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Rank-1 perturbations of cosine functions and semigroups

Wolfgang Arendt [a], Charles J.K. Batty [b,*]

[a] *Abteilung Angewandte Analysis, Universität Ulm, 89069 Ulm, Germany*
[b] *St. John's College, Oxford OX1 3JP, UK*

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Abstract

Let A be the generator of a cosine function on a Banach space X. In many cases, for example if X is a UMD-space, $A+B$ generates a cosine function for each $B \in \mathcal{L}(D((\omega - A)^{1/2}), X)$. If A is unbounded and $1/2 < \gamma \leqslant 1$, then we show that there exists a rank-1 operator $B \in \mathcal{L}(D((\omega - A)^{\gamma}), X)$ such that $A+B$ does not generate a cosine function. The proof depends on a modification of a Baire argument due to Desch and Schappacher. It also allows us to prove the following. If $A+B$ generates a distribution semigroup for each operator $B \in \mathcal{L}(D(A), X)$ of rank-1, then A generates a holomorphic C_0-semigroup. If $A+B$ generates a C_0-semigroup for each operator $B \in \mathcal{L}(D((\omega - A)^{\gamma}), X)$ of rank-1 where $0 < \gamma < 1$, then the semigroup T generated by A is differentiable and $\|T'(t)\| = O(t^{-\alpha})$ as $t \downarrow 0$ for any $\alpha > 1/\gamma$. This is an approximate converse of a perturbation theorem for this class of semigroups.

Keywords: Perturbation; Rank-one; C_0-semigroup; Distribution semigroup; Cosine function; Fractional power

0. Introduction

Let A be the generator of a cosine function. Then A also generates a holomorphic C_0-semigroup. Let $\omega_A \in \mathbb{R}$ such that $(\omega_A, \infty) \subset \varrho(A)$ and $\sup_{\lambda \geqslant \omega_A} \|\lambda R(\lambda, A)\| < \infty$. Then for $\gamma > 0$ and $\omega > \omega_A$, the operator $A_\gamma = (\omega - A)^{\gamma} \in \mathcal{L}(X)$ exists and is invertible. The domain

* Corresponding author.
E-mail addresses: arendt@mathematik.uni-ulm.de (W. Arendt), charles.batty@sjc.ox.ac.uk (C.J.K. Batty).

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$D(A_\gamma)$ is a Banach space for the graph norm, and this space does not depend on the choice of $\omega > \omega_A$ up to equivalent norms. Moreover,

$$D(A_\gamma) \hookrightarrow D(A_\beta)$$

for $0 \leqslant \beta \leqslant \gamma < \infty$. If X is a UMD-space, then $A_{1/2}$ generates a C_0-group on X and $A + B$ also generates a cosine function whenever B is a bounded linear operator from $D(A_{1/2})$ into X (where $D(A_{1/2})$ carries the graph norm) (see [2, Sections 3.14, 3.16]). This property is most important for applications to hyperbolic equations (see [2, Chapter 7]). The aim of this article is to show that $\gamma = \frac{1}{2}$ is best possible for this property. In fact, our main result says the following. Assume that A is unbounded. Let $\frac{1}{2} < \gamma \leqslant 1$. Then there exists an operator $B \in \mathcal{L}(D(A_\gamma), X)$ of rank-1 such that $A + B$ does not generate a cosine function. Moreover, we can choose B of arbitrarily small norm and such that $A + B$ does not generate a k-times integrated cosine function for any $k \in \mathbb{N}$.

Our proof is based on a technique due to Desch and Schappacher [8] who proved that for each generator A of a C_0-semigroup T which is not holomorphic there exists a rank-1 perturbation $B : D(A) \to X$ such that $A + B$ does not generate a C_0-semigroup. We show in Section 3 that B can even be chosen such that $A + B$ does not generate a distribution semigroup. This is done with the help of a generalization and a simplification of the Baire argument of Desch and Schappacher which we establish in Section 1.

Our argument also sheds light on the reason for instability. Rank-1 perturbation may lead to an explosion of the resolvent: the resolvent of $A + B$ cannot have any prescribed growth outside any parabola oriented to the left (in the cosine case) and on any right half-plane in the semigroup case.

In Section 4 we consider (rank-1) perturbations $B : D(A_\gamma) \to X$ of the generator A of a C_0-semigroup T where $0 < \gamma < 1$. We exhibit a perturbation theorem for a class of differentiable semigroups first considered by Crandall and Pazy [6]. Conversely, if $A + B$ generates a C_0-semigroup for each such B, then we show that T belongs to that class of semigroups.

1. Rank-1 perturbations

Let A be a closed linear operator on a Banach space X. Then the domain $D(A)$ of A is a Banach space for the graph norm $\|x\|_A := \|x\| + \|Ax\|$. Let $C \in \mathcal{L}(D(A), X)$ be a bounded linear operator (where $D(A)$ carries the graph norm). Given $a \in X$, $b^* \in X^*$ we consider the perturbation $B \in \mathcal{L}(D(A), X)$ of A given by

$$Bx = b^*(Cx)a \quad \big(x \in D(A)\big).$$

We denote this operator B by ab^*C.

Lemma 1.1. *Let $a \in X$, $b^* \in X^*$, $\lambda \in \varrho(A)$. Then $\lambda \in \varrho(A + ab^*C)$ if and only if*

$$b^*CR(\lambda, A)a \neq 1.$$

In that case

$$R(\lambda, A + ab^*C)x = R(\lambda, A)x + \frac{b^*CR(\lambda, A)x}{1 - b^*CR(\lambda, A)a} \cdot R(\lambda, A)a \tag{1.1}$$

for all $x \in X$. In particular,

$$R(\lambda, A + ab^*C)a \cdot \big(1 - b^*CR(\lambda, A)a\big) = R(\lambda, A)a. \tag{1.2}$$

Proof. Let $\lambda \in \varrho(A + ab^*C)$. Let $x \in X$, $y = R(\lambda, A + ab^*C)x$. Then $x = (\lambda - A)y - ab^*Cy$. Hence $y = R(\lambda, A)x + cR(\lambda, A)a$ where $c = b^*Cy$. Consequently, $x = (\lambda - A - ab^*C)y = x - (b^*CR(\lambda, A)x) \cdot a + c \cdot a - (cb^*CR(\lambda, A)a) \cdot a$. This implies that $b^*CR(\lambda, A)a \neq 1$ and

$$c = \frac{b^*CR(\lambda, A)x}{1 - b^*CR(\lambda, A)a}.$$

Thus one implication is proved. We omit the easy proof of the other one. □

For $\delta > 0$, $p \in X$, $q^* \in X^*$ we let

$$B_\delta(p) := \big\{x \in X\colon\ \|x - p\| \leqslant \delta\big\},$$
$$B_\delta(p, q^*) := \big\{(a, b^*) \in X \times X^*\colon\ \|a - p\| \leqslant \delta,\ \|b^* - q^*\| \leqslant \delta\big\}$$

be closed balls in X and $X \times X^*$, respectively.

Lemma 1.2. *Let $\Omega \subset \varrho(A)$ be non-empty. Assume that*

$$\sup_{\lambda \in \Omega} \big\|CR(\lambda, A)x\big\| < \infty \tag{1.3}$$

for all x in a dense subspace Y of X, but

$$\sup_{\lambda \in \Omega} \big\|CR(\lambda, A)\big\| = \infty. \tag{1.4}$$

Let $\delta > 0$, $p \in X$, $q^* \in X^*$. Then there exist $(a, b^*) \in B_\delta(p, q^*)$, $\lambda \in \Omega$ such that

$$b^*CR(\lambda, A)a = 1.$$

Proof. By the Uniform Boundedness Principle, there exists $b^* \in B_\delta(q^*)$ such that $\sup_{\lambda \in \Omega} \|b^*CR(\lambda, A)\| = \infty$, and there exists $a_2 \in X$ such that $\sup_{\lambda \in \Omega} |b^*CR(\lambda, A)a_2| = \infty$. By (1.3) there exists $a_1 \in B_{\delta/2}(p)$ such that $\sup_{\lambda \in \Omega} \|CR(\lambda, A)a_1\| < \infty$. Let

$$a_3 = \frac{1 - b^*CR(\lambda, A)a_1}{b^*CR(\lambda, A)a_2} \cdot a_2,$$

where $\lambda \in \Omega$ is chosen such that $\|a_3\| \leqslant \delta/2$. Then $b^*CR(\lambda, A)a_3 = 1 - b^*CR(\lambda, A)a_1$. Hence $b^*CR(\lambda, A)a = 1$ where $a = a_1 + a_3 \in B_\delta(p)$. □

In order to formulate the main result of this section we let $\Omega_n \subset \mathbb{C}$ be arbitrary non-empty sets ($n \in \mathbb{N}$) and $g_n : \Omega_n \to (0, \infty)$ be arbitrary functions (which measure the growth of resolvents).

Theorem 1.3. *Let A be a closed operator on X, $C : D(A) \to X$ be a bounded operator, $\varepsilon > 0$. Assume that $\Omega_n \subset \varrho(A)$ and*

$$\sup_{\lambda \in \Omega_n} \left\| CR(\lambda, A)x \right\| < \infty$$

for all x in a dense subspace Y_n of X and all $n \in \mathbb{N}$. Assume that for each $(a, b^*) \in B_\varepsilon(0,0)$ there exists $n \in \mathbb{N}$ such that $\Omega_n \subset \varrho(A + ab^*C)$ and

$$\left\| R(\lambda, A + ab^*C) \right\| \leqslant g_n(\lambda) \quad (\lambda \in \Omega_n). \tag{1.5}$$

Then there exists $m \in \mathbb{N}$ such that

$$\sup_{\lambda \in \Omega_m} \left\| CR(\lambda, A) \right\| < \infty. \tag{1.6}$$

Proof. Let

$$F_n = \left\{ (a, b^*) \in B_\varepsilon(0,0)\colon \ \left\| R(\lambda, A)a \right\| \leqslant g_n(\lambda) \cdot \|a\| \cdot \left| 1 - b^* CR(\lambda, A)a \right| \text{ for all } \lambda \in \Omega_n \right\}.$$

Then F_n is closed and it follows from (1.5) and Lemma 1.1 that $B_\varepsilon(0,0) = \bigcup_{n \in \mathbb{N}} F_n$. By Baire's theorem there exist $p \in X$, $q^* \in X^*$, $\delta > 0$ and $m \in \mathbb{N}$ such that $B_\delta(p, q^*) \subset F_m$. It follows that $b^* CR(\lambda, A)a \neq 1$ for all $\lambda \in \Omega_m$ and all $(a, b^*) \in B_\delta(p, q^*)$. Now Lemma 1.2 implies (1.6). □

2. Perturbation of cosine generators

Generators of cosine functions can conveniently be characterized by Laplace transforms (see [2, Section 3.14]). An operator A on a Banach space X *generates a cosine function* if there exist a strongly continuous function $\cos : \mathbb{R}_+ \to \mathcal{L}(X)$ and some $M \geqslant 0$, $\sigma \geqslant 1$, satisfying

$$\left\| \int_0^t \cos(s)x \, ds \right\| \leqslant M e^{(\sigma - 1)t} \|x\| \quad (t \geqslant 0, x \in X) \tag{2.1}$$

and

$$\lambda^2 \in \varrho(A) \quad \text{and} \quad \lambda R\big(\lambda^2, A\big)x = \int_0^\infty e^{-\lambda t} \cos(t)x \, dt \quad (x \in X) \tag{2.2}$$

whenever $\operatorname{Re} \lambda \geqslant \sigma$. In that case, the function cos is unique and is called *the cosine function generated by A*. Note that condition (2.1) ensures that the integral in (2.2) converges in the improper sense. However, it follows that cos is even exponentially bounded, i.e.,

$$\left\| \cos(t) \right\| \leqslant M' e^{\sigma' t} \quad (t \geqslant 0)$$

for some $M' \geqslant 1$, $\sigma' \in \mathbb{R}$. If A generates a cosine function, then $D(A)$ is dense and for $x, y \in X$ the function $u(t) = \cos(t)x + \int_0^t \cos(s)y\,ds$ is a mild solution of the second order Cauchy problem

$$(\mathrm{P}_2) \quad \begin{cases} u''(t) = Au(t) & (t \geqslant 0), \\ u(0) = x, \quad u'(0) = y \end{cases}$$

which is the motivation for studying cosine functions.

More generally, one defines the following concept [1]. Let $k \in \mathbb{N}_0$. We say that A generates a *k-times integrated cosine function* on X if there exists a strongly continuous function $S : \mathbb{R}_+ \to \mathcal{L}(X)$ and some $M \geqslant 0$, $\sigma \geqslant 1$ satisfying

$$\left\| \int_0^t S(s)x\,ds \right\| \leqslant Me^{(\sigma-1)t}\|x\| \quad (t \geqslant 0, x \in X) \tag{2.3}$$

and

$$\lambda^2 \in \varrho(A) \quad \text{and} \quad \lambda^{-k+1}R\big(\lambda^2, A\big)x = \int_0^\infty e^{-\lambda t}S(t)x\,dt \quad (x \in X) \tag{2.4}$$

whenever $\operatorname{Re}\lambda \geqslant \sigma$. Then S is called the *k-times integrated cosine function generated by A*. Thus cosine functions are the same as 0-times integrated cosine functions, and 1-times integrated cosine functions are sine functions (see [2, Section 3.15]). Moreover, if A generates a k-times integrated cosine function S_k, then it generates the $(k+1)$-times integrated cosine function S_{k+1} given by

$$S_{k+1}(t)x = \int_0^t S_k(s)x\,ds \quad (x \in X),$$

and S_{k+1} is exponentially bounded. We remark that the fact that A generates a k-times integrated cosine function can be reformulated in terms of well-posedness of (P_2) (see Keyantuo [14, Chapter 2, Section 4]); the smaller k is, the less regular the initial values x, y can be chosen in order to obtain a solution of (P_2).

Example 2.1. It is shown in [12,15] (see also [2, Theorem 7.3.1]) that the Laplacian Δ, with domain $W^{2,p}(\mathbb{R}^N)$, generates a k-times integrated cosine function on $L^p(\mathbb{R}^N)$, where $1 < p < \infty$, if and only if $k \geqslant (N-1)|\frac{1}{p} - \frac{1}{2}|$.

Now assume that A generates a k-times integrated cosine function S. Replacing k by $k+1$ if necessary, we may assume that S is exponentially bounded, i.e.

$$\big\|S(t)\big\| \leqslant Me^{(\sigma-1)t} \quad (t \geqslant 0),$$

where $M \geqslant 0$, $\sigma \geqslant 1$. Then by (2.4) we have for $\operatorname{Re}\lambda \geqslant \sigma$,

$$\left\|\lambda^{-k+1}R\left(\lambda^2, A\right)\right\| \leqslant M\int_0^\infty e^{(\sigma-1)t}e^{-\operatorname{Re}\lambda t}\,dt \leqslant \frac{M}{\operatorname{Re}\lambda-\sigma+1}.$$

Thus

$$\left\|R\left(\lambda^2, A\right)\right\| \leqslant M|\lambda|^{k-1} \quad (\operatorname{Re}\lambda \geqslant \sigma).$$

Consequently, the resolvent of A is polynomially bounded on

$$\Omega_\sigma := \left\{\lambda^2\colon \operatorname{Re}\lambda \geqslant \sigma\right\} = \left\{\xi + i\eta\colon \eta \in \mathbb{R}, \xi \geqslant \sigma^2 - \frac{\eta^2}{4\sigma^2}\right\}$$

which is the exterior of a horizontal parabola.

Now we wish to consider perturbations $B \in \mathcal{L}(D(A_\gamma), X)$ where $0 < \gamma \leqslant 1$, $\omega > \sigma^2$ and $A_\gamma := (\omega - A)^\gamma$ is a fractional power. If $k = 0$, then A generates a holomorphic C_0-semigroup and the definition of A_γ is standard (see [2, Section 3.8] for example). If $k > 0$ and A is densely defined, the fractional powers A_γ can be defined by [18, Definition 1.11] or [7, Section 5] since the resolvent of $\omega - A$ is polynomially bounded on a sector. However, we shall need certain properties of fractional powers (for example, that $D(A_\gamma) \subset D(A)$) which are standard for the case $k = 0$ but do not appear to be known for $k > 0$. Therefore we shall now assume either that $\gamma = 1$ or that A generates a cosine function, although we shall allow the possibility that $A + B$ generates an integrated cosine function. Then the operator A_γ is closed and $D(A_\gamma) = D((\omega_1 - A)^\gamma)$ whenever $\omega_1 > \sigma^2$. For $a \in X$, $b^* \in X^*$ we consider the rank-1 perturbation $B : D(A_\gamma) \to X$ given by

$$Bx = b^*(A_\gamma x)a \quad \left(x \in D(A_\gamma)\right)$$

which we denote by $B = ab^*A_\gamma$. Now we can formulate the main result of this section.

Theorem 2.2. *Assume that either A is the generator of a cosine function and $1/2 < \gamma \leqslant 1$, or that A is the generator of a k-times integrated cosine function for some $k \in \mathbb{N}_0$ and $\gamma = 1$. Let $A_\gamma = (\omega - A)^\gamma$ where ω is large enough. Let $\varepsilon > 0$. Assume that for each $a \in X$, $b^* \in X^*$ satisfying $\|a\| \leqslant \varepsilon$, $\|b^*\| \leqslant \varepsilon$ there exists $\ell \in \mathbb{N}$ such that $A + ab^*A_\gamma$ generates an ℓ-times integrated cosine function. Then A is bounded.*

We need the following two lemmas. We do not claim originality but we include proofs for the convenience of the reader.

Lemma 2.3. *Let A be an operator such that the resolvent exists and is polynomially bounded outside a ball. Then A is bounded.*

Proof. With the help of the spectral projection associated with the bounded spectrum we reduce the problem to the case where $\varrho(A) = \mathbb{C}$ and $R(\lambda, A)$ is polynomially bounded. By elementary complex function theory, $R(\lambda, A)$ is a polynomial. Then $(-1)^{n-1}(n-1)!R(\lambda, A)^n = (d/d\lambda)^{n-1}R(\lambda, A) = 0$ for some $n \in \mathbb{N}$. Since $R(\lambda, A)$ is injective we conclude that $X = \{0\}$. □

Lemma 2.4. (See [9, Theorem 2].) *Let $A_\gamma = (\omega - A)^\gamma$, where $0 < \gamma < 1$ and $\omega > \omega_A$. Let Ω be a subset of $\varrho(A)$ whose closure does not contain* 0. *The following are equivalent*:

(i) $\sup_{\lambda\in\Omega} \|\lambda^\gamma R(\lambda, A)\| < \infty$;
(ii) $\sup_{\lambda\in\Omega} \|A_\gamma R(\lambda, A)\| < \infty$.

Proof. Let $x \in X$. By the moment inequality [10, Theorem II.5.34],

$$\|A_\gamma R(\lambda, A)x\| \leqslant \|(\omega - A)R(\lambda, A)x\|^\gamma \|R(\lambda, A)x\|^{1-\gamma}.$$

Hence,

$$\|A_\gamma R(\lambda, A)\| \leqslant \|(\omega - \lambda)R(\lambda, A) + I\|^\gamma \|R(\lambda, A)\|^{1-\gamma}, \tag{2.5}$$

and it follows that (i) implies (ii).

Also by the moment inequality,

$$\|R(\lambda, A)x\| \leqslant \|A_\gamma R(\lambda, A)x\|^{1-\gamma} \|A_\gamma R(\omega, A)R(\lambda, A)x\|^\gamma.$$

Hence

$$\|R(\lambda, A)\| \leqslant \|A_\gamma R(\lambda, A)\|^{1-\gamma} \left(\frac{\|A_\gamma (R(\lambda, A) - R(\omega, A))\|}{|\lambda - \omega|}\right)^\gamma,$$

and it follows that (ii) implies (i). □

Proof of Theorem 2.2. Let A be the generator of a k-times integrated cosine function. Then for suitable $M, \sigma \geqslant 1$, $r \in \mathbb{N}$ one has $\Omega_0 := \{\lambda^2\colon \operatorname{Re}\lambda \geqslant \sigma\} \subset \varrho(A)$ and

$$\|R(\mu, A)\| \leqslant M|\mu|^r \quad (\mu \in \Omega_0).$$

Take $\omega > \sigma^2$. Then $\omega - A$ is invertible and $(\omega - A)^\alpha$ is a bounded operator whenever $\alpha \leqslant 0$. Let

$$\Omega_n := \{\lambda^2\colon \operatorname{Re}\lambda \geqslant \sigma + n\}, \quad g_n(\lambda) = n(1 + |\lambda|)^n \quad (\lambda \in \Omega_n)$$

for $n \in \mathbb{N}$. Under the assumptions of the theorem, for all $(a, b^*) \in B_\varepsilon(0, 0)$ there exists $n \in \mathbb{N}$ such that $\Omega_n \subset \varrho(A + ab^*A_\gamma)$ and $\|R(\lambda, A + ab^*A_\gamma)\| \leqslant g_n(\lambda)$ $(\lambda \in \Omega_n)$. For $x \in D(A^r)$,

$$R(\lambda, A)x = \lambda^{-r} R(\lambda, A)A^r x + \sum_{m=0}^{r-1} \lambda^{-(m+1)} A^m x,$$

so $R(\lambda, A)x$ is bounded on Ω_0. Hence $A_\gamma R(\lambda, A)x = (\omega - A)^{\gamma-1} R(\lambda, A)(\omega - A)x$ is bounded on Ω_0 for all $x \in D(A^{r+1})$ which is dense in X. By Theorem 1.3 there exists $m \in \mathbb{N}$ such that $\sup_{\lambda\in\Omega_m} \|A_\gamma R(\lambda, A)\| < \infty$. By Lemma 2.4,

$$c := \sup_{\lambda\in\Omega_m} \|\lambda^\gamma R(\lambda, A)\| < \infty.$$

Now let $\lambda \in \partial\Omega_m$, i.e., $\lambda = \xi + i\eta$, where $\eta \in \mathbb{R}$ and $\xi = (\sigma+m)^2 - \frac{\eta^2}{4(\sigma+m)^2}$. Let $\mu = \xi + i\eta_1$ where $|\eta_1| \leqslant |\eta|$. Write $\mu - A = (I - (\lambda-\mu)R(\lambda, A))(\lambda - A)$. Then,

$$\begin{aligned}\big\|(\lambda-\mu)R(\lambda,A)\big\| &\leqslant |\eta|\big\|R(\lambda,A)\big\| \leqslant c\frac{|\eta|}{|\lambda|^\gamma}\\ &= c|\eta|\bigg[\bigg((\sigma+m)^2 - \frac{\eta^2}{4(\sigma+m)^2}\bigg)^2 + \eta^2\bigg]^{-\gamma/2}\\ &\leqslant 1/2\end{aligned}$$

if $|\eta|$ is sufficiently large. Here we use that $\gamma > \frac{1}{2}$. Thus there exists $\xi_0 > 0$ such that, for $\lambda = \xi + i\eta \in \partial\Omega_m$ with $\xi \leqslant -\xi_0$, one has $\mu = \xi + i\eta_1 \in \varrho(A)$ and $\|R(\mu, A)\| \leqslant 2\|R(\lambda, A)\|$ whenever $|\eta_1| \leqslant |\eta|$.

Since $|\lambda| \leqslant \alpha|\xi| \leqslant \alpha|\mu|$ for some constant α independent of λ we conclude that $R(\mu, A)$ is polynomially bounded in the region $\{\mu \in \mathbb{C}\colon \operatorname{Re}\mu \leqslant -\xi_0\} \setminus \Omega_m$. Since $R(\mu, A)$ is polynomially bounded on Ω_m we deduce from Lemma 2.3 that A is bounded. □

Remark 2.5. In the proof of Theorem 2.2, it was not important that the functions g_n were polynomially bounded, although it was important that $R(\lambda, A)$ is polynomially bounded. The proof shows the following. Suppose that A is unbounded, and $\omega - A$ is sectorial and $R(\lambda, A)$ exists and is polynomially bounded outside a parabola $\Omega_m := \{\lambda^2\colon \operatorname{Re}\lambda \geqslant m\}$. Let $g_n : \Omega_n \to (0,\infty)$ be any functions and let $\gamma > 1/2$. Then there exist $a \in X$, $b^* \in X^*$ and $\lambda_n \in \Omega_n$ ($n \in \mathbb{N}$) such that either $\lambda_n \in \sigma(A + ab^*A_\gamma)$ or $\|R(\lambda_n, A + ab^*A_\gamma)\| \geqslant g_n(\lambda)$.

Similar remarks apply to Theorems 3.1 and 4.3.

3. Perturbation of distribution semigroups

The property of generating a holomorphic C_0-semigroup is stable under small perturbations. In fact, if A generates a holomorphic C_0-semigroup then so does $A + B$ for each compact $B : D(A) \to X$, see [2, Theorem 3.7.25] or [8]. Desch and Schappacher [8] showed that the property of generating a C_0-semigroup is not stable under small perturbations unless the given semigroup is already holomorphic. Our general perturbation result of Section 1 allows us to generalize the Desch–Schappacher result to a much larger class, characterizing generators of holomorphic C_0-semigroups among the class of all generators of distribution semigroups.

The concept of a *distribution semigroup* was introduced by Lions [16]. It is equivalent to the notion of *local k-times integrated semigroup* introduced in [3] which can be formulated precisely in terms of the well-posedness of the Cauchy problem defined by A. Here we use the following characterization in terms of the resolvent. A densely defined operator A generates a distribution semigroup if and only if there exists $k \in \mathbb{N}$ such that A generates a local k-times integrated semigroup, or equivalently if and only if there exists an *exponential region*

$$E(\alpha,\beta) := \big\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant \beta,\ |\operatorname{Im}\lambda| \leqslant e^{\alpha \operatorname{Re}\lambda}\big\} \quad \text{where } \beta \in \mathbb{R},\ \alpha \geqslant 0,$$

such that $E(\alpha, \beta) \subset \varrho(A)$ and $R(\lambda, A)$ is polynomially bounded on $E(\alpha, \beta)$.

With the help of this characterization we can prove the following.

Theorem 3.1. *Let A be a densely defined operator on a Banach space X and let $\varepsilon > 0$. Assume that for each $a \in X$, $b^* \in X^*$ satisfying $\|a\| \leqslant \varepsilon$, $\|b^*\| \leqslant \varepsilon$, the operator $A + ab^*A$ generates a distribution semigroup. Then A generates a holomorphic C_0-semigroup.*

Proof. Since A generates a distribution semigroup there exist $\alpha > 0$, $\beta \in \mathbb{R}$, $c \geqslant 0$ and $\ell \in \mathbb{N}$ such that $E(\alpha, \beta) \subset \varrho(A)$ and $\|R(\lambda, A)\| \leqslant c(1 + |\lambda|)^\ell$ for all $\lambda \in E(\alpha, \beta)$. Let

$$\Omega_n = E(\alpha + n, \beta + n) = \left\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant \beta + n,\ |\operatorname{Im}\lambda| \leqslant e^{(\alpha+n)\operatorname{Re}\lambda}\right\}$$

and let $g_n(\lambda) = (c + n)(1 + |\lambda|)^{\ell+n}$ where $n \in \mathbb{N}$. The assumption implies that for each $(a, b^*) \in B_\varepsilon(0, 0)$ there exists $n \in \mathbb{N}$ such that $\Omega_n \subset \varrho(A + ab^*A)$ and $\|R(\lambda, A + ab^*A)\| \leqslant g_n(\lambda)$ $(\lambda \in \Omega_n)$. For $x \in D(A^\ell)$,

$$R(\lambda, A)x = \lambda^{-\ell} R(\lambda, A)A^\ell x + \sum_{m=0}^{\ell-1} \lambda^{-(m+1)} A^m x,$$

so $R(\lambda, A)x$ is bounded on $E(\alpha, \beta)$. Thus $AR(\lambda, A)x$ is bounded on $E(\alpha, \beta)$ for all $x \in D(A^{\ell+1})$ which is a dense subspace. By Theorem 1.3 there exists $m \in \mathbb{N}$ such that

$$M := \sup_{\lambda \in \Omega_m} \left\|\lambda R(\lambda, A)\right\| < \infty.$$

By the von Neumann expansion we obtain $\theta \in (0, \pi/2)$, $M' \geqslant 0$ such that the following holds: if a half-line $L = \{re^{i\gamma}\colon r \geqslant r_0\}$ is in $\varrho(A)$ and $\|\lambda R(\lambda, A)\| \leqslant M$ on L, then also $L_\theta := \{re^{i\psi}\colon r \geqslant r_0,\ |\psi - \gamma| \leqslant \theta\} \subset \varrho(A)$ and $\|\lambda R(\lambda, A)\| \leqslant M'$ on L_θ. Since the boundary of Ω_m becomes arbitrarily steep, we find $\omega \geqslant \alpha$ such that

$$\Omega := \left\{\omega + re^{i\psi}\colon r \geqslant 0,\ |\psi| \leqslant \frac{\pi}{2} + \frac{\theta}{2}\right\} \subset \varrho(A)$$

and

$$M'' := \sup_{\lambda \in \Omega} \left\|\lambda R(\lambda, A)\right\| < \infty.$$

Consequently, A generates a holomorphic C_0-semigroup. □

4. Fractional perturbation of semigroup generators

In this section, we combine techniques from the previous sections to show that if there exists $\gamma > 0$ such that $A + ab^*A_\gamma$ generates a C_0-semigroup for every $a \in X$, $b^* \in X^*$, then the semigroup T generated by A belongs to a class considered by Crandall and Pazy [6]. That is, T is immediately differentiable and its derivative $AT(t)$ satisfies

$$\left\|AT(t)\right\| = O\left(t^{-\alpha}\right) \quad \text{as } t \downarrow 0 \tag{4.1}$$

for some $\alpha > 0$. It was shown in [6,9] that this is equivalent to the property that

$$\left\|R(\omega + is, A)\right\| = O\left(|s|^{-\beta}\right) \quad \text{as } |s| \to \infty \tag{4.2}$$

for some $\beta > 0$ and $\omega > \omega_0(T)$, the exponential growth bound of T. Indeed, (4.1) implies (4.2) for $\beta = 1/\alpha$ and any $\omega > \omega_0(T)$. On the other hand, (4.2) implies (4.1) for any $\alpha > 1/\beta$. By a standard Neumann series argument, (4.2) for one value of ω implies that, for any $\omega' \in \mathbb{R}$, $R(\omega' + is, A)$ exists for all real s with $|s|$ sufficiently large and $\|R(\omega' + is, A)\| = O(|s|^{-\beta})$ as $|s| \to \infty$. Moreover, Lemma 2.4 shows that (4.2) is equivalent to the property that

$$\sup_{s\in\mathbb{R}} \bigl\| A_\beta R(\omega + is, A)\bigr\| < \infty. \tag{4.3}$$

This class of semigroups arises naturally when considering differentiability of solutions of inhomogeneous Cauchy problems [6] and of delay differential equations [4,5]. Note that the case when $\alpha = 1$ in (4.1) and the case when $\beta = 1$ in (4.2) each correspond to T being a holomorphic semigroup [2, Corollary 3.7.18, Theorem 3.7.19], [10, Theorem II.4.6], so we are really interested in the case when $\alpha > 1$ and $0 < \beta < 1$.

We will first show that this class of generators is invariant under suitable fractionally bounded perturbations. This result fits naturally between the standard results for bounded perturbations of C_0-semigroups and relatively bounded perturbations of holomorphic semigroups (see [10, Corollary III.2.14]). We are very grateful to Markus Haase for enabling us to complete the proof of this result.

Proposition 4.1. *Let A be the generator of a C_0-semigroup T and suppose that A satisfies* (4.2) *for some $\beta > 0$. Let $B \in \mathcal{L}(D(A_\gamma), X)$ where $0 < \gamma < \beta$. Then $A + B$ generates a C_0-semigroup. Moreover, $\|R(\omega + is, A + B)\| = O(|s|^{-\beta})$ as $|s| \to \infty$.*

Proof. In this proof, c will denote a constant which may vary from place to place.

Choose $\alpha \in (\beta^{-1}, \gamma^{-1})$. Then (4.1) holds, so

$$\bigl\|AT(t)\bigr\| \leqslant ct^{-\alpha} \quad (0 < t \leqslant 1).$$

Let $x \in X$. By the moment inequality [10, Theorem II.5.34],

$$\bigl\|A_\gamma T(t)x\bigr\| \leqslant \bigl\|(\omega - A)T(t)x\bigr\|^\gamma \bigl\|T(t)x\bigr\|^{1-\gamma} \leqslant ct^{-\alpha\gamma}\|x\|.$$

Hence,

$$\int_0^1 \bigl\|BT(t)\bigr\|\,dt < \infty.$$

It follows from [13, Corollary 1, p. 400] (see also [10, Theorem III.3.14]) that $A + B$ generates a C_0-semigroup S.

Let $\lambda = \omega + is$. By (2.5) and (4.2),

$$\bigl\|BR(\lambda, A)\bigr\| \leqslant c\bigl\|A_\gamma R(\lambda, A)\bigr\| \leqslant c|\lambda|^{\gamma-\beta}.$$

Hence $\|BR(\lambda, A)\| \leqslant 1/2$ whenever $|s|$ is sufficiently large. For such s, it follows from the identity $\lambda - (A + B) = (I - BR(\lambda, A))(\lambda - A)$ that $\lambda \in \varrho(A + B)$ and

$$\bigl\|R(\lambda, A + B)\bigr\| = \bigl\|R(\lambda, A)\bigl(I - BR(\lambda, A)\bigr)^{-1}\bigr\| \leqslant 2\bigl\|R(\lambda, A)\bigr\| \leqslant c|s|^{-\beta}. \qquad \square$$

If A generates a holomorphic semigroup and $B : D(A) \to X$ is compact, then $A + B$ also generates a holomorphic semigroup [8, Theorem 1], [2, Theorem 3.7.25]. One might expect that if A generates a semigroup and satisfies (4.2) and $B : D(A_\beta) \to X$ is compact, then $A+B$ should also be a generator. We do not know whether this is the case, but we have the following partial results when B is of rank-1.

Proposition 4.2. *Let A be the generator of a C_0-semigroup on a Banach space X, and suppose that A satisfies* (4.2) *for some $\beta > 0$. Let $B = ab^* A_\beta$, where $a \in X$ and $b^* \in X^*$.*

(1) *There exists $r \geqslant 0$ such that $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant \omega, |\lambda| \geqslant r\} \subset \varrho(A + B)$ and $\|R(\omega + is, A + B)\| = O(|s|^{-\beta})$ as $|s| \to \infty$.*
(2) *If X is a Hilbert space then $A + B$ generates a C_0-semigroup.*

Proof. (1) By (4.2), an estimate

$$\big\|R(\lambda, A)\big\| \leqslant \frac{c}{|\lambda|^\beta} \tag{4.4}$$

holds when $\operatorname{Re}\lambda = \omega$. By a standard Neumann series estimate and changing the value of c, we can arrange that (4.4) holds when $\omega \leqslant \operatorname{Re}\lambda \leqslant \omega + c^{-1}|\operatorname{Im}\lambda|^\beta$.

Choose ω' so that $\omega_0(T) < \omega' < \omega$. Then $\|R(\lambda, A)\| \leqslant c'/(\operatorname{Re}\lambda - \omega')$ when $\operatorname{Re}\lambda > \omega$. If $\operatorname{Re}\lambda > \omega + c^{-1}|\operatorname{Im}\lambda|^\beta$, then

$$|\lambda|^\beta \leqslant 2^{\beta/2} \max\big(\operatorname{Re}\lambda, |\operatorname{Im}\lambda|\big)^\beta \leqslant c''(\operatorname{Re}\lambda - \omega').$$

Changing the value of c again if necessary, we obtain that (4.4) holds whenever $\operatorname{Re}\lambda \geqslant \omega$.

By Lemma 2.4, $\|A_\beta R(\lambda, A)\|$ is uniformly bounded for $\operatorname{Re}\lambda \geqslant \omega$. As $|\lambda| \to \infty$, $\|A_\beta R(\lambda, A)x\| \to 0$ first for $x \in D(A_\beta)$ and then by density for $x \in X$, and in particular for $x = a$. Now Lemma 1.1 shows that, for $|\lambda|$ sufficiently large, $\lambda \in \varrho(A + B)$ and

$$R(\lambda, A + B) = R(\lambda, A) + Q(\lambda),$$

where $Q(\lambda)$ is a bounded (rank-1) operator and $\|Q(\lambda)\| \leqslant c\|R(\lambda, A)a\|$ for some constant c. In particular, this shows that

$$\big\|R(\lambda, A + B)\big\| \leqslant \big(1 + c\|a\|\big)\big\|R(\lambda, A)\big\| = O\big(|\lambda|^{-\beta}\big)$$

as $|\lambda| \to \infty$.

(2) Now suppose that X is a Hilbert space. Replacing A by $A - \omega$, we may assume that T is bounded (by K, say). For any $x \in X$ and $\sigma > 0$, Plancherel's theorem gives

$$\int_{-\infty}^{\infty} \big\|R(\sigma + is, A)x\big\|^2 \, ds = 2\pi \int_0^\infty e^{-2\sigma t} \big\|T(t)x\big\|^2 \, dt \leqslant \frac{\pi K^2}{\sigma} \|x\|^2.$$

Moreover, for σ sufficiently large,

$$\int_{-\infty}^{\infty} \left\| Q(\sigma + is)x \right\|^2 ds \leqslant \int_{-\infty}^{\infty} c^2 \left\| R(\sigma + is, A)a \right\|^2 \|x\|^2 ds \leqslant \frac{\pi c^2 \|a\|^2 K^2}{\sigma} \|x\|^2.$$

Hence,

$$\int_{-\infty}^{\infty} \left\| R(\sigma + is, A + B)x \right\|^2 ds \leqslant \frac{\pi (1 + c\|a\|)^2 K^2}{\sigma} \|x\|^2.$$

Similarly,

$$\int_{-\infty}^{\infty} \left\| R(\sigma + is, A)^* x \right\|^2 ds = 2\pi \int_{0}^{\infty} e^{-2\sigma t} \left\| T(t)^* x \right\|^2 dt \leqslant \frac{\pi K^2}{\sigma} \|x\|^2,$$

$$\int_{-\infty}^{\infty} \left\| Q(\sigma + is)^* x \right\|^2 ds \leqslant \int_{-\infty}^{\infty} c^2 \left\| R(\sigma + is, A)a \right\|^2 \|x\|^2 ds \leqslant \frac{\pi c^2 \|a\|^2 K^2}{\sigma} \|x\|^2,$$

$$\int_{-\infty}^{\infty} \left\| R(\sigma + is, A + B)^* x \right\|^2 ds \leqslant \frac{\pi (1 + c\|a\|)^2 K^2}{\sigma} \|x\|^2.$$

Now the result follows from [11, Theorem 2] or [17, Theorems 1.1, 4.1]. □

The next result is the converse of Proposition 4.1.

Theorem 4.3. *Let A generate a C_0-semigroup T. Let $0 < \gamma \leqslant 1$, $A_\gamma = (\omega - A)^\gamma$ where ω is large enough. Let $\varepsilon > 0$. Assume that for each $a \in X$, $b^* \in X^*$ satisfying $\|a\| \leqslant \varepsilon$, $\|b^*\| \leqslant \varepsilon$, the perturbed operator $A + ab^* A_\gamma$ generates a C_0-semigroup. Then* (4.2) *holds for $\beta = \gamma$. In particular, T is immediately differentiable and $\|AT(t)\| = O(t^{-\alpha})$ as $t \downarrow 0$, for any $\alpha > 1/\gamma$.*

Proof. There exists ω such that $\Omega_0 := \{\lambda\colon \operatorname{Re}\lambda \geqslant \omega\} \subset \varrho(A)$ and $R(\lambda, A)$ is bounded on Ω_0. For $x \in D(A)$, $A_\gamma R(\lambda, A)x$ is bounded on Ω_0. For $n \in \mathbb{N}$, let $\Omega_n = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant n + \omega\}$, $g_n(\lambda) = n$ $(\lambda \in \Omega_n)$. The assumptions of the present theorem imply those of Theorem 1.3 with $C = A_\gamma$, so it follows that there exists $m \in \mathbb{N}$ such that $\sup_{\lambda \in \Omega_m} \|A_\gamma R(\lambda, A)\| < \infty$. Thus, we have established (4.3) and hence (4.2). □

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