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Addendum

Vector-valued holomorphic functions revisited

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One of the statements, Corollary 3.7, in [1] is erroneous. In fact, the hypothesis that W is a separating subspace of X' has to be replaced by the stronger hypothesis that W is almost norming. The arguments are implicitely in [1]. We give precise statements and proofs. Let X be a Banach space and W a *separating subspace* of the dual space X' of X, i.e., for all $x \in X \setminus \{0\}$ there exists $\varphi \in W$ such that $\varphi(x) \neq 0$. Then we may identify X with a subspace of W', the dual space of W. To say that W is *almost norming* means by definition that

$$\|x\|_W := \sup\{|\varphi(x)| : \varphi \in W, \|\varphi\| \leq 1\}$$

is an equivalent norm on X. This is equivalent to saying that X is closed in W'. By a result of Davis and Lindenstrauss [1, Remark 1.2] each separating subspace of X' is almost norming if and only if $\dim X''/X < \infty$. Now we first formulate and prove the corrected version of [1, Corollary 3.7].

Theorem 1. *Let $\Omega \subset \mathbb{C}$ be open and connected. Let $A \subset \Omega$ have a limit point in Ω and let $h : A \to X$ be a function. Assume that there exist $c \geq 0$, an almost norming subspace W of X' and a family $\{H_\varphi : \varphi \in W\}$ of holomorphic functions on Ω such that*

$$H_\varphi(z) = \langle \varphi, h(z) \rangle \quad (\varphi \in W, z \in A)$$

and

$$|H_\varphi(z)| \leq c\|\varphi\| \quad (\varphi \in W, z \in \Omega) .$$

Then h has a holomorphic extension to Ω with values in X.

Proof. Define the mapping $H : \Omega \to W'$ by $\langle H(z), \varphi\rangle = H_\varphi(z)$. Then $\|H(z)\| \le c$ for all $z \in \Omega$. It follows from [1, Theorem 2.2] that H is holomorphic. Since X is a closed subspace of W' and $H(z) \in X$ for all $z \in A$ it follows from the Uniqueness Theorem [1, Theorem 2.2] that $H(z) \in X$ for all $z \in \Omega$. □

Next we show that in Theorem 1 one may not replace almost norming by the weaker property of separating in general.

Theorem 2. *Let $W \subset X'$ be a separating subspace which is not almost norming. Then there exists a functions $f : \mathbb{D} \to X$ such that $\varphi \circ f$ is holomorphic for all $\varphi \in W$,*

$$|\varphi(f(z))| \le c\|\varphi\| \quad \text{for all} \quad z \in \mathbb{D}, \quad \varphi \in W$$

and some constant $c \ge 0$. But f is not holomorphic.

Proof. Since W is separating, X is a subspace of W' and the embedding is continuous. However, X is not closed in W' since W is not almost norming. By [1, Theorem 1.6] there exists a function $g : \mathbb{D} \to X$ which is not holomorphic such that $g : \mathbb{D} \to W'$ is holomorphic. Then g is not holomorphic on $r\mathbb{D}$ with values in X for some $0 < r < 1$. Define $f(z) = g(rz) \quad (z \in \mathbb{D})$. Then $f : \mathbb{D} \to X$ is not holomorphic. But f as a function with values in W' is holomorphic and bounded. Hence $|\varphi(f(z))| \le c\|\varphi\|$ for all $\varphi \in W$, $z \in \mathbb{D}$ and some $c \ge 0$. □

We emphasize that [1, Corollary 3.8] is correct as it is formulated. The Krein-Smulyan Theorem can be used to show that the weak hypothesis of separating does suffice here.

We recall the statement:

Corollary. *Let Y be a Banach space continuously embedded into X. Let $f : \Omega \to X$ be holomorphic. Assume that for each $z \in \Omega$ there exists an open bounded set $\omega \subset \Omega$ such that $z \in \omega$, $\bar\omega \subset \Omega$, $f(v) \in Y$ for all $v \in \partial\omega$, and $\sup_{v\in\partial\omega} \|f(v)\|_Y < \infty$. Then $f(z) \in Y$ for all $z \in \Omega$ and f is holomorphic if it is considered as a function with values in Y.*

Proof of the Corollary. Let $A = \{z \in \Omega : f(z) \in Y\}$. Then $W := \{\varphi \in Y' : \exists\, f_\varphi : \Omega \to \mathbb{C}$ holomorphic such that $f_\varphi(z) = \varphi(f(z))$ for all $z \in A\}$ is a subspace of Y' which contains the separating space $\{\psi_{|Y} : \psi \in X'\}$. Thus W is $\sigma(Y', Y)$-dense in Y'. We claim that $W = Y'$. By the Krein-Smulyan Theorem it suffices to show that

$$W_1 := \{\varphi \in W : \|\varphi\| \le 1\} \text{ is } \sigma(Y', Y) - \text{closed}.$$

Let (φ_i) be a net in W_1 converging to φ with respect to $\sigma(Y', Y)$. It follows from the maximum principle and the hypothesis that $(f_{\varphi_i}(z))_{i\in I}$ is locally bounded. Now Vitali's Theorem [1, Theorem 2.1] implies that $\lim_i f_{\varphi_i}(z) = g(z)$ exists for all $z \in \Omega$ and defines a holomorphic function $g : \Omega \to \mathbb{C}$. Clearly, $g(z) = \lim_i \varphi_i(f(z)) = \varphi(f(z))$ whenever $z \in A$. Thus $\varphi \in W_1$. We have shown that $W = Y'$. It follows from the maximum principle and the hypothesis that $\sup_{z\in\bar\omega} |f_\varphi(z)| \le$

$\|\varphi\| \sup_{z \in \partial\omega} \|f(z)\|_Y$ whenever ω is open, $\bar{\omega} \subset \Omega$ and $\partial\omega \subset A$. Now we can apply Theorem 1 above. □

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References

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