Rank-1 perturbations of cosine functions and semigroups

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Abstract

Let \( A \) be the generator of a cosine function on a Banach space \( X \). In many cases, for example if \( X \) is a UMD-space, \( A + B \) generates a cosine function for each \( B \in \mathcal{L}(D((\omega - A)^{1/2}), X) \). If \( A \) is unbounded and \( 1/2 < \gamma \leq 1 \), then we show that there exists a rank-1 operator \( B \in \mathcal{L}(D((\omega - A)\gamma), X) \) such that \( A + B \) does not generate a cosine function. The proof depends on a modification of a Baire argument due to Desch and Schappacher. It also allows us to prove the following. If \( A + B \) generates a distribution semigroup for each operator \( B \in \mathcal{L}(D(A), X) \) of rank-1, then \( A \) generates a holomorphic \( C_0 \)-semigroup. If \( A + B \) generates a \( C_0 \)-semigroup for each operator \( B \in \mathcal{L}(D((\omega - A)^{\gamma}), X) \) of rank-1 where \( 0 < \gamma < 1 \), then the semigroup \( T \) generated by \( A \) is differentiable and \( \|T'(t)\| = O(t^{-\alpha}) \) as \( t \downarrow 0 \) for any \( \alpha > 1/\gamma \). This is an approximate converse of a perturbation theorem for this class of semigroups.

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0. Introduction

Let \( A \) be the generator of a cosine function. Then \( A \) also generates a holomorphic \( C_0 \)-semigroup. Let \( \omega_A \in \mathbb{R} \) such that \((\omega_A, \infty) \subset \varrho(A)\) and \( \sup_{\lambda \geq \omega_A} \|R(\lambda, A)\| < \infty \). Then for \( \gamma > 0 \) and \( \omega > \omega_A \), the operator \( A_\gamma = (\omega - A)^{\gamma} \in \mathcal{L}(X) \) exists and is invertible. The domain

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$D(A_\gamma)$ is a Banach space for the graph norm, and this space does not depend on the choice of $\omega > \omega_\Lambda$ up to equivalent norms. Moreover,

$$D(A_\gamma) \hookrightarrow D(A_\beta)$$

for $0 \leq \beta \leq \gamma < \infty$. If $X$ is a UMD-space, then $A_{1/2}$ generates a $C_0$-group on $X$ and $A + B$ also generates a cosine function whenever $B$ is a bounded linear operator from $D(A_{1/2})$ into $X$ (where $D(A_{1/2})$ carries the graph norm) (see [2, Sections 3.14, 3.16]). This property is most important for applications to hyperbolic equations (see [2, Chapter 7]). The aim of this article is to show that $\gamma = \frac{1}{2}$ is best possible for this property. In fact, our main result says the following.

Assume that $A$ is unbounded. Let $\frac{1}{2} < \gamma \leq 1$. Then there exists an operator $B \in \mathcal{L}(D(A_\gamma), X)$ of rank-1 such that $A + B$ does not generate a cosine function. Moreover, we can choose $B$ of arbitrarily small norm and such that $A + B$ does not generate a $k$-times integrated cosine function for any $k \in \mathbb{N}$.

Our proof is based on a technique due to Desch and Schappacher [8] who proved that for each generator $A$ of a $C_0$-semigroup $T$ which is not holomorphic there exists a rank-1 perturbation $B : D(A) \to X$ such that $A + B$ does not generate a $C_0$-semigroup. We show in Section 3 that $B$ can even be chosen such that $A + B$ does not generate a distribution semigroup. This is done with the help of a generalization and a simplification of the Baire argument of Desch and Schappacher which we establish in Section 1.

Our argument also sheds light on the reason for instability. Rank-1 perturbation may lead to an explosion of the resolvent: the resolvent of $A + B$ cannot have any prescribed growth outside any parabola oriented to the left (in the cosine case) and on any right half-plane in the semigroup case.

In Section 4 we consider (rank-1) perturbations $B : D(A_\gamma) \to X$ of the generator $A$ of a $C_0$-semigroup $T$ where $0 < \gamma < 1$. We exhibit a perturbation theorem for a class of differentiable semigroups first considered by Crandall and Pazy [6]. Conversely, if $A + B$ generates a $C_0$-semigroup for each such $B$, then we show that $T$ belongs to that class of semigroups.

1. Rank-1 perturbations

Let $A$ be a closed linear operator on a Banach space $X$. Then the domain $D(A)$ of $A$ is a Banach space for the graph norm $\|x\|_A := \|x\| + \|Ax\|$. Let $C \in \mathcal{L}(D(A), X)$ be a bounded linear operator (where $D(A)$ carries the graph norm). Given $a \in X, b^* \in X^*$ we consider the perturbation $B \in \mathcal{L}(D(A), X)$ of $A$ given by

$$Bx = b^*(Cx)a \quad (x \in D(A)).$$

We denote this operator $B$ by $ab^*C$.

**Lemma 1.1.** Let $a \in X, b^* \in X^*, \lambda \in \varrho(A)$. Then $\lambda \in \varrho(A + ab^*C)$ if and only if

$$b^*C R(\lambda, A)a \neq 1.$$

In that case

$$R(\lambda, A + ab^*C)x = R(\lambda, A)x + \frac{b^*C R(\lambda, A)x}{1 - b^*C R(\lambda, A)a} \cdot R(\lambda, A)a$$

(1.1)
for all \( x \in X \). In particular,
\[
R(\lambda, A + ab^*C)a \cdot (1 - b^*CR(\lambda, A)a) = R(\lambda, A)a.
\]

**Proof.** Let \( \lambda \in \varrho(A + ab^*C) \). Let \( x \in X \), \( y = R(\lambda, A + ab^*C)x \). Then \( x = (\lambda - A)y - ab^*Cy \). Hence \( y = R(\lambda, A)x + cR(\lambda, A)a \) where \( c = b^*Cy \). Consequently, \( x = (\lambda - A - ab^*C)y = x - (b^*CR(\lambda, A)x) \cdot a + c \cdot a = (cb^*CR(\lambda, A)a) \cdot a \). This implies that \( b^*CR(\lambda, A)a \neq 1 \) and
\[
c = \frac{b^*CR(\lambda, A)x}{1 - b^*CR(\lambda, A)a}.
\]
Thus one implication is proved. We omit the easy proof of the other one. \( \square \)

For \( \delta > 0, p \in X, q^* \in X^* \) we let
\[
B_\delta(p) := \{x \in X: \|x - p\| \leq \delta\},
\]
\[
B_\delta(p, q^*) := \{(a, b^*) \in X \times X^*: \|a - p\| \leq \delta, \|b^* - q^*\| \leq \delta\}
\]
be closed balls in \( X \) and \( X \times X^* \), respectively.

**Lemma 1.2.** Let \( \Omega \subset \varrho(A) \) be non-empty. Assume that
\[
\sup_{\lambda \in \Omega} \|CR(\lambda, A)x\| < \infty \quad (1.3)
\]
for all \( x \) in a dense subspace \( Y \) of \( X \), but
\[
\sup_{\lambda \in \Omega} \|CR(\lambda, A)\| = \infty. \quad (1.4)
\]
Let \( \delta > 0, p \in X, q^* \in X^* \). Then there exist \((a, b^*) \in B_\delta(p, q^*)\), \( \lambda \in \Omega \) such that
\[
b^*CR(\lambda, A)a = 1.
\]

**Proof.** By the Uniform Boundedness Principle, there exists \( b^* \in B_\delta(q^*) \) such that \( \sup_{\lambda \in \Omega} \|b^*CR(\lambda, A)\| = \infty \), and there exists \( a_2 \in X \) such that \( \sup_{\lambda \in \Omega} |b^*CR(\lambda, A)a_2| = \infty \). By (1.3) there exists \( a_1 \in B_{\delta/2}(p) \) such that \( \sup_{\lambda \in \Omega} \|CR(\lambda, A)a_1\| < \infty \). Let
\[
a_3 = \frac{1 - b^*CR(\lambda, A)a_1}{b^*CR(\lambda, A)a_2} \cdot a_2,
\]
where \( \lambda \in \Omega \) is chosen such that \( \|a_3\| \leq \delta/2 \). Then \( b^*CR(\lambda, A)a_3 = 1 - b^*CR(\lambda, A)a_1 \). Hence \( b^*CR(\lambda, A)a = 1 \) where \( a = a_1 + a_3 \in B_\delta(p) \). \( \square \)

In order to formulate the main result of this section we let \( \Omega_n \subset \mathbb{C} \) be arbitrary non-empty sets \((n \in \mathbb{N})\) and \( g_n: \Omega_n \to (0, \infty) \) be arbitrary functions (which measure the growth of resolvents).
Theorem 1.3. Let $A$ be a closed operator on $X$, $C : D(A) \to X$ be a bounded operator, $\varepsilon > 0$. Assume that $\Omega_n \subset \varrho(A)$ and

$$\sup_{\lambda \in \Omega_n} \left\| CR(\lambda, A)x \right\| < \infty$$

for all $x$ in a dense subspace $Y_n$ of $X$ and all $n \in \mathbb{N}$. Assume that for each $(a, b^*) \in B_\varepsilon(0, 0)$ there exists $n \in \mathbb{N}$ such that $\Omega_n \subset \varrho(A + ab^*C)$ and

$$\left\| R(\lambda, A + ab^*C) \right\| \leq g_n(\lambda) \quad (\lambda \in \Omega_n).$$

Then there exists $m \in \mathbb{N}$ such that

$$\sup_{\lambda \in \Omega_m} \left\| CR(\lambda, A) \right\| < \infty.$$ (1.6)

Proof. Let

$$F_n = \{(a, b^*) \in B_\varepsilon(0, 0) : \left\| R(\lambda, A)a \right\| \leq g_n(\lambda) \cdot \|a\| \cdot \|1 - b^*CR(\lambda, A)a\| \text{ for all } \lambda \in \Omega_n\}.$$  

Then $F_n$ is closed and it follows from (1.5) and Lemma 1.1 that $B_\varepsilon(0, 0) = \bigcup_{n \in \mathbb{N}} F_n$. By Baire’s theorem there exist $p \in X$, $q^* \in X^*$, $\delta > 0$ and $m \in \mathbb{N}$ such that $B_\delta(p, q^*) \subset F_m$. It follows that $b^*CR(\lambda, A)a \neq 1$ for all $\lambda \in \Omega_m$ and all $(a, b^*) \in B_\delta(p, q^*)$. Now Lemma 1.2 implies (1.6). □

2. Perturbation of cosine generators

Generators of cosine functions can conveniently be characterized by Laplace transforms (see [2, Section 3.14]). An operator $A$ on a Banach space $X$ generates a cosine function if there exist a strongly continuous function $\cos : \mathbb{R}_+ \to \mathcal{L}(X)$ and some $M \geq 0$, $\sigma \geq 1$, satisfying

$$\left\| \int_0^t \cos(s)x \, ds \right\| \leq Me^{(\sigma - 1)t}\|x\| \quad (t \geq 0, x \in X)$$ (2.1)

and

$$\lambda^2 \in \varrho(A) \quad \text{and} \quad \lambda R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} \cos(t)x \, dt \quad (x \in X)$$ (2.2)

whenever $\Re \lambda \geq \sigma$. In that case, the function $\cos$ is unique and is called the cosine function generated by $A$. Note that condition (2.1) ensures that the integral in (2.2) converges in the improper sense. However, it follows that $\cos$ is even exponentially bounded, i.e.,

$$\|\cos(t)\| \leq M'e^{\sigma't} \quad (t \geq 0)$$
for some $M' \geq 1$, $\sigma' \in \mathbb{R}$. If $A$ generates a cosine function, then $D(A)$ is dense and for $x, y \in X$ the function $u(t) = \cos(t)x + \int_0^t \cos(s)y \, ds$ is a mild solution of the second order Cauchy problem

$$(P_2) \quad \begin{cases} u''(t) = Au(t) & (t \geq 0), \\ u(0) = x, & u'(0) = y \end{cases}$$

which is the motivation for studying cosine functions.

More generally, one defines the following concept [1]. Let $k \in \mathbb{N}_0$. We say that $A$ generates a $k$-times integrated cosine function on $X$ if there exists a strongly continuous function $S : \mathbb{R}_+ \to \mathcal{L}(X)$ and some $M \geq 0, \sigma \geq 1$ satisfying

$$\left\| \int_0^t S(s)x \, ds \right\| \leq Me^{(\sigma-1)t}\|x\| \quad (t \geq 0, x \in X) \quad (2.3)$$

and

$$\lambda^2 \in \varrho(A) \quad \text{and} \quad \lambda^{-k+1} R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} S(t)x \, dt \quad (x \in X) \quad (2.4)$$

whenever $\Re \lambda \geq \sigma$. Then $S$ is called the $k$-times integrated cosine function generated by $A$. Thus cosine functions are the same as 0-times integrated cosine functions, and 1-times integrated cosine functions are sine functions (see [2, Section 3.15]). Moreover, if $A$ generates a $k$-times integrated cosine function $S_k$, then it generates the $(k+1)$-times integrated cosine function $S_{k+1}$ given by

$$S_{k+1}(t)x = \int_0^t S_k(s)x \, ds \quad (x \in X),$$

and $S_{k+1}$ is exponentially bounded. We remark that the fact that $A$ generates a $k$-times integrated cosine function can be reformulated in terms of well-posedness of $(P_2)$ (see Keyantuo [14, Chapter 2, Section 4]); the smaller $k$ is, the less regular the initial values $x, y$ can be chosen in order to obtain a solution of $(P_2)$.

**Example 2.1.** It is shown in [12,15] (see also [2, Theorem 7.3.1]) that the Laplacian $\Delta$, with domain $W^{2,p}(\mathbb{R}^N)$, generates a $k$-times integrated cosine function on $L^p(\mathbb{R}^N)$, where $1 < p < \infty$, if and only if $k \geq (N-1)|\frac{1}{p} - \frac{1}{2}|$.

Now assume that $A$ generates a $k$-times integrated cosine function $S$. Replacing $k$ by $k+1$ if necessary, we may assume that $S$ is exponentially bounded, i.e.

$$\left\| S(t) \right\| \leq Me^{(\sigma-1)t} \quad (t \geq 0),$$

where $M \geq 0, \sigma \geq 1$. Then by (2.4) we have for $\Re \lambda \geq \sigma$, 

$$\| \lambda^{-k+1} R(\lambda^2, A) \| \leq M \int_0^\infty e^{(\sigma-1)t} e^{-\text{Re} \lambda t} \, dt \leq \frac{M}{\text{Re} \lambda - \sigma + 1}.$$ 

Thus

$$\| R(\lambda^2, A) \| \leq M |\lambda|^{k-1} \quad (\text{Re} \lambda \geq \sigma).$$

Consequently, the resolvent of $A$ is polynomially bounded on

$$\Omega_\sigma := \{ \lambda^2 : \text{Re} \lambda \geq \sigma \} = \{ \xi + i\eta : \eta \in \mathbb{R}, \xi \geq \sigma^2 - \frac{\eta^2}{4\sigma^2} \}$$

which is the exterior of a horizontal parabola.

Now we wish to consider perturbations $B \in \mathcal{L}(D(A_\gamma), X)$ where $0 < \gamma \leq 1$, $\omega > \sigma^2$ and $A_\gamma := (\omega - A)^\gamma$ is a fractional power. If $k = 0$, then $A$ generates a holomorphic $C_0$-semigroup and the definition of $A_\gamma$ is standard (see [2, Section 3.8] for example). If $k > 0$ and $A$ is densely defined, the fractional powers $A_\gamma$ can be defined by [18, Definition 1.11] or [7, Section 5] since the resolvent of $\omega - A$ is polynomially bounded on a sector. However, we shall need certain properties of fractional powers (for example, that $D(A_\gamma) \subseteq D(A)$) which are standard for the case $k = 0$ but do not appear to be known for $k > 0$. Therefore we shall now assume either that $\gamma = 1$ or that $A$ generates a cosine function, although we shall allow the possibility that $A + B$ generates an integrated cosine function. Then the operator $A_\gamma$ is closed and $D(A_\gamma) = D((\omega_1 - A)^\gamma)$ whenever $\omega_1 > \sigma^2$. For $a \in X$, $b^* \in X^*$ we consider the rank-1 perturbation

$$Bx = b^*(A_\gamma x)a \quad (x \in D(A_\gamma))$$

which we denote by $B = ab^* A_\gamma$. Now we can formulate the main result of this section.

**Theorem 2.2.** Assume that either $A$ is the generator of a cosine function and $1/2 < \gamma \leq 1$, or that $A$ is the generator of a $k$-times integrated cosine function for some $k \in \mathbb{N}_0$ and $\gamma = 1$. Let $A_\gamma = (\omega - A)^\gamma$ where $\omega$ is large enough. Let $\varepsilon > 0$. Assume that for each $a \in X$, $b^* \in X^*$ satisfying $\|a\| \leq \varepsilon$, $\|b^*\| \leq \varepsilon$ there exists $\ell \in \mathbb{N}$ such that $A + ab^* A_\gamma$ generates an $\ell$-times integrated cosine function. Then $A$ is bounded.

We need the following two lemmas. We do not claim originality but we include proofs for the convenience of the reader.

**Lemma 2.3.** Let $A$ be an operator such that the resolvent exists and is polynomially bounded outside a ball. Then $A$ is bounded.

**Proof.** With the help of the spectral projection associated with the bounded spectrum we reduce the problem to the case where $\varphi(A) = \mathbb{C}$ and $R(\lambda, A)$ is polynomially bounded. By elementary complex function theory, $R(\lambda, A)$ is a polynomial. Then $(-1)^{n-1}(n-1)!R(\lambda, A)^n = (d/d\lambda)^{n-1} R(\lambda, A) = 0$ for some $n \in \mathbb{N}$. Since $R(\lambda, A)$ is injective we conclude that $X = \{0\}$. □
Lemma 2.4. (See [9, Theorem 2].) Let \( A^\gamma = (\omega - A)^\gamma \), where \( 0 < \gamma < 1 \) and \( \omega > \omega_A \). Let \( \Omega \) be a subset of \( \varrho(A) \) whose closure does not contain \( 0 \). The following are equivalent:

(i) \( \sup_{\lambda \in \Omega} \| \lambda^\gamma R(\lambda, A) \| < \infty \);
(ii) \( \sup_{\lambda \in \Omega} \| A^\gamma R(\lambda, A) \| < \infty \).

**Proof.** Let \( x \in X \). By the moment inequality [10, Theorem II.5.34],

\[
\| A^\gamma R(\lambda, A) x \| \leq \| (\omega - A) R(\lambda, A) x \|^{\gamma} \| R(\lambda, A) x \|^{1-\gamma}.
\]

Hence,

\[
\| A^\gamma R(\lambda, A) \| \leq \| (\omega - \lambda) R(\lambda, A) + I \|^{\gamma} \| R(\lambda, A) \|^{1-\gamma},
\]

and it follows that (i) implies (ii).

Also by the moment inequality,

\[
\| R(\lambda, A) x \| \leq \| A^\gamma R(\lambda, A) x \|^{1-\gamma} \| A^\gamma R(\omega, A) R(\lambda, A) x \|^{\gamma}.
\]

Hence

\[
\| R(\lambda, A) \| \leq \| A^\gamma R(\lambda, A) \|^{1-\gamma} \left( \frac{\| A^\gamma (R(\lambda, A) - R(\omega, A)) \|}{|\lambda - \omega|} \right)^{\gamma},
\]

and it follows that (ii) implies (i). \( \Box \)

**Proof of Theorem 2.2.** Let \( A \) be the generator of a \( k \)-times integrated cosine function. Then for suitable \( M, \sigma \geq 1 \), \( r \in \mathbb{N} \) one has \( \Omega_0 := \{ \lambda^2 : \text{Re} \lambda \geq \sigma \} \subset \varrho(A) \) and

\[
\| R(\mu, A) \| \leq M |\mu|^r \quad (\mu \in \Omega_0).
\]

Take \( \omega > \sigma^2 \). Then \( \omega - A \) is invertible and \((\omega - A)^\alpha \) is a bounded operator whenever \( \alpha \leq 0 \). Let

\[
\Omega_n := \{ \lambda^2 : \text{Re} \lambda \geq \sigma + n \}, \quad g_n(\lambda) = n(1 + |\lambda|)^n \quad (\lambda \in \Omega_n)
\]

for \( n \in \mathbb{N} \). Under the assumptions of the theorem, for all \((a, b^*) \in B_e(0, 0)\) there exists \( n \in \mathbb{N} \) such that \( \Omega_n \subset \varrho(A + ab^* A^\gamma) \) and \( \| R(\lambda, A + ab^* A^\gamma) \| \leq g_n(\lambda) \quad (\lambda \in \Omega_n) \). For \( x \in D(A^r) \),

\[
R(\lambda, A) x = \lambda^{-r} R(\lambda, A) A^r x + \sum_{m=0}^{r-1} \lambda^{-(m+1)} A^m x,
\]

so \( R(\lambda, A) x \) is bounded on \( \Omega_0 \). Hence \( A^\gamma R(\lambda, A) x = (\omega - A)^{-1} R(\lambda, A) (\omega - A) x \) is bounded on \( \Omega_0 \) for all \( x \in D(A^{r+1}) \) which is dense in \( X \). By Theorem 1.3 there exists \( m \in \mathbb{N} \) such that \( \sup_{\lambda \in \Omega_m} \| A^\gamma R(\lambda, A) \| < \infty \). By Lemma 2.4,

\[
c := \sup_{\lambda \in \Omega_m} \| \lambda^{\gamma} R(\lambda, A) \| < \infty.
\]
Now let $\lambda \in \partial \Omega_m$, i.e., $\lambda = \xi + i \eta$, where $\eta \in \mathbb{R}$ and $\xi = (\sigma + m)^2 - \frac{\eta^2}{4(\sigma + m)^2}$. Let $\mu = \xi + i \eta_1$ where $|\eta_1| \leq |\eta|$. Write $\mu - A = (I - (\lambda - \mu)R(\lambda, A))(\lambda - A)$. Then,

$$\| (\lambda - \mu)R(\lambda, A) \| \leq |\eta| \| R(\lambda, A) \| \leq c \frac{|\eta|}{|\lambda|^\gamma}$$

$$= c|\eta|\left[\left( (\sigma + m)^2 - \frac{\eta^2}{4(\sigma + m)^2} \right)^2 + \eta^2 \right]^{\gamma/2}$$

$$\leq 1/2$$

if $|\eta|$ is sufficiently large. Here we use that $\gamma > \frac{1}{2}$. Thus there exists $\xi_0 > 0$ such that, for $\lambda = \xi + i \eta \in \partial \Omega_m$ with $\xi \leq -\xi_0$, one has $\mu = \xi + i \eta_1 \in \sigma(\lambda)$ and $\| R(\mu, A) \| \leq 2 \| R(\lambda, A) \|$ whenever $|\eta_1| \leq |\eta|$.

Since $|\lambda| \leq \alpha|\xi| \leq \alpha|\mu|$ for some constant $\alpha$ independent of $\lambda$ we conclude that $R(\mu, A)$ is polynomially bounded in the region $\{ \mu \in \mathbb{C}: \text{Re} \mu \leq -\xi_0 \} \setminus \Omega_m$. Since $R(\mu, A)$ is polynomially bounded on $\Omega_m$ we deduce from Lemma 2.3 that $A$ is bounded. \qed

**Remark 2.5.** In the proof of Theorem 2.2, it was not important that the functions $g_n$ were polynomially bounded, although it was important that $R(\lambda, A)$ is polynomially bounded. The proof shows the following. Suppose that $A$ is unbounded, and $\omega - A$ is sectorial and $R(\lambda, A)$ exists and is polynomially bounded outside a parabola $\Omega_m := \{ \lambda^2: \text{Re} \lambda \geq m \}$. Let $g_n: \Omega_m \rightarrow (0, \infty)$ be any functions and let $\gamma > 1/2$. Then there exist $a \in X$, $b^* \in X^*$ and $\lambda_n \in \Omega_n (n \in \mathbb{N})$ such that either $\lambda_n \in \sigma(A + ab^*A_{\gamma})$ or $\| R(\lambda_n, A + ab^*A_{\gamma}) \| \geq g_n(\lambda)$. Similar remarks apply to Theorems 3.1 and 4.3.

### 3. Perturbation of distribution semigroups

The property of generating a holomorphic $C_0$-semigroup is stable under small perturbations. In fact, if $A$ generates a holomorphic $C_0$-semigroup then so does $A + B$ for each compact $B: D(A) \rightarrow X$, see [2, Theorem 3.7.25] or [8]. Desch and Schappacher [8] showed that the property of generating a $C_0$-semigroup is not stable under small perturbations unless the given semigroup is already holomorphic. Our general perturbation result of Section 1 allows us to generalize the Desch–Schappacher result to a much larger class, characterizing generators of holomorphic $C_0$-semigroups among the class of all generators of distribution semigroups.

The concept of a *distribution semigroup* was introduced by Lions [16]. It is equivalent to the notion of *local k-times integrated semigroup* introduced in [3] which can be formulated precisely in terms of the well-posedness of the Cauchy problem defined by $A$. Here we use the following characterization in terms of the resolvent. A densely defined operator $A$ generates a distribution semigroup if and only if there exists $k \in \mathbb{N}$ such that $A$ generates a local $k$-times integrated semigroup, or equivalently if and only if there exists an *exponential region*

$$E(\alpha, \beta) := \{ \lambda \in \mathbb{C}: \text{Re} \lambda \geq \beta, |\text{Im} \lambda| \leq e^{\alpha \text{Re} \lambda} \}$$

where $\beta \in \mathbb{R}, \alpha \geq 0$,

such that $E(\alpha, \beta) \subset \sigma(A)$ and $R(\lambda, A)$ is polynomially bounded on $E(\alpha, \beta)$.

With the help of this characterization we can prove the following.
Theorem 3.1. Let $A$ be a densely defined operator on a Banach space $X$ and let $\varepsilon > 0$. Assume that for each $a \in X$, $b^* \in X^*$ satisfying $\|a\| \leq \varepsilon$, $\|b^*\| \leq \varepsilon$, the operator $A + ab^*A$ generates a distribution semigroup. Then $A$ generates a holomorphic $C_0$-semigroup.

Proof. Since $A$ generates a distribution semigroup there exist $\alpha > 0$, $\beta \in \mathbb{R}$, $c \geq 0$ and $\ell \in \mathbb{N}$ such that $E(\alpha, \beta) \subset \varrho(A)$ and $\|R(\lambda, A)\| \leq c(1 + |\lambda|)^\ell$ for all $\lambda \in E(\alpha, \beta)$. Let

$$\Omega_n = E(\alpha + n, \beta + n) = \{\lambda \in \mathbb{C}: \text{Re} \lambda \geq \beta + n, |\text{Im} \lambda| \leq e^{(\alpha + n) \text{Re} \lambda}\}$$

and let $g_n(\lambda) = (c + n)(1 + |\lambda|)^{\ell+n}$ where $n \in \mathbb{N}$. The assumption implies that for each $(a, b^*) \in B_\varepsilon(0, 0)$ there exists $n \in \mathbb{N}$ such that $\Omega_n \subset \varrho(A + ab^*A)$ and $\|R(\lambda, A + ab^*A)\| \leq g_n(\lambda)$ ($\lambda \in \Omega_n$). For $x \in D(\mathcal{A}^\ell)$,

$$R(\lambda, A)x = \lambda^{-\ell} R(\lambda, A)A^\ell x + \sum_{m=0}^{\ell-1} \lambda^{-(m+1)} A^m x,$$

so $R(\lambda, A)x$ is bounded on $E(\alpha, \beta)$. Thus $AR(\lambda, A)x$ is bounded on $E(\alpha, \beta)$ for all $x \in D(\mathcal{A}^{\ell+1})$ which is a dense subspace. By Theorem 1.3 there exists $m \in \mathbb{N}$ such that

$$M := \sup_{\lambda \in \Omega_m} \|\lambda R(\lambda, A)\| < \infty.$$

By the von Neumann expansion we obtain $\theta \in (0, \pi/2)$, $M' \geq 0$ such that the following holds: if a half-line $L = \{re^{i\gamma}: r \geq r_0\}$ is in $\varrho(A)$ and $\|\lambda R(\lambda, A)\| \leq M$ on $L$, then also $L_\theta := \{re^{i\psi}: r \geq r_0, |\psi - \gamma| \leq \theta\} \subset \varrho(A)$ and $\|\lambda R(\lambda, A)\| \leq M'$ on $L_\theta$. Since the boundary of $\Omega_m$ becomes arbitrarily steep, we find $\omega \geq \alpha$ such that

$$\Omega := \left\{\omega + re^{i\psi}: r \geq 0, |\psi| \leq \frac{\pi}{2} + \frac{\theta}{2}\right\} \subset \varrho(A)$$

and

$$M'' := \sup_{\lambda \in \Omega} \|\lambda R(\lambda, A)\| < \infty.$$

Consequently, $A$ generates a holomorphic $C_0$-semigroup. \hfill \Box

4. Fractional perturbation of semigroup generators

In this section, we combine techniques from the previous sections to show that if there exists $\gamma > 0$ such that $A + ab^*A_\gamma$ generates a $C_0$-semigroup for every $a \in X$, $b^* \in X^*$, then the semigroup $T$ generated by $A$ belongs to a class considered by Crandall and Pazy [6]. That is, $T$ is immediately differentiable and its derivative $AT(t)$ satisfies

$$\|AT(t)\| = O(t^{-\alpha}) \quad \text{as } t \downarrow 0$$

for some $\alpha > 0$. It was shown in [6,9] that this is equivalent to the property that

$$\|R(\omega + is, A)\| = O(|s|^{-\beta}) \quad \text{as } |s| \to \infty$$

(4.2)
for some $\beta > 0$ and $\omega > \omega_0(T)$, the exponential growth bound of $T$. Indeed, (4.1) implies (4.2) for $\beta = 1/\alpha$ and any $\omega > \omega_0(T)$. On the other hand, (4.2) implies (4.1) for any $\alpha > 1/\beta$. By a standard Neumann series argument, (4.2) for one value of $\omega$ implies that, for any $\omega' \in \mathbb{R}$, $R(\omega' + is, A)$ exists for all real $s$ with $|s|$ sufficiently large and $\|R(\omega' + is, A)\| = O(|s|^{-\beta})$ as $|s| \to \infty$. Moreover, Lemma 2.4 shows that (4.2) is equivalent to the property that

$$\sup_{s \in \mathbb{R}} \left\| A^{\beta} R(\omega + is, A) \right\| < \infty. \quad (4.3)$$

This class of semigroups arises naturally when considering differentiability of solutions of inhomogeneous Cauchy problems [6] and of delay differential equations [4,5]. Note that the case when $\alpha = 1$ in (4.1) and the case when $\beta = 1$ in (4.2) each correspond to $T$ being a holomorphic semigroup [2, Corollary 3.7.18, Theorem 3.7.19], [10, Theorem II.4.6], so we are really interested in the case when $\alpha > 1$ and $0 < \beta < 1$.

We will first show that this class of generators is invariant under suitable fractionally bounded perturbations. This result fits naturally between the standard results for bounded perturbations of $C_0$-semigroups and relatively bounded perturbations of holomorphic semigroups (see [10, Corollary III.2.14]). We are very grateful to Markus Haase for enabling us to complete the proof of this result.

**Proposition 4.1.** Let $A$ be the generator of a $C_0$-semigroup $T$ and suppose that $A$ satisfies (4.2) for some $\beta > 0$. Let $B \in \mathcal{L}(D(A^{\gamma}), X)$ where $0 < \gamma < \beta$. Then $A + B$ generates a $C_0$-semigroup. Moreover, $\|R(\omega + is, A + B)\| = O(|s|^{-\beta})$ as $|s| \to \infty$.

**Proof.** In this proof, $c$ will denote a constant which may vary from place to place.

Choose $\alpha \in (\beta^{-1}, \gamma^{-1})$. Then (4.1) holds, so

$$\left\| AT(t) \right\| \leq c t^{-\alpha} \quad (0 < t \leq 1).$$

Let $x \in X$. By the moment inequality [10, Theorem II.5.34],

$$\left\| A^{\gamma} T(t) x \right\| \leq \| \omega - A \| T(t) x \| T(t) x \|^{1-\gamma} \leq c t^{-\alpha \gamma} \| x \|.$$

Hence,

$$\int_0^1 \left\| B T(t) \right\| dt < \infty.$$

It follows from [13, Corollary 1, p. 400] (see also [10, Theorem III.3.14]) that $A + B$ generates a $C_0$-semigroup $S$.

Let $\lambda = \omega + is$. By (2.5) and (4.2),

$$\left\| B R(\lambda, A) \right\| \leq c \left\| A^{\gamma} R(\lambda, A) \right\| \leq c |\lambda|^{\gamma-\beta}.$$

Hence $\|BR(\lambda, A)\| \leq 1/2$ whenever $|s|$ is sufficiently large. For such $s$, it follows from the identity $\lambda - (A + B) = (I - BR(\lambda, A))(\lambda - A)$ that $\lambda \in \rho(A + B)$ and

$$\left\| R(\lambda, A + B) \right\| = \left\| R(\lambda, A) \left( I - BR(\lambda, A) \right)^{-1} \right\| \leq 2 \left\| R(\lambda, A) \right\| \leq c |s|^{-\beta}. \quad \square$$
If \( A \) generates a holomorphic semigroup and \( B : D(A) \to X \) is compact, then \( A + B \) also generates a holomorphic semigroup [8, Theorem 1], [2, Theorem 3.7.25]. One might expect that if \( A \) generates a semigroup and satisfies (4.2) and \( B : D(A_\beta) \to X \) is compact, then \( A + B \) should also be a generator. We do not know whether this is the case, but we have the following partial results when \( B \) is of rank-1.

**Proposition 4.2.** Let \( A \) be the generator of a \( C_0 \)-semigroup on a Banach space \( X \), and suppose that \( A \) satisfies (4.2) for some \( \beta > 0 \). Let \( B = ab^*A_\beta \), where \( a \in X \) and \( b^* \in X^* \).

1. There exists \( r \geq 0 \) such that \( \{ \lambda \in \mathbb{C} : \text{Re}\lambda \geq \omega, |\lambda| \geq r \} \subset \mathcal{Q}(A + B) \) and \( \|R(\omega + is, A + B)\| = O(|s|^{-\beta}) \) as \( |s| \to \infty \).
2. If \( X \) is a Hilbert space then \( A + B \) generates a \( C_0 \)-semigroup.

**Proof.** (1) By (4.2), an estimate

\[
\left\| R(\lambda, A) \right\| \leq \frac{c}{|\lambda|^\beta}
\]  

holds when \( \text{Re}\lambda = \omega \). By a standard Neumann series estimate and changing the value of \( c \), we can arrange that (4.4) holds when \( \omega \leq \text{Re}\lambda \leq \omega + c^{-1}|\text{Im}\lambda|^\beta \).

Choose \( \omega' \) so that \( \omega_0(T) < \omega' < \omega \). Then \( \|R(\lambda, A)\| \leq c'/(|\text{Re}\lambda - \omega'|) \) when \( \text{Re}\lambda > \omega \). If \( \text{Re}\lambda > \omega + c^{-1}|\text{Im}\lambda|^\beta \), then

\[
|\lambda|^\beta \leq 2^{\beta/2} \max(\text{Re}\lambda, |\text{Im}\lambda|)^\beta \leq c''(\text{Re}\lambda - \omega').
\]

Changing the value of \( c \) again if necessary, we obtain that (4.4) holds whenever \( \text{Re}\lambda \geq \omega \).

By Lemma 2.4, \( \|A_\beta R(\lambda, A)\| \) is uniformly bounded for \( \text{Re}\lambda \geq \omega \). As \( |\lambda| \to \infty \), \( \|A_\beta R(\lambda, A)x\| \to 0 \) first for \( x \in D(A_\beta) \) and then by density for \( x \in X \), and in particular for \( x = a \). Now Lemma 1.1 shows that, for \( |\lambda| \) sufficiently large, \( \lambda \in \mathcal{Q}(A + B) \) and

\[
R(\lambda, A + B) = R(\lambda, A) + Q(\lambda),
\]

where \( Q(\lambda) \) is a bounded (rank-1) operator and \( \|Q(\lambda)\| \leq c\|R(\lambda, A)a\| \) for some constant \( c \). In particular, this shows that

\[
\left\| R(\lambda, A + B) \right\| \leq (1 + c\|a\|)\left\| R(\lambda, A) \right\| = O(|\lambda|^{-\beta})
\]

as \( |\lambda| \to \infty \).

(2) Now suppose that \( X \) is a Hilbert space. Replacing \( A \) by \( A - \omega \), we may assume that \( T \) is bounded (by \( K \), say). For any \( x \in X \) and \( \sigma > 0 \), Plancherel’s theorem gives

\[
\int_{-\infty}^{\infty} \left\| R(\sigma + is, A)x \right\|^2 \, ds = 2\pi \int_{0}^{\infty} e^{-2\sigma t} \left\| T(t)x \right\|^2 \, dt \leq \frac{\pi K^2}{\sigma} \|x\|^2.
\]
Moreover, for $\sigma$ sufficiently large,
\[ \int_{-\infty}^{\infty} \| Q(\sigma + is)x \|^2 ds \leq \int_{-\infty}^{\infty} c^2 \| R(\sigma + is, A)a \|^2 \| x \|^2 ds \leq \frac{\pi c^2 \| a \|^2 K^2}{\sigma} \| x \|^2. \]

Hence,
\[ \int_{-\infty}^{\infty} \| R(\sigma + is, A + B)x \|^2 ds \leq \frac{\pi (1 + c \| a \|) K^2}{\sigma} \| x \|^2. \]

Similarly,
\[ \int_{-\infty}^{\infty} \| R(\sigma + is, A)^*x \|^2 ds = 2\pi \int_{0}^{\infty} e^{-2\sigma t} \| T(t)^*x \|^2 dt \leq \frac{\pi K^2}{\sigma} \| x \|^2, \]
\[ \int_{-\infty}^{\infty} \| Q(\sigma + is)^*x \|^2 ds \leq \int_{-\infty}^{\infty} c^2 \| R(\sigma + is, A)a \|^2 \| x \|^2 ds \leq \frac{\pi c^2 \| a \|^2 K^2}{\sigma} \| x \|^2, \]
\[ \int_{-\infty}^{\infty} \| R(\sigma + is, A + B)^*x \|^2 ds \leq \frac{\pi (1 + c \| a \|) K^2}{\sigma} \| x \|^2. \]

Now the result follows from [11, Theorem 2] or [17, Theorems 1.1, 4.1].

The next result is the converse of Proposition 4.1.

**Theorem 4.3.** Let $A$ generate a $C_0$-semigroup $T$. Let $0 < \gamma \leq 1$, $A^\gamma = (\omega - A)^\gamma$ where $\omega$ is large enough. Let $\varepsilon > 0$. Assume that for each $a \in X$, $b^* \in X^*$ satisfying $\| a \| \leq \varepsilon$, $\| b^* \| \leq \varepsilon$, the perturbed operator $A + ab^* A^\gamma$ generates a $C_0$-semigroup. Then (4.2) holds for $\beta = \gamma$. In particular, $T$ is immediately differentiable and $\| AT(t) \| = O(t^{-\alpha})$ as $t \downarrow 0$, for any $\alpha > 1/\gamma$.

**Proof.** There exists $\omega$ such that $\Omega_0 := \{ \lambda : \text{Re} \lambda \geq \omega \} \subset \sigma(A)$ and $R(\lambda, A)$ is bounded on $\Omega_0$. For $x \in D(A)$, $A^\gamma R(\lambda, A)x$ is bounded on $\Omega_0$. For $n \in \mathbb{N}$, let $\Omega_n = \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq n + \omega \}$, $g_n(\lambda) = n \ (\lambda \in \Omega_n)$. The assumptions of the present theorem imply those of Theorem 1.3 with $C = A^\gamma$, so it follows that there exists $m \in \mathbb{N}$ such that $\sup_{\lambda \in \Omega_m} \| A^\gamma R(\lambda, A) \| < \infty$. Thus, we have established (4.3) and hence (4.2).

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References


