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Maximal L^p-regularity for parabolic and elliptic equations on the line

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Dedicated to Giuseppe Da Prato on the occasion of his 70th birthday

Abstract. Let A be a closed operator on a Banach space X. We study maximal L^p-regularity of the problems

$$\begin{aligned} u'(t) &= Au(t) + f(t) \quad \text{and} \\ u''(t) &= Au(t) + f(t) \end{aligned}$$

on the line. The results are used to solve quasilinear parabolic and elliptic problems on the line.

1. Introduction

In the seminal paper [DPG75] of 1975 Da Prato and Grisvard studied in a systematic way invertibility of the sum of two sectorial operators. One important example, which can be treated in this way, is the initial value problem

$$\begin{cases} u' &= Au(t) + f(t) \qquad (t \in [0, \tau]) \\ u(0) &= 0 \end{cases} \tag{1.1}$$

where A is a sectorial operator on a Banach space X. If instead of the initial value problem one is interested in solving the problem

$$u'(t) = Au(t) + f(t) \qquad (t \in \mathbb{R}) \tag{1.2}$$

on the entire line then one needs to extend the results of Da Prato and Grisvard to the sum of commuting bisectorial operators (instead of sectorial operators). This has been done in [AB05]. The sum method gives also results on maximal regularity, in particular in the sense of Hölder continuous functions by interpolation methods. For L^p-maximal regularity Dore and Venni [DV87] proved their famous result based on functional calculus in 1987. It could be applied to the sum of commuting sectorial operators and in particular to the initial value problem (1.1). In the same year 1987, Mielke [Mie87] investigated problem (1.2) on the line and characterized maximal L^p-regularity on Hilbert spaces. Our first goal in this paper is to extend Mielke's results to UMD-spaces (and in particular L^q-spaces, $1 < q < \infty$) with the help of the recent operator-valued Fourier multiplier theorem due to Weis (see [Wei01a], [Wei01b], [KW04], [DHP01]). It turns out that problem (1.2) is maximal L^p-regular (i.e., for each $f \in L^p(\mathbb{R}, X)$ there exists a unique solution $u \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D(A))$) if and only if A is R-bisectorial and invertible. In Section 3 we then apply the result to study the second order problem

$$u''(t) = Au(t) + f(t) \tag{1.3}$$

by considering a suitable system. The results on maximal L^p-regularity are then used to solve quasilinear equations of the type

$$u' = A(u)u + f \text{ and} \tag{1.4}$$

$$u'' = A(u)u + f \tag{1.5}$$

on the real line. If $A(u)$ is an elliptic operator, then (1.4) is a parabolic and (1.5) is an elliptic equation on a cylindrical domain. As example we consider $A(u) = -m(u)\Delta_q u$ where Δ_q is the Dirichlet Laplacian on $L^q(\Omega)$ and $m : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous.

2. Maximal L^p-regularity of the first order equation on the line

Let A be a closed operator on X. We consider the problem

$$u'(t) = Au(t) + f(t) \quad (t \in \mathbb{R}). \tag{2.1}$$

DEFINITION 2.1. Let $1 < p < \infty$. Problem (2.1) is *maximal L^p-regular* if for all $f \in L^p(\mathbb{R}, X)$ there exists a unique $u \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D(A))$ solving (2.1). Here we consider $D(A)$ as a Banach space for the graph norm.

Note that problem (2.1) is maximal L^p-regular if and only if the corresponding problem with A replace by $-A$ is so.

REMARK 2.2. Recall that $W^{1,p}(\mathbb{R}, X)$ consists of those functions $u \in L^p(\mathbb{R}, X)$ for which there exists $u' \in L^p(\mathbb{R}, X)$ such that

$$-\int_{\mathbb{R}} u(t)\varphi'(t)dt = \int_{\mathbb{R}} u'(t)\varphi(t)dt$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$. Thus, if $u \in L^p(\mathbb{R}, D(A))$ is a *weak solution* of (2.1), i.e., if

$$-\int_{\mathbb{R}} u(t)\varphi'(t)dt = \int_{\mathbb{R}} (Au(t) + f(t))\varphi(t)dt \tag{2.2}$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$, then $u \in W^{1,p}(\mathbb{R}, X)$ and $u' = Au + f$. Thus (2.1) is maximal L^p-regular if and only if for all $f \in L^p(\mathbb{R}, X)$ there exists a unique weak solution $u \in L^p(\mathbb{R}, D(A))$ of (2.1).

If (2.1) is maximal L_p-regular, then it follows from the Closed Graph Theorem that there exists a constant $c > 0$ such that

$$\|u\|_{W^{1,p}(\mathbb{R},X)} + \|u\|_{L^p(\mathbb{R},D(A))} \leq c\|f\|_{L^p(\mathbb{R},X)} \tag{2.3}$$

whenever $f \in L^p(\mathbb{R}, X)$ and u is the solution of (2.1).

DEFINITION 2.3. An operator A is called *bisectorial* if

$$i\mathbb{R} \setminus \{0\} \subset \varrho(A) \text{ and } \sup_{s\in\mathbb{R}\setminus\{0\}} \|sR(is, A)\| < \infty.$$

If the set

$$\{sR(is, A) : s \in \mathbb{R} \setminus \{0\}\}$$

is even R-bounded, then we call A *R-bisectorial*.

THEOREM 2.4. *Assume that X is a UMD-space. Let $1 < p < \infty$. The following assertions are equivalent.*

(i) *Problem (2.1) is maximal L^p-regular;*
(ii) *A is R-bisectorial and invertible.*

If X is a Hilbert space, then a family of operators in $\mathcal{L}(X)$ is R-bounded if and only if it is bounded and thus, A is R-bisectorial if and only if A is bisectorial. In the Hilbert space case Theorem 2.4 is due to Mielke [Mie87] who also proved that in arbitrary Banach spaces maximal L^p-regularity implies that A is bisectorial and invertible. If X is not isomorphic to a Hilbert space, then R-boundedness is strictly stronger than boundedness (see [AB02]). We refer to [CPSW00]. [KW04], [DHP01], [Wei01a], [Wei01b] for more information on R-boundedness and the definition of UMD-spaces. The proof of Theorem 2.4 is based on the vector-valued Fourier transform. The implication $(ii) \Rightarrow (i)$ is due to Schweiker [Sch00]. It follows from Weis' multiplier theorem. For the converse implication we use the result by Clément-Prüss that each operator-valued L^p-Fourier multiplier is R-bounded [CP01], [KW04]. We now give a detailed proof of Theorem 2.4.

By $\mathcal{S}(\mathbb{R}, X)$ we denote the *Schwartz space* of all smooth rapidly decreasing functions on $\mathbb{R}$ with values in X. The *Fourier transform*

$$\begin{aligned} \mathcal{S}(\mathbb{R}, X) &\to \mathcal{S}(\mathbb{R}, X) \\ f &\mapsto \hat{f} \end{aligned}$$

given by

$$\hat{f}(s) = \int_{\mathbb{R}} e^{-ist} f(t)dt$$

is an isomorphism. Denote by $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}), X)$ the space of all *tempered distributions*. Then the Fourier transform $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}, X)$ is defined by

$$\langle \mathcal{F}u, \varphi \rangle = \langle u, \hat{\varphi} \rangle \quad (u \in \mathcal{S}'(\mathbb{R}, X), \varphi \in \mathcal{S}(\mathbb{R})) .$$

If we identify $\mathcal{S}(\mathbb{R}, X)$ with a subspace of $\mathcal{S}'(\mathbb{R}, X)$ by letting

$$\langle u, \varphi \rangle = \int_{\mathbb{R}} u(t)\varphi(t)dt \quad (\varphi \in \mathcal{S}(\mathbb{R})) ,$$

for all $u \in \mathcal{S}(\mathbb{R}, X)$, then $\hat{u} = \mathcal{F}u$, i.e.,

$$\int_{\mathbb{R}} u(t)\hat{\varphi}(t)dt = \int_{\mathbb{R}} \hat{u}(s)\varphi(s)ds \tag{2.4}$$

for all $u \in \mathcal{S}(\mathbb{R}, X), \varphi \in \mathcal{S}(\mathbb{R})$. Thus $\mathcal{F} : \mathcal{S}'(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, X)$ is an isomorphism extending the isomorphism $u \mapsto \hat{u}$ on $\mathcal{S}(\mathbb{R}, X)$. We refer to [Am95] for all these properties.

Next we characterize solutions by the Fourier transform. Let

$$\mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) := \{f \in \mathcal{S}(\mathbb{R}, X) : \hat{f} \in \mathcal{D}(\mathbb{R}, X)\}$$

where $\mathcal{D}(\mathbb{R}, X)$ is the space of all infinitely differentiable functions $\varphi : \mathbb{R} \to X$ of compact support.

PROPOSITION 2.5. *Assume that* $i\mathbb{R} \subset \varrho(A)$. *Let* $1 < p < \infty$. *Let* $f \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X)$ *and* $u \in L^p(\mathbb{R}, D(A))$. *The following assertions are equivalent.*

(i) $u \in W^{1,p}(\mathbb{R}, X)$ *and* u *is a solution of (2.1);*
(ii) $u \in \mathcal{S}(\mathbb{R}, D(A))$ *and* $\hat{u}(s) = R(is, A)\hat{f}(s) \qquad (s \in \mathbb{R})$.

Proof. (ii) ⇒ (i). One has $\hat{u}'(s) = is\hat{u}(s)$ and $\widehat{Au}(s) = A\hat{u}(s)$ hence $(u' - Au)^\wedge(s) = \hat{f}(s)$ for all $s \in \mathbb{R}$. Consequently, $u' - Au = f$.
(i) ⇒ (ii). Let $u \in L^p(\mathbb{R}, D(A)) \cap W^{1,p}(\mathbb{R}, X)$ be a solution of (2.1). Recall that

$$W^{1,p}(\mathbb{R}, X) \subset C_0(\mathbb{R}, X) := \{u : \mathbb{R} \to X \ : \ u \text{ is continuous and } \lim_{|t|\to\infty} u(t) = 0\} .$$

Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(s)R(is, A)\hat{f}(s)ds = \\
&\lim_{n\to\infty}\int_{\mathbb{R}} \varphi(s)R(is, A)\int_{-n}^{n} f(t)e^{-ist}dtds = \\
&\lim_{n\to\infty}\int_{\mathbb{R}} \varphi(s)R(is, A)\int_{-n}^{n} (u'(t) - Au(t))e^{-ist}dtds = \\
&\lim_{n\to\infty}\int_{\mathbb{R}} \varphi(s)R(is, A)\{\int_{-n}^{n} e^{-ist}[isu(t) - Au(t)]dt + u(n)e^{-isn} - u(-n)e^{isn}\}ds = \\
&\lim_{n\to\infty}\int_{\mathbb{R}} \varphi(s)\int_{-n}^{n} e^{-ist}u(t)dtds = \\
&\lim_{n\to\infty}\int_{-n}^{n} u(t)\int_{\mathbb{R}} \varphi(s)e^{-ist}dsdt = \\
&\int_{\mathbb{R}} u(t)\hat{\varphi}(t)dt\ .
\end{aligned}$$

Recall that we may identify $L^p(\mathbb{R}, D(A))$ with a subspace of $\mathcal{S}'(\mathbb{R}, D(A))$ by letting

$$\langle v, \varphi\rangle = \int_{\mathbb{R}} v(t)\varphi(t)dt$$

for $v \in L^p(\mathbb{R}, D(A))$, $\varphi \in \mathcal{S}(\mathbb{R})$. Thus, the identity above, says that $\mathcal{F}u = R(i\cdot, A)\hat{f}(\cdot) \in \mathcal{D}(\mathbb{R}, D(A))$. Hence $u \in \mathcal{S}(\mathbb{R}, D(A))$. □

Next we formulate a special case of Weis' multiplier theorem.

We need the notion of operator-valued multipliers. For our purposes it suffices to consider C^∞-functions.

DEFINITION 2.6. Let X, Y be Banach spaces, $1 < p < \infty$. A function $M \in C^\infty(\mathbb{R}, \mathcal{L}(X, Y))$ is an $L^p(\mathbb{R}, X) - L^p(\mathbb{R}, Y)$ *multiplier* if there exists a bounded operator $T : L^p(\mathbb{R}, X) \to L^p(\mathbb{R}, Y)$ such that for all $f \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X)$

$$Tf \in \mathcal{S}(\mathbb{R}, Y) \text{ and } (Tf)^\wedge(s) = M(s)\hat{f}(s) \qquad (s \in \mathbb{R})\ . \tag{2.5}$$

Note that the operator is uniquely determined by (2.5) since $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X)$ is dense in $L^p(\mathbb{R}, X)$.

The following operator-valued version of Michlin's Theorem is due to Weis [Wei01a], see also [KW04].

THEOREM 2.7. *Let X, Y be UMD-spaces. Let $M \in C^\infty(\mathbb{R}, \mathcal{L}(X, Y))$ such that the sets*

$$\{M(s) : s \in \mathbb{R}\} \quad \textit{and} \quad \{sM'(s) : s \in \mathbb{R}\}$$

are both R-bounded in $\mathcal{L}(X, Y)$. Then M is an $L^p(\mathbb{R}, X) - L^p(\mathbb{R}, Y)$ multiplier for $1 < p < \infty$.

Proof of Theorem 2.4. (ii)⇒(i) Assume that $i\mathbb{R} \subset \varrho(A)$ and that the set $\{sR(is, A) : s \in \mathbb{R}\}$ is R-bounded. Consider the mapping $M \in C^\infty(\mathbb{R}, \mathcal{L}(X, D(A)))$ given by $M(s) = R(is, A)$. We show that M satisfies the hypothesis of Theorem 2.7. For this we have to show that $\{N(s) : s \in \mathbb{R}\}$ and $\{sN'(s) : s \in \mathbb{R}\}$ are R-bounded in $\mathcal{L}(X)$ where $N(s) = AR(is, A)$. Since $N(s) = isR(is, A) - I$ and $sN'(s) = isR(is, A) + s^2R(is, A)^2$, this follows from the assumption and the fact that the composition of R-bounded sets in R-bounded [KW04, I.2.8 p.88]. By Theorem 2.7 there exists a bounded operator

$$T : L^p(\mathbb{R}, X) \to L^p(\mathbb{R}, D(A))$$

such that for $f \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X)$, $u := Tf \in \mathcal{S}(\mathbb{R}, D(A))$ and $\hat{u}(s) = R(is, A)\hat{f}(s)$. By Proposition 2.5 it follows that u is a solution of (2.1). Moreover,

$$\|u\|_{L^p(\mathbb{R}, D(A))} \leq \|T\| \, \|f\|_{L^p(\mathbb{R}, X)}.$$

Now let $f \in L^p(\mathbb{R}, X)$ be arbitrary. Then there exist $f_n \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X)$ such that $f_n \to f$ in $L^p(\mathbb{R}, X)$. Let $u_n = Tf_n$. Then $u_n \to u$ in $L^p(\mathbb{R}, D(A))$. For $\varphi \in \mathcal{D}(\mathbb{R})$, one has

$$-\int_{\mathbb{R}} u_n(t)\varphi'(t)dt = \int_{\mathbb{R}} (Au_n(t) + f_n(t))\varphi(t)dt \, .$$

Letting $n \to \infty$ shows that u is a weak solution of (2.1). By Remark 2.1 it follows that $u \in W^{1,p}(\mathbb{R}, X)$. We have shown existence. It remains to prove uniqueness. For this let $u \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D(A))$ such that

$$u'(t) = Au(t) \qquad \text{a.e.}$$

Note that $u \in C_0(\mathbb{R}, X)$. Consider the *Carleman transform* $\tilde{u}$ of u given by

$$\tilde{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t}u(t)dt & (\mathrm{Re}\lambda > 0) \\ -\int_{-\infty}^0 e^{-\lambda t}u(t)dt & (\mathrm{Re}\lambda < 0) \, . \end{cases}$$

Then $\tilde{u} : \mathbb{C} \setminus i\mathbb{R} \to X$ is holomorphic. Let $x = u(0)$. Then $\int_0^t u(s)ds \in D(A)$ and $u(t) = x + A\int_0^t u(s)ds$ for all $t \geq 0$ since $u'(t) = Au(t)$ a.e. It follows that for $\lambda \in \varrho(A) \setminus i\mathbb{R}$, $\tilde{u}(\lambda) \in D(A)$ and $\tilde{u}(\lambda) = R(\lambda, A)x$. Since $i\mathbb{R} \subset \varrho(A)$, it follows that $\tilde{u}$ has an entire extension. By [Prü93, Proposition 05, p. 22] this implies that $u = 0$.

$(i) \Rightarrow (ii)$ Assume that Problem (2.1) is maximal L^p-regular. Then, by Mielke's result [Mie87, Satz 2.2.] (see also the comments following Theorem 2.4), one has $i\mathbb{R} \subset \varrho(A)$. In view of Remark 2.2 the Closed Graph Theorem shows that there exists a bounded operator $T : L^p(\mathbb{R}, X) \to L^p(\mathbb{R}, D(A))$ such that for $f \in L^p(\mathbb{R}, X)$, the function $u = Tf$ is the solution of (2.1). If $f \in \mathcal{F}^{-1}\mathbb{D}(\mathbb{R}, X)$, then it follows from Proposition 2.5 that $Tf \in \mathcal{S}(\mathbb{R}, D(A))$ and $(Tf)^\wedge(s) = R(is, A)\hat{f}(s) \quad (s \in \mathbb{R})$. Thus the function M with values in $\mathcal{L}(X, D(A))$ given by $M(s) = R(is, A)$ is an $L^p(\mathbb{R}, X) - L^p(\mathbb{R}, D(A))$ multiplier. It follows from a result of Clément-Prüss [CP01], see also [KW04, 3.13], that the set $\{M(s) : s \in \mathbb{R}\} \subset \mathcal{L}(X, D(A))$ is $R-$bounded. Since $A : D(A) \to X$ is an isomorphism, the set $\{AM(s) : s \in \mathbb{R}\} \subset \mathcal{L}(X)$ is $R-$bounded. This implies the claim since $AM(s) = isR(is, A) - I$. □

3. The second order problem

Let A be a closed operator on a Banach space X.

DEFINITION 3.1. The operator A is called *sectorial* if $(-\infty, 0) \subset \varrho(A)$ and $\sup\limits_{\lambda>0} \|\lambda(\lambda + A)^{-1}\| < \infty$. If the set $\{\lambda(\lambda + A)^{-1} : \lambda > 0\}$ is even $R-$bounded, then we call A *$R-$sectorial*.

Our aim is to study the equation

$$u''(t) = Au(t) + f(t) \qquad (t \in \mathbb{R}) . \tag{3.1}$$

We might treat problem (3.1) in a similar way as the first order problem in Section 2. But we prefer to write (3.1) as a system and to apply the results of Section 2. It turns out that in this way stronger regularity properties are obtained.

REMARK 3.2. Clément and Guerre-Delabrière [CG98] prove results on the equivalence of maximal L^p-regularity for the first and the second order problem on a bounded interval with initial values.

Assume that A is densely defined sectorial and invertible. Then $-A^{1/2}$ generates a bounded holomorphic C_0-semigroup. In particular, $A^{1/2}$ is sectorial as well and invertible. We consider $V := D(A^{1/2})$ as a Banach space with the graph norm. Then $A^{1/2} : D(A^{1/2}) \to X$ is an isomorphism. Consider the Banach space $\mathcal{X} = V \times X$ and the operator $\mathcal{A}$ on $\mathcal{X}$ given by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \; D(\mathcal{A}) = D(A) \times V$$

PROPOSITION 3.3. *If A is R-sectorial, then $\mathcal{A}$ is R-bisectorial.*

Proof. a) Let $\lambda \in \mathbb{C}$ such that $\lambda^2 \in \varrho(A)$. Then one easily checks that $\lambda \in \varrho(\mathcal{A})$ and

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} \lambda R(\lambda^2, A) & R(\lambda^2, A) \\ AR(\lambda^2, A) & \lambda R(\lambda^2, A) \end{pmatrix}.$$

In particular, $i\mathbb{R} \subset \varrho(\mathcal{A})$. In order to show that $\mathcal{A}$ is bisectorial, we let $\lambda = is$ and we have to show that $s^2(s^2 + A)^{-1}$ is bounded in $\mathcal{L}(V)$, $s(s^2 + A)^{-1}$ in $\mathcal{L}(X, V)$, $sA(s^2 + A)^{-1}$ in $\mathcal{L}(V, X)$ and $s^2(s^2 + A)^{-1}$ in $\mathcal{L}(X)$ uniformly in $s \in \mathbb{R}$. The last assertion follows from the hypothesis. Recall that $A^{1/2} : V \to X$ is an isomorphism with inverse $A^{-1/2}$. Thus the first assertion follows from the last one since $A^{-1/2}$ commutes with the resolvent. The second and third assertion signify, that $sA^{1/2}(s^2 + A)^{-1}$ is bounded in $\mathcal{L}(X)$ uniformly in $s \in \mathbb{R}$. This follows from the moment inequality [Paz83, p.7]

$$\|A^{1/2}y\|^2 \leq 4M^2\|Ay\|\ \|y\|$$

($y \in D(A)$) where $M = \sup_{t\geq 0} \|e^{-tA^{1/2}}\|_{\mathcal{L}(X)}$. If A is R-sectorial, then we have to show R-boundedness of the four families above. For the first and the fourth this follows directly as before. For the second and third one, one has to prove R-boundedness of the set $\{sA^{1/2}(s^2 + A)^{-1} : s \in \mathbb{R}\}$ in $\mathcal{L}(X)$ which is [KW01, Lemma 10]. □

Now we can apply Theorem 2.4 to the R-bisectorial operator $\mathcal{A}$ and we obtain the following result.

THEOREM 3.4. *Assume that X is a UMD-space, and that A is R-sectorial and invertible. Let $1 < p < \infty$. Then for each $f \in L^p(\mathbb{R}, X)$ there is a unique $u \in W^{2,p}(\mathbb{R}, X) \cap W^{1,p}(\mathbb{R}, V) \cap L^p(\mathbb{R}, D(A))$ solving* (3.1).

Proof. Since V is isomorphic to X also V and $\mathcal{X} = V \times X$ are UMD-spaces. Consider the function $(0, f) \in L^p(\mathbb{R}, \mathcal{X})$. By Theorem 2.4, there exists a unique $u = (u_1, u_2) \in W^{1,p}(\mathbb{R}, \mathcal{X}) \cap L^p(\mathbb{R}, D(\mathcal{A}))$ such that

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \mathcal{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Thus $u_1 \in W^{1,p}(\mathbb{R}, V) \cap L^p(\mathbb{R}, D(A))$, $u_2 \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, V)$ and $u_1' = u_2$, $u_2' = Au_1 + f$. Consequently, $u_1 \in W^{2,p}(\mathbb{R}, X)$ and $u_1'' = u_2' = Au_1 + f$. Uniqueness follows from Theorem 2.4 or by Proposition 3.5 below. □

Using the Carleman Transform as in Section 2 we obtain actually the following stronger uniqueness result.

PROPOSITION 3.5. *Let A be an operator such that $i\mathbb{R} \subset \varrho(A)$. Let $u \in W^{2,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D(A))$ such that*

$$u'' = Au\ .$$

Then $u = 0$.

Proof. Observe that $u \in C^1(\mathbb{R}, X)$ and $u, u' \in C_0(\mathbb{R}, X)$. Let

$$x = u(0), y = u'(0)\ .$$

Then $u(t) = x + ty + \int_0^t (t-s)u''(s)ds$. Hence $u(t) = x + ty + A\int_s^t (t-s)u(s)ds$. Thus the Carleman Transform $\tilde{u}$ of u (see the proof of Theorem 2.4) satisfies

$$\tilde{u}(\lambda) = \frac{x}{\lambda} + \frac{y}{\lambda^2} + \frac{A\tilde{u}(\lambda)}{\lambda^2}$$

for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. Hence $\tilde{u}(\lambda) = R(\lambda^2, A)(\lambda x + y)$ for $\lambda \in \varrho(A) \setminus i\mathbb{R}$. Since $i\mathbb{R} \subset \varrho(A)$, it follows that $\tilde{u}$ has an entire extension. Hence $u = 0$, as in Theorem 2.4. □

As a corollary we obtain the following interpolation result.

COROLLARY 3.6. *Let X be a UMD-space, $1 < p < \infty$ and assume that A is R-sectorial and invertible. Then*

$$W^{2,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D(A)) \hookrightarrow W^{1,p}(\mathbb{R}, V)\ .$$

Proof. Let $u \in W^{2,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D(A))$. Let $f = u'' - Au$. By Theorem 3.4 there exists $v \in W^{2,p}(\mathbb{R}, X) \cap W^{1,p}(\mathbb{R}, V) \cap L^p(\mathbb{R}, D(A))$ such that $v'' - Av = f$. It follows from Proposition 3.5 that $u = v$. □

REMARK 3.7. (elliptic equations). It depends on the hypotheses on A whether (3.1) is an elliptic or a hyperbolic problem. But Theorem 3.4 is suitable for elliptic problems. For example, let $1 < p, q < \infty$ and let $A = -\Delta_q + I$ on $L^q(\mathbb{R}^N)$ with domain $D(A) = W^{2,q}(\mathbb{R}^N)$. Then A is R-sectorial. Theorem 3.4 asserts that the operator L on $L^p(\mathbb{R}, L^q(\mathbb{R}^N)$ given by $Lu = u - u'' - \Delta u$ with domain

$$D(L) = W^{2,p}(\mathbb{R}, L^q(\mathbb{R}^N)) \cap W^{1,p}(\mathbb{R}, W^{1,q}(\mathbb{R}^N)) \cap L^p(\mathbb{R}, W^{2,q}(\mathbb{R}^N))$$

is invertible. This is a result on maximal regularity for the Laplacian on $\mathbb{R}^{N+1}$.

4. Quasilinear parabolic equations

Let X, D be a Banach space such that D is continuously and densely imbedded into X, we write $D \underset{d}{\hookrightarrow} X$. Let $1 < p < \infty$. We consider the maximal regularity space

$$MR_p := W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D)$$

equipped with the norm

$$\|u\|_{MR_p} := \|u\|_{W^{1,p}(\mathbb{R},X)} + \|u\|_{L^p(\mathbb{R},D)} ,$$

and the trace space

$$Tr_p := \{u(0) : u \in MR_p\}$$

which becomes a Banach space for the norm

$$\|x\|_{Tr_p} = \inf\{\|u\|_{MR_p} : u(0) = x\} .$$

Then

$$D \hookrightarrow Tr_p \hookrightarrow X .$$

In fact, by [Lun95, p. 20],

$$Tr_p = (X, D)_{\frac{1}{p'},p}$$

where $\frac{1}{p'} + \frac{1}{p} = 1$.

LEMMA 4.1. *Let $u \in MR_p$. Then $u \in C_0(\mathbb{R}, Tr_p)$ and*

$$\|u(t)\|_{Tr_p} \leq \|u\|_{MR_p} \qquad (t \in \mathbb{R}) . \tag{4.1}$$

Proof. Let $u \in MR_p$. For $t \in \mathbb{R}$ define $v_t \in MR_p$ by $v_t(s) = u(s + t)$. Thus $u(t) = v_t(0) \in Tr_p$ and $\|u(t)\|_{Tr_p} \leq \|v_t\|_{MR_p} = \|u\|_{MR_p}$. Moreover,

$$\|u(t) - u(t_0)\|_{Tr_p} \leq \|v_t - v_{t_0}\|_{MR_p} \to 0$$

as $t \to t_o$ since the translation group is continuous on $L^p(\mathbb{R}, X)$ and $L^p(\mathbb{R}, D)$. It remains to show that $\|u(t)\|_{Tr_p} \to 0$ as $|t| \to \infty$. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a test function such that $\Phi(t) = 1$ for $|t| \leq 1$. Let $\Phi_n(t) = \Phi(t - n)$, $n \in \mathbb{Z}$. Then by the Dominated Convergence Theorem, $\|\Phi_n u\|_{MR_p} \to 0$ as $|n| \to \infty$. Since for $t \in (n - 1, n + 1)$, $\|u(t)\|_{Tr_p} = \|(\Phi_n u)(t)\|_{Tr_p} \leq \|\Phi_n u\|_{MR_p}$ the claim follows. □

For $r > 0$ denote by $\mathcal{U}_r := \{x \in Tr_p : \|x\|_{Tr_p} \leq r\}$ the closed ball of radius r in Tr_p. Let $r_0 > 0$ and

$$A : \mathcal{U}_{r_0} \to \mathcal{L}(D, X)$$

a Lipschitz continuous function, i.e.,

$$\|A(x) - A(y)\|_{\mathcal{L}(D,X)} \leq L\|x - y\|_{Tr_p} \tag{4.2}$$

for all $x, y \in \mathcal{U}_{r_0}$ and some constant $L \geq 0$. We assume that problem (2.1) with $A = A(0)$ satisfies *maximal L^p-regularity*, i.e. for all $g \in L^p(\mathbb{R}, X)$ there is a unique $u \in MR_p$ such that

$$u' = A(0)u + g\ . \tag{4.3}$$

Denote by M the norm of the solution operator $g \in L^p(\mathbb{R}, X) \mapsto u \in MR_p$. Let $F : \mathbb{R} \times \mathcal{U}_{r_0} \to X$ be a continuous function such that

$$\|F(t, x)\|_x \leq h_1(t)\|x\|_{Tr_p} \quad \text{and} \tag{4.4}$$

$$\|F(t, x) - F(t, y)\|_X \leq h_2(t)\|x - y\|_{Tr_p} \tag{4.5}$$

for all $t \in \mathbb{R}, x, y \in \mathcal{U}_{r_0}$ where $h_1, h_2 \in L^p(\mathbb{R})$ such that

$$\|h_1\|_{L^p(\mathbb{R})} < M^{-1}, \|h_2\|_{L^p(\mathbb{R})} < M^{-1}.$$

THEOREM 4.2. *Under these hypotheses there exist a radius $0 < r \leq r_0$ and $\delta > 0$ such that for each $f \in L^p(\mathbb{R}, X)$ with $\|f\|_{L^p(\mathbb{R},X)} \leq \delta$ there exists a unique $u \in MR_p$ with $\|u\|_{MR_p} \leq r$ satisfying*

$$u'(t) = A(u(t))u(t) + F(t, u(t)) + f(t) \qquad a.e. \quad t \in \mathbb{R}\ . \tag{4.6}$$

Proof. Choose $0 < r \leq r_0$ such that $LMr + M\|h_1\|_{L^p} < 1$ and $2LMr + M\|h_2\|_{L^p} < 1$. Let $\delta = r(M^{-1} - Lr - \|h_1\|_{L^p})$. Let $f \in L^p(\mathbb{R}, X)$ such that $\|f\|_{L^p(\mathbb{R},X)} \leq \delta$. Let $v \in MR_p$, $\|v\|_{MR_p} \leq r$. Consider the function

$$g(t) = (A(v(t)) - A(0))v(t) + F(t, v(t)) + f(t)\ .$$

We claim that $g \in L^p(\mathbb{R}, X)$ and $M\|g\|_{L^p(\mathbb{R},X)} \leq r$. In fact, since by Lemma 4.1, $v \in C_0(\mathbb{R}, Tr_p)$, one has $A(v(\cdot)) - A(0) \in C(\mathbb{R}, \mathcal{L}(D, X))$. Thus g is measurable and

$$\|g(t)\|_X \leq L\|v(t)\|_{Tr_p}\|v(t)\|_D + h_1(t)\|v(t)\|_{Tr_p} + \|f(t)\|_X\ .$$

Since by Lemma 4.1, $\|v(t)\|_{Tr_p} \leq \|v\|_{MR_p} \leq r$, it follows that

$$\begin{aligned} M\|g\|_{L^p(\mathbb{R},X)} &\leq M\{Lr\|v\|_{L^p(\mathbb{R},D)} + \|h_1\|_{L^p} \cdot r + \delta\} \\ &\leq M\{Lr^2 + \|h_1\|_{L^p} \cdot r + \delta\} \\ &\leq r. \end{aligned}$$

Denote by $\Phi(v) := u$ the solution of (4.3) for the inhomogeneity g. Then $\|u\|_{MR_p} \leq M\|g\|_{L^p(\mathbb{R},X)} \leq r$. Thus Φ maps the set $\mathcal{C} := \{v \in MR_p \ : \ \|v\|_{MR_p} \leq r\}$ into itself. We show that Φ is a strict contraction. In fact, let $u_1 = \Phi(v_1)$, $u_2 = \Phi(v_2)$, $v_1, v_2 \in \mathcal{C}$. Then $u_1 - u_2$ is the solution of (4.3) for the inhomogeneity

$$\begin{aligned} g(t) &= (A(v_2(t)) - A(0))(v_1(t) - v_2(t)) - (A(v_1(t) - A(v_2(t))v_1(t) \\ &\quad + F(t, v_1(t)) - F(t, v_2(t)) \, . \end{aligned}$$

As before we estimate

$$\begin{aligned} \|u_1 - u_2\|_{MR_p} &\leq M\|g\|_{L^p(\mathbb{R},X)} \\ &\leq M\{Lr\|v_1 - v_2\|_{L^p(\mathbb{R},D)} + L\|v_1 - v_2\|_{MR_p} \cdot \\ &\quad \|v_1\|_{L^p(\mathbb{R},D)} + \|h_2\|_{L^p} \cdot \|v_2 - v_1\|_{MR_p}\} \\ &\leq M\{2Lr + \|h_2\|_{L^p}\}\|v_1 - v_2\|_{MR_p} \, . \end{aligned}$$

Thus Φ is a strict contraction. By Banach's Fixed Point Theorem there exists a unique fixed point $u \in \mathcal{C}$ of Φ which is exactly the claim. □

COROLLARY 4.3. *Assume that* $A : \mathcal{U}_{r_0} \to \mathcal{L}(D, X)$ *is Lipschitz continuous and* $A(0)$ *satisfies maximal* L^p*-regularity on the real line. Then there exist* $0 < r \leq r_0$ *and* $\delta > 0$ *such that for each* $f \in L^p(\mathbb{R}, X)$ *with* $\|f\|_{L^p(\mathbb{R},X)} \leq \delta$ *there exists a unique* $u \in MR_p$ *satisfying*

$$\begin{cases} \|u\|_{MR_p} \leq r \quad \text{and} \\ u'(t) = A(u(t))u(t) + f(t) \qquad \text{a.e. } \ t \in \mathbb{R} \, . \end{cases} \tag{4.7}$$

REMARK 4.4. In the case of initial value problems existence results for quasilinear equations based on maximal regularity have been obtained by Clément and Li [CL93]. They prove that a solution exists on some time interval $[0, T]$ for sufficiently small time T. Here we consider solutions on the entire line and we have to assume that the inhomogeneity f is sufficiently small.

As application we consider a quasilinear heat equation.

Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume that Ω is *contained in a strip*, i.e., there exist $j_0 \in \{1, \cdots N\}$, $c > 0$ such that $|x_{j_0}| \leq c$ for all $x \in \Omega$. Let $1 < p < \infty$. Consider the space

$$MR := W^{1,p}(\mathbb{R}, L^q(\Omega)) \cap L^p(\mathbb{R}, D(\Delta_q)) \, .$$

THEOREM 4.5. *Assume that* $\frac{N}{2q} < 1 - \frac{1}{p}$. *Let* $m : \mathbb{R} \to \mathbb{R}$ *be Lipschitz continuous on bounded sets such that* $m(0) > 0$. *Then there exist* $r > 0, \delta > 0$ *such that for all*

$f \in L^p(\mathbb{R}, L^q(\Omega))$ with $\|f\|_{L^p(\mathbb{R},L^q(\Omega))} \le \delta$ there exists a unique $u \in MR$ satisfying $\|u\|_{MR} \le r$ such that

$$u' = m(u)\Delta_q u + f \quad \text{on} \quad \mathbb{R} \times \Omega\,. \tag{4.8}$$

Here the Dirichlet Laplacian Δ_2 on $L_2(\Omega)$ is defined by

$$\begin{aligned} D(\Delta_2) &:= \{u \in H_0^1(\Omega) : \Delta u \in L_2(\Omega\} \\ \Delta_2 u &= \Delta u\,. \end{aligned}$$

Then $(e^{t\Delta_2})_{t\ge 0}$ is a symmetric submarkovian C_0-semigroup. Hence there exists a consistent extrapolation C_0-semigroup $(e^{t\Delta_q})_{t\ge 0}$ on $L^q(\Omega)$. Since the semigroup $(e^{t\Delta_q})_{t\ge 0}$ is positive and contractive, it follows that $-\Delta_q$ is R-sectorial by [Wei01b, 4d]. In fact the R-sectorial angle is smaller than $\pi/2$, thus $-\Delta_q$ is R-bisectorial in the sense of Definition 2.3. It follows from Poincaré's inequality [DL88, IV § 7 p. 125] that the spectral bound $s(\Delta_2)$ is negative; in particular $0 \in \varrho(\Delta_2)$. Since the semigroup $(e^{t\Delta_2})_{t\ge 0}$ has a Gaussian upper bound, the spectrum of Δ_q is the same as the one of Δ_2, (see [Are94], [Kun99] or [Are04, 7.4.6]). Thus $0 \in \varrho(A_q)$. It follows from Theorem 2.4 that Problem (2.1) is maximal L^p-regular for $A = m(0)\Delta_q$ and hence also for $A_0 = m(0)\Delta_q$. Now let $X = L^q(\Omega)$, $D = D(\Delta_q)$ endowed with the norm

$$\|v\|_D := \|\Delta_q v\|_X\,.$$

Let $Tr_p = (X, D)_{\frac{1}{p'},p}$ be the trace space as above. Since we do not assume any regularity of the boundary of Ω, the domain D of Δ_q is not a Sobolev space, in general. Instead of Sobolev embedding theorems we use ultracontractivity of the semigroup $(e^{t\Delta_q})_{t\ge 0}$ in order to prove the following embedding.

LEMMA 4.6. *If $\frac{N}{2q} < 1 - \frac{1}{p}$, then*

$$Tr_p \hookrightarrow L_\infty(\Omega)\,.$$

Proof. By the embedding properties of real interpolation spaces [Tri78, p. 25] and the relation of domains of fractional powers of sectorial operators and the real interpolation spaces [Tri78, p.101 (3)] we have for $0 < \theta < \frac{1}{p'} = 1 - \frac{1}{p}$,

$$(L_q(\Omega), D)_{\frac{1}{p'},p} \hookrightarrow D((-\Delta_q)^\theta)\,.$$

Now recall that

$$\|e^{-t\Delta_q}\|_{\mathcal{L}(L^q,L^\infty)} \le ct^{-\frac{N}{2}\frac{1}{q}}e^{\omega t} \qquad (t \ge 0)$$

for some $c > 0, \omega < 0$ since $e^{t\Delta_q} \leq G(t)$, where G denotes the Gaussian semigroup on $L^q(\mathbb{R}^N)$, (cf.[Are04, Section 7.3], [Dav89]). Since

$$(-\Delta_q)^{-\theta} = \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-t\Delta_q} dt$$

we deduce that $(-\Delta^q)^{-\theta}(L^q(\Omega)) \subset L^\infty(\Omega)$ if $\theta > \frac{N}{2q}$ (integrability at zero). Since $D((-\Delta_q)^\theta)$ is the range of $(-\Delta_q)^{-\theta}$ we have proved that $Tr_p \subset L^\infty(\Omega)$. It follows from the Closed Graph Theorem that the embedding is continuous. □

Proof of Theorem 4.5. We consider the function $A : Tr_p \to \mathcal{L}(D, X)$ given by $A(v) = m(v)\Delta_q$. We show that A is Lipschitz continuous on bounded sets. Then the claim follows from Corollary 4.3. By Lemma 4.6, there exists $c > 0$ such that

$$\|v\|_{L_\infty(\Omega)} \leq c\|v\|_{Tr_p} \qquad (v \in Tr_p) .$$

Let $r > 0$. There exists $L > 0$ such that $|m(s) - m(t)| \leq L|s - t|$ if $|s|, |t| \leq c \cdot r$. Thus, if $v_1, v_2 \in Tr_p$ such that $\|v_1\|_{Tr_p} \leq r$, $\|v_2\|_{Tr_p} \leq r$, then for $u \in D$

$$\begin{aligned} \|(A(v_1) - A(v_2))u\|_X \leq \|m(v_1) - m(v_2)\|_{L^\infty(\Omega)} \|\Delta_q u\|_{L^q} &\leq \\ L\|v_1 - v_2\|_{L^\infty(\Omega)} \|u\|_D &\leq \\ Lc\|v_1 - v_2\|_{Tr_p} \|u\|_D &. \end{aligned}$$

We have shown that

$$\|A(v_1) - A(v_2)\|_{\mathcal{L}(D,X)} \leq Lc\|v_1 - v_2\|_{Tr_p}$$

whenever $\|v_1\|_{Tr_p} \leq r$, $\|v_2\|_{Tr_p} \leq r$. □

REMARK 4.7. Theorem 4.5 remains valid if we replace $-\Delta_q$ by an elliptic operator A_q with measurable coefficients. Let $a_{ij} \in L^\infty(\Omega)$ such that

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad (\xi \in \mathbb{R}^N)$$

$x - a.e.$, where $\alpha > 0$. Let A_2 be the operator associated with the closed form

$$a(u, v) = \int_\Omega \sum_{i,j=1}^N a_{ij} D_i u D_j v dx$$

with form domain $H_0^1(\Omega)$. Then $-A_2$ generates a holomorphic C_0-semigroup $(e^{-tA_2})_{t\geq 0}$ which allows Gaussian upper bounds (see [Ouh04], [AtE97], [Are04]). Hence the C_0-semigroup extrapolates in a consistent way to C_0-semigroups $(e^{-tA_q})_{t\geq 0}$ on $L^q(\Omega)$ for $1 \leq q < \infty$. If $1 < q < \infty$, then Theorem 4.5 holds for the operator $-A_q$ instead of Δ_q.

Also differential operators with lower order terms with Dirichlet or other boundary conditions may be considered as in [Ouh04], [AtE97], [Are04]. Then Theorem 4.5 remains valid if the 0-th order term is chosen such that the semigroup is exponentially stable. Gaussian estimates are valid if the domain has the extension property and they imply that A_q satisfies maximal L^p-regularity.

5. Quasilinear elliptic equations on the line

In this paragraph we consider a second order quasilinear equation

$$u'' = A(u, u')u + F(t, u, u') + f(t) \qquad (t \in \mathbb{R}) , \tag{5.1}$$

on a Banach space X. A similar problem on an interval with initial values instead of the real line has been studied by Chill and Srivastava [CS05]. Here we obtain existence and uniqueness results on the entire line for a small inhomogeneity f instead of for small time intervals (as in [CS05]).

Let D be a Banach space such that $D \underset{d}{\hookrightarrow} X$. Let $1 < p < \infty$. Define the maximal regularity space

$$MR_p = W^{2,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D)$$

equipped with the norm

$$\|u\|_{MR_p} = \|u\|_{W^{2,p}(\mathbb{R},X)} + \|u\|_{L^p(\mathbb{R},D)}.$$

The trace space

$$Tr_p := \{(u(0), u'(0)) : u \in MR_p\}$$

is a Banach space for the norm

$$\|(x_0, x_1)\|_{Tr_p} := \inf\{\|u\|_{MR_p} : u(0) = x_0, u'(0) = x_1\}.$$

LEMMA 5.1. *Let* $p \in (1, \infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$. *Then*

$$Tr_p \hookrightarrow (X, D)_{\frac{1}{p'},p} \times X. \tag{5.2}$$

Proof. Let $u \in MR_p$. Then in particular, $u \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D)$ and $u' \in W^{1,p}(\mathbb{R}, X)$. The claim now follows from [Lun95, p. 20]. ☐

LEMMA 5.2. *Let* $u \in MR_p$. *Then* $(u, u') \in C_0(\mathbb{R}, Tr_p)$ *and*

$$\|(u(t), u'(t))\|_{Tr_p} \le \|u\|_{MR_p} \qquad (t \in \mathbb{R}). \tag{5.3}$$

Proof. The proof of Lemma 4.1 carries over to this case. □

For $r > 0$ let $\mathcal{U}_r := \{x_0, x_1) \in Tr_p : \|(x_0, x_1)\|_{Tr_p} \le r\}$. Let $r_0 > 0$ and let

$$A : \mathcal{U}_{r_0} \to \mathcal{L}(D, X)$$

be Lipschitz continuous. We suppose that the second order problem (3.1) for $A(0)$ has maximal L^p-regularity, i.e., for all $f \in L^p(\mathbb{R}, X)$ there is a unique $u \in MR_p$ such that

$$u''(t) = A(0)u(t) + f(t) \qquad (t \in \mathbb{R}) .$$

Denote by M the norm of the operator $f \in L^p(\mathbb{R}, X) \mapsto u \in MR_p$. Let $F : \mathbb{R} \times \mathcal{U}_{r_0} \to X$ be continuous such that

$$\|F(t, x_0, x_1)\|_X \le h_1(t)\|(x_0, x_1)\|_{Tr_p} \tag{5.4}$$

$$\|F(t, x_0, x_1) - F(t, y_0, y_1)\|_X \le h_2(t)\|(x_0, x_1) - (y_0, y_1)\|_{Tr_p} \tag{5.5}$$

for all $(x_0, x_1), (y_0, y_1) \in \mathcal{U}_{r_0}, t \in \mathbb{R}$ where $h_1, h_2 \in L^p(\mathbb{R})$ such that

$$\|h_1\|_{L^p} < M^{-1} , \ \|h_2\|_{L^p} < M^{-1} .$$

THEOREM 5.3. *Under the above assumptions there exist $\delta > 0, 0 < r \le r_0$ such that for all $f \in L^p(\mathbb{R}, X)$ with $\|f\|_{L^p(\mathbb{R},X)} \le \delta$ there exists a unique $u \in MR_p$ satisfying $\|u\|_{MR_p} \le r$ such that*

$$u''(t) = A(u(t), u'(t))u(t) + F(t, u(t), u'(t)) + f(t) \qquad a.e. \ \ t \in \mathbb{R} .$$

The proof is analogous to the one of Theorem 4.2.

COROLLARY 5.4. *Let $A : \mathcal{U}_{r_0} \to \mathcal{L}(D, X)$ be Lipschitz continuous. Assume that the second order problem (3.1) for $A(0)$ has maximal L^p-regularity. Then there exist $\delta > 0, 0 < r \le r_0$ such that for each $f \in L^p(\mathbb{R}, X)$ with $\|f\|_{L^p(\mathbb{R},X)} \le \delta$ there exists a unique $u \in MR_p$ of norm $\|u\|_{MR_p} \le r$ satisfying*

$$u'' = A(u(t), u'(t))u(t) + f(t) \quad t \in \mathbb{R} \quad a.e.$$

Applying Corollary 5.4 to the same operator-valued function as in Section 4 we obtain the following. Let $\Omega \subset \mathbb{R}^N$ be open and contained in a strip. Let $1 < p, q < \infty$ such that $\frac{N}{2q} < 1 - \frac{1}{2p}$. Let $m : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous on bounded sets.

THEOREM 5.5. *Under these assumptions there exists $\delta > 0$ such that for all $f \in L^p(\mathbb{R}, L^q(\Omega))$ of norm $\|f\|_{L^p(\mathbb{R},L^q(\Omega))} \le \delta$ there exists a unique*

$$u \in W^{2,p}(\mathbb{R}, L^q(\Omega)) \cap L^p(\mathbb{R}, D(\Delta_q)) \cap C_0(\mathbb{R}, L^\infty(\Omega))$$

such that

$$u_{tt} + m(u)\Delta u = f \quad on \quad \mathbb{R} \times \Omega .$$

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