space, we have

$$\begin{aligned} B^* f &= -f' \\ D(B) &= \{f \in L^2(\mathbb{R}; X) : \ f' \in L^2(\mathbb{R}; X)\}, \end{aligned}$$

thus $D(B) = D(B^*)$. Therefore for $0 < \theta < 1$

$$D_B(\theta, 2) = D_{B^*}(\theta, 2) = B^\theta_{2,2}(\mathbb{R}; X). \quad (5.7)$$

When $Y = BUC(\mathbb{R}; X)$, for $q = \infty$ and $0 < \theta < 1$, by (5.1)

$$D_B(\theta, \infty) = \{f \in BUC(\mathbb{R}; X) : \ \sup_{t>0} t^{-\theta} \|f(t + \cdot) - f(\cdot)\|_\infty < \infty\} \quad (5.8)$$

is precisely the space $C^\theta_b(\mathbb{R}; X)$ of X-valued bounded and θ-Hölder continuous functions defined on $\mathbb{R}$. By (5.2), an equivalent norm on $D_B(\theta, \infty)$ is given by

$$\|f\|_\infty + \sup_{t>0} t^{-\theta} \|f(t + \cdot) - f(\cdot)\|_\infty.$$

Let A be an invertible operator on X. Assume that there exists $0 < \beta < \frac{\pi}{2}$ such that

$$\Omega'_\beta = \{z \in \mathbb{C} : \ |\frac{\pi}{2} - arg(z)| < \beta \text{ or } |\frac{3\pi}{2} - arg(z)| < \beta\} \cup \{0\} \subset \rho(A) \quad (5.9)$$

and

$$\sup_{z \in \Omega'_\beta} \|zR(z, A)\| < \infty. \quad (5.10)$$

Define the operator $\mathcal{A}$ on $Y = L^p(\mathbb{R}; X)$ (resp. $BUC(\mathbb{R}; X)$) by

$$\begin{aligned} (\mathcal{A}f)(t) &:= A(f(t)), \quad (t \in \mathbb{R}) \\ D(\mathcal{A}) &:= L^p(\mathbb{R}; D(A)) \\ (resp.\ D(\mathcal{A})) &:= BUC(\mathbb{R}; D(A))), \end{aligned}$$

where $D(A)$ is equipped with the graph norm so that it becomes a Banach space.

For $0 < \theta < 1$, when $Y = L^p(\mathbb{R}; X)$, by Fubini's theorem we have

$$D_{\mathcal{A}}(\theta, p) = L^p(\mathbb{R}; D_A(\theta, p)). \tag{5.11}$$

Similarly when $Y = BUC(\mathbb{R}; X)$, we have

$$D_{\mathcal{A}}(\theta, \infty) = BUC(\mathbb{R}; X) \cap B(\mathbb{R}; D_A(\theta, \infty)). \tag{5.12}$$

where $B(\mathbb{R}; D_A(\theta, \infty))$ denotes the space of all bounded $D_A(\theta, \infty)$-valued functions defined on $\mathbb{R}$. For $p = \infty_0$, we have $D_{\mathcal{A}}(\theta, \infty_0) = BUC(\mathbb{R}; D_A(\theta, \infty_0))$. See [10] and [7] for the proofs of similar results. Finally, we see by (5.3), (5.4), (5.9) and (5.10) that $\mathcal{A}$ and B satisfy the assumptions (H_1)-(H_5).

Now consider the evolution equation

$$u' + Au = f \tag{5.13}$$

on $\mathbb{R}$, where $f \in L^p(\mathbb{R}; X)$ (resp. $f \in BUC(\mathbb{R}; X)$). We want to find a solution $u \in W^{1,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$, where $W^{1,p}(\mathbb{R}; X) := \{f \in L^p(\mathbb{R}; X) : \ f' \in L^p(\mathbb{R}; X)\}$ is the first Sobolev space (resp. $u \in BUC^1(\mathbb{R}; X) \cap BUC(\mathbb{R}; D(A))$, where $BUC^1(\mathbb{R}; X) := \{f \in BUC(\mathbb{R}; X) : \ f' \in BUC(\mathbb{R}; X)\}$). If such solution exists, we say that it is *a strict solution* of (5.13). It is known that in general such solution does not exist.

An immediate application of Theorem 4.3 and (5.7) gives the following: when X is a Hilbert space, A a closed operator satisfying (5.9) and (5.10) such that $D(A)$ is dense in X, then for each $f \in L^2(\mathbb{R}; X)$, there exists a unique strict solution $u \in W^{1,2}(\mathbb{R}; X) \cap L^2(\mathbb{R}; D(A))$ of (5.13). This is not new. In fact Mielke [19] showed that for $f \in L^p(\mathbb{R}; X)$, the solution u is in $W^{1,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$, $1 < p < \infty$ (see also Corollary 3.2.10 in [23]). For general Banach spaces, one has to assume that the operator-valued function $t \to itR(it, A)$ is Rademacher bounded on $\mathbb{R}$ to ensure that a strict solution exists in the case $Y = L^p(\mathbb{R}; X)$ (see Schweiker [23, Theorem 3.2.8]).

We will see that a solution of (5.13) in a weaker sense always exists and it is given by Sf, where S is defined by the integral (3.3) using $\mathcal{A}$ and B.

When $Y = BUC(\mathbb{R}; X)$, for $f \in BUC(\mathbb{R}; X)$, a function $u \in BUC(\mathbb{R}; X)$ is called *a strong solution* of (5.13) if, there exist $u_n \in BUC^1(\mathbb{R}; X) \cap BUC(\mathbb{R}; D(A))$ such that $u_n \to u$ and $Au_n + u_n' \to f$ in $BUC(\mathbb{R}; X)$ as $n \to \infty$. Since $\mathcal{A}$ and B satisfy the assumptions (H_1)-(H_5), this is equivalent to say that $u \in D(L)$ and $u = L^{-1}f$, where L is the closure of $\mathcal{A}+B$. By Theorem 3.6, we have $0 \in \rho(L)$ (note that $D(B) = BUC^1(\mathbb{R}; X)$ is dense in $BUC(\mathbb{R}; X)$, in particular $D(\mathcal{A}) + D(B)$ is dense in $BUC(\mathbb{R}; X)$), therefore for every $f \in BUC(\mathbb{R}; X)$, a strong solution of (5.13) exists and it is unique.

For $f \in BUC(\mathbb{R}; X)$, a function $u \in BUC(\mathbb{R}; X)$ is said to be *a mild solution* of (5.13) if, $\int_0^t u(s)ds \in D(A)$ and

$$u(t) - u(0) + A \int_0^t u(s)ds = \int_0^t f(s)ds$$

for all $t \in \mathbb{R}$. We claim that any strong solution of (5.13) is a mild solution. Indeed, Let $f \in BUC(\mathbb{R}; X)$ and let $u \in BUC(\mathbb{R}; X)$ be a strong solution of (5.13). There exist $u_n \in BUC^1(\mathbb{R}; X) \cap BUC(\mathbb{R}; D(A))$, such that $u_n \to u$ and $(\mathcal{A} + B)u_n \to f$ in $BUC(\mathbb{R}; X)$ as $n \to \infty$. For each $n \in \mathbb{N}$, we have $u_n' + Au_n = (\mathcal{A} + B)u_n$. Integrating on the interval $[0, t]$ leads

$$u_n(t) - u_n(0) + A \int_0^t u_n(s)ds = \int_0^t \{(\mathcal{A} + B)u_n\}(s)ds.$$

Letting $n \to \infty$, the closedness of A implies that $\int_0^t u(s)ds \in D(A)$ and

$$u(t) - u(0) + A \int_0^t u(s)ds = \int_0^t f(s)ds,$$

for $t \in \mathbb{R}$. Thus u is a mild solution of (5.13). It is actually shown by Schweiker [22, Theorem 1.1] that under the additional assumption that $D(A)$ is dense in X, for every $f \in BUC(\mathbb{R}; X)$, the mild solution of (5.13) exists and it is unique.

When $Y = L^p(\mathbb{R}; X)$, for $f \in L^p(\mathbb{R}; X)$, a function $u \in L^p(\mathbb{R}; X)$ is said to be *a strong solution* of (5.13) if, there exist $u_n \in W^{1,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$ such that $u_n \to u$ and $u_n' + Au_n \to f$ in $L^p(\mathbb{R}; X)$ as $n \to \infty$. Since $\mathcal{A}$ and B satisfy the assumptions (H_1)-(H_5), this is equivalent to say that $u \in D(L)$ and $u = L^{-1}f$, where L is the closure of $\mathcal{A} + B$. By Theorem 3.6, we have $0 \in \rho(L)$ (note that $D(B) = W^{1,p}(\mathbb{R}; X)$ is dense in $L^p(\mathbb{R}; X)$, in particular $D(\mathcal{A}) + D(B)$ is dense in $L^p(\mathbb{R}; X)$), therefore for every $f \in L^p(\mathbb{R}; X)$, a strong solution of (5.13) exists and it is unique.

For $f \in L^p(\mathbb{R}; X)$, a function $u \in L^p(\mathbb{R}; X)$ is said to be *a mild solution* of (5.13) if, there exists $x \in X$ and $a \in \mathbb{R}$, such that $\int_0^t u(s)ds \in D(A)$ and

$$u(t) + x + A \int_a^t u(s)ds = \int_a^t f(s)ds$$

for almost all $t \in \mathbb{R}$. We claim that any strong solution of (5.13) is a mild solution. Indeed, let $f \in L^p(\mathbb{R}; X)$ and let $u \in L^p(\mathbb{R}; X)$ be a strong solution of (5.13). Then there exist $u_n \in W^{1,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$ such that $u_n \to u$ and $(\mathcal{A} + B)u_n \to f$ in $L^p(\mathbb{R}; X)$ as $n \to \infty$. For all $n \in \mathbb{N}$ and almost all $s \in \mathbb{R}$, we have

$$u_n'(s) + Au_n(s) = (\mathcal{A} + B)u_n(s). \tag{5.14}$$

Since there exists a subsequence u_{n_k} of u_n which converges almost a.e. on $\mathbb{R}$, without loss of generality we can assume that $u_n(a)$ converges to some element $x \in X$ for some $a \in \mathbb{R}$ as $n \to \infty$. Integrating (5.14) on the interval $[a, t]$ leads

$$u_n(t) - u_n(a) + A \int_a^t u_n(s)ds = \int_a^t [(\mathcal{A} + B)u_n](s)ds$$

for almost all $t \in \mathbb{R}$. Letting $n \to \infty$, then by the closedness of A we have $\int_a^t u(s)ds \in D(A)$ and

$$u(t) - x + A\int_a^t u(s)ds = \int_a^t f(s)ds$$

for almost all $t \in \mathbb{R}$. Thus u is a mild solution of (5.13). Moreover, we remark that by [20, Theorem 1.2] at most one mild solution exists.

Immediate applications of Corollary 3.5 and Proposition 3.3 together with the relations (5.6), (5.8), (5.11) and (5.12) give the following result.

Theorem 5.1. *Let X be a Banach space and let A be a closed operator on X satisfying* (5.9) *and* (5.10). *Then*

1) *If $1 \le p < \infty$ and $f \in W^{1,p}(\mathbb{R}; X) + L^p(\mathbb{R}; D(A))$, then the unique strong solution u of* (5.13) *is in $W^{1,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$, i.e. u is a strict solution.*
2) *If $f \in BUC^1(\mathbb{R}; X) + BUC(\mathbb{R}; D(A))$, then the unique strong solution u of* (5.13) *is in $BUC^1(\mathbb{R}; X) \cap BUC(\mathbb{R}; D(A))$, i.e. u is a strict solution.*
3) *If $1 \le p < \infty$ and $f \in L^p(\mathbb{R}; X)$, then the unique strong solution u of* (5.13) *is in $B^\theta_{p,q}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D_A(\theta, p))$ for $0 < \theta < 1$ and $1 \le q \le \infty$.*
4) *If $f \in BUC(\mathbb{R}; X)$, then the unique strong solution u of* (5.13) *is in $C^\theta_b(\mathbb{R}; X) \cap BUC(\mathbb{R}; D_A(\theta, \infty_0))$ for $0 < \theta < 1$.*

An immediate application of Theorem 4.1 and the relations (5.6), (5.8), (5.11) and (5.12) give the following result.

Theorem 5.2. *Let X be a Banach space and let A be a closed operator on X satisfying* (5.9) *and* (5.10). *Then*

1) *If $0 < \theta < 1$, $1 \le p < \infty$, $1 \le q \le \infty$ and $f \in B^\theta_{p,q}(\mathbb{R}, ; X)$, then the unique strong solution u of* (5.13) *is a strict solution and it satisfies $u', Au \in B^\theta_{p,q}(\mathbb{R}; X)$.*
2) *If $0 < \theta < 1$, $1 \le p < \infty$ and $f \in L^p(\mathbb{R}; D_A(\theta, p))$, then the unique strong solution u of* (5.13) *is a strict solution and it satisfies $u' \in L^p(\mathbb{R}; D_A(\theta, p))$, $Au \in L^p(\mathbb{R}; D_A(\theta, p)) \cap B^\theta_{p,p}(\mathbb{R}; X)$.*
3) *If $0 < \theta < 1$ and $f \in C^\theta_b(\mathbb{R}; X)$, then the unique strong solution u of* (5.13) *is a strict solution and it satisfies $u' \in C^\theta_b(\mathbb{R}; X) \cap B(\mathbb{R}; D_A(\theta, \infty_0))$, $Au \in C^\theta_b(\mathbb{R}; X)$.*
4) *If $0 < \theta < 1$ and $f \in BUC(\mathbb{R}; D_A(\theta, \infty_0))$, then the unique strong solution u of* (5.13) *is a strict solution and it satisfies $u' \in BUC(\mathbb{R}; D_A(\theta, \infty_0))$, $Au \in BUC(\mathbb{R}; D_A(\theta, \infty_0)) \cap C^\theta_b(\mathbb{R}; X)$.*

Next we consider the periodic boundary conditions. Let X be a Banach space, $1 \le p < \infty$. Consider the Banach space $Y = L^p_{2\pi}(\mathbb{R}; X)$ (resp. $C_{2\pi}(\mathbb{R}; X)$) the space of X-valued, 2π-periodic measurable functions f on $\mathbb{R}$ such that

$$\|f\|_p = (\int_0^{2\pi} \|f(t)\|^p \frac{dt}{2\pi})^{1/p} < \infty$$

(resp. the space of X-valued and 2π-periodic continuous functions f on $\mathbb{R}$ equipped with the norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$). It is clear that Y equipped with the norm $\|\cdot\|_p$ (resp. $\|\cdot\|_\infty$) becomes a Banach space.

Now consider the operator B on Y defined by

$$\begin{aligned} Bf &:= f' \\ D(B) &:= \{f \in Y : \ f' \in Y\}. \end{aligned}$$

Then B is the generator of the translation group on Y defined by $(T_t f)(s) = f(t+s)$ for $t, s \in \mathbb{R}$. A simple computation shows that $\mathbb{R} \setminus i\mathbb{Z} \subset \rho(B)$ and for $0 < \alpha < \pi/2$

$$\Omega_\alpha = \{z \in \mathbb{C} : \ |\frac{\pi}{2} - arg(z)| \geq \alpha \text{ and } |\frac{3\pi}{2} - arg(z)| \geq \alpha\} \subset \rho(B), \tag{5.15}$$

$$\sup_{z\in\Omega_\alpha} \|zR(z,B)\| < \infty. \tag{5.16}$$

When $Y = L^p_{2\pi}(\mathbb{R}; X)$, by (5.1) for $0 < \theta < 1$ and $1 \leq q \leq \infty$ we have

$$D_B(\theta, q) = \{f \in L^p_{2\pi}(\mathbb{R}; X) : \ \int_0^{+\infty} t^{-\theta q} \|f(t+\cdot) - f(\cdot)\|_p^q \frac{dt}{t} < \infty\}. \tag{5.17}$$

with usual convention when $q = \infty$. By (5.2), an equivalent norm on $D_B(\theta, q)$ is defined by

$$\|f\|_p + (\int_0^{2\pi} t^{-\theta q} \|f(t+\cdot) - f(\cdot)\|_p^q \frac{dt}{t})^{1/q}. \tag{5.18}$$

This shows that $D_B(\theta, q)$ is precisely the X-valued periodic Besov space $B^\theta_{p,q}(\mathbb{T}; X)$ (see [3]).

When $p = 2$ and X is a Hilbert space, we have

$$\begin{aligned} B^* f &= -f' \\ D(B) &= \{f \in L^2_{2\pi}(\mathbb{R}; X) : \ f' \in L^2_{2\pi}(\mathbb{R}; X)\} = W^{1,2}_{2\pi}(\mathbb{R}; X). \end{aligned}$$

In particular, we have $D(B) = D(B^*)$. Therefore for $0 < \theta < 1$

$$D_B(\theta, 2) = D_{B^*}(\theta, 2) = B^\theta_{2,2}(\mathbb{T}; X). \tag{5.19}$$

When $Y = C_{2\pi}(\mathbb{R}; X)$, for $q = \infty$ and $0 < \theta < 1$, by (5.1)

$$D_B(\theta, \infty) = \{f \in C_{2\pi}(\mathbb{R}; X) : \ \sup_{t>0} t^{-\theta} \|f(t+\cdot) - f(\cdot)\|_\infty < \infty\} \tag{5.20}$$

is the space $C^\theta_{2\pi}(\mathbb{R}; X)$ of X-valued 2π-periodic and θ-Hölder continuous functions defined on $\mathbb{R}$. By (5.2), an equivalent norm on $D_B(\theta, \infty)$ is given by

$$\|f\|_\infty + \sup_{t>0} t^{-\theta} \|f(t+\cdot) - f(\cdot)\|_\infty.$$

Let A be a linear invertible operator on X. Assume that $i\mathbb{Z} \subset \rho(A)$ and $\sup_{n\in\mathbb{Z}} \|nR(in, A)\| < \infty$. Then there exists $0 < \beta < \frac{\pi}{2}$ and $\omega > 0$ such that

$$\Omega''_\beta = \{|\frac{\pi}{2} - arg(z)| < \beta \text{ or } |\frac{3\pi}{2} - arg(z)| < \beta\} \cap \{|Im(z)| \geq \omega\} \subset \rho(A) \tag{5.21}$$

and

$$\sup_{z\in\Omega''_\beta} \|zR(z,A)\| < \infty. \tag{5.22}$$

Define the operaor $\mathcal{A}$ on $Y = L^p_{2\pi}(\mathbb{R}; X)$ (resp. $C_{2\pi}(\mathbb{R}; X)$) by

$$\begin{aligned} (\mathcal{A}f)(t) &= A(f(t)), \quad (t \in \mathbb{R}) \\ D(\mathcal{A}) &= L^p_{2\pi}(\mathbb{R}; D(A)) \\ (\textit{resp. } D(\mathcal{A})) &= C_{2\pi}(\mathbb{R}; D(A))), \end{aligned}$$

where $D(A)$ is equipped with the graph norm so that it becomes a Banach space.

For $0 < \theta < 1$, when $Y = L^p_{2\pi}(\mathbb{R}; X)$, by Fubini's theorem we have

$$D_{\mathcal{A}}(\theta, p) = L^p_{2\pi}(\mathbb{R}; D_A(\theta, p)). \tag{5.23}$$

Similarly when $Y = C_{2\pi}(\mathbb{R}; X)$, we have

$$D_{\mathcal{A}}(\theta, \infty) = C_{2\pi}(\mathbb{R}; X) \cap B(\mathbb{R}; D_A(\theta, \infty)), \tag{5.24}$$

where $B(\mathbb{R}; D_A(\theta, \infty))$ denotes the space of all bounded $D_A(\theta, \infty)$-valued functions defined on $\mathbb{R}$. We have also $D_{\mathcal{A}}(\theta, \infty_0) = C_{2\pi}(\mathbb{R}; D_A(\theta, \infty_0))$. See [7] and [10] for the proof of a similar results. Finally, it is easy to verify that $\mathcal{A}$ and B satisfy the assumptions (H_1)-(H_5).

Now consider the evolution equation with periodic boundary condition

$$u' + Au = f, \quad u(0) = u(2\pi) \tag{5.25}$$

on $[0, 2\pi]$, where $f \in L^p_{2\pi}(\mathbb{R}; X)$ (resp. $f \in C_{2\pi}(\mathbb{R}; X)$). We want to find solution $u \in W^{1,p}_{2\pi}(\mathbb{R}; X) \cap L^p_{2\pi}(\mathbb{R}; D(A))$, where $W^{1,p}_{2\pi}(\mathbb{R}; X) := \{f \in L^p_{2\pi}(\mathbb{R}; X) : \ f' \in L^p_{2\pi}(\mathbb{R}; X)\}$ is the first periodic Sobolev space (resp. $u \in C^1_{2\pi}(\mathbb{R}; X) \cap C_{2\pi}(\mathbb{R}; D(A))$, where $C^1_{2\pi}(\mathbb{R}; X) := \{f \in C_{2\pi}(\mathbb{R}; X) : \ f' \in C_{2\pi}(\mathbb{R}; X)\}$). If such solution exists, we say that it is *a strict solution* of (5.25). It is known that in general such solution does not exist. An immediate application of Theorem 4.3 and (5.19) gives the following: when X is a Hilbert space, A a closed operator on X satisfying (5.21) and (5.22) such that $D(A)$ is dense in X, then for $f \in L^2_{2\pi}(\mathbb{R}; X)$, there exists a unique strict solution u of (5.25). This can be also obtained by using Theorem 2.3 in [2]. For general Banach spaces, one has to assume that the set $\{inR(in, A) : n \in \mathbb{Z}\}$ is Rademacher bounded in the case $Y = L^p_{2\pi}(\mathbb{R}; X)$ for the equation (5.25) to have a strict solution [2]. We will see that a solution in a weak sense always exists and it is given by Sf, where S is defined by the integral (3.3) using the operators $\mathcal{A}$ and B.

When $Y = C_{2\pi}(\mathbb{R}; X)$, for $f \in C_{2\pi}(\mathbb{R}; X)$, a function $u \in \mathbb{C}_{2\pi}(\mathbb{R}; X)$ is called *a strong solution* of (5.25) if, there exist $u_n \in C^1_{2\pi}(\mathbb{R}; X) \cap C_{2\pi}(\mathbb{R}; D(A))$ such that $u_n \to u$ and $Au_n + u'_n \to f$ in $C_{2\pi}(\mathbb{R}; X)$ as $n \to \infty$. Since $\mathcal{A}$ and B satisfy the assumptions (H_1)-(H_5), this is equivalent to say that $u \in D(L)$ and $u = L^{-1}f$, where L is the closure of $\mathcal{A} + B$. By Theorem 3.6, we have $0 \in \rho(L)$. Note that $D(B) = C^1_{2\pi}(\mathbb{R}; X)$ is dense in $C_{2\pi}(\mathbb{R}; X)$. In particular $D(\mathcal{A}) + D(B)$ is dense in $C_{2\pi}(\mathbb{R}; X)$. Therefore for every $f \in C_{2\pi}(\mathbb{R}; X)$, a strong solution of (5.25) exists and it is unique.

For $f \in C_{2\pi}(\mathbb{R}; X)$, a function $u \in C_{2\pi}(\mathbb{R}; X)$ is said to be *a mild solution* of (5.25) if, $\int_0^t u(s)ds \in D(A)$ and

$$u(t) - u(0) + A\int_0^t u(s)ds = \int_0^t f(s)ds$$

for all $t \in \mathbb{R}$. A similar argument as in the BUC-case on the line shows that each strong solution of (5.25) is a mild solution. Moreover, mild solutions are unique (cf. [2]).

When $Y = L^p_{2\pi}(\mathbb{R}; X)$, for $f \in L^p_{2\pi}(\mathbb{R}; X)$, a function $u \in L^p_{2\pi}(\mathbb{R}; X)$ is said to be *a strong solution* of (5.25) if, there exist $u_n \in W^{1,p}_{2\pi}(\mathbb{R}; X) \cap L^p_{2\pi}(\mathbb{R}; D(A))$ such that $u_n \to u$ and $u_n' + Au_n \to f$ in $L^p_{2\pi}(\mathbb{R}; X)$ as $n \to \infty$. Since $\mathcal{A}$ and B satisfy the assumptions (H_1)-(H_5), this is equivalent to say that $u \in D(L)$ and $u = L^{-1}f$, where L is the closure of $\mathcal{A} + B$. By Theorem 3.6, we have $0 \in \rho(L)$. Note that $D(B) = W^{1,p}_{2\pi}(\mathbb{R}; X)$ is dense in $L^p_{2\pi}(\mathbb{R}; X)$. In particular $D(\mathcal{A}) + D(B)$ is dense in $L^p_{2\pi}(\mathbb{R}; X)$. Therefore for every $f \in L^p_{2\pi}(\mathbb{R}; X)$, a strong solution of (5.25) exists and it is unique.

For $f \in L^p_{2\pi}(\mathbb{R}; X)$, a function $u \in L^p_{2\pi}(\mathbb{R}; X)$ is said to be *a mild solution* of (5.25) if, there exists $x \in X$, such that $\int_0^t u(s)ds \in D(A)$ and

$$u(t) + x + A\int_0^t u(s)ds = \int_0^t f(s)ds$$

for almost all $t \in \mathbb{R}$. A similar argument as in the L^p-case on the real line shows that each strong solution of (5.25) is a mild solution. Moreover, mild solutions are unique (cf. [2]).

Immediate applications of Corollary 3.5 and Proposition 3.3 together with the relations (5.18), (5.20), (5.23) and (5.24) give the following result.

Theorem 5.3. *Let X be a Banach space and let A be a closed operator on X satisfying $i\mathbb{Z} \subset \rho(A)$ and $\sup_{n\in\mathbb{Z}}\|nR(in, A)\| < \infty$. Then*

1) *If $1 \le p < \infty$ and $f \in W^{1,p}_{2\pi}(\mathbb{R}; X) + L^p_{2\pi}(\mathbb{R}; D(A))$, then the unique strong solution u of* (5.25) *is in $W^{1,p}_{2\pi}(\mathbb{R}; X) \cap L^p_{2\pi}(\mathbb{R}; D(A))$, i.e. u is a strict solution.*
2) *If $f \in C^1_{2\pi}(\mathbb{R}; X) + C_{2\pi}(\mathbb{R}; D(A))$, then the unique strong solution u of* (5.25) *is in $C^1_{2\pi}(\mathbb{R}; X) \cap C_{2\pi}(\mathbb{R}; D(A))$, i.e. u is a strict solution.*
3) *If $1 \le p < \infty$ and $f \in L^p_{2\pi}(\mathbb{R}; X)$, then the unique strong solution u of* (5.25) *is in $B^\theta_{p,q}(\mathbb{T}; X) \cap L^p_{2\pi}(\mathbb{R}; D_A(\theta, p))$ for $0 < \theta < 1$ and $1 \le q \le \infty$.*
4) *If $f \in C_{2\pi}(\mathbb{R}; X)$, then the unique strong solution u of* (5.25) *is in $C^\theta_{2\pi}(\mathbb{R}; X) \cap C_{2\pi}(\mathbb{R}; D_A(\theta, \infty_0))$ for $0 < \theta < 1$.*

An immediate application of Theorem 4.1 and the relations (5.18), (5.20), (5.23) and (5.24) give the following result.

Theorem 5.4. *Let X be a Banach space and let A be a closed operator on X satisfying* (5.21) *and* (5.22). *Then*

1) *If* $0 < \theta < 1,\ 1 \le p < \infty$ *and* $f \in B^{\theta}_{p,q}(\mathbb{T}; X)$*, then the unique strong solution* u *of* (5.25) *is a strict solution and it satisfies* $u', Au \in B^{\theta}_{p,q}(\mathbb{T}; X)$*.*
2) *If* $0 < \theta < 1,\ 1 \le p < \infty$ *and* $f \in L^p_{2\pi}(\mathbb{R}; D_A(\theta, p))$*, then the unique strong solution* u *of* (5.25) *is a strict solution and satisfies* $u' \in L^p_{2\pi}(\mathbb{R}; D_A(\theta, p))$*,* $Au \in B^{\theta}_{p,p}(\mathbb{T}; X) \cap L^p_{2\pi}(\mathbb{R}; D_A(\theta, p))$*.*
3) *If* $0 < \theta < 1$ *and* $f \in C^{\theta}_{2\pi}(\mathbb{R}; X)$*, then the unique strong solution* u *of* (5.25) *is a strict solution and satisfies* $u' \in C^{\theta}_{2\pi}(\mathbb{R}; X) \cap B(\mathbb{R}; D_A(\theta, \infty_0))$*,* $Au \in C^{\theta}_{2\pi}(\mathbb{R}; X)$*.*
4) *If* $0 < \theta < 1$ *and* $f \in C_{2\pi}(\mathbb{R}; D_A(\theta, \infty_0))$*, then the unique strong solution* u *of* (5.25) *is a classical solution and it satisfies* $u' \in C_{2\pi}(\mathbb{R}; D_A(\theta, \infty_0))$*,* $Au \in C^{\theta}_{2\pi}(\mathbb{R}; X) \cap C_{2\pi}(\mathbb{R}; D_A(\theta, \infty_0))$*.*

6. Appendix: Proof of the Separating Curve Lemma

Here we give a proof of Lemma 3.1. Recall that $R = [-a, a] + i[-b, b]$. The sets $S, T \subset \mathbb{C}$ are open such that $R \subset S \cup T$, $S^c \cap T^c = \emptyset$, $\pm a + i[-b, b] \subset S$, $[-a, a] \pm ib \subset T$.

Let $m \in \mathbb{N}$ be large, $\delta_1 = 2a/m$, $\delta_2 = 2b/m$, $a_k = -a + k\delta_1$, $b_k = -b + k\delta_2$ $(k = 0, 1, \cdots, m)$. We will consider curves in the grid

$$G = \cup_{k=0}^m \{(a_k + i[-b, b]) \cup ([-a, a] + ib_k)\}.$$

The number m is chosen so large that

$$[-a, -a + \delta_1] \times i[-b, -b + \delta_2] \subset T \tag{6.1}$$

$$[a, a - \delta_1] \times i[b, b - \delta_2] \subset T \tag{6.2}$$

$$(\delta_1^2 + \delta_2^2)^{1/2} < dist(S^c \cap R, T^c \cap R). \tag{6.3}$$

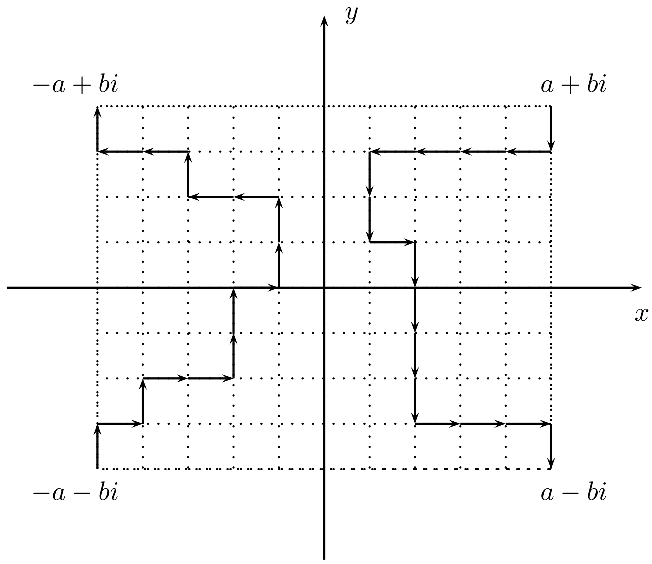
We will consider curves in the grid G. Such a curve Γ can be presented by a finite sequence of vectors $\gamma_1, \cdots, \gamma_n$ in the grid G such that the end point of γ_k coincides with the initial point of γ_{k+1}, where $k = 1, 2, \cdots, n-1$.

Such a curve will be called *admissible*, if the following three conditions are satisfied.

(C1) *Direction on the boundary* ∂R *of* R : Each vector points upwards on $-a + i[-b, b]$, to the right on $[-a, a] - ib$, downwards on $a + i[-b, b]$ and to the left on $[-a, a] + ib$.

(C2) *Closedness to* T^c : Let γ_k be one of the vectors of Γ. Consider the closed rectangle Q_l to the left of γ_k, and the rectangle Q_r to the right of γ_k. Here "left" and "right" are understood with respect to the direction of γ_k. For example, if γ_k points to the right and has end point $c + id$, then $Q_r = [c - \delta_1, c] + i[d - \delta_2, d]$. Then we ask that $Q_l \cap T^c \ne \emptyset$ whenever $Q_l \subset R$ and $Q_r \cap \mathbb{T}^c = \emptyset$ whenever $Q_r \subset R$.

(C3) *Left trun condition* : Let γ_k be a vector of Γ. Consider $Q_r,\ Q_l$ as above and let $\widetilde{Q}_l$ be the closed rectangle above Q_l (following the direction of γ_k) and $\widetilde{Q}_l$ the closed rectangle above Q_l. For example, if γ_k is pointing upwards and has end



point $c+id$, then $\widetilde{Q}_r = [c, c+\delta_1] + i[d, d+\delta_2]$. Assume that $Q_l,\ Q_r,\ \widetilde{Q}_l,\ \widetilde{Q}_r \subset R$. Assume that $\widetilde{Q}_l \cap T^c = \emptyset$ and $\widetilde{Q}_r \cap T^c \neq \emptyset$. Then γ_{k+1} points to the left.

Note that in the situation described in (C3) also a right turn would lead to a prolongation satisfying (C2). Condition (C3) asks the curve to turn left whenever it can. It makes the successor unique.

Now we establish several properties of admissible curves.

(P1) *Unique Prolongation.* Let Γ be an admissible curve whose end point is not $-a+ib$ or $a-ib$. Then Γ has a unique prolongation.

In order to prove this property we have to check all possible cases of the position of the last vector γ_k of Γ.

Case 1: The end point $c+id$ of γ_k lies in the interior of R. Consider the rectangles $Q_l,\ Q_r,\ \widetilde{Q}_l,\ \widetilde{Q}_r$ corresponding to γ_k as defined in (C2) and (C3). Four cases may occur:

Case 1.1: $\widetilde{Q}_l \cap T^c = \emptyset$ and $\widetilde{Q}_r \cap T^c = \emptyset$. Then we let γ_{k+1} point to the left side.

Case 1.2: $\widetilde{Q}_l \cap T^c = \emptyset$ and $\widetilde{Q}_r \cap T^c \neq \emptyset$. Then we let γ_{k+1} point to the left side (according to (C3)).

Case 1.3: $\widetilde{Q}_l \cap T^c \neq \emptyset$ and $\widetilde{Q}_r \cap T^c = \emptyset$. Then we let γ_{k+1} point upwards.

Case 1.4: $\widetilde{Q}_l \cap T^c \neq \emptyset$ and $\widetilde{Q}_r \cap T^c \neq \emptyset$. Then we let γ_{k+1} point to the right.

In each of these cases γ_{k+1} satisfies condition in (C2) and the choice of γ_{k+1} is compulsory.

Case 2: The end point $c+id$ of γ_k lies on ∂R. One checks in a similar way as for the Case 1 that for each of the four segments composing ∂R and each of the two possible position of γ_k (namely, pointing to the boundary or lying entirely in the boundary), there exists a unique prolongation.

(P2) Unique Predecessor: Considering the same cases as in (P1), one checks that a given vector whose initial point is different from $-a-ib$ or $a+ib$ has at most one predecessor.

(P3) Admissible Curves Do Not Cross and Do Not Joint: Property (P2) shows that two different admissible curves cannot immerge to one curve. Condition (C3) implies that two admissible curves cannot cross. We remark however that they may touch in an isolated point as described in (C3).

(P4) Each Admissible Curve Γ *Lies in* $S \cap T$: Recall that $\pm a + i[-b,b] \subset S$ and $[-a,a] \pm ib \subset T$. Conditions (C1) and (C2) imply that $\Gamma \subset T$. In order to show that $\Gamma \subset S$ consider a vector γ_k of Γ and the associated rectangle Q_l. If $Q_l \not\subset R$, then γ_k lies in $\pm a + i[-b,b] \subset S$. If $Q_l \subset R$, then $Q_l \cap T^c \neq \emptyset$ by condition (C2). Now the choice (6.3) of the grid implies that $Q_l \subset S$ and so $\gamma_k \subset S$.

Now we prove the existence of the curves described in the lemma. Let Γ_1 be the admissible curve of maximal length starting at $-a-ib$ with the first vector pointing upwards (which lies in $T \cap S$ by condition (3.1)). Then by (P1) Γ_1 has the endpoint $-a+ib$ or $a-ib$. Analogously, we consider the admissible curve Γ_2 of maximal length starting at $a+ib$ with the vector pointing downwards. Since the two curves Γ_1 and Γ_2 cannot cross, the endpoint of Γ_2 is $a-ib$ if the endpoint of Γ_1 is $-a+ib$, and the endpoint of Γ_2 is $-a+ib$ if the endpoint of Γ_1 is $a-ib$. The curves Γ_1 and Γ_2 may touch in a finite number of points. A small perturbation leads to disjoint curves. This finishes the proof of Lemma 3.1

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