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# Outgrowths of Hardy's Inequality

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*Dedicated to Rainer Nagel on the occasion of his 65th birthday*

## 1. Introduction and main results

Scaling plays an important role in quantum mechanics and other areas of applied mathematics. When a physicist says that "kinetic energy scales like $\lambda^2$", she means the following. The kinetic energy operator is the negative Laplacian, so consider the Laplacian $\Delta$ as a selfadjoint operator on the Hilbert space $L^2(\mathbb{R}^N)$. For $\lambda > 0$ let $U(\lambda)$ be the normalized scaling operator defined by

$$(U(\lambda) f)(x) = \lambda^{\frac{N}{2}} f(\lambda x), \qquad f \in L^2(\mathbb{R}^N),\ x \in \mathbb{R}^N.$$

Then $U(\lambda)$ is unitary on $L^2(\mathbb{R}^N)$ and $U(\lambda)^{-1} = U\left(\frac{1}{\lambda}\right)$. "The Laplacian scales like $\lambda^2$" means

$$U(\lambda)^{-1} \Delta U(\lambda) = \lambda^2 \Delta$$

holds for all $\lambda > 0$, as is easy to verify.

Now we proceed somewhat formally and do not make precise statements about domains. The operator representing multiplication by $\frac{1}{|x|^p}$ scales like $\lambda^p$, in the sense that for

$$(M_p f)(x) = |x|^{-p} f(x), \qquad f \in L^2(\mathbb{R}^N),\ x \in \mathbb{R}^N,$$

we have

$$U(\lambda)^{-1} M_p U(\lambda) = \lambda^p M_p$$

for all $\lambda > 0$. Consequently

$$U(\lambda)^{-1} \left(\Delta + \frac{c}{|x|^2}\right) U(\lambda) = \lambda^2 \left(\Delta + \frac{c}{|x|^2}\right) \tag{1.1}$$

holds for all $\lambda > 0$ and all $c \in \mathbb{R}$. Now let $A_c^0 = \Delta + \frac{c}{|x|^2}$ on $D(A_c^0) = C_c^\infty(\mathbb{R}^N) = \mathcal{D}(\mathbb{R}^N)$ if $N \geq 5$ and $D(A_c^0) = C_c^\infty(\mathbb{R}^N \setminus \{0\}) = \mathcal{D}(\mathbb{R}^N \setminus \{0\})$ if $N \leq 4$. Let $A_c$

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be a selfadjoint extension of $A_c^0$ for which (1.1) holds on the domain of $A_c$. By (1.1) the spectrum of $A_c$ is preserved by multiplication by a positive number. Also,

$$\left(\left(\Delta + \frac{c}{|x|^2}\right)\varphi \middle| \varphi\right) = -\|\nabla\varphi\|^2 + c\int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^2}\,dx$$

is negative for $\varphi \neq 0$ and $supp(\varphi)$ shifted far away from the origin, thus, $\sigma(A_c) \supset (-\infty, 0]$. Thus $\sigma(A_c) = (-\infty, 0]$ or $\sigma(A_c) = \mathbb{R}$. If $(A_c^0\varphi|\varphi) \leq 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, let $-A_c$ be the Friedrichs extension of $-A_c^0$. Then $U(\lambda)^{-1} A_c U(\lambda) = \lambda^2 A_c$ for $\lambda > 0$ and hence $\sigma(A_c) = (-\infty, 0]$.

Hardy's inequality helps to explain how the spectrum of $A_c$ depends on $c$. The usual way to express Hardy's inequality is

$$\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2}\,dx$$

for all $u \in H^1_{loc}(\mathbb{R}^N)$ for which the right hand side is finite. The constant $\left(\frac{N-2}{2}\right)^2$ is maximal in all dimensions (including $N = 2$). Thus, $A_c$ is nonpositive if and only if $c \leq \left(\frac{N-2}{2}\right)^2$.

Thus the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + \frac{c}{|x|^2}u & x \in \mathbb{R}^N,\ t \geq 0 \\ u(x,0) &= f(x) & x \in \mathbb{R}^N, \end{aligned}$$

is well-posed ( for $f \in L^2(\mathbb{R}^N)$) if and only if $c \leq \left(\frac{N-2}{2}\right)^2$, in which case it is governed by a positive contraction semigroup.

It is natural to ask about $L^p$ versions of this result and about other situations in which scaling plays a role and a critical dimension dependent constant appears. The purpose of this paper is to answer these questions.

Our main result is as follows.

THEOREM 1.1. *Let $\beta \in \mathbb{R}$, $1 < p < \infty$, $c \in \mathbb{R}$ and let*

$$Bu = B_{pc\beta}u := \Delta u - \frac{\beta}{|x|}\frac{\partial u}{\partial r} + \frac{c}{|x|^2}u$$

*with domain $\mathcal{D}_0 = \mathcal{D}(\mathbb{R}^N \setminus \{0\})$ if $N \geq 2$, $\mathcal{D}_0 = \mathcal{D}(0,\infty)$ if $N = 1$, acting on $L^p_\beta = L^p\left(\mathbb{R}^N,\ |x|^{-\beta}\,dx\right)$ if $N \geq 2$ and $L^p_\beta = L^p\left((0,\infty),\ |x|^{-\beta}\,dx\right)$ if $N = 1$; here $\frac{\partial u}{\partial r} = \nabla \cdot \frac{x}{|x|}$ is the radial derivative. Then a suitable extension of $B_{pc\beta}$ generates a $(C_0)$ contraction semigroup on $L^p_\beta$ if*

$$c \leq (N-\beta-2)^2\left(\frac{p-1}{p^2}\right) = \left(\frac{N-\beta-2}{2}\right)^2 \frac{4}{pp'} =: K(N,p,\beta). \tag{1.2}$$

*If $2 \leq p < \infty$ and $c > K(N,p,\beta)$, then $B_{pc\beta}$ has no quasidissipative extension on $L^p_\beta$.*

Specializing Theorem 1.1 to $\beta = 0$ yields

COROLLARY 1.2. *Let* $1 < p < \infty$. *Then*

$$A = \Delta + \frac{c}{|x|^2}$$

*on* $\mathcal{D}_0 = \mathcal{D}(\mathbb{R}^N \setminus \{0\})$ *if* $N \geq 2$, $\mathcal{D}_0 = \mathcal{D}(0, \infty)$ *if* $N = 1$, *has an m-dissipative extension on* $L^p$ *if*

$$c \leq (N-2)^2 \left(\frac{p-1}{p^2}\right) =: K(N,p); \tag{1.3}$$

$L^p = L^p(\mathbb{R}^N)$ *if* $N \geq 2$, $L^p = L^p(0, \infty)$ *if* $N = 1$. *For* $c > K(N,p)$,

$$\sup\left\{\operatorname{Re}\langle Au, J_p(u)\rangle : u \in \mathcal{D}(A), \|u\|_p = 1\right\} = \infty$$

*and so A has no quasidississipative extension, provided* $2 \leq p$.

This follows from Theorem 1.1 by taking $\beta = 0$; here $K(N,p) = K(N,p,0)$ and $J_p$ is the duality map of $L^p$. Note that

$$K(N,2,\beta) = \left(\frac{N-2-\beta}{2}\right)^2.$$

Throughout this paper we consider real vector spaces.

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## 2. Hardy's inequality and the inverse square potential

The aim of this section is to study the semigroup generated by $\Delta + \frac{c}{|x|^2}$ for suitable $c$. We let $\mathcal{D}_0 = \mathcal{D}(\mathbb{R}^N \setminus \{0\})$.

THEOREM 2.1. *(Hardy's Inequality)*
*One has*

$$\left(\frac{N-2}{2}\right)^2 \int \frac{|u|^2}{|x|^2}\, dx \leq \int |\nabla u|^2\, dx$$

*for all* $u \in \mathcal{D}_0$. *The constant* $\left(\frac{N-2}{2}\right)^2$ *is optimal.*

Here the integral is taken over $\mathbb{R}^N$ if $N \geq 2$ and over $(0, \infty)$ if $N = 1$.

Theorem 2.1 is well-known. We give a proof of the inequality for the convenience of the reader.

PROOF. Let $u \in \mathcal{D}_0$, $u$ real.
a) Let $N \geq 3$. Then

$$\begin{aligned} u(x)^2 &= -\int_1^\infty \frac{d}{d\lambda} u(\lambda x)^2\, d\lambda \\ &= -2\int_1^\infty u(\lambda x)\ \nabla u(\lambda x) \cdot x\, d\lambda. \end{aligned}$$

Hence

$$\begin{aligned}\int \frac{|u|^2}{|x|^2}\,dx &= -2\int\int_1^\infty \frac{u(\lambda x)}{|x|^2}\nabla u(\lambda x)\cdot x\,d\lambda\,dx \\ &= -2\int_1^\infty \int \frac{\lambda^2 u(y)}{|y|^2}\nabla u(y)\cdot\frac{y}{\lambda}\frac{dy}{\lambda^N}\,d\lambda \\ &= -2\int\int_1^\infty \lambda^{-N+1}\,d\lambda\,\frac{u(y)}{|y|}\nabla u(y)\cdot\frac{y}{|y|}\,dy \\ &= \frac{2}{-N+2}\int\frac{u(y)}{|y|}\nabla u(y)\cdot\frac{y}{|y|}\,dy \\ &\le \frac{2}{N-2}\left(\int\frac{u(y)^2}{|y|^2}\,dy\right)^{\frac12}\left(\int|\nabla u(y)|^2\,dy\right)^{\frac12}.\end{aligned}$$

Hence,

$$\left(\int\frac{u(x)^2}{|x|^2}\,dx\right)^{\frac12}\le\left(\frac{2}{N-2}\right)\left(\int|\nabla u(x)|^2\,dx\right)^{\frac12}.$$

b) For N=1, $u\in\mathcal{D}_0$,

$$u(x)^2=\int_0^1\frac{d}{d\lambda}u(\lambda x)^2\,d\lambda.$$

Now use the same arguments as in part a). ∎

For N=1, $\mathcal{D}_0$ is dense in $H_0^1(\mathbb{R}\setminus\{0\})=\{u\in H^1(\mathbb{R}):u(0)=0\}$, which makes sense since $H_0^1(\mathbb{R})\subset C(\mathbb{R})$. By considering even functions in $H_0^1(\mathbb{R}\setminus\{0\})$, we can write the one dimensional Hardy inequality as

$$\int_0^\infty (u')^2\,dx\ge\frac14\int_0^\infty\frac{u^2}{x^2}\,dx$$

for all $u\in H_0^1:=H_0^1(0,\infty)$.

From now on, integrals (as in (2.1)) will be over $\mathbb{R}^N$ if $N\ge2$ and over $(0,\infty)$ if $N=1$. And for $N=1$, $\mathcal{D}_0$ will henceforth denote $\mathcal{D}(0,\infty)$.

If $N=1$, then $\mathcal{D}_0$ is dense in $H_0^1$ which makes sense, since $H_0^1(0,\infty)\subset C^b[0,\infty)$. If $N\ge2$, then $\mathcal{D}_0$ is dense in $H^1(\mathbb{R}^N)$ (cf. [3, Lemma 2.4]). Consequently, the inequality remains true for all $u\in H^1(\mathbb{R}^N)$ if $N\ge3$ and for all $u\in H_0^1(0,\infty)$ if $N=1$.

We now sketch why the constant in Hardy's inequality is optimal. Let $x\in\mathbb{R}^N$ ($x>0$ if $N=1$) and let $r=|x|$. For $\varepsilon>0$, $a>0$, $b\ge0$, let

$$\phi(r)=\begin{cases}\varepsilon^{-a-b}r^b & if\ \ 0\le r\le\varepsilon\\ r^{-a} & if\ \ \varepsilon\le r\le1\\ 2-r & if\ \ 1\le r\le2\\ 0 & if\ \ 2\le r.\end{cases}$$

Then, letting $\psi(x)=\phi(r)$ and using $|\nabla\psi|^2=(\phi')^2$, we can show that, given $\delta>0$,

$$\int|\nabla\psi|^2<\left[\left(\frac{N-2}{2}\right)^2+\delta\right]\int\frac{\psi^2}{|x|^2}$$

provided that $\varepsilon$, $a$, and $b$ are suitably chosen. (We may choose $b = 0$ for $N \geq 3$, but $b > 0$ is necessary for $N = 1, 2$.) In fact, given $c > \left(\frac{N-2}{2}\right)^2$ and $M > 0$, we may choose $\varepsilon$, $a$, $b$ so that

$$\frac{\int |\nabla \psi|^2 - c \int \frac{|\psi|^2}{|x|^2}}{\int \psi^2} < -M.$$

To see how to choose $\varepsilon$, $a$, $b$ in this way ( in a much more general setting), see [10].

In the following we use the notion of positive linear forms. Given a real Hilbert space $H$, a positive form on $H$ is a bilinear mapping $a : D(a) \times D(a) \to \mathbb{R}$ such that

$$a(x, y) = a(y, x) \qquad x, y \in D(a)$$

and

$$a(x, x) \geq 0 \qquad x \in D(a).$$

Here $D(a)$ is a subspace of $H$, the *domain* of the form $a$. The form is called *closed* if $D(a)$ is complete for the norm $\|u\|_a = \left(\|u\|_H^2 + a(u, u)\right)^{\frac{1}{2}}$. The form is called *closable* if the continuous extension of the injection $D(a) \to H$ to the completion $D(\overline{a})$ of $D(a)$ is injective. In that case $a$ has a continuous extension $\overline{a} : D(\overline{a}) \times D(\overline{a}) \to \mathbb{R}$ that is a positive closed form.

If $a$ is a closed densely defined positive form on $H$, the associated operator $A$ on $H$ is defined by

$$D(A) = \{u \in D(a) : \text{there is a } v \in H \text{ such that}$$

$$a(u, \varphi) = (v | \varphi)_H \text{ for all } \varphi \in D(a)\},$$

$$Au = v.$$

The operator $A$ is selfadjoint and form positive. Thus -$A$ generates a $(C_0)$ contraction semigroup $\left(e^{-tA}\right)_{t \geq 0}$ on $H$.

In the following we let $L^2 := L^2\left(\mathbb{R}^N\right)$ if $N \geq 2$ and $L^2 = L^2(0, \infty)$ if $N = 1$.

If $H = L^2$, then the semigroup $e^{-tA}$ is *submarkovian* (i.e. $0 \leq f \leq 1$ implies $0 \leq e^{-tA} f \leq 1$) if and only if $u \in D(a)$ implies $u \wedge 1 \in D(a)$ and $a\left(u \wedge 1, (u-1)^+\right) \geq 0$. In that case $\left\|e^{-tA} f\right\|_p \leq \|f\|_p$ for all $f \in L^p \cap L^2$. Now for $c \in \mathbb{R}$, consider the positive form $a_c$ on $L^2$ given by

$$a_c(u, v) = \int \nabla u \cdot \nabla v \, dx - c \int \frac{u^2}{|x|^2} dx,$$

$D(a) = H_0^1$, where $H_0^1 := H_0^1\left(\mathbb{R}^N\right)$ if $N \geq 2$ and $H_0^1 := H_0^1(0, \infty)$ if $N = 1$. We investigate the question: "For which real $c$ is the form $a_c$ closed or closable"?

PROPOSITION 2.2. *Let $c < \left(\frac{N-2}{2}\right)^2$. Then the form $a_c$ given by $D(a_c) = H_0^1$,*

$$a_c(u, v) = \int \nabla u \cdot \nabla v \, dx - c \int \frac{uv}{|x|^2} dx,$$

*is positive and closed.*

PROOF. Let $q = c\left(\frac{N-2}{2}\right)^{-2} < 1$. Then by Hardy's inequality

$$\begin{aligned} a_c(u) &= q\left\{\int |\nabla u|^2\,dx - \left(\frac{N-2}{2}\right)^2 \int \frac{u^2}{|x|^2}\,dx\right\} + (1-q)\int |\nabla u|^2\,dx \\ &\geq (1-q)\int |\nabla u|^2\,dx. \end{aligned}$$

Thus, $a_c(u,u) + \|u\|_{L^2}^2 \geq (1-q)\|u\|_{H^1}^2$. This shows that $\|\cdot\|_{a_c}$ is equivalent to $\|\cdot\|_{H^1}$. ∎

For the critical constant we have the following result.

PROPOSITION 2.3. *For $c_N = \left(\frac{N-2}{2}\right)^2$, $N \neq 2$, the form $a_{c_N}$ is closable.*

PROOF. Define the symmetric operator $B$ on $L^2$ by $D(B) = \mathcal{D}_0$,

$$Bu = \Delta u + c_N \frac{u}{|x|^2}.$$

It is well known that the form $b$ given by $b(u,v) = -(Bu|v)_{L^2}$, $D(b) = D(B)$ is closable (and the operator associated with $\overline{b}$ is called the Friedrichs extension of $b$). Observe that $b(u,v) = a_{c_N}(u,v)$ for $u, v \in \mathcal{D}_0$. Since $\mathcal{D}_0$ is dense in $H_0^1$, it follows that $H_0^1 \subset D(\overline{b})$ and $a_{c_N} = \overline{b}$ on $H_0^1 \times H_0^1$. This shows that $a_{c_N}$ is closable and $\overline{a_{c_N}} = b$. ∎

We next show that $a_{c_N}$ is not closed. For simplicity we consider only the case $N = 3$. Then the critical constant is $c_3 = \frac{1}{4}$. We show that

$$D\left(\overline{a_{\frac{1}{4}}}\right) \not\subset L^6(\mathbb{R}^3). \tag{2.2}$$

On the other hand, by Sobolev embedding, $H_0^1 \subset L^6$. Consequently $H_0^1$ is a proper subspace of $D\left(\overline{a_{\frac{1}{4}}}\right)$.

PROPOSITION 2.4. *The form $a_{\frac{1}{4}}$ is not closed and (2.2) holds.*

PROOF. Let $\eta \in \mathcal{D}(\mathbb{R}^3)$ be a test function such that $0 \leq \eta \leq 1$ and $\eta = 1$ in a neighborhood of 0. Let $u(x) = |x|^{-\frac{1}{2}}\eta$. Then $u \in L^2(\mathbb{R}^3)$, but $u \notin L^6(\mathbb{R}^3)$. Let $u_\varepsilon(x) = |x|^{-\frac{1}{2}+\varepsilon}\eta$. Then $u_\varepsilon \in H_0^1$ and $u_\epsilon \longrightarrow u$ in $L^2$.

Since $a_{\frac{1}{4}}$ is closable, it suffices to show that $a_{\frac{1}{4}}(u_{\varepsilon_1} - u_{\varepsilon_2}) \longrightarrow 0$ as $\varepsilon_1$, $\varepsilon_2 \longrightarrow 0$. Let $\delta > 0$ such that $\eta(x) = 1$ for $|x| \leq \delta$.

It suffices to show that

$$c(\varepsilon_1, \varepsilon_2) := \int_{|x|\leq\delta} \left\{|\nabla(u_{\varepsilon_1} - u_{\varepsilon_2})|^2 - \frac{1}{4}\frac{(u_{\varepsilon_1} - u_{\varepsilon_2})^2}{|x|^2}\right\} dx \longrightarrow 0$$

as $\varepsilon_1, \varepsilon_2 \longrightarrow 0+$. We compute

$$\begin{aligned}
4c(\varepsilon_1,\varepsilon_2) \quad : \quad &= \int_{|x|\le\delta} 4\left[\left(-\frac{1}{2}+\varepsilon_1\right)|x|^{\varepsilon_1} - \left(-\frac{1}{2}+\varepsilon_2\right)|x|^{\varepsilon_2}\right]^2 |x|^{-3}\, dx \\
&\quad - \int_{|x|\le\delta} 4\left(|x|^{2\varepsilon_1} - 2|x|^{\varepsilon_1+\varepsilon_2} + |x|^{2\varepsilon_2}\right)|x|^{-3}\, dx \\
&= \int_{|x|\le\delta} \left\{\left(-\varepsilon_1+\varepsilon_1^2\right)|x|^{2\varepsilon_1} + (\varepsilon_1+\varepsilon_2)|x|^{\varepsilon_1+\varepsilon_2}\right\}|x|^{-3}\, dx \\
&\quad + \int_{|x|\le\delta} \left\{-2\varepsilon_1\varepsilon_2|x|^{\varepsilon_1+\varepsilon_2} + \left(-\varepsilon_2+\varepsilon_2^2\right)|x|^{2\varepsilon_2}\right\}|x|^{-3}\, dx \\
&= \int_0^\delta \left\{\left(-\varepsilon_1+\varepsilon_1^2\right)r^{2\varepsilon_1-1} + (\varepsilon_1+\varepsilon_2)r^{\varepsilon_1+\varepsilon_2-1}\right\} dr \\
&\quad + \int_0^\delta \left\{-2\varepsilon_1\varepsilon_2 r^{\varepsilon_1+\varepsilon_2-1} + \left(-\varepsilon_2+\varepsilon_2^2\right)r^{2\varepsilon_2-1}\right\} dr \\
&= \left(-\varepsilon_1+\varepsilon_1^2\right)\frac{\delta^{2\varepsilon_1}}{2\varepsilon_1} + \delta^{\varepsilon_1+\varepsilon_2} - 2\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}\delta^{\varepsilon_1+\varepsilon_2} + \left(-\varepsilon_2+\varepsilon_2^2\right)\frac{\delta^{2\varepsilon_2}}{2\varepsilon_2} \\
&= -\frac{1}{2}\delta^{2\varepsilon_1} + \frac{\varepsilon_1}{2}\delta^{2\varepsilon_1} + \delta^{\varepsilon_1+\varepsilon_2} - 2\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}\delta^{\varepsilon_1+\varepsilon_2} - \frac{1}{2}\delta^{2\varepsilon_2} + \frac{\varepsilon_2}{2}\delta^{2\varepsilon_2} \\
&\longrightarrow 0
\end{aligned}$$

as $\varepsilon_1 \downarrow 0$, $\varepsilon_2 \downarrow 0$. ∎

Next we consider the associated operators and semigroups. In particular, we will show that the critical constant $c_N = \left(\frac{N-2}{2}\right)^2$ is also optimal to associate a semigroup to the operator $\Delta + \frac{c}{|x|^2}$ in a reasonable way.

We denote by $-A_c$ the operator associated with $a_c$ for $c < \left(\frac{N-2}{2}\right)^2$ and with $\overline{a_c}$ for $c = \left(\frac{N-2}{2}\right)^2$. Then $A_c$ is given by

$$A_c u = \Delta u + \frac{c}{|x|^2} u \tag{2.3}$$

on

$$D(A_c) = \left\{u \in H_0^1 : \Delta u + \frac{c}{|x|^2} u \in L^2\right\}$$

in case $c < \left(\frac{N-2}{2}\right)^2$, and

$$D(A_c) = \left\{u \in D\left(\overline{a_{c_N}}\right) : \Delta u + \frac{c}{|x|^2} u \in L^2\right\}$$

in the critical case. Here $\Delta u + \frac{c}{|x|^2} u$ is understood in the sense of $\mathcal{D}_0'$. This follows immediately from the definition of the associated operator.

The semigroup $e^{tA_c}$ may be described as the limit of the semigroups defined by the cut-off potentials. Denote by $\Delta_p$ the generator of the Gaussian semigroup on $L^p := L^p\left(\mathbb{R}^N\right)$ for $N \geq 2$ and the Laplacian with Dirichlet boundary conditions on $L^p := L^p(0,\infty)$ if $N = 1$, $1 \leq p < \infty$ (cf. [8,Theorem 1.4.1]). Then $-\Delta_2$ is associated with $a_0$. Let $A_{c,k} = \Delta_2 + c\left(\frac{1}{x^2} \wedge k\right)$ for $k \in \mathbb{N}$. Then

$$0 \leq e^{tA_{c,k}} \leq e^{tA_{c,k+1}}. \tag{2.4}$$

If $c \leq \left(\frac{N-2}{2}\right)^2$, then it follows from the monotone convergence theorem for forms [18, S14] that

$$\lim_{k \longrightarrow \infty} e^{tA_{c,k}} = e^{tA_c} \qquad \text{strongly in } L^2. \tag{2.5}$$

PROPOSITION 2.5. *If $c > \left(\frac{N-2}{2}\right)^2$, then*

$$\lim_{k \longrightarrow \infty} \left\| e^{tA_{c,k}} \right\|_{\mathcal{L}(L^2)} = \infty \qquad \text{for } t > 0.$$

PROOF. For a selfadjoint operator $A$ on a Hilbert space $H$ one has $\left\| e^{tA} \right\|_{\mathcal{L}(H)} = e^{ts(A)}$ where $s(A) = \sup\{(Au|u)_H : u \in D(A), \ \|u\|_H = 1\}$. This follows from the spectral theorem. In our situation

$$s(A_{c,k}) = \sup\left\{ -\int |\nabla u|^2 \, dx + c \int |u|^2 \left( \frac{1}{|x|^2} \wedge k \right) dx : u \in D(\Delta_2), \ \|u\|_{L^2} = 1 \right\}.$$

It follows from the optimality of the constant $c_N$ in Hardy's inequality that $s(A_{c,k}) \longrightarrow \infty$ as $k \longrightarrow \infty$ if $c > c_N$. ∎

Another way of looking at semigroups associated with the operator $\Delta + \frac{c}{|x|^2}$ is to consider the minimal operator $A_{c,\min}$ on $L^2$ given by

$$\begin{aligned} A_{c,\min}(u) &= \Delta u + \frac{c}{|x|^2} u \\ D(A_{c,\min}) &= \mathcal{D}_0. \end{aligned}$$

THEOREM 2.6. *Let $N \geq 5$.*

*a) Let $c \leq \left(\frac{N-2}{2}\right)^2$. Then $A_c$ generates a positive $(C_0)$ semigroup on $L^2$ which is minimal among all positive $(C_0)$ semigroups generated by an extension of $A_{c,\min}$.*

*b) If $c > \left(\frac{N-2}{2}\right)^2$, then no extension of $A_{c,\min}$ generates a positive $(C_0)$ semigroup on $L^2$.*

For the proof we need the following.

LEMMA 2.7. *Let $N \geq 5$. Then $\mathcal{D}_0$ is a core of $\Delta_2$ (the Laplacian on $L^2(\mathbb{R}^N)$).*

PROOF. It is well-known that $\mathcal{D}(\mathbb{R}^N)$ is a core. Let $u \in \mathcal{D}(\mathbb{R}^N)$. Let $\eta \in C^1(\mathbb{R})$ be such that $0 \leq \eta \leq 1$, $\eta(r) = 1$ for $|r| \geq 2$, $\eta(r) = 0$ for $r \leq 1$, and let $\eta_k(x) = \eta(k|x|)$. It is easy to see that $\eta_k u \longrightarrow u$ and $\Delta(\eta_k u) \longrightarrow \Delta u$ in $L^2(\mathbb{R}^N)$. ∎

PROOF OF THEOREM 2.6. Let $B$ be the generator of a positive $(C_0)$ semigroup $e^{tB}$ such that $A_{c,\min} \subset B$. Consider the semigroup $T_k(t) = e^{t\left(B - c\left(\frac{1}{|x|^2} \wedge k\right)\right)}$ for $c > 0$. Then $0 \leq T_{k+1}(t) \leq T_k(t)$. It follows from [2] or [22] that $\lim_{k \to \infty} T_k(t) = T_\infty(t)$ exists strongly and defines a $(C_0)$ semigroup whose generator we call $B_\infty$. Let $v \in \mathcal{D}_0$. Then there exists $k_0$ such that $B_k v = \Delta v$ for all $k \geq k_0$. Since

$$T_k(t) v - v = \int_0^t T_k(s) B v \, ds,$$

letting $k \longrightarrow \infty$ we see that

$$T_\infty(t) v - v = \int_0^t T_\infty(s) B v \, ds.$$

Thus, $v \in D(B_\infty)$ and $B_\infty v = \Delta v$. Since $N \geq 5$, $\mathcal{D}_0$ is dense in $H^2(\mathbb{R}^N) = D(\Delta_2)$. Thus $\Delta_2$ coincides with $B_\infty$ on a core. Consequently, $\Delta_2 = B_\infty$. Thus $e^{t\Delta_2} \leq e^{t\left(B - c\left(\frac{1}{|x|^2}\wedge k\right)\right)}$ for all $k$. It follows that $e^{tA_{c,k}} \leq e^{tB}$ for all $k$. Now Proposition 2.5 implies that $c \leq \left(\frac{N-2}{2}\right)^2$, and from (2.5) it follows that $e^{tA_c} \leq e^{tB}$ for $t \geq 0$. ∎

By a result of Kalf, Schmincke, Walter and Wüst [12], $\mathcal{D}_0$ is a core of $A_c$ iff $c \leq \left(\frac{N-2}{2}\right)^2 - 1$; see also [18] for dimension $N = 5$.

## 3. The Caffarelli-Kohn-Nirenberg inequality and associated semigroups

Let $\beta \in \mathbb{R}$, $L^2_\beta := L^2\left(\mathbb{R}^N, \ |x|^{-\beta} \ dx\right)$ if $N \geq 2$ and $L^2_\beta := L^2\left((0,\infty), \ |x|^{-\beta} \ dx\right)$ if $N = 1$. Note that $L^2_\beta \subseteq L^2_{loc}\left(\mathbb{R}^N \setminus \{0\}\right)$ if $N \geq 2$ and $L^2_\beta \subseteq L^2_{loc}(0,\infty)$ if $N = 1$. Thus, $L^2_\beta \subset \mathcal{D}_0'$ where $\mathcal{D}_0 = \mathcal{D}\left(\mathbb{R}^N \setminus \{0\}\right)$ if $N \geq 2$ and $\mathcal{D}_0 = \mathcal{D}(0,\infty)$ if $N = 1$. We let $H^1_{00} := \left\{u \in H^1\left(\mathbb{R}^N\right) : \text{supp } u \text{ is compact and } 0 \notin \text{supp } u\right\}$ if $N \geq 2$ and $H^1_{00} := \left\{u \in H^1(0,\infty) : \text{supp } u \text{ is compact, supp } u \subset (0,\infty)\right\}$ if $N = 1$.

Consider the unitary operator $U : L^2_\beta \longrightarrow L^2_0$, $u \longmapsto |x|^{-\frac{\beta}{2}} u$. It maps $H^1_{00}$ onto $H^1_{00}$. Consider the Dirichlet form

$$a(u,v) = \int \nabla u \cdot \nabla v \, dx$$

on $L^2_0$ with domain $H^1_{00}$. We transport the Dirichlet form by $U$ to the space $L^2_\beta$ by defining

$$\begin{aligned} b(u,v) &= \int \nabla\left(|x|^{-\frac{\beta}{2}} u\right) \cdot \nabla\left(|x|^{-\frac{\beta}{2}} v\right) dx \\ &= a(Uu, Uv) \end{aligned}$$

for $u, v \in H^1_{00}$. Then we have

$$\begin{aligned} b(u,v) = &\int \nabla u \cdot \nabla v \, |x|^{-\beta} \ dx \\ &- \left\{\left(\frac{N-2-\beta}{2}\right)^2 - \left(\frac{N-2}{2}\right)^2\right\} \int \frac{uv}{|x|^2} |x|^{-\beta} \ dx \end{aligned} \tag{3.1}$$

for all $u, v \in H^1_{00}$.

PROOF OF (3.1). Let $u$, $v \in H^1_{00}$. Then

$$\begin{aligned} &b(u,v) \\ = &\int \nabla\left(|x|^{-\frac{\beta}{2}} u\right) \cdot \nabla\left(|x|^{-\frac{\beta}{2}} v\right) dx \\ = &\int \left(-\frac{\beta}{2} |x|^{-\left(\frac{\beta}{2}\right)-1} \frac{x}{|x|} u + |x|^{-\frac{\beta}{2}} \nabla u\right) \cdot \left(-\frac{\beta}{2} |x|^{-\left(\frac{\beta}{2}\right)-1} \frac{x}{|x|} v + |x|^{-\frac{\beta}{2}} \nabla v\right) dx \\ = &\int \nabla u \cdot \nabla v \, |x|^{-\beta} dx + \frac{\beta^2}{4} \int uv \, |x|^{-\beta-2} \ dx - \frac{\beta}{2} \int |x|^{-\beta-2} x \cdot \nabla(uv) \ dx \\ = &\int \nabla u \cdot \nabla v \, |x|^{-\beta} dx - \left\{\left(\frac{N-2-\beta}{2}\right)^2 - \left(\frac{N-2}{2}\right)^2\right\} \int \frac{uv}{|x|^2} |x|^{-\beta} \ dx \end{aligned}$$

since by integration by parts,

$$-\frac{\beta}{2}\int |x|^{-\beta-2}\, x\cdot\nabla (uv)\ dx = -\frac{\beta}{2}(\beta+2)\int |x|^{-\beta-2}\, uv\ dx + \frac{\beta}{2}N\int \frac{uv}{|x|^2}|x|^{-\beta}\ dx.$$

∎

Now by Hardy's inequality,

$$\int |\nabla u|^2 |x|^{-\beta}\ dx - \left(\frac{N-2-\beta}{2}\right)^2 \int \frac{u^2}{|x|^2}|x|^{-\beta}\ dx$$

$$= a\left(|x|^{-\frac{\beta}{2}}u, |x|^{-\frac{\beta}{2}}u\right) - \left(\frac{N-2}{2}\right)^2 \int \frac{\left(|x|^{-\frac{\beta}{2}}u\right)^2}{|x|^2}\ dx \geq 0.$$

Thus, we have proved the following.

THEOREM 3.1. *One has*

$$\text{(CKN)}\qquad \left(\frac{N-2-\beta}{2}\right)^2 \int \frac{u^2}{|x|^2}|x|^{-\beta}\ dx \leq \int |\nabla u|^2 |x|^{-\beta}\ dx$$

*for all $u, v \in H^1_{00}$, with optimal constant.*

This is the Caffarelli-Kohn-Nirenberg inequality [7] which we have deduced from Hardy's inequality by similarity. Note that the case of $N = 2$ is included here.

Now for $c \in \mathbb{R}$ we consider the form $b_c$ on $L^2_\beta$ given by

$$b_c(u,v) = \int \nabla u\cdot\nabla v\ |x|^{-\beta}\ dx - c\int uv\ |x|^{-\beta}\ dx$$

with domain $D(b_c) = H^1_{00}$. Then by Theorem 3.1 the form $b_c$ is positive iff $c \leq \left(\frac{N-2-\beta}{2}\right)^2$.

THEOREM 3.2. *Let $\beta \in \mathbb{R}$ and let $c \leq \left(\frac{N-2-\beta}{2}\right)^2$. Then the form $b_c$ is closable and $\mathcal{D}_0$ is a form core. Let $-B_c$ be the operator on $L^2_\beta$ associated with $\overline{b_c}$. Then $\mathcal{D}_0 \subset D(B_c)$ and*

$$B_c u = \Delta u - \frac{\beta}{|x|}\frac{\partial u}{\partial r} + c\frac{u}{|x|^2} \tag{3.2}$$

*for all $u \in \mathcal{D}_0$. The operator $B_c$ generates a positive $(C_0)$ semigroup on $L^2_\beta$ which is submarkovian iff $c \leq 0$.*

PROOF. Letting $\delta = -\left(\frac{N-2}{2}\right)^2 + \left(\frac{N-2-\beta}{2}\right)^2$ we have, by (3.1), $b_\delta = b$ and more generally,

$$a_{c-\delta}(Uu, Uv) = b_c(u,v) \tag{3.3}$$

for all $u, v \in H^1_{00}$. Since $U\mathcal{D}_0 = \mathcal{D}_0$, it follows from the results of Section 2 that $b_c$ is closable and $\mathcal{D}_0$ is a form core. It follows from (3.3) that

$$B_c = U^{-1}A_{c-\delta}U$$

and so

$$e^{tB_c} = U^{-1}e^{tA_{c-\delta}}U$$

for $t \geq 0$. In particular, $e^{tB_c} \geq 0$.

Let $u \in \mathcal{D}_0$, $v = \Delta u - \frac{\beta}{|x|}\frac{\partial u}{\partial r} + c\frac{u}{|x|^2}$. Integration by parts shows that

$$b_c(u,\varphi) = -\int v\varphi\,|x|^{-\beta}\,dx$$

for all $\varphi \in \mathcal{D}_0$. Hence, $u \in B_c$ and $B_c u = v$.

Finally we show that $e^{tB_c}$ is submarkovian iff $c \leq 0$. Let $u \in H^1_{00}$. It follows from [9, p.152] that $(u \wedge 1)\, sgn\ u \in H^1_{00}$ and

$$D_j\left(\left(|u| \wedge 1\right) sgn\ u\right) = D_j u\ 1|_{\{|u|<1\}}$$

where $sgn\ u = \frac{u}{|u|}\ 1_{\{u\neq 0\}}$. Thus

$$\begin{aligned} & b_0\left((1 \wedge |u|)\, sgn\ u,\ (1 \wedge |u|)\, sgn\ u\right) \\ = & \int |\nabla u|^2\, 1|_{\{|u|<1\}}\,|x|^{-\beta}\,dx \leq b_0(u,u). \end{aligned}$$

It follows from [17, Theorem 2.6, Theorem 2.14] that $\left(e^{tB_0}\right)_{t\geq 0}$ is submarkovian.

If $c > 0$, then the semigroup $e^{tB_c}$ is not submarkovian. In fact, let $u \in \mathcal{D}_0$ be such that $(u-1)^+ \neq 0$. Then $D_j(u-1)^+ = D_j u\ 1_{\{u>1\}}$ and $D_j(u \wedge 1) = D_j u \cdot 1_{\{u<1\}}$. Thus

$$\begin{aligned} b_c\left(u \wedge 1,\ (u-1)^+\right) &= \int \nabla(u \wedge 1) \cdot \nabla\left((u-1)^+\right) \\ & \quad - c\int \frac{(u \wedge 1)(u-1)^+}{|x|^2}\,|x|^{-\beta}\,dx \\ &= -c\int \frac{(u-1)^+}{|x|^2}\,|x|^{-\beta}\,dx < 0. \end{aligned}$$

Thus $e^{tB_c}$ is not submarkovian by [17, Corollary 2.17]. ∎

Next we want to describe the domain of $\overline{b_c}$. We denote by

$$H^1_\beta := \left\{u \in L^2_\beta : D_j u \in L^2_\beta,\ j = 1, ..., N\right\}$$

the weighted Sobolev space where $D_j u$ is understood in the sense of $\mathcal{D}_0'$. Then $H^1_\beta$ is a Hilbert space for the scalar product

$$(u|v)_{H^1_\beta} := (u|v)_{L^2_\beta} + \int \nabla u \cdot \nabla v\ |x|^{-\beta}\ dx.$$

$D\left(\overline{b_c}\right)$ is the completion of $\mathcal{D}_0$ with respect to the norm

$$\left(\|u\|^2_{L^2_\beta} + b(u,u)\right)^{\frac{1}{2}} = \|u\|_{H^1_\beta}.$$

Thus, $D\left(\overline{b_c}\right)$ is the closure $H^1_{\beta 0}$ of $\mathcal{D}_0$ in $H^1_\beta$ . It follows from Fatou's Lemma that (CKN) remains true for all $u \in H^1_{\beta 0}$. As in the proof of Proposition 2.2, one deduces that for $0 < c < \left(\frac{N-2-\beta}{2}\right)^2$, one has $D\left(\overline{b_c}\right) = H^1_{\beta 0}$ and

$$\overline{b_c}(u,v) = \int \nabla u \cdot \nabla v\ |x|^{-\beta}\ dx - c\int \frac{|u|^2}{|x|^2}\,|x|^{-\beta}\ dx.$$

We summarize these facts as

PROPOSITION 3.3. *One has* $D\left(\overline{b}\right) = H^1_{\beta 0}$ *and*

$$\text{(CKN)}\qquad \left(\frac{N-2-\beta}{2}\right)^2 \int \frac{u^2}{|x|^2} |x|^{-\beta}\, dx \le \int |\nabla u|^2 |x|^{-\beta}\, dx$$

*for all* $u \in H^1_{\beta 0}$. *If* $0 \le c \le \left(\frac{N-2-\beta}{2}\right)^2$, *then* $D\left(\overline{b_c}\right) = H^1_{\beta 0}$.

For $N \ge 3$ and $\beta \ne N-2$, we have the following

PROPOSITION 3.4. *If* $N \ge 3$ *and* $\beta \ne N-2$, *then* $H^1_\beta = |x|^{\frac{\beta}{2}} H^1 = H^1_{\beta 0}$.

PROOF. Let $u \in H^1$, $v = |x|^{\frac{\beta}{2}} u$. Then $v \in L^2_\beta$ and $D_j v = \frac{\beta}{2} |x|^{\frac{\beta}{2}-1} \frac{x_j}{|x|} u + |x|^{\frac{\beta}{2}} D_j u$ in $\mathcal{D}_0'$. It follows from Hardy's inequality that $|x|^{\frac{\beta}{2}-1} u \in L^2_\beta$. Hence $D_j v \in L^2_\beta$ and $v \in H^1_\beta$.

Conversely, let $v \in H^1_\beta$, $u = |x|^{-\frac{\beta}{2}} v$. Then $u \in L^2$ and $D_j u = -\frac{\beta}{2} |x|^{-\frac{\beta}{2}-1} \frac{x_j}{|x|} v + |x|^{-\frac{\beta}{2}} D_j v$. It follows from $(CKN)$ that $|x|^{-\frac{\beta}{2}-1} v \in L^2$. Since $v \in H^1_\beta$, one has $|x|^{-\frac{\beta}{2}} D_j v \in L^2$. Thus, $D_j u \in H^1$.

We have shown that $H^1_\beta = |x|^{\frac{\beta}{2}} H^1$. The above computation also shows that $\|v\|_{H^1_\beta}$ and $\left\| |x|^{-\frac{\beta}{2}} v \right\|_{H^1}$ are equivalent norms on $H^1_\beta$. Since $|x|^{-\frac{\beta}{2}} \mathcal{D}_0 = \mathcal{D}_0$ and since $\mathcal{D}_0$ is dense in $H^1$ it follows that $\mathcal{D}_0$ is also dense in $H^1_\beta$. ∎

Next we consider realizations of the semigroup $\left(e^{tB_c}\right)$ on $L^p_\beta := L^p\left(\mathbb{R}^N, |x|^{-\beta} dx\right)$ if $N \ge 2$ and $L^p_\beta := L^p\left((0,\infty), |x|^{-\beta} dx\right)$ if $N = 1$. Here $p \in [1,\infty)$ will have to be restricted to a certain interval containing 2.

Let us write $B = B_0$. Since $\left(e^{tB}\right)_{t\ge 0}$ is submarkovian, there exists a consistent family

$$\left(e^{tB^{(p)}}\right)_{t\ge 0}$$

of $(C_0)$ semigroups on $L^p_\beta$, $1 \le p < \infty$ with $B^{(2)} = B$. Consider the bounded potentials

$$V_k(x) = \inf\left\{\frac{1}{|x|^2}, k\right\}.$$

Then $e^{t\left(B^{(p)}+cV_k\right)}$ is a $(C_0)$ semigroup on $L^p_\beta$ for all $c > 0$, $1 \le p < \infty$ . Moreover,

$$0 \le e^{t\left(B^{(p)}+cV_k\right)} \le e^{t\left(B^{(p)}+cV_{k+1}\right)}.$$

Now let $c < \left(\frac{N-2-\beta}{2}\right)^2$. Let $1 \le p_-(c) < 2 < p_+(c) \le \infty$ be such that

$$[p_-(c), p_+(c)] = \left\{p \in [1,\infty] : c \le \frac{4}{pp'}\left(\frac{N-2-\beta}{2}\right)^2\right\}.$$

Note that $1 < p_-(c) < p_+(c) < \infty$ if $c > 0$, and $p_-(c) = 1$, $p_+(c) = \infty$ if $c \le 0$. Observe that $p \in [p_-(c), p_+(c)]$ iff $p' \in [p_-(c), p_+(c)]$. By $B_{c,\min}$ we denote the operator given by $D(B_{c,\min}) = \mathcal{D}_0$ ,

$$B_{c,\min} u = \Delta u - \frac{\beta}{|x|}\frac{\partial u}{\partial r} + \frac{c}{|x|^2} u$$

which acts on all $L^p_\beta$, $1 \le p < \infty$.

THEOREM 3.5. *Let* $p \in [p_-(c), p_+(c)]$, $p < \infty$. *Then* $T_{p,c}(t) := \text{strong}\lim_{k\longrightarrow\infty} e^{t\left(B^{(p)}+cV_k\right)}$ *exists in* $\mathcal{L}\left(L^p_\beta\right)$ *and defines a* $(C_0)$ *semigroup* $T_{p,c}$ *on* $L^p_\beta$. *Its generator* $B^{(p)}_c$ *is an extension of* $B_{c,\min}$. *In addition* $B^{(2)}_c = B_c$.

It clearly follows from Theorem 3.5 that the semigroup $T_{p,c}$ are consistent, i.e.,

$$T_{p,c}(t) f = T_{q,c}(t) f$$

for all $t \ge 0$, $f \in L^p_\beta \cap L^q_\beta$, $p_-(c) \le p, q \le p_+(c)$. For the proof of Theorem 3.5 we will use the following simple argument for increasing semigroups referring to Voigt [22], [23] for further information and developments.

PROPOSITION 3.6. *Let* $T_k$ *be* $(C_0)$ *contraction semigroups on* $L^p$, $1 \le p < \infty$, *such that* $0 \le T_k(t) \le T_{k+1}(t)$. *Then* $T(t) f = \lim_{k\longrightarrow\infty} T_k(t) f$ *exists for all* $f \in L^p$, $t \ge 0$, *and defines a* $(C_0)$ *semigroup* $T$.

PROOF. The strong limit exists by the Beppo Levi Theorem. Then $T(t) \in \mathcal{L}(L^p)$ and $T(t+s) = T(t)T(s)$ for $t, s \ge 0$. It remains to prove strong continuity. Let $t_n \downarrow 0$, $0 \le f \in L^p$. We have to show that $f_n := T(t_n) f \longrightarrow f$ as $n \longrightarrow \infty$. Let $g_n := T_1(t_n) f$. Then $0 \le g_n \le f_n$ and $g_n \longrightarrow f$ as $n \longrightarrow \infty$. Moreover, $\|g_n\|_{L^p} \le \|f\|_{L^p}$.

a) Let $p = 1$. Then

$$\int (f_n - g_n)\, dx + \int g_n\, dx = \int f_n\, dx \le \|f\|_{L^1}.$$

Since $(f_n - g_n) \ge 0$ and $\int g_n\, dx \longrightarrow \|f\|_{L^1}$, it follows that $\|f_n - g_n\|_{L^1} = \int (f_n - g_n)\, dx \longrightarrow 0$ as $n \longrightarrow \infty$. Since $g_n \longrightarrow f$ in $L^1$, also $f_n \longrightarrow f$ in $L^1$.

b) Let $1 < p < \infty$. It suffices to show that each subsequence of $(f_n)$ has a subsequence converging to $f$ in $L^p$. Since $L^p$ is reflexive, we may assume that $f_n$ converges weakly to a function $h \in L^p$ (consider a subsequence otherwise). Since $g_n \le f_n$ and $g_n \longrightarrow f$ it follows that $f \le h$. Hence $\|f\|_{L^p} \le \|h\|_{L^p}$. Since $L^p$is uniformly convex, this implies that $f_n$ converges strongly to $h$. It follows that $\|h\|_{L^p} \le \|f\|_{L^p}$. Since $f \le h$, this implies that $f = h$. ∎

PROOF OF THEOREM 3.5. Let $p \in [p_-(c), p_+(c)]$, i.e. $c \le \frac{4}{pp'}\left(\frac{N-2-\beta}{2}\right)^2$. We show that $B^{(p)} + cV_k$ is dissipative for all $k \in \mathbb{N}$. Let $u \in D\left(B^{(p)}\right)$. We have to show that

$$\left\langle B^{(p)}u + cV_k u, |u|^{p-1} sgn\ u\right\rangle \le 0.$$

By a result of Liskevich-Semenov [17, Theorem 3.9] one has $u|u|^{\frac{p}{2}-1} \in D\left(\bar{b}\right)$ and

$$\left\langle B^{(p)}u, |u|^{p-1} sgn\ u\right\rangle \le -\frac{4}{pp'}\bar{b}\left(u|u|^{\frac{p}{2}-1}, u|u|^{\frac{p}{2}-1}\right).$$

Hence by (CKN) in Proposition 3.3,

$$\left\langle B^{(p)}u, |u|^{p-1} sgn\ u\right\rangle \le -\frac{4}{pp'}\left(\frac{N-2-\beta}{2}\right)^2 \int \frac{|u|^p}{|x|^2}|x|^{-\beta}\, dx.$$

Since

$$\begin{aligned} c\left\langle V_k u, |u|^{p-1} \, sgn \; u\right\rangle &= c\int_{|x|^2\geq\frac{1}{k}} \frac{u}{|x|^2}\,|u|^{p-1}\,sgn\; u\;|x|^{-\beta}\;dx \\ &= c\int_{|x|^2\geq\frac{1}{k}} \frac{|u|^p}{|x|^2}\,|x|^{-\beta}\;dx, \end{aligned}$$

and the claim follows. By Proposition 3.6, there exists a $(C_0)$ semigroup $T_{c,p}$ on $L^p_\beta$ such that $T_{c,p}(t) = \lim_{k\longrightarrow\infty} e^{t\left(B^{(p)}+cV_k\right)}$ strongly. Denote the generator of $T_{c,p}$ by $B_c^{(p)}$. Let $u \in \mathcal{D}_0$. Then there exists $k_0 \in \mathbb{N}$ such that $u(x) = 0$ if $|x|^2 \leq \frac{1}{k_0}$. Hence

$$\left(B^{(p)} + cV_k\right)u = \left(B^{(p)} + c\frac{1}{|x|^2}\right)u = B_{c,\min}u =: v$$

for all $k \geq k_0$.

Observe that

$$e^{t\left(B^{(p)}+cV_k\right)}u - u = \int_0^t e^{s\left(B^{(p)}+cV_k\right)}v\;ds$$

for all $k \geq k_0$. Passing to the limit as $k \longrightarrow \infty$, we obtain

$$T_{c,p}(t)\,u - u = \int_0^t T_{c,p}(s)\,v\;ds \qquad (t \geq 0).$$

This implies that $u \in D\left(B_c^{(p)}\right)$ and $B_c^{(p)}u = v$. We have shown that $B_{c,\min} \subset B_c^{(p)}$. It follows from the convergence theorem for an increasing sequence of closed positive forms [18,Theorem S.14, p.373] that $B_c^{(2)} = B_c$. ∎

Finally, we describe the behavior of $B_c^{(p)}$ by rescaling. For $\lambda > 0$, the operator $U_\lambda$ given by

$$(U_\lambda f)(x) = \lambda^{\frac{N-\beta}{p}} f(\lambda x)$$

is an isometric isomorphism on $L^p_\beta$ which is unitary if $p = 2$. Moreover, $U_\lambda H^1_{00} = H^1_{00}$ and

$$b_c(U_\lambda u, U_\lambda v) = \lambda^2 b_c(u,v) \tag{3.4}$$

for all $u, v \in H^1_{00}$ (as one easily shows). It follows that $U_\lambda D\left(\overline{b_c}\right) = D\left(\overline{b_c}\right)$ and that (3.4) remains valid for all $u, v \in D\left(\overline{b_c}\right)$. This implies that

$$U_\lambda B_{c\lambda}^{-1} = \lambda^2 B_c \qquad (\lambda > 0) \tag{3.5}$$

by the definition of the associated operator. This implies that

$$U_\lambda e^{tB_c^{(2)}} U_\lambda^{-1} = e^{t\lambda^2 B_c^{(2)}} \qquad (t > 0) \tag{3.6}$$

for all $\lambda > 0$. By consistency it follows that

$$U_\lambda e^{tB_c^{(p)}} U_\lambda^{-1} = e^{t\lambda^2 B_c^{(p)}} \qquad (t > 0) \tag{3.7}$$

for all $\lambda > 0$, $p \in [p_-(c), p_+(c)]$, $p < \infty$, whenever $c < \left(\frac{N-2-\beta}{2}\right)^2$.

From these rescaling properties we deduce the following result on the spectrum.

PROPOSITION 3.7. (a) *For* $p = 2$ *one has* $\sigma(B_c) = (-\infty, 0]$ *for all* $c \leq \left(\frac{N-2-\beta}{2}\right)^2$.

(b) *Let* $c < \left(\frac{N-2-\beta}{2}\right)^2$. *Then* $\sigma\left(B_c^{(p)}\right) \cap \mathbb{R} = (-\infty, 0]$, *for all* $p \in [p_-(c), p_+(c)]$, $p < \infty$.

PROOF. (a) It follows from (3.5) that $\sigma(B_c) = \lambda^2 \sigma(B_c)$ for all $\lambda > 0$. Since $\sigma(B_c) \neq \emptyset$, the claim follows.

(b) Let $p \in [p_-(c), p_+(c)]$, $p < \infty$. We first show that $\sigma\left(B_c^{(p)}\right) \neq \emptyset$. In fact, if $\sigma\left(B_c^{(p)}\right) = \emptyset$, then it follows from Weis' Theorem [4, Theorem 5.3.6] that $\omega\left(B_c^{(p)}\right) = -\infty$, where $\omega$ denotes the growth bound (or type) of the semigroup generated by $B_c^{(p)}$. But then it follows from the Riesz-Thorin Theorem that $\omega\left(B_c^{(2)}\right) = -\infty$, which is impossible by (a).

The spectral bound

$$s\left(B_c^{(p)}\right) = \sup\left\{\operatorname{Re}\mu : \mu \in \sigma\left(B_c^{(p)}\right)\right\}$$

is finite. It follows from (3.7) and (a) that $s\left(B_c^{(p)}\right) = 0$ and $(-\infty, 0] \subset \sigma\left(B_c^{(p)}\right)$. ∎

REMARK 3.8. *We refer to [21], [20], [16] and [1] for results on p-independence of the spectra of consistent semigroups. In general the spectrum of the generator of a symmetric submarkovian semigroup may depend on* $p \in [1, \infty)$. *An interesting example is the Neumann Laplacian on irregular domains (see Kunstmann [14]).*

Finally, we show that the constant $\frac{4}{pp'}\left(\frac{N-2-\beta}{2}\right)^2$ in Theorem 3.5 is optimal. For $\beta = 0$ this result has been proved by Vogt [21, Example 3.31].

THEOREM 3.9. *Let* $2 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, *and let* $c > \frac{4}{pp'}\left(\frac{N-2-\beta}{2}\right)^2$. *Then the operator* $B_{c,\min}$ *is not quasidissipative. Thus no extension of* $B_{c,\min}$ *generates a* $(C_0)$ *semigroup* $T$ *on* $L^p_\beta$ *satisfying* $\|T(t)\|_{\mathcal{L}(L^p_\beta)} \leq e^{\omega t}$ *for* $t \geq 0$ *and for some* $\omega \in \mathbb{R}$.

Recall that an operator $A$ on a Banach space $X$ is dissipative iff

$$\|x\| \leq \|x - tAx\| \qquad x \in D(A),\ t \geq 0.$$

An operator $A$ is quasidissipative iff $A - \omega I$ is dissipative for some $\omega \in \mathbb{R}$. Let $U : X \longrightarrow X$ be an isometric isomorphism. Then this characterization of dissipativity shows that $A$ is dissipative iff $UAU^{-1}$ is dissipative. If $A$ is dissipative, then also $\lambda A$ is dissipative for all $\lambda > 0$, and $A - \omega$ is dissipative for all $\omega \geq 0$. Conversely, let $\omega \geq 0$ and assume that $\lambda A - \omega$ is dissipative for all $\lambda > 0$. Then it follows that $A$ is dissipative.

PROOF OF THEOREM 3.9. Let $\omega \geq 0$. Assume that $B_{c,\min} - \omega$ is dissipative in $L^p_\beta$. Let $\lambda > 0$, $\widetilde{U_\lambda} = const \cdot U_\lambda$ where the constant is chosen in such a way that $\widetilde{U_\lambda}$ is an isometric isomorphism on $L^p_\beta$. Then $\lambda^2 B_{c,\min} - \omega = \widetilde{U_\lambda}\left(B_{c,\min} - \omega\right)\widetilde{U_\lambda}^{-1}$ is dissipative. Since $\lambda > 0$ is arbitrary, it follows that $B_{c,\min}$ is dissipative.

Thus, we merely have to prove that $B_{c,\min}$ is not dissipative in $L^p_\beta$. Let us assume that $B_{c,\min}$ is dissipative. Let $u \in \mathcal{D}_0$. It follows that

$$0 \geq \left\langle \Delta u - \frac{\beta}{|x|}\frac{\partial u}{\partial r} + c\frac{u}{|x|^2}, |u|^{p-1}\, sgn\ u \right\rangle . \tag{3.8}$$

We compute, since $p \geq 2$,

$$\begin{aligned}
\left\langle \Delta u, |u|^{p-1}\, sgn\ u \right\rangle &= \int \Delta u \left( |u|^{p-1}\, sgn\ u \right) |x|^{-\beta}\ dx \\
&= -\int |\nabla u|^2 (p-1) |u|^{p-2} |x|^{-\beta}\ dx \\
&\quad - \int \sum_{j=1}^{N} D_j u\, |u|^{p-1}\, sgn\ u \cdot (-\beta) |x|^{-\beta-1} \frac{x_j}{|x|}\ dx \\
&= -(p-1) \int |\nabla u|^2 |x|^{p-2} |x|^{-\beta}\ dx \\
&\quad + \int \frac{\beta}{|x|}\frac{\partial u}{\partial r} sgn\ u\, |u|^{p-1} |x|^{-\beta}\ dx.
\end{aligned}$$

Thus it follows from (3.8) that

$$\begin{aligned}
c\int \frac{|u|^p}{|x|^2} |x|^{-\beta}\ dx &= \left\langle c\frac{u}{|x|^2}, |u|^{p-1}\, sgn\ u \right\rangle \\
&\leq (p-1) \int |\nabla u|^2 |u|^{p-2} |x|^{-\beta}\ dx.
\end{aligned}$$

In order to rewrite the last expression we compute

$$\nabla |u|^{\frac{p}{2}} = \frac{p}{2} |u|^{\frac{p}{2}-1} \nabla u \frac{u}{|u|},$$

hence

$$\left| \nabla |u|^{\frac{p}{2}} \right|^2 = \frac{p^2}{4} |u|^{p-2} |\nabla u|^2 .$$

So we obtain

$$c\int \frac{|u|^p}{|x|^2} |x|^{-\beta}\ dx \leq \frac{4}{p^2}(p-1) \int \left| \nabla |u|^{\frac{p}{2}} \right|^2 |x|^{-\beta}\ dx.$$

Hence

$$c\int \frac{|u|^2}{|x|^2} |x|^{-\beta}\ dx \leq \frac{4}{pp'} \int \left| \nabla u^{\frac{p}{2}} \right|^2 |x|^{-\beta}\ dx \tag{3.9}$$

for $u \in \mathcal{D}_{0+}$. By replacing $u$ by $\lambda_\varepsilon * u$ where $\{\lambda_\varepsilon\}_{\varepsilon>0}$ is a Friedrichs mollifier, we deduce that (3.9) remains true for all $u \in H^1_{00} \cap L^\infty_+$. Here we again use that $p \geq 2$.

Let $r = \frac{p}{2}$. We define

$$\varphi_n(s) = s \wedge ns^r \quad (s \geq 0), \qquad u_n = \varphi_n \circ u.$$

Then $u_n^{\frac{1}{r}} \in H^1_{00}$. So (3.9) implies

$$c\int \frac{|u_n|^2}{|x|^2} |x|^{-\beta}\ dx \leq \frac{4}{pp'} \int |\nabla u_n|^2\ |x|^{-\beta}\ dx. \tag{3.10}$$

Letting $n \longrightarrow \infty$, since $|\nabla u_n|^2 \leq \frac{p^2}{4} |\nabla u|^2$ for all $n \in \mathbb{N}$, we deduce that

$$c \int \frac{|u|^2}{|x|^2} |x|^{-\beta} \, dx \leq \frac{4}{pp'} \int |\nabla u|^2 \, |x|^{-\beta} \, dx \tag{3.11}$$

for all $u \in H^1_{00} \cap L^\infty_+$. Applying (3.11) to $u \wedge n$ instead of $u$ and letting $n \longrightarrow \infty$, we see that (3.11) remains true for all $u \in H^1_{00+}$. Finally, applying (3.11) to $|u|$ instead of $u$ it follows that (3.11) holds for all $u \in H^1_{00}$. Now Theorem 3.1 implies $c \leq \frac{4}{pp'} \left(\frac{N-2-\beta}{2}\right)^2$. ∎

We remark that even though for $0 < c < \left(\frac{N-2-\beta}{2}\right)^2$ the interval $[p_-(c), p_+(c)]$ is optimal for obtaining quasicontractive (equivalently, contractive) extrapolating semigroups to $L^p_\beta$ for all $p \in [p_-(c), p_+(c)]$, one might still have extensions which are not quasicontractive. In fact, for $\beta = 0$, it is shown by Vogt [21, Example 3.31] that for $p \in \left(\frac{N}{N-2} p_-(c), \frac{N}{N-2} p_+(c)\right)$, and $N \geq 3$, such an extrapolation semigroup on $L^p(\mathbb{R}^N)$ exists.

## References


[1] W. Arendt, *Gaussian estimates and interpolation of the spectrum in $L^p$*, Differential and Integral Equations, **7** (1994) 1153-1168.

[2] W. Arendt and C. Batty, *Abosrption semigroups and Dirichlet boundary conditions,* Math. Ann. **295** (1993), 427-448.

[3] W. Arendt and C. Batty, *Exponential stability of a diffusion equation with absorption*, Differential and Integral Equations, **6** (1993), 1009-1024.

[4] W. Arendt, C. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems,* Birkhäuser, Basel 2001.

[5] P. Baras and J. Goldstein, *The heat equation with a singular potential,* Trans. Amer. Math. Soc., **284** (1984), 121-139.

[6] P. Baras and J. Goldstein, *Remarks on the inverse square potential in quantum mechanics,* North Holland Math. Stud., **92** (1984), 31-35.

[7] L. Caffarelli, R. V. Kohn and L. Nirenberg, *First order interpolation inequality with weights,* Compositio Math. **53** (1984), 259-275.

[8] E. B. Davies, *Heat Kernels and Spectral Theory,* Oxford University Press 1989.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer, Berlin 1998.

[10] G. R. Goldstein, J. A. Goldstein and I. Kombe, *The power potential and nonexistence of positive solutions*, Proc. 2004 Cortona Conf., to appear.

[11] J. A. Goldstein and I. Kombe, *Instantaneous blow-up*, Contemporary Math **327** (2003), 141-150.

[12] H. Kalf, U.-W. Schmincke, J. Walter and R. Wüst, *On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials,* Lecture Notes in Math., Vol. 448, Springer, Berlin (1975), 182-226.

[13] H. Kalf and J. Walter, *Strongly singular potentials and essential self-adjointness of singular elliptic operators in $C_0^\infty(\mathbb{R}^N \setminus \{0\})$*, J. Functional Analysis **10** (1972), 114-130.

[14] P. C. Kunstmann, *$L^p$-spectral properties of the Neumann Laplacian on horns, comets, and stars,* Math. Z. **242** (2002), 183-201.

[15] V. Liskevich and Yu. Semenov, *Some inequalities for submarkovian generators and their applications to the perturbation theory,* Proc. Amer. Math. Soc. **119** (1993), 1171-1177.

[16] V. Liskevich, A. Sobol and H. Vogt, *On the $L_p$-theory of $C_0$-semigroups associated with second order elliptic operators, II*, J. Functional Analysis **193** (2002), 55-76.

[17] E.M. Ouhabaz, *Analysis of Heat Equations on Domains,* LMS Monograph Series **31**, Princeton University Press, Princeton 2004.

[18] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol. I Functional Analysis,* 2nd edition, Academic Press, London 1980.

[19] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol. II Fourier Analysis, Self-Adjointness*, Academic Press, London 1975.

[20] Z. Sobol and H. Vogt, *On the $L_p$−theory of $C_0$−semigroups associated with second order elliptic operators, I*, J. Functional Analysis **193** (2002), 24-54.

[21] H. Vogt, *$L_p$−properties of second order elliptic differential equations,* PhD thesis, Dresden 2001.

[22] J. Voigt, *Absorption semigroups, their generators and Schrödinger semigroups,* J. Functional Analysis **67** (1986), 167-205.

[23] J. Voigt, *Absorption semigroups,* J. Operator Theory **20** (1988), 117-131.

[24] J. Voigt, *One-parameter semigroups acting simultaneoously on different $L_p$−spaces,* Bull. Soc. Roy. Sci. Liège **61** (1992), 465-470.

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