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Hölder's inequality for perturbations of positive semigroups by potentials

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Abstract

The objective of this note is to give an estimate for a positive perturbed semigroup in terms of the free one. Here we consider perturbation by a potential and the estimate is given by a pointwise Hölder inequality. As a consequence it is shown that ultracontractivity and Gaussian upper bounds are preserved by such perturbations.

Keywords: Schrödinger semigroups

1. Introduction

Motivated by the Schrödinger equation, a by now classical subject is perturbation of semigroups by a potential, i.e., a multiplication operator V. More precisely, we consider a positive C_0-semigroup $(e^{tA})_{t\geqslant 0}$ on a space $L^r(\Omega)$ and consider an admissible positive potential V, i.e., we assume that $e^{t(A+V_n)}$ converges strongly to a C_0-semigroup which we denote symbolically by $(e^{t(A+V)})_{t\geqslant 0}$ where $V_n = \inf\{n, V\}$. Perturbations by admissible potentials have been studied systematically by Voigt [11,12]. Here we prove the following

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pointwise Hölder inequality

$$e^{t(A+V)}f \leqslant \left(e^{t(A+pV)}f\right)^{1/p}\left(e^{tA}f\right)^{1/p'}, \tag{1}$$

where

$$1 \leqslant p < \infty, \quad \frac{1}{p}+\frac{1}{p'}=1, \quad 0 \leqslant f \in L^r(\Omega).$$

If the semigroup is given by a stochastic process and the Feynman–Kac formula is valid, then this is just the classical Hölder's inequality (see [10]). However, in the general case the proof is more involved: we use Trotter's formula and techniques from positive semigroups. The Hölder's inequality (1) has interesting consequences. Even though the semigroup $e^{t(A+V)}$ is larger than e^{tA} (in the sense of positive (by which we mean positivity preserving) operators), several properties are preserved. If e^{tA} is ultracontractive, so is $e^{t(A+V)}$ and if e^{tA} admits Gaussian upper bounds so does $e^{t(A+V)}$. The classical case is if A is the Laplacian and V is in the Kato class. But we may also replace the Laplacian by a general elliptic operator with measurable coefficients or even by an operator of the form

$$A = \Delta - \sum_{j=1}^{d} b_j D_j - V_0 \tag{2}$$

with unbounded drift (see [1]). In both cases Gaussian estimates hold. This implies that a positive V in the Kato class is admissible also for these operators and the perturbed semigroup admits upper Gaussian bounds. Note that Gaussian estimates have important consequences concerning regularity and spectrum (see [2,8]).

We should explain the choice of the sign: if $A = \Delta$ is the Laplacian, we consider potentials $V \geqslant 0$ here. Then $0 \leqslant e^{t\Delta} \leqslant e^{t(\Delta+V)}$ in the sense of positive (i.e., positivity preserving) operators. Moreover, $e^{t(\Delta+V_1)} \leqslant e^{t(\Delta+V_2)}$ if $0 \leqslant V_1 \leqslant V_2$ and V_2 is admissible. This monotonicity property is used throughout.

2. Hölder's inequality for potentials

Let $E = L^r(\Omega)$, $1 \leqslant r < \infty$, where (Ω, Σ, μ) is a σ-finite measure space. Let A be the generator of a positive C_0-semigroup $(e^{tA})_{t\geqslant 0}$ on E. Then for $V \in L^\infty(\Omega)$ the operator $A+V$ (given by $(A+V)f = Af + Vf$ on $D(A+V) = D(A)$) generates a C_0-semigroup $(e^{t(A+V)})_{t\geqslant 0}$ given by Trotter's formula

$$e^{t(A+V)}f = \lim_{n\to\infty}\left(e^{\frac{t}{n}A}e^{\frac{t}{n}V}\right)^n f \tag{3}$$

for all $f \in E$.

Now assume that $0 \leqslant f \in E$. Then the following Hölder's inequality holds.

Proposition 2.1. *Let* $1 \leqslant p < \infty$. *Then*

$$e^{t(A+V)}f \leqslant \left(e^{t(A+pV)}f\right)^{1/p}\left(e^{tA}f\right)^{1/p'} \tag{4}$$

for all $0 \leqslant f \in E$, $t \geqslant 0$ *where* $\frac{1}{p}+\frac{1}{p'}=1$.

For the proof we use the following.

Lemma 2.2. *Let $S:(0,\infty)\to\mathfrak{B}(L^\infty(\Omega))$ be a function such that*

$$S(t+s)=S(t)S(s)\quad (t,s>0)$$

and $S(t)\geqslant 0$ for all $t\geqslant 0$. Let $V\in L^\infty(\Omega)$, $1\leqslant p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$. Then for $t>0$, $n\in\mathbb{N}$

$$\left(S\left(\frac{t}{n}\right)e^{\frac{t}{n}V}\right)^n f\leqslant\left(\left(S\left(\frac{t}{n}\right)e^{\frac{t}{n}pV}\right)^n f\right)^{1/p}\cdot\big(S(t)f\big)^{1/p'} \tag{5}$$

a.e. for $f\in E_+$.

Note that (5) is an inequality between measurable functions on Ω. Here we denote by E_+ the cone of those functions in $E=L^r(\Omega)$ which are $\geqslant 0$ almost everywhere. We do not assume any continuity of the semigroup S. This will be important in the proof of Proposition 2.1.

Proof of Lemma 2.2. By the Gelfand–Naimark theorem [9, 11.18] there exists a compact space K and an algebra isomorphism $J:C(K)\to L^\infty(\Omega)$. In particular, for $f\in C(K)$ one has $f(x)\geqslant 0$ for all $x\in K$ if and only if $(Jf)(y)\geqslant 0$ μ-a.e. Thus, in order to prove the lemma we may replace $L^\infty(\Omega)$ by $C(K)$.

Let $x\in K$, $t>0$. Then $\mu_{t,x}:=S(t)'\delta_x$ defines a positive Radon measure on K, where $S(t)'$ denotes the adjoint of $S(t)$. Let $0\leqslant f\in C(K)$. Then

$$\begin{aligned}
&\big(\big(S(t/n)e^{\frac{t}{n}V}\big)^n f\big)(x)\\
&\quad=\int_K e^{\frac{t}{n}V(y_n)}\int_K e^{\frac{t}{n}V(y_{n-1})}\cdots\\
&\qquad\times\int_K e^{\frac{t}{n}V(y_1)}f(y_1)\,d\mu_{\frac{t}{n},y_2}(y_1)\cdots d\mu_{\frac{t}{n},y_n}(y_{n-1})\,d\mu_{\frac{t}{n},x}(y_n)\\
&\quad=\int_K\cdots\int_K e^{\frac{t}{n}(V(y_n)+\cdots+V(y_1))}f(y_1)\,d\mu_{\frac{t}{n},y_2}(y_1)\cdots d\mu_{\frac{t}{n},x}(y_n)\\
&\quad\leqslant\left[\int_K\cdots\int_K e^{\frac{t}{n}p(V(y_n)+\cdots+V(y_1))}f(y_1)\,d\mu_{\frac{t}{n},y_2}(y_1)\cdots d\mu_{\frac{t}{n},x}(y_n)\right]^{1/p'}\\
&\qquad\cdot\left[\int_K\cdots\int_K 1^{p'}f(y_1)\,d\mu_{\frac{t}{n},y_2}(y_1)\cdots d\mu_{\frac{t}{n},x}(y_n)\right]^{1/p}\\
&\quad=\left\{\left[\left(S\left(\frac{t}{n}\right)e^{\frac{t}{n}pV}\right)^n f\right](x)\right\}^{1/p}\cdot\big\{\big(S(t)f\big)(x)\big\}^{1/p'}.\qquad\square
\end{aligned}$$

Proof of Proposition 2.1. Let $D(A)_+=E_+\cap D(A)$. Since $\lim_{\lambda\to\infty}\lambda R(\lambda,A)f=f$ (with $R(\lambda,A)=(\lambda-A)^{-1}$) in E for all $f\in E$, it follows that $D(A)_+$ is dense in E_+.

Thus it suffices to prove (2.2) for all $f \in D(A)_+$. Let $f \in D(A)_+$. Let $\lambda > \omega(A)$ (the type of the semigroup $(e^{tA})_{t\geqslant 0}$). There exists $v \in E$ such that $f = R(\lambda, A)v$. Let $w \in E$ such that $|v| \leqslant w$ and $w(x) > 0$ a.e. Then $u := R(\lambda, A)w > 0$ a.e. In fact, if not there exists $0 \leqslant f \in E'$, the dual space of E, $f \neq 0$ such that $\langle u, f\rangle = 0$.

Then $\langle Au, f\rangle = \lim_{t\to 0} \frac{\langle T(t)u, f\rangle}{t} \geqslant 0$.

Thus $0 < \langle w, f\rangle = \langle \lambda u, f\rangle - \langle Au, f\rangle = -\langle Au, f\rangle \leqslant 0$, a contradiction.

Note that

$$e^{tA}u = e^{tA}\int_0^\infty e^{-\lambda s}e^{sA}w\,ds = \int_0^\infty e^{-\lambda s}e^{(t+s)A}w\,ds = e^{\lambda t}\int_t^\infty e^{-\lambda r}e^{rA}w\,dr$$
$$\leqq e^{\lambda t}R(\lambda, A)w = e^{\lambda t}u.$$

Thus the space

$$E_u := \{h \in E\colon |h| \leqslant m \cdot u \text{ for some } m \in \mathbb{N}\}$$

is invariant under e^{tA} for all $t \geqslant 0$. The mapping $\phi : L^\infty(\Omega) \to E_u$, $g \to gu$ is an isomorphism.

Moreover, $e^{tV}\phi = \phi e^{tV}$ $(t \geqslant 0)$.

Let $S(t)g = \frac{1}{u}e^{tA}(ug)$ for $g \in L^\infty(\Omega)$. Then $S(t)S(s) = S(t+s)$ for $s, t > 0$ and $(S(\frac{t}{n})e^{\frac{t}{n}V})^n g = \frac{1}{u}(e^{\frac{t}{n}A}e^{\frac{t}{n}V})^n(ug)$ for $g \in L^\infty(\Omega)$, $t > 0$, $n \in \mathbb{N}$. Thus, by Lemma 2.2, for $g = \frac{f}{u}$,

$$\begin{aligned}
\big(e^{\frac{t}{n}A}e^{\frac{t}{n}V}\big)^n f &= u\big(S(t/n)e^{\frac{t}{n}V}\big)^n g \\
&\leqslant u\big(\big(S(t/n)e^{\frac{t}{n}pV}\big)^n g\big)^{1/p} \cdot \big(S(t)g\big)^{1/p'} \\
&= u\left(\frac{1}{u}\big(e^{\frac{t}{n}A}e^{\frac{t}{n}pV}\big)^n ug\right)^{1/p}\left(\frac{1}{u}e^{tA}(ug)\right)^{1/p'} \\
&= uu^{-1/p}\big(\big(e^{\frac{t}{n}A}e^{\frac{t}{n}pV}\big)^n f\big)^{1/p}u^{-1/p'}\big(e^{tA}f\big)^{1/p'} \\
&= \big(\big(e^{\frac{t}{n}A}e^{\frac{t}{n}pV}\big)^n f\big)^{1/p}\big(e^{tA}f\big)^{1/p'}
\end{aligned}$$

which proves the claim. □

Definition 2.3. A measurable function $V : \Omega \to [0, \infty]$ is called *admissible* (with respect to A) if for $t \geqslant 0$, $f \in E$,

$$\lim_{n\to\infty} e^{t(A+V_n)}f =: S(t)f$$

exists in E and defines a C_0-semigroups S on E. Here $V_n(x) = \inf\{V(x), n\}$. In that case we denote by $A + V$ the generator of S and write $e^{t(A+V)} := S(t)$, $t \geqslant 0$.

Admissible potentials were studied systematically by Voigt [11,12]. Below we will consider some concrete examples.

Now we can formulate the general version of Hölder's inequality for admissible positive functions, which follows immediately from Proposition 2.1.

Theorem 2.4 (The potential Hölder's inequality). *Let $E = L^r(\Omega)$, $1 \leqslant r < \infty$, with a σ-finite measure space (Ω, Σ, μ).*

Let $V : \Omega \to [0, \infty]$ be a measurable $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Assume that pV is admissible. Let $f \in E_+$. Then

$$e^{t(A+V)} f \leqslant \left(e^{t(A+pV)} f\right)^{1/p} \left(e^{tA} f\right)^{1/p'} \tag{6}$$

μ-a.e.

We note a consequence in terms of norms instead of pointwise inequalities.

Corollary 2.5. *Let $V : \Omega \to [0, \infty]$ be measurable, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Assume that pV is admissible. Then*

$$\left\|e^{t(A+V)} f\right\|_r \leqslant \left\|e^{t(A+pV)} f\right\|_r^{1/p} \cdot \left\|e^{tA} f\right\|_r^{1/p'}$$

for all $0 \leqslant f \in L^r(\Omega)$. Consequently,

$$\left\|e^{t(A+V)}\right\|_{\mathfrak{L}(L^r)} \leqslant \left\|e^{t(A+pV)}\right\|_{\mathfrak{L}(L^r)}^{1/p} \cdot \left\|e^{tA}\right\|_{\mathfrak{L}(L^r)}^{1/p'}.$$

Proof. Let $q_1 = p \cdot r$, $q_2 = p' \cdot r$. Then $q_1 > 1$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r}$. Applying Hölder's inequality to (6) we obtain for $0 \leqslant f \in L^r(\Omega)$,

$$\begin{aligned}\left\|e^{t(A+V)} f\right\|_r &\leqslant \left\|\left(e^{t(A+pV)} f\right)^{1/p}\right\|_{q_1} \cdot \left\|\left(e^{tA} f\right)^{1/p'}\right\|_{q_2} \\ &= \left\|e^{t(A+pV)} f\right\|_r^{1/p} \left\|e^{tA} f\right\|_r^{1/p'}. \qquad \square\end{aligned}$$

3. Ultracontractivity and Gaussian estimates

Let $(e^{tA})_{t \geqslant 0}$ be a positive C_0-semigroup on $E = L^r(\Omega)$ where $1 \leqslant r < \infty$ is fixed. Assume that the semigroup is *ultracontractive of asymptotic dimension* $d > 0$, i.e., there exists a constant $c > 0$ such that

$$\left\|e^{tA}\right\|_{\mathfrak{B}(L^r, L^\infty)} \leqslant c t^{-\frac{d}{2}\frac{1}{r}} \quad (0 < t \leqslant 1). \tag{7}$$

We choose c such that also

$$\left\|e^{tA}\right\|_{\mathfrak{B}(L^r)} \leqslant c \quad (0 < t \leqslant 1). \tag{8}$$

By the Riesz–Thorin theorem it follows from (7), (8) that for $r \leqslant q \leqslant \infty$,

$$\left\|e^{tA}\right\|_{\mathfrak{B}(L^r, L^q)} \leqslant c t^{-\frac{d}{2}(\frac{1}{r} - \frac{1}{q})} \quad (0 < t \leqslant 1). \tag{9}$$

Proposition 3.1. *Let $V : \Omega \to [0, \infty]$ be measurable and $p > r$ such that pV is admissible. Let*

$$c_p := \sup_{0 < t \leqslant 1} \left\|e^{t(A+pV)}\right\|_{\mathfrak{B}(L^r)}.$$

Then

$$\left\|e^{t(A+V)}\right\|_{\mathfrak{B}(L^r,L^p)} \leqslant c^{1/p'} c_p^{1/p} t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{p})} \quad (0 < t \leqslant 1). \tag{10}$$

Proof. *First case*: $r = 1$. By the potential Hölder's inequality, Theorem 2.4, for $0 \leqslant f \in L^1$,

$$\begin{aligned} e^{t(A+V)} f &\leqslant \left(e^{t(A+pV)} f\right)^{1/p} \left(e^{tA} f\right)^{1/p'} \\ &\leqslant \left(e^{t(A+pV)} f\right)^{1/p} c^{1/p'} t^{-\frac{d}{2}\frac{1}{p'}} \|f\|_1^{1/p'}. \end{aligned}$$

Hence

$$\begin{aligned} \left\|e^{t(A+V)} f\right\|_{L^p} &\leqslant \left\|e^{t(A+pV)} f\right\|_{L^1}^{1/p} c^{1/p'} t^{-\frac{d}{2}(1-\frac{1}{p})} \|f\|_1^{1/p'} \\ &\leqslant c_p^{1/p} c^{1/p'} t^{-\frac{d}{2}(1-\frac{1}{p})} \|f\|_1 \end{aligned}$$

for $0 < t \leqslant 1$.

Second case: $r > 1$. Let $q = p \cdot r' - r'$. Then $r < p < q$. We now use (8) and (9).

Let $0 \leqslant f \in L^r$, $0 < t \leqslant 1$.

By the potential Hölder's inequality, $(e^{t(A+V)} f)^p \leqslant (e^{t(A+pV)} f)(e^{tA} f)^{\frac{p}{p'}}$.

Since $\frac{p}{p'} \cdot r' = q$, it follows from Hölder's inequality that

$$\begin{aligned} \int_\Omega \left(e^{t(A+V)} f\right)^p &\leqslant \left\|e^{t(A+pV)} f\right\|_r \left(\int_\Omega \left(e^{tA} f\right)^q\right)^{1/r'} \\ &\leqslant c_p \|f\|_r \left\|e^{tA} f\right\|_q^{q/r'} \\ &\leqslant c_p \|f\|_r c^{q/r'} t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})\frac{q}{r'}} \|f\|_r^{q/r'}. \end{aligned}$$

Hence, for $0 < t \leqslant 1$,

$$\begin{aligned} \left\|e^{t(A+pV)} f\right\|_p &\leqslant c_p^{1/p} \|f\|_r^{1/p} c^{1/p'} t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})\frac{1}{p'}} \|f\|_r^{1/p'} \\ &\leqslant c_p^{1/p} \|f\|_r c^{1/p'} t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{p})}. \qquad \square \end{aligned}$$

We recall the following converse of the Riesz–Thorin theorem (see [1, 7.3.2]), due to Coulhon [13].

Extrapolation Theorem 3.2. *Assume that* (9) *holds where $r < q < \infty$. Assume furthermore that*

$$\sup_{0<t\leqslant 1} \left\|e^{tA}\right\|_{\mathfrak{B}(L^\infty)} < \infty.$$

Then there exists a constant $c' \geqslant 0$ such that

$$\left\|e^{tA}\right\|_{\mathfrak{B}(L^r,L^\infty)} \leqslant c' t^{-\frac{d}{2r}} \quad (0 < t \leqslant 1).$$

Applying this Extrapolation Theorem to the semigroup $(e^{t(A+V)})_{t\geqslant 0}$, we deduce from Proposition 3.1 the following.

Theorem 3.3. *Let $(e^{tA})_{t\geqslant 0}$ be an ultracontractive positive C_0-semigroup of asymptotic dimension $d>0$ on $E=L^r(\Omega)$. Let $V:\Omega\to[0,\infty]$ be measurable such that pV is admissible for some $p>r$.*

Assume that

$$\sup_{0<t\leqslant 1}\left\|e^{t(A+V)}\right\|_{\mathfrak{B}(L^\infty)}<\infty. \tag{11}$$

Then $(e^{t(A+V)})_{t\geqslant 0}$ is ultracontractive of asymptotic dimension d.

Alternatively, instead of (11) we may assume that the constants c_p of Proposition 3.1 can be controlled:

Proposition 3.4. *Let the assumption of Proposition 3.1 be satisfied, but assume that pV is admissible for all $p>0$. If $\overline{\lim}_{p\to\infty}c_p^{1/p}<\infty$, then $(e^{t(A+V)})_{t\geqslant 0}$ is ultracontractive of asymptotic dimension d.*

This follows directly from Proposition 3.1.

Next we assume that $r=1$.

Then the ultracontractivity assumption (7) means that e^{tA} is given by a kernel $K_t^A\in L^\infty(\Omega\times\Omega)$ via

$$\left(e^{tA}f\right)(x)=\int_\Omega K_t^A(x,y)f(y)\,d\mu(y)\quad x\text{-a.e.},\ t>0,$$

where

$$0\leqslant K_t^A(x,y)\leqslant ct^{-d/2}\quad (0<t\leqslant 1),$$

and consequently

$$0\leqslant K_t^A(x,y)\leqslant c_1t^{-d/2}e^{\omega t}\quad (t>0)$$

for some $\omega>0$ and some constant $c_1>0$.

Now we want to estimate the kernel of $e^{t(A+V)}$ by the free kernel K_t^A.

We assume now that $\Omega\subset\mathbb{R}^d$ is open and that μ is the Lebesgue measure.

Lemma 3.5. *Let $k_1,k_2:\Omega\to[0,\infty)$ be measurable and bounded such that*

$$\int_\Omega k_1(y)f(y)\,d\mu(y)\leqslant\left(\int_\Omega k_2(y)f(y)\,d\mu(y)\right)^{1/q}$$

for all $0\leqslant f\in L^1(\Omega)$, $\|f\|_1\leqslant 1$ where $1\leqslant q<\infty$.

Then $k_1(y)\leqslant k_2(y)^{1/q}$, μ-a.e.

Proof. By Lebesgue's Differentiation Theorem (see, e.g., [6, 1.7.1]),

$$k_1(x)=\lim_{r\downarrow 0}\frac{1}{|B_r(x)|}\int_{B_r(x)}k_1(y)\,d\mu(y)\leqq\lim_{r\downarrow 0}\left(\frac{1}{|B_r(x)|}\int_{B_r(x)}k_2(y)\,d\mu(y)\right)^{1/q}$$
$$=k_2(x)^{1/q}\quad x\text{-a.e.}\qquad\square$$

Now we continue to assume (7), (8) for $r = 1$. Let $0 \leqslant V$ be measurable such that pV is admissible for some $p > 1$. Assume (11). Thus $(e^{t(A+V)})_{t\geqslant 0}$ is an ultracontractive C_0-semigroup on $L^1(\Omega)$ by Theorem 3.3. Thus $e^{t(A+V)}$ is given by a kernel K_t^{A+V}. By the potential Hölder's inequality for $0 \leqslant f \in L^1(\Omega)$ with $\|f\|_1 \leqslant 1$ we have

$$\int_\Omega K_t^{A+V}(x,y) f(y)\,dy = \big(e^{t(A+V)} f\big)(x) \leqslant \big(e^{t(A+pV)} f\big)^{1/p}(x)\big(e^{tA} f\big)^{1/p'}$$

$$\leqslant ct^{-\frac{d}{2p}} \bigg(\int_\Omega K_t^A(x,y) f(y)\,dy\bigg)^{1/p'}.$$

It follows from Lemma 3.5 that for almost all $x \in \Omega$

$$K_t^{A+V}(x,y) \leqslant ct^{-\frac{d}{2p}} K_t^A(x,y)^{1/p'}$$

for almost all $y \in \Omega$, $0 < t \leqslant 1$.

By Fubini's theorem this means that

$$K_t^{A+V}(x,y) \leqslant ct^{-\frac{d}{2p}} K_t^A(x,y)^{1/p'} \tag{12}$$

for almost all $(x,y) \in \Omega \times \Omega$.

Now we assume that $(e^{tA})_{t\geqslant 0}$ admits a *Gaussian estimate* (see [2,3]), i.e., there exists a constant such that

$$K_t^A(x,y) \leqslant \text{const} \cdot t^{-\frac{d}{2}} e^{-|x-y|^2/bt} \tag{13}$$

$(0 < t \leqslant 1)$ for some $b > 0$. Then it follows from (12) that

$$K_t^{A+V}(x,y) \leqslant \text{const} \cdot t^{-\frac{d}{2}} e^{-|x-y|^2/p'bt}$$

$(0 < t \leqslant 1)$. Thus we have proved the following.

Theorem 3.6. *Assume that $(e^{tA})_{t\geqslant 0}$ satisfies a Gaussian estimate. Let $V : \Omega \to \mathbb{R}$ be measurable such that pV is admissible for some $p > 1$ and $\sup_{0<t\leqslant 1} \|e^{t(A+V)}\|_{\mathfrak{B}(L^\infty)} < \infty$. Then $(e^{t(A+V)})_{t\geqslant 0}$ also satisfies a Gaussian estimate.*

In Section 5 we will show by an example that in Theorem 3.6 it does not suffice that V is admissible.

We remark that the Gaussian estimate (13) implies that every operator e^{tA} is given by a kernel K_t^A such that

$$0 \leqslant K_t^A(x,y) \leqslant \text{const} \cdot t^{-\frac{d}{2}} e^{-|x-y|^2/bt} e^{\omega t}$$

for all $t \geqslant 0$ and some $\omega \geqslant 0$. Moreover, Gaussian estimates have interesting consequences for regularity and spectral behaviour (see [2,3,8]).

4. Schrödinger operators

We first consider the Gaussian semigroup $(U(t))_{t\geqslant 0}$ on $L^1(\mathbb{R}^d)$ given by

$$\big(U(t)f\big)(x) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/2t} f(y)\,dy.$$

The generator is $\frac{1}{2}\Delta_1$, where Δ_1 is the Laplacian on $L^1(\mathbb{R}^d)$ with domain $D(\Delta_1) = \{f \in L^1(\mathbb{R}^d)\colon\ \Delta f \in L^1(\mathbb{R}^d)\}$. Following Voigt [11,12], we denote by

$$\hat{K}_d = \left\{ V \in L_{1,\mathrm{loc}}\colon\ \|V\|_{\hat{K}_d} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int_{|x-y|\leqslant 1} \big|g_d(x-y)\big|\big|V(x)\big|\,dy < \infty \right\}$$

the *extended Kato class*, and for $V \in \hat{K}_d$

$$c_d(V) := \lim_{\alpha\downarrow 0}\left(\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int_{|x-y|\leqslant\alpha} \big|g_d(x-y)\big|\big|V(x)\big|\,dy \right),$$

where

$$g_d(x) = \begin{cases} |x| & \text{if } d=1,\\ \frac{1}{\pi}\ln(x) & \text{if } d=2,\\ -\frac{\Gamma(d/2)}{(d-2)\pi^{d/2}}|x|^{-d/2} & \text{if } d\geqslant 3. \end{cases}$$

The *Kato class* K_d is then defined by $K_d := \{V \in \hat{K}_d\colon\ c_d(V) = 0\}$. These two classes of functions can be alternatively described as follows (see [11, 5.1]). A measurable function V is in $\hat{K}_d$ if and only if $Vf \in L^1(\mathbb{R}^d)$ for all $f \in D(A_1)$. Moreover, $K_d = \{V \in \hat{K}_d\colon\ \|VR(\lambda, A_1)\|_{\mathfrak{L}(L^1)} \to 0\ (\lambda \to \infty)\}$. Here we identify V with the multiplication operator $f \mapsto V\cdot f$ from $D(\Delta_1)$ into $L^1(\mathbb{R}^d)$.

Theorem 4.1. *Let $0 \leqslant V \in \hat{K}_d$ and $c_d(V) < 1$. Then $\frac{1}{2}\Delta_1 + V$ with domain $D(\Delta_1)\cap D(V)$ generates a positive C_0-semigroup which admits Gaussian estimates.*

Proof. It follows from [11, Remarks 5.2(b)] that pV is admissible for $p > 1$ such that $c_d(pV) < 1$. Moreover, by, e.g., [5, Theorem 2.9],

$$\sup_{0<t\leqslant 1} \big\| e^{t(\frac{1}{2}\Delta_1+V)}\big\|_{\mathfrak{B}(L^\infty)} < \infty.$$

Now it follows from Theorem 3.10 that the semigroup $(e^{t(\frac{1}{2}\Delta_1+V)})_{t\geqslant 0}$ admits Gaussian estimates. [11, Theorem 5.3] implies that the generator of the semigroup $(e^{t(\frac{1}{2}\Delta_1+V)})_{t\geqslant 0}$ is the operator $\frac{1}{2}\Delta_1 + V$ with domain $D(\frac{1}{2}\Delta_1 + V) = \{f \in D(\Delta_1)\colon\ \int |f|V\,dx < \infty\}$. $\square$

Next we will assume that $(e^{tA})_{t\geqslant 0}$ is a positive C_0-semigroup on $L^1(\Omega)$ satisfying a Gaussian estimate, where $\Omega \subset \mathbb{R}^d$ is an open set. This is the case if A is a uniformly elliptic operator with measurable coefficients. More precisely, let $a_{ij}, b_i, c_i, c_0 \in L^\infty(\Omega)$, $i, j = 1, \dots, d$, be real coefficients and assume that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geqslant \alpha|\xi|^2 \quad (x \in \Omega)$$

for all $\xi \in \mathbb{R}^n$, where $\alpha > 0$. Consider the form

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij} D_i u D_j v + \sum_{i=1}^{d} (b_i D_i v u + c_i D_i u v + c_0 u v) \right) dx$$

with form domain $D(a) = H_0^1(\Omega)$ or $D(a) = H^1(\Omega)$.

Denote by A_2 the associated operator on $L^2(\Omega)$. In the case where $D(a) = H_0^1(\Omega)$ we do not assume any regularity condition for Ω, if $D(a) = H^1(\Omega)$, then we assume the *extension property* (i.e., for each $u \in H^1(\Omega)$ there exists $\tilde{u} \in H^1(\mathbb{R}^d)$, such that $\tilde{u}|_\Omega = u$, see [1, 7.3.6]). Then the semigroup $(e^{-tA_2})_{t \geqslant 0}$ admits Gaussian estimates. This was proved in [3] for $b_i, c_i \in W^{1,\infty}$ and in the general case by Daners [4], where in the case of $D(a) = H^1(\Omega)$ a stronger version of the extension property is assumed. Ouhabaz [7] showed recently that the weak form of the extension property suffices (see also [8]).

Since $(e^{-tA_2})_{t \geqslant 0}$ satisfies Gaussian estimates, the semigroup extrapolates to $L^p(\Omega)$ and we denote by $(e^{-tA_p})_{t \geqslant 0}$ the extrapolated semigroup, $1 \leqslant p < \infty$. Now we consider a potential $0 \leqslant V \in K_d$. Because of the Gaussian estimate, the potential V is not only admissible for the Laplacian on $L^1(\mathbb{R}^d)$ but also for the general elliptic operator considered here. In fact, we have the following result.

Theorem 4.2. *Let $0 \leqslant V \in K_d$. Then V is admissible for $-A_1$ and the semigroup $(e^{-t(A_1+V)})_{t \geqslant 0}$ has a Gaussian estimate.*

Proof. (a) Denote by G the Gaussian semigroup on $L^1(\mathbb{R}^d)$ with generator $\frac{1}{2}\Delta_1$. Then the hypothesis implies that $e^{-tA_1} \leqslant cG(bt)e^{\omega t}$ for all $t > 0$ and for some $\omega \in \mathbb{R}_+$, $c > 0$, $b > 0$ (see [3]). Taking Laplace transforms we see that $R(\lambda, A_1) \leqslant cR((\lambda - \omega)/b, \frac{1}{2}\Delta_1)$. Since $\lim_{\lambda \to \infty} \|VR(\lambda, \frac{1}{2}\Delta_1)\|_{\mathfrak{L}(L^1)} = 0$, by [11, 4.7 and 2.1(b)], it follows that also $\lim_{\lambda \to \infty} \|VR(\lambda, A_1)\|_{\mathfrak{L}(L^1)} = 0$. Then by [12, 4.7, 4.5], pV is admissible with respect to A_1 for all $p > 0$.

(b) In order to apply Theorem 3.6, we have to show that

$$\sup_{0<t\leqslant 1} \left\| e^{t(A_1+V)} \right\|_{\mathfrak{B}(L^\infty)} < \infty.$$

For this, since V is admissible, it suffices to show that $\|e^{t(A_1+V_n)} f\|_{\mathfrak{B}(L^\infty)} \leqslant c$ for all $n \in \mathbb{N}$ and all $0 \leqslant f \in L^1 \cap L^\infty$, $\|f\|_\infty \leqslant 1$. Consider the adjoint form a^* given by $a^*(u,v) = \overline{a(v,u)}$ with form domain $D(a^*) = D(a)$. The associated operator is A_2^*, the adjoint of A_2. Then the adjoint of $A_2 + V_n$ is $A_2^* + V_n$. Since V is admissible also for A_1^* (the negative generator of the extrapolated semigroup of $(e^{-tA_2})_{t \geqslant 0}$ to L^1) it follows from (a) that

$$\left\| e^{t(-A_2^*+V_n)} g \right\|_{L^1} \leqslant c$$

for all $f \in L^1 \cap L^\infty$, $\|f\|_{L^1} \leqslant 1$, $n \in \mathbb{N}$, $t \geqslant 0$. This implies the claim by duality. □

5. Unbounded drift

As a further application we consider Schrödinger semigroups with unbounded drift. Let $b = (b_1, \dots, b_d) : \mathbb{R}^d \to \mathbb{R}^d$ be C^1 functions and let $V_0 : \mathbb{R}^d \to [0, \infty)$ be c continuous. Let $1 \leqslant r < \infty$. Assume that V_0 satisfies the condition

$$\frac{\operatorname{div} b}{r} \leqslant V_0 + c \tag{14}$$

for some constant $c \geqslant 0$. Define the maximal operator

$$A_{r,\max} u = \Delta u - \sum_{j=1}^{d} b_j D_j u - V_0 u$$

on $L^r(\mathbb{R}^d)$ with domain

$$D(A_{r,\max}) := \left\{ u \in W^{1,r}_{\mathrm{loc}}\left(\mathbb{R}^d\right) \colon\ A_{r,\max} u \in L^r\left(\mathbb{R}^d\right) \right\}.$$

Then it is shown in [1] that $A_{r,\max}$ has a (unique) restriction A_r which generates a minimal positive C_0-semigroup on $L^r(R^d)$. Note that the drift functions b_j may be unbounded. In that case the potential V_0 is needed for compensation. Now assume that V_0 satisfies the stronger condition

$$\operatorname{div} b \leqslant \beta V_0 + c \tag{15}$$

for some $\beta < 1$ and some $c \geqslant 0$, and also

$$\left| b(x) \right| \leqslant \gamma V_0(x)^{1/2} \quad \left(x \in \mathbb{R}^d \right). \tag{16}$$

Then it is shown in [1] that the C_0-semigroup $(e^{tA_r})_{t \geqslant 0}$ admits Gaussian estimates. Consequently, it has an extrapolation semigroup $(e^{tA_1})_{t \geqslant 0}$ on $L^1(\mathbb{R}^d)$. The previous results allow us to add a positive potential V. In fact, as in Section 4 we obtain the following result.

Theorem 5.1. *Let $0 \leqslant V \in K_d$. Then V is admissible for A_1 and the semigroup*

$$\left(e^{t(A_1 + V)} \right)_{t \geqslant 0}$$

admits an upper Gaussian bound.

The proof is the same as for Theorem 4.2. Note that for the duality argument we need the complete operator (also with coefficients c_j as in the case considered above). We refer to [1, Section 5].

Note added in proof

We are grateful to V. Liskevich who pointed out that Hölder's inequality (6) appears in a more special situation in: Y. Semenov, Stability of L^p-spectrum of generalized Schrödinger operators and equivalence of Green's function, Int. Math. Res. Not. 12 (1997) 573–593, inequality (6.2).

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