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FORMS, FUNCTIONAL CALCULUS, COSINE FUNCTIONS AND PERTURBATION

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Abstract. In this article we describe properties of unbounded operators related to evolutionary problems. It is a survey article which also contains several new results. For instance we give a characterization of cosine functions in terms of mild well-posedness of the Cauchy problem of order 2, and we show that the property of having a bounded H^∞-calculus is stable under rank-1 perturbations whereas the property of being associated with a closed form and the property of generating a cosine function are not.

Introduction. Many second-order elliptic differential operators can be realized on L^2-spaces by means of closed quadratic forms (see [Ev, Chapter 6]). Typically the space is $L^2(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$, and the domain V of the form is a Sobolev space such as $H^1(\Omega)$ or $H^1_0(\Omega)$. The domain of the associated operator A is more difficult to identify but it often happens that the domain of the square root of A coincides with the form domain V. Kato [Kat] initiated a study of closed forms and the associated operators as an abstract approach to such differential operators. If one takes a fixed inner product, one can characterize the operators which are associated with forms by means of a condition that the numerical range of the operators should be contained in a suitable sector. If one allows changes of the inner product the class of operators associated with forms becomes much wider, so it is useful to study their properties modulo similarity.

The notion of a bounded H^∞-calculus of a sectorial operator was introduced by McIntosh [McI3] in work on singular integral operators but it has subsequently proved to be very important for questions of maximal regularity in evolution equations (see [KW] for an extended survey). Not every sectorial operator on Hilbert space has such a calculus,

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and remarkably it turns out that the class of operators on Hilbert space which have a bounded H^∞-calculus on a sector of angle less than $\pi/2$ is exactly the same as the class of operators associated with forms, modulo similarity (Theorem 2.5).

Cosine functions were first studied by Fattorini (see [Fa3]) and Kisyński [Ki] as the second-order analogues of C_0-semigroups. Indeed the second-order Cauchy problem

$$u''(t) = Au(t) \quad (t \geq 0)$$

is well-posed if and only if A generates a cosine function (see Theorem 5.3). A remarkable recent result of Haase [Ha1] and Crouzeix [Cr] is that generators of cosine functions on Hilbert space can be characterized by a condition that the numerical range, with respect to some inner product, is contained inside a parabola (Theorem 5.11).

Perturbation theory is an important tool for studying differential operators, where more complicated operators may be regarded as perturbations of simpler operators. Abstract perturbation theory may then allow a more general case to be reduced to a simpler case. It is standard to regard the lower-order terms of a differential operator as a perturbation of the principal part A which is relatively bounded with respect to a fractional power of A. Here we are interested more in A-bounded perturbations which are of finite rank or relatively compact.

In this article we describe some of the connections between these topics. The emphasis is on Hilbert spaces but we state results for Banach spaces where appropriate. The article is mostly a survey of some known results but it includes some new results. For example, we show that bounded H^∞-calculus is stable under A-bounded perturbations of finite rank (Theorem 4.1), but association with a form, for a fixed scalar product, is not (Theorem 3.8). Generation of a cosine function is also not stable under these perturbations (Theorem 5.9) but we refer to [AB] for the proof. We do not attempt to give a complete survey of any of the individual topics or to give a full historical account, and broader recent surveys may be found in [Ar] and [KW].

1. Forms. Let H, V be complex Hilbert spaces such that $V \underset{d}{\hookrightarrow} H$, i.e., V is continuously embedded into H with dense image. Let $a : V \times V \to \mathbb{C}$ be a continuous sesquilinear form which is *closed*, i.e.,

$$\operatorname{Re} a(u,u) + \omega(u \,|\, u)_H \geq \alpha \|u\|_V^2 \quad (u \in V) \tag{1.1}$$

holds for some $\alpha > 0$, $\omega \in \mathbb{R}$. Here $(\ |\ )_H$ denotes the scalar product of H. We call V the *domain* of the form. We can associate with a an operator A on H by

$$\begin{aligned} D(A) &= \{u \in V : \text{there exists } v \in H \text{ such that } a(u,\varphi) = (v \,|\, \varphi)_H \text{ for all } \varphi \in V\}, \\ Au &= v. \end{aligned}$$

We write $A \sim a$ and say that A *is associated with* a. More precisely, we may write $A \sim a$ on $(H, (\ |\ )_H)$. In this situation it is always the case that $D(A)$ is dense in H and $-A$ generates a holomorphic C_0-semigroup T on H. If $\omega = 0$, i.e., if the form a is *coercive*, then the semigroup T is exponentially stable. We refer to [Kat, Chapter VI] for the general theory of closed forms and the associated operators.

EXAMPLE 1.1 (The Laplacian with Dirichlet boundary conditions). Let $\Omega \subset \mathbb{R}^N$ be open, $H = L^2(\Omega)$ with the usual inner product, $V = H_0^1(\Omega)$ and $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$. Then a is a closed form. Let $A \sim a$. It is not difficult to see that

$$D(A) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}, \qquad Au = -\Delta u,$$

where Δu is understood in the sense of distributions [ABHN, Theorem 7.2.1].

DEFINITION 1.2. Let A be an operator on $(H, (\,|\,)_H)$. We say that A is *associated with a form* if there exists a closed form a such that $A \sim a$.

It is easy to characterize those operators which are associated with a form on $(H, (|)_H)$. For $0 < \theta < \pi$ we shall denote by $\Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}$ the sector of half-angle θ with vertex 0, and by $\Sigma_\theta + \omega$ the corresponding sector with vertex ω where $\omega \in \mathbb{R}$. For an operator A, we let $W(A)$ be the *numerical range* of A:

$$W(A) = \{(Ax, x)_H : x \in D(A), \|x\|_H = 1\}.$$

THEOREM 1.3 ([Kat, Theorem VI.2.7]). *An operator A on $(H, (\,|\,)_H)$ is associated with a form if and only if there exist $\theta \in (0, \pi/2)$ and $\omega \in \mathbb{R}$ such that $W(A) \subset \Sigma_\theta + \omega$ and the range of $A - \omega$ is H.*

Definition 1.2 and the definition of $W(A)$ depend on the choice of the scalar product, and Theorem 1.3 characterizes the operators associated with a form for a fixed scalar product. If we consider a fixed form but we change to an equivalent scalar product $(\,|\,)_1$ on H then we obtain a different operator.

EXAMPLE 1.4. Let $A \sim a$ on $L^2(\Omega)$ when $L^2(\Omega)$ has the usual scalar product. Let $m \in L^\infty(\Omega, \mathbb{R})$ such that $\inf_{x\in\Omega} m(x) > 0$. Consider the equivalent scalar product

$$(u\,|\,v)_1 = \int_\Omega \frac{u\bar{v}}{m}\, dx$$

on $L^2(\Omega)$. Then the operator mA is associated with a on $(L^2(\Omega), (\,|\,)_1)$. Here we use m also to denote the bounded operator of multiplication by the function m.

It is natural to ask which operators A on H are associated with a closed form on $(H, (\,|\,)_1)$ whatever the equivalent scalar product is. This occurs if A is bounded because we may take the form $a(u,v) = (Au\,|\,v)_1$. Matolcsi has recently answered the question by showing that only bounded operators have this property.

THEOREM 1.5 ([Mat]). *Let A be an operator on H. Assume that for each equivalent scalar product $(\,|\,)_1$ on H there exists a closed form a on H such that $A \sim a$ on $(H, (\,|\,)_1)$. Then A is bounded.*

Thus, given an unbounded operator A, we can always find a bad scalar product so that $W(A)$ (with respect to this scalar product) is not contained in any sector. Another natural and more interesting question is which operators are associated with a closed form with respect to some equivalent scalar product.

DEFINITION 1.6. A densely defined operator A on H is called *form-similar* if there exists an equivalent scalar product $(\,|\,)_1$ on H and a closed form a on H such that $A \sim a$ on $(H, (\,|\,)_1)$.

Thus our question is: *which operators on H are form-similar?* We shall describe the answer in the next section.

2. Functional calculus. In this section we shall describe the answer to the question posed at the end of Section 1. For this we shall recall the notion of a bounded functional calculus of a sectorial operator A. For many purposes, for instance for questions of regularity of solutions of Cauchy problems, it does not matter whether A or the shifted operator $A+\omega$ is considered. If $-A$ generates the semigroup T then $-(A+\omega)$ generates the semigroup $(e^{-\omega t}T(t))_{t\geq 0}$. The class of form-similar operators is also invariant under these shifts. If $A \sim a$ on a Hilbert space H, then $A+\omega$ is associated with the closed form a_ω given by $a_\omega(u,v) = a(u,v) + \omega(u \mid v)_H$. So we build such invariance into our definitions.

In the following, $\overline{\Sigma}_\theta$ is the closure of the sector Σ_θ for $0 < \theta < \pi$, and $\overline{\Sigma}_0 = \mathbb{R}_+ = [0,\infty)$. For an operator A, let $R(\lambda, A) = (\lambda - A)^{-1}$ for λ in the resolvent set $\varrho(A) = \mathbb{C} \setminus \sigma(A)$ of A.

The following use of the term "sectorial operator" is the one which has become standard (apart from varying conventions whether A is required to be densely defined, injective or invertible); it can be found in many books on semigroup theory and related topics. However it differs from Kato [Kat] who used the term for operators whose numerical range is contained in a sector $\Sigma_\theta + \omega$ with $\theta < \pi/2$.

DEFINITION 2.1. Let A be an operator on a Banach space X.

a) Let $0 \leq \theta < \pi$. We say that A is *sectorial of angle* θ and write $A \in \mathrm{Sec}(\theta)$ if $\sigma(A) \subset \overline{\Sigma}_\theta$ and $\sup_{\lambda \notin \overline{\Sigma}_{\theta'}} \|\lambda R(\lambda, A)\| < \infty$ for each $\theta' > \theta$.
b) We say that A is *quasisectorial of angle* θ and write $A \in \mathrm{QSec}(\theta)$ if there exists $\omega \in \mathbb{R}$ such that $A + \omega \in \mathrm{Sec}(\theta)$.
c) We say that A is *sectorial* (resp., *quasisectorial*) and we write $A \in \mathrm{Sec}$ (resp., $A \in \mathrm{QSec}$) if $A \in \mathrm{Sec}(\theta)$ (resp., $A \in \mathrm{QSec}(\theta)$) for some $\theta \in (0,\pi)$.

When $A \in \mathrm{Sec}$, we let

$$\vartheta_S(A) := \inf\{\theta : A \in \mathrm{Sec}(\theta)\}$$

be the *sectorial angle* of A. If X is reflexive and $A \in \mathrm{Sec}$, then $D(A)$ is dense in X [Ha2, Proposition 2.1.1].

Note that if $\sigma(A) \subset \overline{\Sigma}_\theta$ and $\sup_{\lambda \notin \overline{\Sigma}_\theta} \|\lambda R(\lambda, A)\| < \infty$, then a standard argument with a Neumann series expansion shows that there exists $\theta' < \theta$ such that $\sigma(A) \subset \Sigma_{\theta'} \cup \{0\}$ and $\sup_{\lambda \notin \Sigma_{\theta'} \cup \{0\}} \|\lambda R(\lambda, A)\| < \infty$. However, Definition 2.1 a) has been set up in such a way that $A \in \mathrm{Sec}(\vartheta_S(A))$.

It is clear that if $A \in \mathrm{Sec}(\theta)$ then $A + \omega \in \mathrm{Sec}(\theta)$ for all $\omega \geq 0$.

When $A \in \mathrm{QSec}$, we let

$$\vartheta_{QS}(A) = \inf\{\theta : A \in \mathrm{QSec}(\theta)\} = \lim_{\omega \to \infty} \vartheta_S(A+\omega).$$

If $A \in \mathrm{Sec}$, then one can define fractional powers A^γ for $\gamma \in \mathbb{R}$. We refer to [MS] for a full account, but we shall only need the basic properties in the case when A is sectorial and invertible and these can be found in many books on semigroup theory and in the appendix of [KW]. For $\gamma > 0$, one defines

$$A^{-\gamma} = \frac{1}{2\pi i} \int_\Gamma z^{-\gamma} R(z, A)\, dz, \tag{2.1}$$

where Γ is a path of the form $\{re^{\pm i\theta} : r \geq \delta\} \cup \{\delta e^{i\varphi} : |\varphi| \leq \theta\}$, for θ and $\delta > 0$ chosen so that $\sigma(A)$ lies to the right of Γ. The integral is absolutely convergent in $\mathcal{L}(X)$. Then $\{A^{-\gamma} : \gamma \geq 0\}$ is a semigroup, and it is a C_0-semigroup if A is densely defined. Moreover, $A^{-\gamma}$ is injective and one defines $A^\gamma = (A^{-\gamma})^{-1}$ with domain equal to the range of $A^{-\gamma}$. If $\gamma\vartheta_S(A) < \pi$, then $A^\gamma \in \mathrm{Sec}$ and $\vartheta_S(A^\gamma) = \gamma\vartheta_S(A)$. If $0 < \gamma < 1$, then $D(A) \subset D(A^\gamma)$ and there is a constant c (depending on γ) such that

$$\|A^\gamma x\| \leq c\|Ax\|^\gamma \|x\|^{1-\gamma} \quad (x \in D(A)).$$

Hence A^γ is small relative to A in the sense that, for every $\varepsilon > 0$, there exists b (depending on γ and ε) such that

$$\|A^\gamma x\| \leq \varepsilon\|Ax\| + b\|x\| \quad (x \in D(A)). \tag{2.2}$$

See [KW, Theorem 15.14] or [MS, Lemma 3.1.7].

It is well known that $-A$ generates a holomorphic C_0-semigroup T if and only if A is densely defined and $A \in \mathrm{QSec}(\theta)$ for some $\theta < \pi/2$. Moreover, T is a bounded holomorphic semigroup if and only if $A \in \mathrm{Sec}(\theta)$ for some $\theta < \pi/2$, and then T is defined by a contour integral of the form

$$T(t) = \exp(-tA) := \frac{1}{2\pi i} \int_\Gamma \exp(-tz) R(z, A)\, dz \tag{2.3}$$

where $\Gamma = \{re^{i\theta'} : r \geq \delta\} \cup \{\delta e^{i\varphi} : \theta' \leq \varphi \leq 2\pi - \theta'\}$ for $\theta < \theta' < \pi/2$ and $\delta > 0$.

We record here two simple, well known facts which we shall need later.

PROPOSITION 2.2. *Let A be a densely defined operator on a Banach space X, and suppose that $A \in \mathrm{Sec}(\theta)$ for some $\theta \in [0, \pi)$. Then*

$$\lim_{|\lambda| \to \infty, \lambda \notin \Sigma_{\theta'}} \|AR(\lambda, A)x\| = 0$$

for all $x \in X$, $\theta' \in (\theta, \pi)$.

Proof. It is clear that this holds for $x \in D(A)$, and then it follows for $x \in X$ by density and the uniform boundedness of $AR(\lambda, A)$ for $\lambda \notin \Sigma_{\theta'}$. ∎

PROPOSITION 2.3. *Let a be a closed form on a Hilbert space H, with domain V, and let $A \sim a$. Let A_V be the part of A in V, so that*

$$D(A_V) = \{u \in D(A) : Au \in V\}, \qquad A_V u = Au.$$

Then $A_V \in \mathrm{QSec}(\theta)$ for some $\theta < \pi/2$.

Proof. This is proved in [Ou, Theorem 1.55] and [Ta, Lemma 3.6.1] for the corresponding operator $A_{V'}$ on the dual space V'. However, A_V is similar to $A_{V'}$ under the isomorphism $A_{V'} + \omega : V \to V'$, for suitable ω, by the Lax-Milgram Theorem. ∎

Now we define the functional calculus. Here $H^\infty(\Sigma_\theta)$ is the Banach algebra of all bounded holomorphic functions f on Σ_θ, with $\|f\|_\infty = \sup\{|f(z)| : z \in \Sigma_\theta\}$.

DEFINITION 2.4. Let A be an operator on a Banach space X.

a) Let $\theta \in (0,\pi)$. We say that A has a *bounded H^∞-calculus of angle θ*, and write simply $A \in H^\infty(\theta)$, if $\sigma(A) \subset \overline{\Sigma}_\theta$ and there exists a bounded algebra homomorphism

$$H^\infty(\Sigma_\theta) \to \mathcal{L}(X), \qquad f \mapsto f(A),$$

satisfying the following two properties:

(i) $r_\lambda(A) = R(\lambda, A)$ for all $\lambda \in \mathbb{C} \setminus \overline{\Sigma}_\theta$, where $r_\lambda(z) = 1/(\lambda - z)$;

(ii) If (f_n) is a sequence in $H^\infty(\Sigma_\theta)$ such that $\sup_n \|f_n\|_\infty < \infty$ and $f_n(z) \to f(z)$ as $n \to \infty$ for each $z \in \Sigma_\theta$, then $f_n(A) \to f(A)$ in the strong operator topology.

b) We say that A has a *bounded quasi-H^∞-calculus of angle θ*, and we write $A \in QH^\infty(\theta)$, if there exists $\omega \in \mathbb{R}$ such that $A + \omega \in H^\infty(\theta)$.

c) We say that A has a *bounded H^∞-calculus*, and we write $A \in H^\infty$, if there exists $\theta \in (0,\pi)$ such that $A \in H^\infty(\theta)$.

d) We say that A has a *bounded quasi-H^∞-calculus*, and we write $A \in QH^\infty$, if there exists $\omega \in \mathbb{R}$ such that $A + \omega \in H^\infty$.

When $A \in H^\infty$, we let

$$\vartheta_H(A) = \inf\{\theta : A \in H^\infty(\theta)\}$$

be the *H^∞-angle* of A.

We remark that our Definition 2.4 a) is not arranged so that $A \in H^\infty(\vartheta_H(A))$, in contrast to Definition 2.1 a). This difference is of no significance for the purposes of this article.

We refer to [Ha2] and [KW] for full accounts of the notion of bounded H^∞-calculus, and we mention here only those aspects which are most relevant to our purpose.

In Definition 2.4 a), the homomorphism $f \mapsto f(A)$ is uniquely determined by the two properties (i) and (ii) [Ha2, Proposition 5.3.9], [KW, Remark 9.7]. If $f(z) = p(z)/q(z)$ where $p(z), q(z)$ are complex polynomials such that the zeros of $q(z)$ all lie in $\mathbb{C} \setminus \overline{\Sigma}_\theta$ and the degree of $q(z)$ is greater than the degree of $p(z)$, then $f(A) = p(A)q(A)^{-1}$. Hence, if $A \in H^\infty(\theta)$ there is a constant C such that

$$\left\|p(A)q(A)^{-1}\right\| \le C \sup\{|p(z)q(z)^{-1}| : z \in \Sigma_\theta\} \tag{2.4}$$

for all such $p(z), q(z)$. In particular taking $p(z) = 1$ and $q(z) = \lambda - z$ shows that $A \in \operatorname{Sec}(\theta)$. The smallest constant C such that (2.4) holds coincides with the operator norm of $f \mapsto f(A)$, and it is known as the *$H^\infty(\Sigma_\theta)$-constant* of A. If the functional calculus takes the constant function 1 to the identity operator then A has dense domain and dense range [Ha2, Proposition 5.3.9].

Suppose that A has dense domain and dense range. If (2.4) holds, it then follows that $A \in H^\infty(\theta)$ [Ha2, Proposition 5.3.4]. In particular, this means that condition (ii) could be omitted from the definition that $A \in H^\infty(\theta)$ in Definition 2.4 in this case, although it may still be needed in order to define $f(A)$ uniquely. There is an explicit way to extend the homomorphism $f \mapsto f(A)$ from the space of rational functions to the whole of $H^\infty(\Sigma_\theta)$.

Indeed, suppose that $f \in H^\infty(\Sigma_\theta)$ and there exists $\alpha > 0$ such that $f(z) = O(|z|^{-\alpha})$ as $|z| \to \infty$, and either A is invertible or $f(z) = O(|z|^\alpha)$ as $z \to 0$. Then $f(A)$ is given by the absolutely convergent integral

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz, \tag{2.5}$$

where $\Gamma = \left\{re^{\pm i\theta'} : r \geq 0\right\}$, for $\theta > \theta' > \vartheta_S(A)$, and the contour is taken in the downward direction. If A is invertible, then the contour can be diverted to the right of the origin and (2.5) holds without any assumption on $f(z)$ as $z \to 0$. In particular, the functional calculus is consistent both with the definition (2.1) of fractional powers when $f(z) = (\omega + z)^{-\gamma}$ for some $\omega > 0$ and $\gamma > 0$, and with the definition (2.3) of the semigroup when $\vartheta_S(A) < \pi/2$ and $f(z) = \exp(-tz)$.

If $A \in H^\infty(\theta)$ and $\omega \geq 0$, then $A + \omega \in H^\infty(\theta)$ with $f(A + \omega) = g(A)$, where $g(z) = f(z + \omega)$.

When $A \in QH^\infty$, we let

$$\vartheta_{QH}(A) = \inf\{\theta : A \in QH^\infty(\theta)\} = \lim_{\omega \to \infty} \vartheta_{QH}(A + \omega).$$

For concrete examples of operators with bounded H^∞-calculus and sectorial operators without it, we refer to [KW, Section 10], [CDMY], [MY].

Now we give the answer to the question of Section 1.

THEOREM 2.5. *Let $-A$ be a closed operator on a Hilbert space H. The following assertions are equivalent:*

(i) *A is form-similar;*
(ii) *A is densely defined and $A \in QH^\infty(\theta)$ for some $\theta < \pi/2$;*
(iii) *$A \in QH^\infty$ and $-A$ generates a holomorphic C_0-semigroup.*

This result was proved in [ABH] based on a result of Le Merdy [LM] characterizing semigroups on Hilbert space which are similar to contraction semigroups. A more direct proof can be found in Haase's thesis [Ha1] and in his book [Ha2, Corollary 7.3.10]. In fact the proofs there show that a sectorial operator A is form-similar if and only if A has "bounded imaginary powers" but on Hilbert space that is equivalent to $A \in H^\infty$, and then $\vartheta_H(A) = \vartheta_S(A)$ (see [ADM], [Ar, Theorem 4.4.10], [Ha2, Theorem 7.3.1]).

Given a Schauder basis in a Banach space X which is not unconditional, one can construct a diagonal operator A on X such that $-A$ generates a holomorphic C_0-semigroup but $A \notin QH^\infty$ (see [Ar, Section 4.5], for example). In particular, such an example can be given on a Hilbert space. However it seems that so far no operator on Hilbert space arising naturally from a model has been found with these properties.

3. Compact perturbation of sectorial operators. Let A be a closed operator on a Banach space X. Then $D(A)$ is a Banach space for the graph norm $\|x\| + \|Ax\|$. We will always consider this Banach space when we talk about a bounded or compact operator $B : D(A) \to X$. We first state the following perturbation result. The proof is a modification of [ABHN, Theorem 3.7.25], which we include for completeness.

THEOREM 3.1. *Let A be a densely defined operator and let $\theta \in (0, \pi)$. Assume that $\sigma(A) \subset \overline{\Sigma}_\theta$ and $\sup_{\lambda \notin \overline{\Sigma}_\theta} \|\lambda R(\lambda, A)\| < \infty$. Let $B : D(A) \to X$ be compact. Then there exists $\omega \in \mathbb{R}_+$ such that $\sigma(A+B) \subset \Sigma_\theta - \omega$ and $\sup_{\lambda \notin \Sigma_\theta - \omega} \|\lambda R(\lambda, A+B)\| < \infty$.*

Proof. We may assume that $0 \in \varrho(A)$ and $\sup_{\lambda \notin \Sigma_\theta} \|\lambda R(\lambda, A)\| < \infty$, considering $A + \varepsilon$ otherwise. Since $AR(\lambda, A) = \lambda R(\lambda, A) - I$, it follows that $M := \sup_{\lambda \notin \Sigma_\theta} \|R(\lambda, A)\|_{\mathcal{L}(X, D(A))} < \infty$. By Proposition 2.2, $AR(\lambda, A)x \to 0$ as $|\lambda| \to \infty, \lambda \notin \Sigma_\theta$, for each $x \in X$. By the Banach-Steinhaus Theorem this convergence is uniform on compact subsets. Thus

$$\|AR(\lambda, A)B\|_{\mathcal{L}(D(A), X)} \to 0 \quad \text{as} \quad |\lambda| \to \infty, \lambda \notin \Sigma_\theta.$$

Hence we may choose $r \geq 0$ such that $\|R(\lambda, A)B\|_{\mathcal{L}(D(A))} \leq \frac{1}{2}$ whenever $\lambda \notin \Sigma_\theta, |\lambda| \geq r$. Then $(I - R(\lambda, A)B)$ is invertible in $\mathcal{L}(D(A))$ and

$$\|(I - R(\lambda, A)B)^{-1}\|_{\mathcal{L}(D(A))} \leq 2.$$

Consequently, $\lambda - A - B = (\lambda - A)(I - R(\lambda, A)B)$ is invertible in $\mathcal{L}(X)$ and

$$R(\lambda, A+B) = (I - R(\lambda, A)B)^{-1} R(\lambda, A),$$
$$\|R(\lambda, A+B)\|_{\mathcal{L}(X, D(A))} \leq 2\|R(\lambda, A)\|_{\mathcal{L}(X, D(A))} \leq 2M.$$

Since $A + B$ is bounded from $D(A)$ to X, this implies that

$$\|\lambda R(\lambda, A+B)\|_{\mathcal{L}(X)} = \|I + (A+B)R(\lambda, A+B)\|_{\mathcal{L}(X)} \leq 1 + \|A+B\|_{\mathcal{L}(D(A), X)} 2M$$

for $\lambda \notin \Sigma_\theta, |\lambda| \geq r$. There exists $\omega \in \mathbb{R}_+$ such that $\{\lambda : |\lambda| \leq r\} \subset \Sigma_\theta - \omega$ and the proof is complete. ∎

REMARK 3.2. Theorem 3.1 remains true if the assumption that $B : D(A) \to X$ is compact is replaced by the assumption that $B : D(A^\gamma) \to X$ is bounded for some $0 \leq \gamma < 1$. This follows from (2.2) by a standard argument (see [ABHN, Theorem 3.7.23]).

COROLLARY 3.3. *Suppose that A is densely defined and quasisectorial, and $B : D(A) \to X$ is compact. Then $A + B$ is quasisectorial and $\vartheta_{QS}(A+B) = \vartheta_{QS}(A)$.*

The following result is a counterpart to Theorem 3.1. It follows from [AB, Theorem 1.3].

THEOREM 3.4. *Let A be a densely defined operator, and let $\theta \in (0, \pi)$. Assume that, for each $B \in \mathcal{L}(D(A), X)$ of rank-1, there exists ω such that $\sigma(A+B) \subset \Sigma_\theta - \omega$ and $\sup_{\lambda \notin \Sigma_\theta - \omega} \|R(\lambda, A+B)\| < \infty$. Then there exists $\theta' < \theta$ such that $A \in \mathrm{QSec}(\theta')$.*

A particular case is the following.

COROLLARY 3.5 (Desch-Schappacher). *Let A be the generator of a C_0-semigroup T. Assume that $A + B$ generates a C_0-semigroup for all $B \in \mathcal{L}(D(A), X)$ of rank 1. Then T is holomorphic.*

Proof. Since $A + B$ generates a C_0-semigroup, its resolvent exists and is bounded on a right half-plane. Then Theorem 3.4 implies that $-A \in \mathrm{QSec}(\theta')$ for some $\theta' < \pi/2$, so T is holomorphic. ∎

REMARK 3.6. Corollary 3.5 was proved by Desch and Schappacher [DS] using a Baire argument. This result was generalized in [AB]. In fact a much more general version of Theorem 3.4 is true. Suppose that A is densely defined and $\sigma(A) \subset \Sigma_\theta$ where $\theta \in [0, \pi)$.

Let $g_n : \mathbb{C} \to \mathbb{R}_+$ be arbitrary functions satisfying $g_n(\lambda) \geq \|R(\lambda, A)\|$ on $\mathbb{C} \setminus \Sigma_\theta$. Assume that for each rank-1 operator $B \in \mathcal{L}(D(A), X)$ there exists $n \in \mathbb{N}$ such that $\sigma(A+B) \subset \Sigma_\theta + n$ and $\|R(\lambda, A+B)\| \leq g_n(\lambda) \quad (\lambda \in \mathbb{C} \setminus (\Sigma_\theta + n))$. Then there exists $\theta' < \theta$ such that $A \in \mathrm{QSec}(\theta')$.

We note the following special case of Theorem 3.1 which is the counterpart of Corollary 3.5.

COROLLARY 3.7. *Let A be the generator of a holomorphic C_0-semigroup and let $B : D(A) \to X$ be compact. Then $A+B$ generates a holomorphic C_0-semigroup.*

Corollary 3.7 is no longer true if we consider the smaller class of operators associated with a form on a Hilbert space. We shall now make this more precise.

Consider an operator A on a Hilbert space H which is associated with a closed form $a : V \times V \to \mathbb{C}$ where $V \underset{d}{\hookrightarrow} H$. Let $\varphi, u \in H$. Then

$$Bx := (Ax \mid \varphi)_H u$$

defines a rank-1 operator $B \in \mathcal{L}(D(A), X)$ which we denote by

$$B = (\varphi \otimes u)A.$$

We suppose that $V \neq H$, i.e., A is unbounded.

THEOREM 3.8. *Let $\varphi \in H \backslash V$. Then there exists $u \in V$ such that the operator $A + (\varphi \otimes u)A$ is not associated with a form.*

For the proof we need the following lemma. Since A is associated with a form, there exists N such that $(n+A)^{-1}$ exists whenever $n \geq N$.

LEMMA 3.9. *Let $\varphi \in H$ such that $((nA(n+A)^{-1}v \mid \varphi)_H)_{n \geq N}$ is bounded for each $v \in V$. Then $\varphi \in V$.*

Proof. Let $\varphi_n = n(n+A^*)^{-1}\varphi$. Since A^* is sectorial and densely defined, $\varphi_n \to \varphi$ in H by Proposition 2.2.
a) We show that $\sup_{n \geq N} |a(v, \varphi_n)| < \infty$ for all $v \in V$. In fact, recall that the adjoint A^* of A is associated with a^* where $a^*(u, v) = \overline{a(v, u)}$ [Ta, Theorem 2.2.2]. Thus

$$a(v, \varphi_n) = \overline{a^*(n(n+A^*)^{-1}\varphi, v)} = \overline{(nA^*(n+A^*)^{-1}\varphi \mid v)_H} = (nA(n+A)^{-1}v \mid \varphi)_H.$$

b) Let $\omega \in \mathbb{R}$, $\alpha > 0$ such that $\mathrm{Re}\, a(u, u) + \omega(u \mid u)_H \geq \alpha \|u\|_V^2 \quad (u \in V)$. Let $\psi \in V^*$. By the Lax-Milgram Theorem there exists $v \in V$ such that $\psi(w) = \lambda_0(w \mid v)_H + a^*(w, v) \quad (w \in V)$. Thus a) shows that $(\varphi_n)_{n \in \mathbb{N}}$ is weakly bounded in V. Since V is reflexive and continuously embedded into H and since $\varphi_n \to \varphi$ in H it follows that $\varphi \in V$. ∎

Proof of Theorem 3.8. Since $\varphi \notin V$, it follows from Lemma 3.9 that there exists $v \in V$ such that $(nA(n+A)^{-1}v \mid \varphi)_H$ is unbounded. Let $x_{n_k} := n_k(n_k + A)^{-1}v$ be a subsequence such that $(Ax_{n_k} \mid \varphi)_H = r_k e^{i\theta_k}$ is such that $r_k \to \infty$ and $\theta_k \to \theta$. By Propositions 2.3 and 2.2, $x_{n_k} \to v$ in V. Let $u = -e^{-i\theta}v$. Then

$$(Ax_{n_k} \mid \varphi)_H (u \mid x_{n_k})_H = -r_{n_k} e^{i\theta_k} e^{-i\theta} (v \mid x_{n_k})_H.$$

Let $(Ax_{n_k} \mid x_{n_k})_H = \rho_k e^{i\varphi_k}$. Then $|\rho_k| = |a(x_{n_k}, x_{a_k})| \leq c$ for some $c \geq 0$ since (x_{n_k}) is bounded in V and a is continuous on V. Thus

$$((A + (\varphi \otimes u)A)x_{n_k} \mid x_{n_k})_H = \rho_k e^{i\varphi_k} - r_k e^{i(\theta_k - \theta)}(v \mid x_{n_k})_H.$$

This shows that the numerical range of $A + (\varphi \otimes u)A$ is not contained in any sector $\Sigma_{\theta'} + \omega$, so $A + (\varphi \otimes u)A$ is not associated with a form, by Theorem 1.3. ∎

On the other hand we shall see in the next section that form-similar operators are stable under finite rank perturbation and even under nuclear perturbation.

4. Nuclear perturbation of H^∞-calculus. Let A be a sectorial operator on a Banach space X with bounded H^∞-calculus. We shall give here some positive results showing that certain perturbations of A also have bounded H^∞-calculus. When X is a Hilbert space, Theorem 2.5 shows that such results establish stability of the class of form-similar operators.

The following is an easy result (see [AHS], [KW, Proposition 13.1]).

PROPOSITION 4.1. *Let $A \in H^\infty$ and $B \in \mathcal{L}(D(A^\gamma), X)$ where $0 \leq \gamma < 1$. Then $A + B \in QH^\infty$.*

Now we want to give a result of this type for the case when $\gamma = 1$, but one has to impose further assumptions on B, since the result clearly fails if $B = -2A$, for example. A natural assumption is that $B : D(A) \to X$ is compact (see Theorem 3.1 and Corollary 3.7). We consider here only the case when $B : D(A) \to X$ is a nuclear operator and we give a direct proof using the H^∞-calculus directly. Since this work was carried out, N.J. Kalton [Kal] has obtained a stronger result but his method uses other characterizations of bounded H^∞-calculus. He has also shown that the result is not true for arbitrary compact operators $B : D(A) \to X$ when X is a Hilbert space but it is true for arbitrary compact operators when X is an L^1-space.

Let C be a nuclear operator on X, so that

$$C = \sum_{n=1}^{\infty} \varphi_n \otimes u_n$$

where $\varphi_n \in X^*$, $u_n \in X$,

$$(\varphi_n \otimes u_n)(x) = (x, \varphi_n)u_n \qquad (x \in X), \qquad \sum_{n=1}^{\infty} \|\varphi_n\| \, \|u_n\| < \infty.$$

Let $B = CA$, and consider $A + B : D(A) \to X$.

THEOREM 4.2. *Suppose that A is densely defined, invertible, and sectorial with a bounded H^∞-calculus on some sector Σ_θ. Let $B = CA$ where C is a nuclear operator on X, and suppose that $A + B$ is invertible and sectorial. Then $A + B$ has a bounded H^∞-calculus, and*

$$\vartheta_H(A + B) \leq \max(\vartheta_H(A), \vartheta_S(A + B)), \qquad \vartheta_H(A) \leq \max(\vartheta_H(A + B), \vartheta_S(A)).$$

COROLLARY 4.3. *Suppose that A is densely defined and $A \in QH^\infty$. Let $B : D(A) \to X$ be nuclear. Then $A + B \in QH^\infty$ and $\vartheta_{QH}(A + B) = \vartheta_{QH}(A)$.*

Proof of Theorem 4.2. Let $\theta_1 > \max(\vartheta_H(A), \vartheta_S(A+B))$ and choose θ_2, θ_3 such that

$$\theta_1 > \theta_2 > \theta_3 > \max(\vartheta_H(A), \vartheta_S(A+B)).$$

Consider the contours $\Gamma_j = \{re^{\pm i\theta_j} : r \geq 0\}$ $(j = 2, 3)$, taken in the downward direction. Since $B : D(A) \to X$ is compact, Theorem 3.1 shows that $\|R(\lambda, A+B)\|_{\mathcal{L}(X, D(A))}$ is bounded for $\lambda \in \mathbb{C} \setminus \Sigma_{\theta_3}$ and $|\lambda|$ sufficiently large. Since $\lambda \mapsto R(\lambda, A+B)$ is continuous from Γ_2 to $\mathcal{L}(X, D(A))$,

$$M := \sup_{\lambda \in \Gamma_2} \|R(\lambda, A+B)\|_{\mathcal{L}(X, D(A))} < \infty.$$

Let $f \in H^\infty(\Sigma_{\theta_1})$ (if desired, one may assume also that f is rational with no poles in $\overline{\Sigma}_{\theta_1}$). For $z \in \Sigma_{\theta_3}$, let $g(z) = (1+z)^{-1/2} f(z)$, and

$$S(z) = \frac{1}{2\pi i} \int_{\Gamma_2} g(\lambda) A R(\lambda, A+B) \frac{d\lambda}{\lambda - z}.$$

The integral defining $S(z)$ is absolutely convergent and

$$\|S(z)\| \leq \frac{M\|f\|_\infty}{2\pi} \int_{\Gamma_2} \frac{|d\lambda|}{|1+\lambda|^{1/2} |\lambda - z|}.$$

Let $\lambda = re^{\pm i\theta_2} \in \Gamma_2$. If $r \geq 2|z|$, then

$$|\lambda - z| \geq r - |z| \geq \frac{1}{3}(r + |z|).$$

If $0 \leq r \leq 2|z|$, then

$$|\lambda - z| \geq |z| \sin(\theta_2 - \theta_3) \geq \frac{\sin(\theta_2 - \theta_3)}{3}(r + |z|).$$

Also

$$|1 + \lambda| = r \left|1/r + e^{\pm i\theta_2}\right| \geq r \sin \theta_2.$$

Hence

$$\|S(z)\| \leq c_0 \|f\|_\infty \int_0^\infty \frac{dr}{r^{1/2}(r + |z|)} = \frac{c_1 \|f\|_\infty}{|z|^{1/2}} \int_0^\infty \frac{ds}{s^{1/2}(1+s)} = c_2 \frac{\|f\|_\infty}{|z|^{1/2}}$$

for some c_0, c_1, c_2 depending on θ_2, θ_3, M. Take $\delta > 0$ such that $\{\lambda : |\lambda| \leq \delta\} \subset \rho(A+B)$ and let $\tilde{\Gamma}_2$ be the contour $\{re^{\pm i\theta_2} : r \geq \delta\} \cup \{\delta e^{i\theta} : -\theta_2 \leq \theta \leq \theta_2\}$. By Cauchy's Theorem,

$$S(z) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_2} g(\lambda) A R(\lambda, A+B) \frac{d\lambda}{\lambda - z} + g(z) A R(z, A+B) \quad (z \in \Sigma_{\theta_3}, |z| < \delta).$$

The right-hand side stays bounded as $z \to 0$. Hence there is a constant c_3 such that

$$\|S(z)\| \leq \frac{c_3 \|f\|_\infty}{|1+z|^{1/2}} \quad (z \in \Sigma_{\theta_3}).$$

For $x \in X$, $\varphi \in X^*$, $z \in \Sigma_{\theta_3}$, let

$$g_{x,\varphi}(z) = \langle S(z)x, \varphi \rangle, \qquad h_{x,\varphi}(z) = (1+z)^{1/2} g_{x,\varphi}(z).$$

Then $g_{x,\varphi}, h_{x,\varphi} \in H^\infty(\Sigma_{\theta_3})$ and

$$g_{x,\varphi}(A) = (1+A)^{-1/2} h_{x,\varphi}(A), \qquad \|h_{x,\varphi}(A)\| \leq c_4 \|f\|_\infty \|x\| \, \|\varphi\|,$$

where c_4 depends on c_3 and the $H^\infty(\Sigma_{\theta_3})$-constant of A. Now

$$\begin{aligned} g_{x,\varphi}(A) &= \frac{1}{2\pi i}\int_{\Gamma_3} \langle S(z)x,\varphi\rangle R(z,A)\,dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_3} \frac{1}{2\pi i}\int_{\Gamma_2} g(\lambda)\langle AR(\lambda,A+B)x,\varphi\rangle \frac{d\lambda}{\lambda - z} R(z,A)\,dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_2} \left(\frac{1}{2\pi i}\int_{\Gamma_3} \frac{R(z,A)}{\lambda - z}\,dz\right) g(\lambda)\langle AR(\lambda,A+B)x,\varphi\rangle\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_2} g(\lambda)\langle AR(\lambda,A+B)x,\varphi\rangle R(\lambda,A)\,d\lambda \end{aligned}$$

by Fubini's and Cauchy's Theorems. Also

$$g(A) = \frac{1}{2\pi i}\int_{\Gamma_2} g(\lambda)R(\lambda,A)\,d\lambda, \qquad g(A+B) = \frac{1}{2\pi i}\int_{\Gamma_2} g(\lambda)R(\lambda,A+B)\,d\lambda,$$

where the integrals are absolutely convergent. For $x \in X$,

$$\begin{aligned} g(A+B)x - g(A)x &= \frac{1}{2\pi i}\int_{\Gamma_2} g(\lambda)R(\lambda,A)BR(\lambda,A+B)x\,d\lambda \\ &= \sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{\Gamma_2} g(\lambda)\langle AR(\lambda,A+B)x,\varphi_n\rangle R(\lambda,A)u_n\,d\lambda \\ &= \sum_{n=1}^{\infty} g_{x,\varphi_n}(A)u_n = (1+A)^{-1/2}\sum_{n=1}^{\infty} h_{x,\varphi_n}(A)u_n. \end{aligned}$$

Here the interchange of integration and summation is justified by Lebesgue's Series Theorem.

Let $T_n x = h_{x,\varphi_n}(A)u_n$. Then $\|T_n x\| \le c_4\|f\|_\infty\|x\|\,\|\varphi_n\|\,\|u_n\|$, so that $\|T_n\| \le c_4\|f\|_\infty\|\varphi_n\|\,\|u_n\|$. Thus $\sum_{n=1}^\infty T_n$ is absolutely convergent and

$$\left\|\sum_{n=1}^{\infty} T_n\right\| \le c_4\|f\|_\infty \sum_{n=1}^{\infty}\|\varphi_n\|\,\|u_n\|. \tag{4.1}$$

Take n fixed, and let (x_k) be a sequence in X with $\|x_k\| \le 1$. Montel's Theorem shows that there is a subsequence (x_{k_r}) such that

$$k(\lambda) := \lim_{r\to\infty} \langle AR(\lambda,A+B)x_{k_r},\varphi_n\rangle$$

exists for all $\lambda \in \Gamma_2$. By Lebesgue's Theorem $\lim_{r\to\infty} g_{x_{k_r},\varphi_n}(z)$ exists for each $z \in \Sigma_{\theta_3}$. Hence $\lim_{r\to\infty} h_{x_{k_r},\varphi_n}(z)$ exists. By the property (ii) in Definition 2.4 a), $\lim_{r\to\infty} T_n x_{k_r}$ exists. Thus T_n is compact, and hence $K_f := \sum_{n=1}^\infty T_n$ is compact. We have

$$g(A+B) = g(A) + (1+A)^{-1/2}K_f.$$

Now take $f = 1$, so that $g(A) = (1+A)^{-1/2}$. Then we obtain that

$$(1+A+B)^{-1/2} = (1+A)^{-1/2}(1+K_1).$$

This shows that $D((1+A+B)^{1/2}) \subset D((1+A)^{1/2})$ and

$$(1+A)^{1/2}(1+A+B)^{-1/2} = I + K_1 \tag{4.2}$$

where K_1 is compact. Since $(1+A)^{1/2}(1+A+B)^{-1/2}$ is injective, Fredholm theory implies that it is invertible. Thus

$$D((1+A+B)^{1/2}) = D((1+A)^{1/2}).$$

Now, for general rational, bounded f,

$$f(A+B) = (1+A+B)^{1/2}g(A+B) = (1+A+B)^{1/2}(1+A)^{-1/2}(f(A)+K_f). \tag{4.3}$$

It now follows from (4.1) that

$$\|f(A+B)\| \le \|(1+A+B)^{1/2}(1+A)^{-1/2}\|\Big(c_5 + c_4\sum_{n=1}^{\infty}\|\varphi_n\|\,\|u_n\|\Big)\|f\|_\infty$$

where c_5 is the $H^\infty(\Sigma_{\theta_1})$-constant of A. Thus $A+B$ has bounded $H^\infty(\Sigma_{\theta_1})$-calculus, whenever

$$\theta_1 > \max(\vartheta_H(A), \vartheta_S(A+B)).$$

This shows that

$$\vartheta_H(A+B) \le \max(\vartheta_H(A), \vartheta_S(A+B)).$$

Finally, note that

$$A = (A+B) + (-CA(A+B)^{-1})(A+B).$$

We can apply the above with A replaced by $A+B$ and C by $-CA(A+B)^{-1}$, which is nuclear, to deduce that

$$\vartheta_H(A) \le \max(\vartheta_H(A+B), \vartheta_S(A)). \ \blacksquare$$

Note that (4.2) and (4.3) show that, for any $f \in H^\infty(\Sigma_\theta)$ where $\theta > \vartheta_H(A)$,

$$f(A+B) = (I+K_1)^{-1}(f(A)+K_f),$$

where K_1 and K_f are compact. Hence,

$$f(A+B) - f(A) = (I+K_1)^{-1}(K_f - K_1 f(A))$$

which is compact. Moreover, (4.1) shows that

$$\|f(A+B) - f(A)\| \to 0 \quad \text{as} \quad \|C\|_\pi \to 0,$$

where $\|C\|_\pi$ is the nuclear norm of C.

Proof of Corollary 4.3. Take θ_1, θ_2 such that $\pi > \theta_1 > \theta_2 > \vartheta_{QH}(A)$. Since $B : D(A) \to X$ is compact, Theorem 3.1 shows that there exists ω such that $A+\omega$ is invertible and sectorial, $A+\omega \in H^\infty(\theta_2)$, $A+B+\omega$ is invertible, and $A+B+\omega \in \mathrm{Sec}(\theta_2)$. Let $\tilde{A} = A+\omega$, and $C = B(A+\omega)^{-1}$ which is nuclear on X. By Theorem 4.2, $A+B+\omega = \tilde{A}+C\tilde{A} \in H^\infty(\theta_1)$. So $A+B \in QH^\infty$ and $\vartheta_{QH}(A+B) \le \vartheta_{QH}(A)$. The reverse inequality follows by replacing A by $A+B$ and B by $-B$. $\blacksquare$

5. Cosine functions. Let X be a Banach space. A strongly continuous function $C : \mathbb{R} \to \mathcal{L}(X)$ is called a *cosine function* if $C(0) = I$ and

$$C(t+s) + C(t-s) = 2C(t)C(s) \tag{5.1}$$

whenever $s, t \in \mathbb{R}$. Taking $t = 0$ shows that any cosine function is an even function.

We shall refer primarily to [ABHN, Section 3.14] for standard facts about cosine functions. Other texts include [Go, Section II.8] and [Fa3].

In [ABHN, Definition 3.14.2] a cosine function was defined to be a strongly continuous function $C : \mathbb{R}_+ \to \mathcal{L}(X)$ satisfying (5.1) for $t \geq s \geq 0$, and it was implicitly assumed that the even extension of C is a cosine function as defined above. This is true but it is not immediately obvious and we take the opportunity to close that gap here.

PROPOSITION 5.1. *Let $C : \mathbb{R}_+ \to \mathcal{L}(X)$ be a strongly continuous function satisfying* (5.1). *Then*

(i) $C(s)C(t) = C(t)C(s)$ *whenever* $s, t \geq 0$;
(ii) *The even extension of C is a cosine function.*

Proof. Let $t \geq 0$. Replacing s and t by $t/2$ in (5.1) gives $C(t) = 2C(t/2)^2 - I$. A simple induction shows that, for each $n \geq 1$, $C(t)$ is a polynomial in $C(t/2^n)$, so they commute.

Now suppose that $C(t)$ commutes with $C(rt/2^n)$ for $r = 0, 1, \ldots, k$, for some $k \geq 1$. Since

$$C\left(\frac{(k+1)t}{2^n}\right) = 2C\left(\frac{kt}{2^n}\right)C\left(\frac{t}{2^n}\right) - C\left(\frac{(k-1)t}{2^n}\right),$$

it follows that $C(t)$ commutes with $C((k+1)t/2^n)$.

By induction, $C(t)$ commutes with $C(rt/2^n)$ for all integers $r, n \geq 1$. Now (i) holds by strong continuity of C. Then (ii) follows easily. ∎

The *generator* A of a cosine function C is defined by

$$\text{(5.2)} \qquad \begin{cases} D(A) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{2}{t^2}(C(t)x - x) \text{ exists} \right\}, \\ Ax = \lim_{t \downarrow 0} \frac{2}{t^2}(C(t)x - x). \end{cases}$$

Then A is closed and densely defined, and A generates a holomorphic C_0-semigroup of angle $\pi/2$. In particular, $-A \in \mathrm{QSec}(\theta)$ for every $\theta > 0$, and one can define the fractional powers $(\omega - A)^\gamma$ for large $\omega \in \mathbb{R}$ and all $\gamma \in \mathbb{R}$.

EXAMPLES 5.2. 1. Suppose that B generates a C_0-group $\{U(t) : t \in \mathbb{R}\}$ on a Banach space X, and let $A = B^2$. Then it easy to verify that A generates a cosine function C on X given by

$$C(t) = \frac{1}{2}(U(t) + U(-t)).$$

2. Another standard fact is that if A generates a cosine function C_0, then $A - \omega$ also generates a cosine function C_ω, for each $\omega \in \mathbb{R}$. A representation of C_ω in terms of C_0 is given in [Go, Remark II.8.11].

It is remarkable that, in many Banach spaces, all cosine functions arise from the two processes in Examples 5.2. See Theorem 5.7.

Cosine functions were introduced by Sova [So], Da Prato and Giusti [DG] and Fattorini [Fa1] as an approach to abstract second order Cauchy problems analogous to the semigroup approach to first order problems. We describe the connection here in terms of mild solutions of second order problems.

Let A be a closed operator on a Banach space X. For $x, y \in X$ consider the problem

$$P_2(x,y)\begin{cases} u''(t) = Au(t) \quad (t \geq 0), \\ u(0) = x \\ u'(0) = y. \end{cases}$$

A *mild solution* of $P_2(x,y)$ is a function $u \in C(\mathbb{R}_+, X)$ such that $\int_0^t (t-s)u(s)\,ds \in D(A)$ and

$$u(t) = x + ty + A\int_0^t (t-s)u(s)\,ds \tag{5.3}$$

for all $t \geq 0$.

Suppose that A is closed and densely defined, and $u \in C(\mathbb{R}_+, X)$. Then u is a mild solution of $P_2(x,y)$ if and only if, for each $x^* \in X^*$,

$$\frac{d^2}{dt^2}\langle u(t), x^*\rangle = \langle u(t), A^*x^*\rangle \qquad (t \geq 0), \tag{5.4}$$

$$\frac{d}{dt}\Big|_{t=0}\langle u(t), x^*\rangle = \langle y, x^*\rangle \quad \text{and} \quad u(0) = x. \tag{5.5}$$

This is easily seen using [ABHN, Proposition B.10].

If A generates a cosine function C, then $P_2(x,y)$ has a unique mild solution given by $u(t) = C(t)x + \int_0^t C(s)y\,ds$. We refer to [ABHN, Corollary 3.14.8] for this. Here we will show the converse. For classical solutions with continuous dependence on the data, such a result was given by Fattorini [Fa1], [Fa3, Theorem II.1.1].

THEOREM 5.3. *Let A be a closed operator. Assume that for each $x \in X$ there exists a unique mild solution of $P_2(x,0)$. Then A generates a cosine function.*

Proof. Let $x \in X$ and let u_x be the mild solution of $P_2(x,0)$. Since $u_x(0) = x$ and $\int_0^t (t-s)u_x(s)\,ds \in D(A)$, it follows that

$$x = \frac{d^2}{dt^2}\Big|_{t=0}\left(\int_0^t (t-s)u_x(s)\,ds\right) \in \overline{D(A)}.$$

Thus $\overline{D(A)} = X$.

Consider the mapping $\phi : x \mapsto u_x$ from X into $C(\mathbb{R}_+, X)$, where $C(\mathbb{R}_+, X)$ carries the Fréchet topology of uniform convergence on compact subsets of $\mathbb{R}_+$. Since A is closed, it follows from the well-posedness assumption that ϕ has a closed graph and hence ϕ is continuous. Consequently, for each $t \geq 0$, there exists $C(t) \in \mathcal{L}(X)$ such that $C(t)x = u_x(t)$ for all $x \in X$. The function $C : \mathbb{R}_+ \to \mathcal{L}(X)$ is strongly continuous and $C(0) = I$. We let $C(-t) = C(t)$ for $t < 0$ and we shall show that (5.1) holds.

First we observe from (5.4) that

$$\frac{d^2}{dt^2}\langle C(t)x, x^*\rangle = \langle C(t)x, A^*x^*\rangle$$

for all $t \in \mathbb{R}$, $x \in X$, $x^* \in D(A^*)$, since we consider the even extension of u_x to $\mathbb{R}$. Now let $x \in X$ and $s \in \mathbb{R}$. Let $v(t) = \frac{1}{2}(C(s+t)x + C(s-t)x)$. Then for $x^* \in D(A^*)$,

$$\frac{d^2}{dt^2}\langle v(t), x^*\rangle = \langle v(t), A^*x^*\rangle \quad (t \in \mathbb{R}),$$

$$\frac{d}{dt}\Big|_{t=0}\langle v(t), x^*\rangle = 0 \quad \text{and} \quad v(0) = C(s)x.$$

From the uniqueness assumption it follows that $v(t) = C(t)C(s)x$ for $t \geq 0$ and then for arbitrary $t \in \mathbb{R}$ since both sides are even. Thus C is a cosine function.

Denote by B the generator of C. We shall show that $A = B$. Let $x \in D(B)$. As $t \downarrow 0$,

$$\frac{2}{t^2}\int_0^t (t-s)C(s)x\,ds \to x,$$

$$A\left(\frac{2}{t^2}\int_0^t (t-s)C(s)x\,ds\right) = \frac{2}{t^2}(C(t)x - x) \to x.$$

Since A is closed, it follows that $x \in D(A)$ and $Ax = Bx$. We have shown that $B \subset A$.

In order to show the converse inclusion, note first that there exists $\lambda \in \mathbb{R}$ such that $\lambda^2 \in \varrho(B)$ [ABHN, Proposition 3.14.4]. In particular, $\lambda^2 - B$ is surjective. Since $\lambda^2 - B \subset \lambda^2 - A$, it suffices to show that $\lambda^2 - A$ is injective in order to conclude that $\lambda^2 - A = \lambda^2 - B$, and so $A = B$. Let $(\lambda^2 - A)x = 0$. Then $u(t) := (\cosh \lambda t)x$ is a solution of $P_2(x, 0)$, so $C(t)x = (\cosh \lambda t)x$. This implies that $Bx = \lambda^2 x$. Hence $x = 0$ since $\lambda^2 \in \varrho(B)$. The proof is complete. ∎

REMARK 5.4 (weak well-posedness). Let A be a closed, densely defined operator on a Banach space X. Call a function $u \in C(\mathbb{R}_+, X)$ a *weak solution* of $P_2(x, y)$ if $u(0) = x$ and (5.4), (5.5) hold. Then Theorem 5.3 shows that A generates a cosine function if and only if for each $x \in X$ there exists a unique weak solution of $P_2(x, 0)$. This is analogous to the characterization of generators of C_0-semigroups given by Ball [Ba].

There is a characterization of generators of cosine functions in terms of estimates for derivatives of the function $\lambda \mapsto \lambda R(\lambda^2, A)$ on a real interval (ω, ∞) which was independently discovered in [So], [DG] and [Fa1] (see [ABHN, Theorem 3.15.3]). Although this is analogous to the Hille-Yosida Theorem for semigroups, it is impossible to apply it in practice. Instead we shall describe a more useful way to characterize generators of cosine functions.

Write $P_2(x, y)$ as a system introducing the matrix

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}.$$

Then formally, u is a solution of $P_2(x, y)$ if and only if $u = u_1$ where

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \mathcal{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \; u_1(0) = x, \; u_2(0) = y.$$

A rigorous result realizing this idea is the following.

THEOREM 5.5 ([ABHN, Theorem 3.14.11]). *A closed operator A on a Banach space X generates a cosine function if and only if there exists a Banach space W satisfying $D(A) \subset W \hookrightarrow X$ such that the operator $\mathcal{A}$ with domain $D(\mathcal{A}) = D(A) \times W$ generates a C_0-semigroup on $W \times X$.*

It is a remarkable result due to Kisyński [Ki] that the space W is uniquely determined by these properties.

THEOREM 5.6 (Kisyński, [ABHN, Theorem 3.14.11]). *Let A be the generator of a cosine function C on a Banach space X, and let W be as in Theorem 5.5. Then*

$$W = \{x \in X : C(\cdot)x \text{ is continuously differentiable}\}$$

equipped with an appropriate norm.

We call the space $W \times X$ the *phase space* of the cosine function. It is important for several reasons. For example, if $x \in D(A)$ and $y \in W$, then the mild solution u of $P_2(x, y)$ is actually a classical solution, i.e., $u \in C^2(\mathbb{R}, X), u(t) \in D(A)$ for all $t \in \mathbb{R}$ and $u''(t) = Au(t)$ $(t \geq 0)$ [ABHN, Corollary 3.14.12].

If X is a UMD-space, there is a very simple description of cosine functions and their phase spaces due to Fattorini [Fa2]. We refer to [Fr] for the definition and properties of a UMD-space, but we note that any Hilbert space, and any L^p-space for $1 < p < \infty$, is a UMD-space.

THEOREM 5.7 (Fattorini, [ABHN, Corollary 3.16.8]). *Let A be an operator on a* UMD-*space. The following are equivalent:*

(i) *A generates a cosine function;*
(ii) *There exist a generator B of a C_0-group and $\omega \geq 0$ such that $A = B^2 + \omega$.*

When these conditions are satisfied, one may choose ω such that $\omega - A$ is sectorial, and then $(\omega - A)^{1/2}$ generates a C_0-group and the space W of Theorem 5.5 coincides with $D((\omega - A)^{1/2})$, with a norm equivalent to the graph norm.

In general Banach spaces, Theorem 5.7 is not true [Fa3, Example 8.2]. Moreover, $W = D((\omega - A)^{1/2})$ if and only if $(\omega - A)^{1/2}$ generates a C_0-group on X [Go, Theorem II.8.8, Remark II.8.9].

Next, we discuss perturbations of cosine functions. From Theorem 5.5 we obtain an immediate corollary which generalizes Example 5.2(2).

COROLLARY 5.8. *Let A be the generator of a cosine function on a Banach space X with phase space $W \times X$. Let $B \in \mathcal{L}(W, X)$. Then $A + B$ generates a cosine function.*

Proof. The operator $\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ with domain $D(\mathcal{A}) = D(A) \times W$ generates a C_0-semigroup on $W \times X$. Since $\mathcal{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$ is bounded on $W \times X$, $\mathcal{A} + \mathcal{B} = \begin{pmatrix} 0 & I \\ A+B & 0 \end{pmatrix}$ with domain $D(A) \times W$ also generates a C_0-semigroup on $W \times X$. Thus by Theorem 5.5, $A+B$ generates a cosine function and the phase space is $W \times X$ again. ∎

In many models generators of cosine functions appear as perturbations of self-adjoint operators on Hilbert space. Thus it is Corollary 5.8 and a more complicated dual version which allow one to treat a general class of hyperbolic equations (see [ABHN, Chapter 7]).

Suppose that A generates a cosine function and $W = D((\omega - A)^{1/2})$ (see Theorem 5.7). One may ask whether one can ever replace this space W in the perturbation result Corollary 5.8 by a smaller space $D((\omega - A)^{\gamma})$ for some $\gamma > \frac{1}{2}$. The answer is negative.

THEOREM 5.9 ([AB, Theorem 2.1]). *Let A be the generator of a cosine function on a Banach space X. Let $\omega \in \mathbb{R}$ such that $\omega - A$ is sectorial and let $1/2 < \gamma \leq 1$. If A is unbounded, then there exists a rank-1 operator $B \in \mathcal{L}(D((\omega - A)^\gamma), X)$ such that $A + B$ does not generate a cosine function.*

Now we consider generators of cosine functions on a Hilbert space H, or, equivalently up to addition of a scalar, squares of generators of C_0-groups on H (see Theorem 5.7). We wish to consider the relation of the generation property to the concepts considered in Sections 1 and 2, and the natural sign convention is now to take $-A$ to be the generator of a cosine function so that A is a quasi-sectorial operator. In many examples, the operator A is defined via a closed form

$$a : V \times V \to \mathbb{C} \quad \text{where} \quad V \underset{d}{\hookrightarrow} H.$$

The following open question relates to this situation.

PROBLEM 5.10. Suppose that A is associated with a closed form a on a Hilbert space H and that $-A$ generates a cosine function on H. Does it follow that the space $W = D((A + \omega)^{1/2})$ of Theorem 5.5 coincides with the domain V of the form a?

In other words, we ask whether the phase space of the cosine function has the natural form $V \times H$. This is useful to know because it is the space V which is given naturally in applications. Indeed, H is frequently $L^2(\Omega)$ with the natural scalar product, where $\Omega \subset \mathbb{R}^N$ is open, and V is the Sobolev space $H^1(\Omega)$ or $H_0^1(\Omega)$ in many cases. However, the domain of the square root may be difficult to identify.

Problem 5.10 is a special case of Kato's famous square root problem for operators A associated with coercive forms a. By [Ar, Theorem 5.5.2], the question can be reformulated as: *Under the assumptions of Problem 5.10, is $A_V \in QH^\infty$?* See Proposition 2.3.

By a counterexample of McIntosh [McI1] (see also [AT, Section 0, Theorem 6]) we know that the domain of $A^{1/2}$ and the domain of a may be different in general, but in many cases they coincide. In particular, this occurs for second-order elliptic differential operators in divergence form (see [AHLMT] and the survey article [Ar]). Another special case with a positive answer is mentioned in Remark 5.12.

In Problem 5.10 a fixed scalar product on H is assumed. However, there are striking results if we allow equivalent scalar products. In fact, if $-A$ generates a cosine function, then A is always form-similar. This result is due to Haase ([Ha1, Corollary 5.18] and [Ha2, Corollary 7.4.6]) who showed further that a scalar product can always be chosen in such a way that the numerical range $W(A)$ of A is contained in the interior of a horizontal parabola $P_\omega := \{\lambda^2 : \operatorname{Re}\lambda \geq \omega\}$. It is most remarkable that the converse is also true: Crouzeix [Cr] showed recently that $-A$ generates a cosine function if A is densely defined, $W(A)$ lies in a parabola P_ω and $\varrho(A) \setminus P_\omega \neq \emptyset$. Thus we may formulate the following generation result for cosine functions on Hilbert spaces.

THEOREM 5.11. *Let A be a densely defined operator on a Hilbert space H. The following are equivalent:*

(i) *$-A$ generates a cosine function;*

(ii) *there exist an equivalent scalar product* $(\ |\ )_1$ *on* H *and* $\omega \in \mathbb{R}$ *such that* $W_1(A) \subset P_\omega$ *and* $\varrho(A) \setminus P_\omega \neq \emptyset$, *where* $W_1(A)$ *denotes the numerical range of* A *with respect to* $(\ |\ )_1$.

One may compare Theorem 5.11 with the Lumer-Phillips Theorem which says that an operator $-A$ on a Hilbert space H generates a C_0-semigroup if the numerical range of A is contained in a half-plane $\mathbb{C}_\omega := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \omega\}$ and $\varrho(A) \setminus \mathbb{C}_\omega \neq \emptyset$. However, in contrast to cosine generators, the Lumer-Phillips Theorem does not characterise semigroup generators on Hilbert space even if one allows equivalent scalar products. Indeed, there exists an operator A on H such that $-A$ generates a C_0-semigroup on H but there is no equivalent scalar product on H such that the corresponding numerical range $W_1(A)$ of A is contained in a half-plane. See the remarks following Theorem 2.5.

In contrast to the real generation theorem [ABHN, Theorem 3.15.3] for cosine functions which involves estimates for all powers of the resolvent, Theorem 5.11 looks more promising for applications.

In the context of Theorem 5.11 we add a remark concerning Problem 5.10 on the coincidence of W and V.

REMARK 5.12. Assume that A is an operator on a Hilbert space H such that $W(A) \subset P_\omega$ and $\varrho(A) \setminus P_\omega \neq \emptyset$ for some $\omega \in \mathbb{R}$. Then we know from Theorem 5.11 that $-A$ generates a cosine function. Since the parabola P_ω lies in a sector, it follows from Theorem 1.3 that A is associated with a closed form $a : V \times V \to \mathbb{C}$ where $V \underset{d}{\hookrightarrow} H$. Moreover in this case, where $W(A)$ is inside a parabola, McIntosh [McI2] has shown that Problem 5.10 has a positive answer, i.e., $V \times H$ is the phase space.

Finally in this section we observe that generators of cosine functions on Hilbert space have bounded H^∞-calculus. Indeed, the following is a corollary of Theorems 5.11, 1.3 and 2.5.

COROLLARY 5.13. *Let* $-A$ *be the generator of a cosine function on a Hilbert space. Then* $A \in QH^\infty$.

PROBLEM 5.14. Does Corollary 5.13 hold on a larger class of Banach spaces, for instance, on UMD-spaces?

6. Hierarchic properties and perturbation. In the following we give a resumé of the preceding results by putting together a list of hierarchic properties which a closed operator may possess with respect to evolution. We then give a list explaining how these properties behave under perturbation. We refer to Lions [Li] and [AEK] for the definition of distribution semigroups which generalizes that of C_0-semigroups.

Hierarchic list. Let A be an operator on a Banach space X. Then the following hold.

1. a) A generates a cosine function.

$$\Downarrow$$

b) A generates a holomorphic C_0-semigroup.

$$\Downarrow$$

c) A generates a C_0-semigroup.

$\Downarrow$

d) A generates a distribution semigroup.

2. Assume that X is a Hilbert space. Then

a) A generates a cosine function.

$\Downarrow$ (Corollary 5.13)

e) $-A \in QH^\infty$.

$\Updownarrow$ (Theorem 2.5)

f) $-A$ is form-similar.

$\Downarrow$

b) A generates a holomorphic C_0-semigroup.

Now we give a summary of results which show how the properties a)–f) are preserved or destroyed by diverse kinds of perturbations.

Perturbation results. Let X be a Banach space, and let A be an operator on X such that $A \in \text{QSec}$ or $-A \in \text{QSec}$. Choose ω such that $\omega + A \in \text{Sec}$ or $\omega - A \in \text{Sec}$. For $0 < \gamma \leq 1$, let $A_\gamma = (\omega + A)^\gamma$ or $A_\gamma = (\omega - A)^\gamma$, and let $D(A_\gamma)$ be the domain of A_γ equipped with the graph norm.

a) Let A be the generator of a cosine function and let $0 < \gamma \leq 1$. Then $A+B$ generates a cosine function for all (rank-1) $B \in \mathcal{L}(D(A_\gamma), X)$ only if either $\gamma \leq \frac{1}{2}$ or A is bounded. If X is a UMD-space, the converse also holds. (Corollary 5.8, Theorem 5.9)

b) Let A be the generator of a holomorphic C_0-semigroup. Then $A + B$ generates a holomorphic C_0-semigroup for each compact $B : D(A) \to X$ and each bounded $B \in \mathcal{L}(D(A_\gamma), X)$ for $0 < \gamma < 1$. (Corollary 3.7, [ABHN, Theorem 3.7.23])

c,d) Let A be the generator of a distribution semigroup. Then $A + B$ generates a distribution semigroup for each rank-1 operator $B \in \mathcal{L}(D(A), X)$ only if A generates a holomorphic C_0-semigroup. (Corollary 3.5, [AB, Theorem 3.1])

e) Let $A \in QH^\infty$. Then $A + B \in QH^\infty$ for each nuclear $B : D(A) \to X$ and each bounded $B \in \mathcal{L}(D(A_\gamma), X), 0 < \gamma < 1$. (Theorem 4.2, Proposition 4.1)

Now let $X = H$ be a Hilbert space.

f) Let A be form-similar. Then $A + B$ is form-similar for each nuclear $B : D(A) \to H$ and each bounded $B : D(A_\gamma) \to H$ for $0 < \gamma < 1$. (Theorem 2.5 and e))

f′) Let A be a closed operator. Then $A + B$ is associated with a form for each $B : D(A) \to H$ of rank-1 if and only if A is bounded. (Theorem 3.8)

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