Positivity (2005) 9:357–367
DOI 10.1007/s11117-005-3558-1

Local Operators and Forms

WOLFGANG ARENDT and SONJA THOMASCHEWSKI
Abteilung Angewandte Analysis, Universität Ulm, 89069 Ulm, Germany. (E-mail: arendt@mathematik.uni-ulm.de; E-mail: thomasch@mathematik.uni-ulm.de)

Received: 10 September 2004; accepted 8 December 2004

Abstract. Let $a\colon V \times V \to \mathbb{R}$ be a continuous, coercive form where V is a Hilbert space, densely and continuously embedded into $L^2(\Omega)$. Denote by T the associated semigroup on $L^2(\Omega)$. We show that T consists of multiplication operators if and only if V is a sublattice with normal cone and

$$a(u^+, u^-) = 0 \quad (u \in V).$$

We also prove a vector-valued version of this result. For this we characterize multiplication operators $M\colon L^p(\Omega, E) \to L^p(\Omega, E)$ by locality. If Ω has no atoms, we show that each local, linear mapping is automatically continuous.

Let Ω be an open subset in $\mathbb{R}^N$. The Sobolev space $H^1(\Omega) := \{u \in L^2(\Omega)\colon D_j u \in L^2(\Omega),\ j = 1, \dots, N\}$ is a Hilbert space and a vector lattice such that $\|u\|_{H^1} = \|\,|u|\,\|_{H^1}$. However, the positive cone is not normal, so $H^1(\Omega)$ is not a Banach lattice. The space $H^1(\Omega)$ occurs as form domain for the Laplacian (with Neumann boundary conditions). Our aim is to show that this situation is typical for domains of forms generating positive semigroups. They are Hilbert spaces which are vector lattices such that the absolute value is continuous. However the cone is normal if and only if the semigroup consists of multiplication operators.

Semigroups associated with (closed) forms play an important role for parabolic equations and in potential theory. We refer to the monographs Dautray–Lions [3, 4], Fukushima, Oshima, Takeda [6], Ma, Röckner [9], Lions [8], Ouhabaz [14] and Tanabe [17]. Positivity of the associated semigroup is characterized by the Beurling–Deny criterium. In the non-symmetric case, which we also consider, this criterion is due to Ouhabaz [12] who also proved an invariance criterion for closed convex sets which we will use (Ouhabaz [13]). In the context of nonautonomous Cauchy problems the vector-valued space $L^2(\Omega, E)$ has to be considered, where E is a Hilbert space (cf. Lions [8]). Also in this more general context we characterize forms which lead to semigroups of multiplication operators.

However, here we can no longer use the positive cone. Moreover, we need the following abstract characterization of multiplication operators. Let E be a Banach space and $1 \leqslant p \leqslant \infty$. We say that an operator T on $L^p(\Omega, E)$ is **local** if $(Tf)(\omega) = 0$ a.e. on $\{\omega \in \Omega : f(\omega) = 0\}$ for each $f \in L^p(\Omega, E)$. We show, as in the scalar case, that local operators are multiplication operators. In fact, the set of all local operators is a Banach algebra which is isometrically isomorphic to $L^\infty(\Omega, \mathcal{L}_s(E))$, where $\mathcal{L}_s(E)$ denotes the space of all bounded operators on E with the strong operator topology. We also give a result on automatic continuity: If the measure space (Ω, Σ, μ) is non-atomic, then each local linear mapping is bounded. Such results on automatic continuity are known in the scalar case. In particular, Yuri Abramovich [1] (see also [2]) proved that even all disjointness preserving operators on a Banach lattice are automatically continuous (see also de Pagter [15] for a short proof).

1. Local Forms and Multiplication Operators

Let V, H be real Hilbert spaces such that V is continuously and densely embedded into H. We write $V \underset{d}{\hookrightarrow} H$. Let $a: V \times V \to \mathbb{R}$ be a continuous bilinear form such that

$$a(u,u) \geqslant 0 \quad (u \in V) \quad \text{and} \quad a(u,u) + \|u\|_H^2 \geqslant \alpha \|u\|_V^2$$

for all $u \in V$ and some $\alpha > 0$. Denote by A the operator on H associated with a, i.e., A is given by

$$\begin{cases} D(A) = \{u \in V : \exists\, f \in H : a(u, \varphi) = (f, \varphi)_H \text{ for all } \varphi \in V\} \\ Au \quad = f. \end{cases}$$

Then $-A$ generates a contractive C_0-semigroup $T = (T(t))_{t \geqslant 0}$ on H. We recall the following invariance criterion due to Ouhabaz [12] and [14, Theorem 2.2].

PROPOSITION 1.1. *Let $K \subset H$ be closed and convex and let $P: H \to K$ be the orthogonal projection onto K. The following are equivalent.*

(i) $T(t)K \subset K \quad (t \geqslant 0)$;

(ii) $u \in V$ *implies* $Pu \in V$ *and* $a(u, u - Pu) \geqslant 0$.

Now let (Y, Σ, μ) be a σ-finite measure space and let $H = L^2(Y, \mu)$.

COROLLARY 1.2. *The semigroup T is positive if and only if*

$$u \in V \quad \textit{implies} \quad |u| \in V \quad \textit{and} \quad a(|u|, |u|) \leqslant a(u, u).$$

EXAMPLE 1.3 (Neumann Laplacian). Let $\Omega \subset \mathbb{R}^n$ be open, $H = L^2(\Omega)$ (with Lebesgue measure), $V = H_0^1(\Omega)$ and $a(u, v) = \int_\Omega \nabla u \nabla v dx$. Then the operator A associated with a is given by

$$\begin{cases} D(A) = \{u \in H_0^1(\Omega) \colon \Delta u \in L^2(\Omega)\} \\ Au \quad = -\Delta u. \end{cases}$$

The semigroup T generated by $-A$ is positive in virtue of Corollary 1.2.

EXAMPLE 1.4 (Multiplication Operator). Let $m \colon Y \to [0, \infty)$ be measurable, $V = L^2(Y, (1+m)d\mu)$, $a(u, v) = \int_Y uvm d\mu$. Then the associated operator A is given by

$$\begin{cases} D(A) = \{u \in V \colon mu \in L^2(Y, \mu)\} \\ Au \quad = m \cdot u \end{cases}$$

and $T(t)f = e^{-mt} f$.

Assume now that the semigroup generated by A is positive. Then V is a sublattice of $L^2(Y, \mu)$ in view of Corollary 1.2. We show that the lattice operations are continuous.

PROPOSITION 1.5. *Assume that T is positive. Then the mapping $u \mapsto |u| \colon V \to V$ is continuous.*

Proof. We may assume that $(u|v)_V := a(u, v) + (u, v)_H$ since this defines an equivalent scalar product on V. Moreover,

$$\| |u| \|_V \leqslant \|u\|_V \quad (u \in V) \tag{1.1}$$

by (Corollary 1.2). Let $u \in V$. It suffices to show that for each sequence $(u_n)_{n \in \mathbb{N}}$ converging to u in V there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} |u_{n_k}| = |u|$ in V. It follows from (1.1) that the sequence $(|u_n|)_{n \in \mathbb{N}}$ is bounded in V. Hence passing to a subsequence we may assume that $(|u_n|)_{n \in \mathbb{N}}$ converges weakly in V. Since $|v_n| \to |u|$ in $L^2(Y, \mu)$, it follows that $(|u_n|)_{n \in \mathbb{N}}$ converges weakly to $|u|$ in V. Now (1.1) implies that the convergence is strong. □

We continue to assume that the semigroup T is positive. We say that the form a is **local** if

$$a(u^+, u^-) = 0 \quad \text{for all } u \in V.$$

This is equivalent to

$$a(|u|, |u|) = a(u, u) \quad \text{for all } u \in V.$$

Thus $\|u\|_V = \||u|\|_V$. In both Examples 1.3 and 1.4, the form a is local (cf. Davies [5]).

Denote by $V_+ = \{u \in V : u \geqslant 0\}$ the positive cone in V. Recall that V_+ is called **normal** if for each $u \in V_+$ the order interval $\{v \in V : 0 \leqslant v \leqslant u\}$ is bounded in V. It is well-known that V_+ is normal if and only if

$$V' = V'_+ - V'_+$$

where $V'_+ = \{\varphi \in V' : \varphi(u) \geqslant 0 \text{ for all } u \in V_+\}$, see [16, Ch. V 3.3. Corollary 3 and Remark].

In Example 1.3, the cone $H^1(\Omega)_+$ is not normal since order intervals may contain functions with derivatives of arbitrarily large norm. However, the cone V_+ in Example 1.4 is normal. In fact, this is the only case as the following theorem shows. A sublattice J of $L^2(Y, \mu)$ is called an **ideal** of $L^2(Y, \mu)$ if for each $u \in J_+$, $f \in L^2(Y, \mu)$,

$$0 \leqslant f \leqslant u \quad \text{implies} \quad f \in J.$$

THEOREM 1.6. *Assume that the semigroup T is positive. The following assertions are equivalent.*

(i) *There exists a measurable function $m : Y \to [0, \infty)$ such that*

$$T(t)f = e^{-tm} f \quad (t \geqslant 0, f \in L^2(Y, \mu)).$$

(ii) *The form a is local and V is an ideal in $L^2(Y, \mu)$.*

(iii) *The form a is local and the cone V_+ is normal.*

For the proof we use Zaanen's [19] characterization of bounded multiplication operators. We include a short proof which will be extended to the vector-valued case in Section 2.

PROPOSITION 1.7 (Zaanen). *Let $S \in \mathcal{L}(L^p(Y, \mu)), 1 \leqslant p \leqslant \infty$. The following assertions are equivalent.*

(i) *There exists* $m \in L^\infty(Y,\mu)$ *such that* $Sf = mf \quad (f \in L^p(Y,\mu))$;
(ii) S *is* **local**, *i.e.,* $(Sf)(x) = 0$ *a.e. on* $\{x \in Y : f(x) = 0\}$ *for all* $f \in L^p(Y,\mu)$.

Proof. (ii) $\Rightarrow$ (i). 1. Assume that $\mu(Y) < \infty$.
(a) One has $T(1_A f) = 1_A Tf$ for all $f \in L^p(Y,\mu)$ and for each measurable set $A \subset Y$. This follows easily from (ii).
(b) Let $m = T1_Y$. We claim that $m \in L^\infty(Y,\mu)$. Let $c > 0$. Assume that $A := \{y \in Y : |m(y)| > c\}$ has positive measure. Then

$$c\mu(A)^{1/p} = \|c1_A\|_p \leqslant \|1_A m\|_p = \|1_A T1\|_p = \|T1_A\|_p \leqslant \|T\|\mu(A)^{1/p}.$$

Hence $c \leqslant \|T\|$.
(c) It follows from a) that $Tf = mf$ for each simple function $f = \sum_{i=1}^n c_i 1_{A_i}$, $A_i \in \Sigma$. Since simple functions are dense in $L^p(Y,\mu)$, the claim (i) follows.

2. Since (Y,μ) is σ-finite, there exists $h \in L^1(Y,\mu)$ such that $h(y) > 0$ a.e. Then $f \mapsto \frac{f}{h^{1/p}}$ defines an isomorphism from $L^p(Y,\mu)$ onto $L^p(Y, h\mu)$. Now the claim follows from 1. This completes the proof of (ii) $\Rightarrow$ (i). The converse is obvious. □

Proof of Theorem 1.6. (iii) $\Rightarrow$ (ii). Let $u \in V_+$, $f \in L^2(Y,\mu)$, $0 \leqslant f \leqslant u$. There exist $u_m \in V$ such that $u_m \to f$ in $L^2(Y,\mu)$ as $m \to \infty$. Since V is a sublattice, $v_m := (u_m \wedge u) \vee 0 \in V$ and $v_m \to f$ in $L^2(Y,\mu)$ as $m \to \infty$. Since $0 \leqslant v_m \leqslant u$ and since the cone V_+ is normal, it follows that $(v_m)_{m\in\mathbb{N}}$ is bounded in V. By reflexivity, there exists a subsequence $(v_{m_k})_{k\in\mathbb{N}}$ which converges weakly in V to an element $v \in V$. Since $\lim_{k\to\infty} v_{m_k} = f$ in $L^2(Y,\mu)$, it follows that $f = v \in V$.

(ii) $\Rightarrow$ (i). Let $f \in L^2(Y,\mu) = H$, $B := \{x \in Y : f(x) \neq 0\}$, $B^c = Y \backslash B$. Then $J := \{g \in L^2(Y,\mu) : g(x) = 0 \text{ a.e. on } Y\backslash B\}$ is a closed subspace (in fact, an ideal) of $L^2(Y,\mu)$. The orthogonal projection P of $L^2(Y,\mu)$ onto J is given by $Ph = 1_B \cdot h$.

Now let $u \in V$. Since V is an ideal, it follows that $1_B u \in V$. Moreover, since a is local, one has $a(1_B u, 1_{B^c} u) = 0$. Thus $a(u, u - Pu) = a(u, 1_{B^c} u) = a(1_B u + 1_{B^c} u, 1_{B^c} u) \geqslant 0$. It follows from Proposition 1.1 that $T(t)J \subset J$. Since $R(1,A)g = \int_0^\infty e^{-t} T(t) g dt$ it follows that $R(1,A)J \subset J$. By Proposition 1.7 there exists $m_1 \in L^\infty(Y,\mu)$ such that

$$R(1,A)g = m_1 g. \tag{1.2}$$

Since $R(1,A) \geqslant 0$ and $\|R(1,A)\| \leqslant 1$, it follows that $0 \leqslant m_1(x) \leqslant 1$ a.e. Since $R(1,A)$ is injective, it follows that $m_1(x) > 0$ a.e. Let $m(x) = 1 - \frac{1}{m_1(x)} (x \in Y)$. Then $m : Y \to [0,\infty]$ is measurable and finite a.e. From (1.2) we deduce that $D(A) = \{g \in L^2(Y, \mu) : mg \in L^2(Y, \mu)\}$ and $Ag = mg$ for all $g \in D(A)$. This implies (i).

The implication (i) $\Rightarrow$ (iii) is trivial. □

COROLLARY 1.8. *Let F be a Hilbert space such that $F \underset{d}{\hookrightarrow} L^2(Y, \mu)$. Assume that*

(a) *$u \in F$ implies $|u| \in F$;*
(b) *$(u^+ \mid u^-)_F = 0$ for all $u \in F$;*
(c) *the cone in F in normal.*

Then there exists a measurable function $m: Y \to [\delta, \infty)$ where $\delta > 0$ such that $F = L^2(Y, m d\mu)$.

proof. The bilinear form $a(u, v) = (u \mid v)_F$ on $F \times F$ is continuous, local and satisfies $a(u, u) \geqslant \delta \|u\|_H^2$ for all $u \in F$ and some $\delta > 0$. Now the claim follows from Theorem 1.6 and Example 1.4. □

CONCLUDING REMARKS 1.9. It is very frequent that the form domain is a sublattice and the form is local. All elliptic operators of second order on $L^2(\mathbb{R}^N)$ occur in this way. However, if we replace the form domain by the domain of the generator, a much more special situation occurs. To be more precise, consider a semigroup T on $L^p(Y)$, $1 \leqslant p < \infty$ with generator A. Then $D(A)$ is a sublattice and A is local (i.e., $(Af)(y) = 0$ a.e. on $\{y \in Y: f(y) = 0\}$) if and only if each $T(t)$ is a lattice homomorphism for all $t \geqslant 0$. This is a result due to Nagel–Uhlig [11]. Observe that a semigroup associated with a form is always holomorphic [17]. This leads us to formulate the following problem.

Problem. Let T be a holomorphic semigroup of lattice homomorphisms. Does it follow that T consists of multiplication operators?

The shift semigroup on $L^p(0, 1)$ consists of lattice homomorphisms and is finally norm continuous, but not holomorphic.

2. Operator-Valued Multiplication Operators

Let (Ω, Σ, μ) be a σ-finite measure space and let E be a separable real or complex Banach space. In this section we characterize multiplication operators on $L^p(\Omega, E)$, $1 \leqslant p \leqslant \infty$.

DEFINITION 2.1. By $\mathcal{L}_s(E)$ we denote the space of all bounded linear operators on E provided with the strong operator topology. Then we define

$$L^\infty(\Omega, \mathcal{L}_s(E)) := \{M: \Omega \to \mathcal{L}(E): M(\cdot)x \in L^\infty(\Omega, E) \text{ for all } x \in E\}.$$

One can show that $L^\infty(\Omega, \mathcal{L}_s(E))$ is a Banach algebra for pointwise algebra operations and the essential supremum norm

$$\|M\| := \operatorname*{ess\,sup}_{\omega\in\Omega} \|M(\omega)\|_{\mathcal{L}(E)}.$$

Let $M \in L^\infty(\Omega, \mathcal{L}_s(E))$. Then for $f \in L^p(\Omega, E)$, the mapping Mf given by

$$(Mf)(\omega) = M(\omega)f(\omega) \quad (\omega \in \Omega)$$

is in $L^p(\Omega, E)$ and $\mathcal{M}_M f := Mf$ defines a bounded operator $\mathcal{M}_M \in \mathcal{L}(L^p(\Omega, E))$. We call $\mathcal{M}_M$ a **multiplication operator**. Such operators can be characterized by **locality** in the following sense.

DEFINITION 2.2. Let $1 \leqslant p \leqslant \infty$. A linear mapping $T: L^p(\Omega, E) \to L^p(\Omega, E)$ is called **local** if for all $f \in L^p(\Omega, E)$,

$$(Tf)(\omega) = 0 \text{ a.e. on } \{\omega \in \Omega; f(\omega) = 0\}.$$

It is easy to see that T is local if and only if

$$T(1_A f) = 1_A Tf \quad (f \in L^p(\Omega, E), A \in \Sigma). \tag{2.1}$$

THEOREM 2.3. *Let $1 \leqslant p < \infty$ and let $T \in \mathcal{L}(L^p(\Omega, E))$ be a local operator. Then there exists $M \in L^\infty(\Omega, \mathcal{L}_s(E))$ such that $T = \mathcal{M}_M$.*

Proof. As in the proof of Proposition 1.8 we can assume that $\mu(\Omega) < \infty$. For $x \in E$ let $M_x = T(1_\Omega \cdot x)$, where $(1_A \cdot x)(\omega) = 1_A(\omega)x$ for all $\omega \in \Omega, A \in \Sigma, x \in E$. Then $M_x \in L^\infty(\Omega, E)$ and $\|M_x\| \leqslant \|T\|\|x\|$ (cf. the proof of Proposition 1.8). Now we choose a dense countable subset F of E such that $F + F \subset F$ and $(\mathbb{Q} + i\mathbb{Q})F \subset F$. Then we find a null set $N \in \Sigma$ such that

$$\|M_x(\omega)\| \leqslant \|T\|\|x\| \tag{2.2}$$
$$M_{\lambda x+\mu y}(\omega) = \lambda M_x(\omega) + \mu M_y(\omega) \tag{2.3}$$

for all $\omega \in \Omega\backslash N, x, y \in F, \lambda, \mu \in \mathbb{Q} + i\mathbb{Q}$. For $x \in E$ we define $\bar{M}_x(\omega) = \lim_{n\to\infty} M_{x_n}(\omega)$ for all $\omega \in \Omega\backslash N$ where $x_n \in F$ and $\lim_{n\to\infty} x_n = x$. This limit exists and is independent of the sequence because of (2.2), (2.3). We let $\bar{M}_x(\omega) = 0$ for all $\omega \in N, x \in E$. Then for each $\omega \in \Omega$ the mapping $x \mapsto \bar{M}_x(\omega)$ is linear and bounded. Moreover, $\bar{M}_x(\cdot) \in L^\infty(\Omega, E)$ for all $x \in E$. Now we define $M \in L^\infty(\Omega, \mathcal{L}_s(E))$ by $M(\omega)x := \bar{M}_x(\omega)$. Then $\|M\| \leqslant \|T\|$. Moreover, $(Tf)(\omega) = M(\omega)f(\omega)$ a.e. if $f = 1_A \cdot x$. Since such functions form a total subset of $L^p(\Omega, E)$, the proof is complete. □

COROLLARY 2.4. *The mapping $M \mapsto \mathcal{M}_M$ is an isometric algebra isomorphism from $L^\infty(\Omega, \mathcal{L}_s(E))$ onto the space of all local operators on $L^p(\Omega, E)$ where $1 \leqslant p < \infty$.*

We omit the details of the proof and refer to Thomaschewski [18] for this and other results on local operators.

REMARK 2.5. One cannot replace $\mathcal{L}_s(E)$ by $\mathcal{L}(E)$. To give an example, consider a C_0-semigroup T on a Banach space E which is not continuous on $(0, \infty)$ for the operator norm. Then $T: (0, \infty) \to \mathcal{L}(E)$ is not measurable by Hille–Phillips [7, p. 305]. But $\mathcal{M}_T$ given by $(M_T f)(t) = T(t)f(t)$ $(t \geqslant 0, f \in L^p((0, \infty); E))$ defines a multiplication operator on $L^p((0, \infty); E)$. □

Next we show a result on automatic continuity (see Abramovich [1], de Pagter [15] and Abramovich, Veksler, Kaldunov [2] for the scalar case).

THEOREM 2.6. *Assume that (Ω, Σ, μ) has no atoms. Let $1 \leqslant p < \infty$ and let $T: L^p(\Omega, E) \to L^p(\Omega, E)$ be linear and local. Then T is continuous.*

Proof. By the argument given in the proof of Proposition 1.8 we may assume that $\mu(\Omega) < \infty$. By a *nontrivial set* we understand a measurable subset Ω_1 of Ω such that $\mu(\Omega_1) > 0$. Recall that $1_{\Omega_1} Tf = T(1_{\Omega_1} f)$ for all $f \in L^p(\Omega; E)$. We let $T_{\Omega_1} = T_{|L^p(\Omega_1, E)}$. Assume that T is unbounded.

(a) We show that for all $M > 0$ there exists a nontrivial set Ω_1 such that $\|T_{\Omega_1}\| > M$ and $T_{\Omega_1^c}$ is unbounded.

In fact, otherwise there exists $M > 0$ such that for each nontrivial set Ω_1 such that $T_{\Omega_1^c}$ is unbounded one has $\|T_{\Omega_1}\| \leqslant M$. Let $c := \sup\{\mu(\Omega_1): \Omega_1$ is nontrivial such that $T_{\Omega_1^c}$ is unbounded$\}$. Observe that for nontrivial sets $\Omega_1, \Omega_2, \|T_{\Omega_j}\| \leqslant M, j = 1, 2$, implies that $\|T_{\Omega_1 \cup \Omega_2}\| \leqslant M$. This follows from the fact that for measurable $U \subset \Omega, f \in L^p(\Omega, E)$ one has $\|f\|_p^p = \|f 1_U\|_p^p + \|f 1_{U^c}\|_p^p$. Hence there exist nontrivial sets $\Omega_n \subset \Omega$ such that $\Omega_n \subset \Omega_{n+1}, \|T_{\Omega_n}\| \leqslant M, T_{\Omega_n^c}$ is unbounded and $\lim_{n\to\infty} \mu(\Omega_n) = c$. Let $\tilde{\Omega} = \bigcup_{n\in\mathbb{N}} \Omega_n$. Then $\|T_{\tilde{\Omega}}\| \leqslant M$ and $\mu(\tilde{\Omega}) = c$. Let $V \subset \Omega \backslash \tilde{\Omega}$ be nontrivial. Then T_V is unbounded (because otherwise also $\|T_{\tilde{\Omega}\cup V}\| \leqslant M$. But $\mu(V \cup \tilde{\Omega}) > \mu(\tilde{\Omega})$ contradicting the definition of c). Since the measure space has no atoms, there exists a nontrivial $V_1 \subset V$ such that $V \backslash V_1$ is nontrivial. Then T_{V_1} and $T_{V_1^c}$ are unbounded, a contradiction.

(b) By (a) we can construct inductively nontrivial pairwise disjoint sets Ω_n such that $\|T_{\Omega_n}\| \geqslant 2^n$ and such that $T_{(\Omega_1 \cup \cdots \cup \Omega_n)^c}$ is unbounded. Thus there exist $f_n \in L^p(\Omega, E)$ such that $f_n = 1_{\Omega_n} f_n$, $\|f_n\|_p \leqslant 2^{-n}$, $\|T_{\Omega_n} f_n\| \geqslant 1$. Let $f = \sum_{n=1}^{\infty} f_n$.
Then

$$\begin{aligned}\|Tf\|_p^p &= \Big\| \sum_{k=1}^{n} Tf_k \Big\|_p^p + \Big\| T \sum_{k=n+1}^{\infty} f_k \Big\|_p^p \\ &\geqslant \Big\| \sum_{k=1}^{n} Tf_k \Big\|_p^p = \sum_{k=1}^{n} \Big\| Tf_k \Big\|_p^p \geqslant n,\end{aligned}$$

for all $n \in \mathbb{N}$, which is impossible. Thus the assumption that T is unbounded leads to a contradiction.

EXAMPLE 2.7. Assume that $A \subset \Omega$ is an atom with $\mu(A) > 0$. Thus, if $B \subset A$ is measurable, then $\mu(B) = 0$ or $\mu(A \backslash B) = 0$. If $\dim E = \infty$, then there exists $S: E \to E$ linear and unbounded. Define

$$(Tf)(\omega) = \begin{cases} Sf(\omega) & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Then $T: L^p(\Omega, E) \to L^p(\Omega, E)$ is local and unbounded.

3. Local forms in the vector-valued case

Let (Ω, Σ, μ) be a σ-finite measure space and H a separable complex Hilbert space. Let $\mathcal{H} = L^2(\Omega; H)$ and let V be a Hilbert space such that $V \underset{d}{\hookrightarrow} \mathcal{H}$. Let $a: V \times V \to \mathbb{C}$ be a continuous sesquilinear form which is **coercive**, i.e.,

$$\operatorname{Re} a(u, u) \geqslant \alpha \|u\|_V^2 \quad (u \in V)$$

for some $\alpha > 0$. Denote by A the operator associated with a. Then $-A$ generates a C_0-semigroup T on $\mathcal{H}$.

THEOREM 3.1. *The following assertions are equivalent:*

(i) *Each operator $T(t)$ is a multiplication operator;*

(ii) *if* $u \in V$ *and* $B \subset \Omega$ *is measurable, then* $1_B u, 1_{B^c} u \in V$ *and*

$$a(1_B u, 1_{B^c} u) = 0.$$

Proof. Let $B \subset \Omega$ be measurable. Let $K = \{f \in L^2(\Omega, H): f = 0$ on Ba.e.$\}$. The orthogonal projection P onto K is given by $Pg = 1_{B^c} g$.

(i) $\Rightarrow$ (ii). If $T(t)$ is a multiplication operator, then $T(t)K \subset K$. It follows from Proposition 1.1 that for $u \in V$ one has $1_{B^c} u \in V$ and

$$a(u, 1_B u) \geqslant 0.$$

Consequently, for all $r \in \mathbb{R}$,

$$0 \leqslant a(1_B u + r 1_{B^c} u, 1_B u) = a(1_B u, 1_B u) + r a(1_{B^c} u, 1_B u).$$

This implies that $a(1_{B^c} u, 1_B u) = 0$.

(ii) $\Rightarrow$ (i). It follows from (ii) that for $u \in V$ one has $Pu \in V$ and

$$a(u, u - Pu) = a(u, 1_B u) = a(1_B u, 1_B u) + a(1_{B^c} u, 1_B u) = a(1_B u, 1_B u) \geqslant 0.$$

Now Proposition 1.1 implies that $T(t)K \subset K \quad (t \geqslant 0)$. Theorem 2.3 implies that $T(t)$ is a multiplication operator. □

A typical example is given as follows. Let H be a separable complex Hilbert space and V a Hilbert space such that $V \underset{d}{\hookrightarrow} H$. Let $a : [0, \tau] \times V \times V \to \mathbb{C}$ be a mapping such that for all $t \in [0, \tau]$

$$|a(t, u, v)| \leqslant M \|u\|_V \|v\|_V \quad (u, v \in V);$$
$$\mathrm{Re}\, a(t, u, u) \geqslant \alpha \|u\|_V^2 \quad (u \in V);$$
$$a(t, \cdot, \cdot): V \times V \to \mathbb{C} \quad \text{is sesquilinear;}$$

and such that $a(\cdot, u, v)$ is measurable on $[0, \tau]$ for all $u, v \in V$. Let

$$\mathcal{V} = L^2((0, \tau); V), \mathcal{H} = L^2((0, \tau); H)$$

and

$$\underline{a}(\underline{u}, \underline{v}) = \int_0^\tau a(s, \underline{u}(s), \underline{v}(s))\, ds \text{ for all } \underline{u}, \underline{v} \in \mathcal{V}.$$

Then $\underline{a}: \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is continuous and coercive, $\mathcal{V} \underset{d}{\hookrightarrow} \mathcal{H}$. The associated operator A on $\mathcal{H}$ generates a C_0-semigroup of multiplication operators on $\mathcal{H}$. We refer to Thomaschewski [18] for further details and applications to the non-autonomous Cauchy problem (see Lions [8], Dautray–Lions [4] for the classical theory).

References

1. Yu. A. Abramovich: Multiplicative representation of the operators preserving disjointness, *Indag. Math., Proc. Nederl. Akad. Sci.* **45** (1983).
2. Yu. A. Abramovich, Veksler, A. I. Kaldunov, A. V.: On operators preserving disjointness, *Soviet Math. Dokl.* **20** (1979), 1089–1093.
3. Dautray, R. Lions, J.L.: *Mathematical Analysis and Numerical Methods in Science and Technology*, Vol 2, Functional and variational methods, Springer, 1988.
4. Dautray, R. Lions, J.L.: *Mathematical Analysis and Numerical Methods in Science and Technology*, Vol 5, Evolution Problems I, Springer, 1992.
5. Davies, E.B.: *Heat Kernels and Spectral Theory*, Cambridge University Press, 1989.
6. Fukushima, M. Oshima, Y. Takeda, M.: *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Studies in Mathematics **19** (1994).
7. Hille, E. Phillips, R.S.: *Functional Analysis and Semigroups*, rev. ed., American Mathematical Society Colloquium Publications Vol. 31 1957.
8. Lions, J.L.: *Equations Différentielles Opérationnelles et Probèmes aux Limites*, Springer (1961).
9. Ma, Z.M. Röckner, M.: *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*, Universitext, Springer, 1992.
10. Nagel R.(ed.): *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Mathematics 1184, Springer (1986).
11. Nagel, R. Uhlig, N.: An abstract Kato inequality for generators of positive operator semigroups on Banach lattices, *J. Operator Th.* **6** (1981), 113–123.
12. Ouhabaz, E.M.: L^{∞}-contractivity of semigroups generated by sectorial forms, *J. London Math. Soc.* **46** (1992), 529–542.
13. Ouhabaz, E.M.: Invariance of closed convex sets and domination criteria for semigroups, *Potential Anal.* **5** (1996), 611–625.
14. Ouhabaz, E.M.: Analysis of Heat Equations on Domains, *Princeton University Press Oxford* 2005.
15. de Pagter, B.: A note on disjointness preserving operators, *Proc. Amer. Math. Soc.* **90**, (1984), 543–550.
16. Schaefer, H.H.: *Topological Vector Spaces*, Springer, Berlin 1971.
17. Tanabe, H.: *Equations of Evolutions*, Pitman, 1974.
18. Thomaschewski, S.: *Form Methods for Autonomous and Non-Autonomous Cauchy Problems*, PhD-thesis, Ulm 2003.
19. Zaanen, A.C.: Examples of orthomorphisms, *J. Approx. Theory* **13** (1975), 192–204.