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L^p-maximal regularity for non-autonomous evolution equations

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Abstract

Let $A : [0, \tau] \to \mathcal{L}(D, X)$ be strongly measurable and bounded, where D, X are Banach spaces such that $D \hookrightarrow X$. We assume that the operator $A(t)$ has maximal regularity for all $t \in [0, \tau]$. Then we show under some additional hypothesis (viz. relative continuity) that the non-autonomous problem

$$(P) \quad \dot{u} + A(t)u = f \quad \text{a.e. on } (0, \tau), \qquad u(0) = x,$$

is well-posed in L^p; i.e. for all $f \in L^p(0, \tau; X)$ and all $x \in (X, D)_{1/p^*,p}$ there exists a unique $u \in W^{1,p}(0, \tau; X) \cap L^p(0, \tau; D)$ solution of (P), where $1 < p < \infty$. If the operators $A(t)$ are accretive, we show that conversely, well-posedness of (P) implies that $A(t)$ has maximal regularity for all $t \in [0, \tau]$. We also consider the non-autonomous second order problem

$$\ddot{u} + B(t)\dot{u} + A(t)u = f \quad \text{a.e. on } (0, \tau), \qquad u(0) = x, \qquad \dot{u}(0) = y,$$

for which we prove similar regularity and perturbation results.

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0. Introduction

In this article we study L^p-maximal regularity for non-autonomous first order and second order Cauchy problems.

In order to explain these concepts, let X and D be two Banach spaces such that D is continuously and densely embedded into X. We say that a single operator $A \in \mathcal{L}(D, X)$ has *L^p-maximal regularity* ($p \in (1, \infty)$) if for every $f \in L^p(0, \tau; X)$ there exists a unique $u \in W^{1,p}(0, \tau; X) \cap L^p(0, \tau; D)$ such that

$$\dot{u} + Au = f \quad \text{a.e. on } (0, \tau), \qquad u(0) = 0. \tag{0.1}$$

The property of L^p-maximal regularity has been studied intensively in the recent years due to its applications to proving existence, uniqueness and regularity of solutions of linear and especially nonlinear evolution equations; see [4–6,11,12,15,25] for abstract results and their applications.

If A is not constant but if $A : [0, \tau] \to \mathcal{L}(D, X)$ is a bounded and strongly measurable function, then L^p-maximal regularity of A is defined similarly as above, the problem (0.1) now being a non-autonomous first order Cauchy problem. The L^p-maximal regularity of the non-autonomous Cauchy problem is less well understood. Hieber and Monniaux [20,21], Štrkalj [33] and Portal and Štrkalj [29] proved L^p-maximal regularity assuming Acquistapace–Terreni conditions on A and L^p-maximal regularity for every $A(t)$. Their approach goes back to the operator sum method of Da Prato and Grisvard [13] and Acquistapace and Terreni [1] but it also uses kernel estimates or the concept of R-boundedness; the time regularity of A is rather strong but their results have the advantage that the domains of the $A(t)$ may depend on t.

More recently, Prüss and Schnaubelt [30] and Amann [3] proved L^p-maximal regularity assuming only that A is continuous and that $A(t)$ has L^p-maximal regularity for every $t \in [0, \tau]$.

In this article, we prove L^p-maximal regularity assuming only that A is bounded, strongly measurable and relatively continuous and that $A(t)$ has L^p-maximal regularity for every $t \in [0, \tau]$. In the application to a non-autonomous diffusion equation which we describe in Section 4, this weaker regularity assumption means that the lower order coefficients need only be measurable in time.

In addition to L^p-maximal regularity, we prove well-posedness of the initial value problem

$$\dot{u} + A(t)u = 0 \quad \text{a.e. on } (s, \tau), \qquad u(s) = x,$$

in certain real interpolation spaces, where $s \in [0, \tau]$, and if the $A(t)$ are in addition accretive, then we actually prove well-posedness of the initial value problem in X itself. Regularity of the solutions or of the associated evolution families (see [8] for this concept) is described in Sections 2 and 3. Note that here our proofs are more direct than those of the corresponding results in [30].

Finally, we study also L^p-maximal regularity of second order Cauchy problems. Let D_A and D_B be two Banach spaces which embed continuously and densely into X. We say that the couple (A, B) of two bounded and strongly measurable functions $A : [0, \tau] \to \mathcal{L}(D_A, X)$ and

$B : [0, \tau] \to \mathcal{L}(D_B, X)$ has *L^p-maximal regularity* if for every $f \in L^p(0, \tau; X)$ there exists a unique $u \in W^{2,p}(0, \tau; X) \cap L^p(0, \tau; D_A)$ such that $\dot{u} \in L^p(0, \tau; D_B)$ and

$$\ddot{u} + B(t)\dot{u} + A(t)u = f \quad \text{a.e. on } (0, \tau), \qquad u(0) = \dot{u}(0) = 0.$$

The concept of L^p-maximal regularity of the second order Cauchy problem is more recent and has been studied in [9] in the autonomous case. In Section 5, we prove L^p-maximal regularity for the non-autonomous Cauchy problem using similar ideas than for the first order problem. But here the resolvent estimates we need are more difficult to obtain and need new ideas. An application to a non-autonomous, strongly damped wave equation is described in Section 6.

1. Perturbation of maximal regularity

Let X and D be two Banach spaces such that D is continuously and densely embedded into X. We write $D \hookrightarrow_d X$.

Let $A \in \mathcal{L}(D, X)$.

Definition 1.1. Let $p \in (1, \infty)$. We say that A has *L^p-maximal regularity* (and write $A \in \mathcal{MR}_p$) if for some bounded interval (a, b) and all $f \in L^p(a, b; X)$ there exists a unique $u \in W^{1,p}(a, b; X) \cap L^p(a, b; D)$ such that

$$\dot{u} + Au = f \quad \text{a.e. on } (a, b), \qquad u(a) = 0. \tag{1.1}$$

Recall, that $W^{1,p}(a, b; X) \subset C([a, b]; X)$ so that the condition $u(a) = 0$ in the above equation makes sense.

It is known that the property of L^p-maximal regularity is independent of the bounded interval (a, b), and if $A \in \mathcal{MR}_p$ for some $p \in (1, \infty)$ then $A \in \mathcal{MR}_p$ for all $p \in (1, \infty)$ [31,7,25]. Hence, we can write $A \in \mathcal{MR}$ for short.

It is also known that if $A \in \mathcal{MR}$ then $-A$, seen as an unbounded operator on X, generates a holomorphic C_0-semigroup $(e^{-tA})_{t \geqslant 0}$ on X [17,25]. The converse is true if X is a Hilbert space [16]. Then $A \in \mathcal{MR}$ if and only if $-A$ generates a holomorphic semigroup. However, this equivalence is restricted to Hilbert spaces, at least in the class of all Banach spaces with unconditional basis [23]. On the other hand, there are large classes of operators which are known to have the property of maximal regularity (see [15] and also the survey article [4]).

Now we fix $p \in (1, \infty)$. By

$$MR(a, b) := W^{1,p}(a, b; X) \cap L^p(a, b; D)$$

we denote the *maximal regularity space* which is a Banach space for the norm

$$\|u\|_{MR} = \|u\|_{W^{1,p}(a,b;X)} + \|u\|_{L^p(a,b;D)}.$$

Moreover, we consider the *trace space* $Tr := \{u(a)\colon\ u \in MR(a, b)\}$ with the norm

$$\|x\|_{Tr} = \inf\bigl\{\|u\|_{MR}\colon\ x = u(a)\bigr\}.$$

The space Tr is isomorphic to the real interpolation space $(X, D)_{\frac{1}{p^*},p}$ where $\frac{1}{p^*} + \frac{1}{p} = 1$ [26, Chapter 1]. In particular, Tr does not depend on the choice of the interval. We also note that

$$MR(a,b) \underset{d}{\hookrightarrow} C\big([a,b], Tr\big).$$

If $A \in \mathcal{MR}$, then for every $x \in Tr$ the homogeneous problem

$$\dot{u} + Au = 0 \quad \text{a.e. on } (a,b), \qquad u(a) = x \tag{1.2}$$

has the unique solution $u(t) = e^{-(t-a)A}x \in MR(a,b)$. Clearly, the condition $x \in Tr$ is necessary for u to belong $MR(a,b)$. The sufficiency will be proved below in a more general context (Proposition 1.3).

It will be convenient to formulate the property of maximal regularity in terms of the closedness of the sum of two operators (see Clément [11] for more information of this aspect). For this, consider first the operator $\mathcal{B}$ on $L^p(a,b;X)$ given by

$$\begin{aligned} D(\mathcal{B}) &= \big\{u \in W^{1,p}(a,b;X)\colon\ u(a) = 0\big\}, \\ \mathcal{B}u &= \dot{u}. \end{aligned} \tag{1.3}$$

Then $-\mathcal{B}$ generates the shift semigroup $(e^{-t\mathcal{B}})_{t\geqslant 0}$ given by

$$\big(e^{-t\mathcal{B}}u\big)(s) = \begin{cases} u(s-t), & 0 \leqslant t \leqslant s-a, \\ 0, & t > s-a. \end{cases}$$

Now assume that $-A$ generates a C_0-semigroup $(e^{-tA})_{t\geqslant 0}$ on X (where $A \in \mathcal{L}(D,X)$ is the given operator, seen as an unbounded operator on X). Consider the multiplication operator $\mathcal{A}$ on $L^p(a,b;X)$ given by

$$\begin{aligned} D(\mathcal{A}) &= L^p(a,b;D), \\ (\mathcal{A}u)(s) &= Au(s), \quad s \in (a,b). \end{aligned}$$

Then $-\mathcal{A}$ generates the C_0-semigroup $(e^{-t\mathcal{A}}u)(s) = e^{-tA}u(s)$ $(s \in (a,b))$. The shift semigroup $(e^{-t\mathcal{B}})_{t\geqslant 0}$ and the multiplication semigroup $(e^{-t\mathcal{A}})_{t\geqslant 0}$ commute and the product

$$\big(e^{-t\mathcal{B}}e^{-t\mathcal{A}}\big)_{t\geqslant 0} \tag{1.4}$$

defines a C_0-semigroup on $L^p(a,b;X)$ whose generator is the closure of $-(\mathcal{A}+\mathcal{B})$. In fact, $D(\mathcal{A}) \cap D(\mathcal{B})$ is dense and invariant by the product semigroup and so a core of its generator. Since the product semigroup is nilpotent, the closure of $\mathcal{A}+\mathcal{B}$ has empty spectrum.

Now assume that $A \in \mathcal{MR}$. This is equivalent to saying that the sum $\mathcal{A}+\mathcal{B}$ is closed. We denote this sum by L_A for short. Thus $-L_A$ is the generator of the semigroup (1.4) and has empty spectrum. We have

$$\begin{aligned} D(L_A) &= \big\{u \in MR(a,b)\colon\ u(a) = 0\big\}, \\ L_A u &= \dot{u} + Au. \end{aligned} \tag{1.5}$$

The inverse L_A^{-1} of L_A gives the solution of the maximal regularity problem. For $f \in L^p(a,b;X)$, $u = L_A^{-1} f$ is the unique solution in $MR(a,b)$ of the inhomogeneous problem (1.1).

Fix $\tau > 0$. For each subinterval $(a,b) \subset (0,\tau)$ we may consider the operator L_A on $L^p(a,b;X)$. We do not use different notations for these operators in order to keep notations simple. We need the following uniform estimate.

Lemma 1.2. *Assume that $A \in \mathcal{MR}$. There exists a constant $M \geqslant 0$ such that*

$$\left\|(\lambda + L_A)^{-1}\right\|_{\mathcal{L}(L^p(a,b;X),MR(a,b))} \leqslant M \quad \text{and} \quad \left\|(1+\lambda)(\lambda + L_A)^{-1}\right\|_{\mathcal{L}(L^p(a,b;X))} \leqslant M,$$

for all intervals $(a,b) \subset (0,\tau)$ and all $\lambda \geqslant 0$.

Proof. Since $-L_A$ generates a C_0-semigroup on $L^p(0,\tau;X)$ and has empty spectrum one has

$$\sup_{\lambda \geqslant 0} \left\|(\lambda + L_A)^{-1}\right\|_{\mathcal{L}(L^p(0,\tau;X),MR(0,\tau))} < \infty$$

and

$$\sup_{\lambda \geqslant 0} \left\|(1+\lambda)(\lambda + L_A)^{-1}\right\|_{\mathcal{L}(L^p(0,\tau;X))} < \infty.$$

Let $(a,b) \subset (0,\tau)$ be any subinterval, and let $\lambda \geqslant 0$. Let $f \in L^p(a,b;X)$. Extend f by 0 to $(0,\tau)$. Let $u \in MR(0,\tau)$ such that

$$\dot{u} + \lambda u + Au = f \quad \text{a.e. on } (0,\tau), \qquad u(0) = 0.$$

Since $f(t) = 0$ on $(0,a)$, it follows from unique solvability of (1.1) on $(0,a)$ that $u = 0$ on $[0,a]$. This shows that $(\lambda + L_{A,a,b})^{-1}$ is the restriction of $(\lambda + L_{A,0,\tau})^{-1}$ where $L_{A,a,b}$ is the operator L_A on $L^p(a,b;X)$. $\square$

Now we prove the perturbation result. We consider the given operator $A \in \mathcal{L}(D,X)$, a fixed $p \in (1,\infty)$ and $\tau > 0$.

Proposition 1.3. *Assume that $A \in \mathcal{MR}$. Let $(a,b) \subset (0,\tau)$ and $B : (a,b) \to \mathcal{L}(D,X)$ be strongly measurable. Suppose that there exists $\eta \geqslant 0$ such that*

$$\left\|B(t)x\right\|_X \leqslant \frac{1}{2M}\|x\|_D + \eta\|x\|_X \tag{1.6}$$

for all $x \in D$, $t \in (a,b)$, where M is the constant in Lemma 1.2. Then for all $f \in L^p(a,b;X)$, $x \in (X,D)_{\frac{1}{p^*},p}$ there exists a unique $u \in MR(a,b)$ satisfying

$$\dot{u} + Au + B(t)u = f \quad \text{a.e. on } (a,b), \qquad u(a) = x. \tag{1.7}$$

Proof. (a) Let $\lambda \in \mathbb{C}$. Assume that for each $g \in L^p(a,b;X)$ there exists a unique $v \in MR(a,b)$ satisfying

$$\dot{v} + (A+\lambda)v + B(t)v = g \quad \text{a.e. on } (a,b), \qquad v(a) = 0. \tag{1.8}$$

Then also (1.7) has a unique solution if $x=0$. In fact, let $u \in MR(a,b)$, $v(t)=e^{-\lambda t}u(t)$. Then u satisfies (1.7) with $x=0$ if and only if v satisfies (1.8) for $g(t)=e^{-\lambda t}f(t)$.

(b) We assume that $x=0$. Consider the operator $\tilde{B} \in \mathcal{L}(MR(a,b), L^p(a,b;X))$ given by $(\tilde{B}u)(t)=B(t)u(t)$. Then

$$\|\tilde{B}u\|_{L^p(a,b;X)} \leqslant \left(\int_a^b \big\|B(t)u(t)\big\|_X^p\,dt\right)^{\frac{1}{p}} \leqslant \frac{1}{2M}\|u\|_{L^p(a,b;D)} + \eta\|u\|_{L^p(a,b;X)}.$$

Consider the operator $L=L_A$ on $L^p(a,b;X)$, as defined in (1.5). Then, by Lemma 1.2,

$$\begin{aligned}
&\big\|\tilde{B}(\lambda+L)^{-1}f\big\|_{L^p(a,b;X)}\\
&\quad\leqslant \frac{1}{2M}\big\|(\lambda+L)^{-1}f\big\|_{L^p(a,b;D)} + \eta\big\|(\lambda+L)^{-1}f\big\|_{L^p(a,b;X)}\\
&\quad\leqslant \frac{1}{2M}\big\|(\lambda+L)^{-1}f\big\|_{MR(a,b)} + \eta\big\|(\lambda+L)^{-1}f\big\|_{L^p(a,b;X)}\\
&\quad\leqslant \frac{1}{2}\|f\|_{L^p(a,b;X)} + \frac{\eta M}{1+\lambda}\|f\|_{L^p(a,b;X)}
\end{aligned}$$

for all $\lambda \geqslant 0$. Hence, we find $\lambda \geqslant 0$ such that

$$\big\|\tilde{B}(\lambda+L)^{-1}\big\|_{\mathcal{L}(L^p(a,b;X))} \leqslant \frac{3}{4}.$$

Thus, the operator $I+\tilde{B}(\lambda+L)^{-1}$ on $L^p(a,b;X)$ is invertible. It follows that also $\lambda+L+\tilde{B} = (I+\tilde{B}(\lambda+L)^{-1})(\lambda+L) \in \mathcal{L}(D(L), L^p(a,b;X))$ is invertible. This means that the problem (1.8) has a unique solution for every $g \in L^p(a,b;X)$. Hence, the problem (1.7) has a unique solution for every $f \in L^p(a,b;X)$, if $x=0$.

(c) Let $x \in (X,D)_{\frac{1}{p^*},p}$. Then there exists $w \in MR(a,b)$ such that $w(a)=x$. By (b), there exists a unique $v \in MR(a,b)$ solution of

$$\dot{v} + \big(A+B(t)\big)v = -\dot{w} - \big(A+B(t)\big)w + f \quad \text{a.e. on } (a,b), \qquad v(a)=0.$$

Putting $u := v+w$, we have proved the existence for (1.7). Uniqueness follows from (b). □

2. The non-autonomous first order problem

Let X and D be two Banach spaces such that $D \hookrightarrow_d X$.

Fix $\tau > 0$, and let $A : [0,\tau] \to \mathcal{L}(D,X)$ be a bounded and strongly measurable function.

Definition 2.1. Let $p \in (1, \infty)$. We say that the function A has L^p*-maximal regularity* (and we write $A \in \mathcal{MR}_p(0, \tau)$) if for every $f \in L^p(0, \tau; X)$ there exists a unique $u \in MR(0, \tau)$ such that

$$\dot{u} + A(t)u = f \quad \text{a.e. on } (0, \tau), \qquad u(0) = 0. \tag{2.1}$$

We show that maximal regularity on every subinterval of $(0, \tau)$ implies the well-posedness of the homogeneous equation with initial values in the trace space and thus the existence of an evolution family on *Tr* associated with A.

Lemma 2.2. *Assume that $A \in \mathcal{MR}_p(0, \tau')$ for every $0 < \tau' \leqslant \tau$. Then for every $x \in Tr$ and every $s \in [0, \tau)$ there exists a unique $u \in MR(s, \tau)$ such that*

$$\dot{u} + A(t)u = 0 \quad \textit{a.e. on } (s, \tau), \qquad u(s) = x. \tag{2.2}$$

Moreover, if for fixed $x \in Tr$ we denote by u_s the solution of the above problem and if $s_n \to s$ then

$$\lim_{n\to\infty} \|u_{s_n} - u_s\|_{MR(s\vee s_n, \tau)} = 0.$$

Proof. *Uniqueness:* Let $u_1, u_2 \in MR(s, \tau)$ be two solutions of (2.2). Define $v = u_1 - u_2$ and extend this function by 0 on $[0, s)$. Then v is a solution of (2.1) for the right-hand side $f = 0$, and therefore, by maximal regularity, $v = 0$.

Existence: Let $w \in MR(0, \tau)$ be such that $w(0) = x$. Let $w_s(t) := w(t - s)$ for $t \in [s, \tau]$ and let

$$f_s(t) := \begin{cases} 0 & \text{if } 0 \leqslant t < s, \\ -\dot{w}_s(t) - A(t)w_s(t) & \text{if } s \leqslant t \leqslant \tau. \end{cases}$$

Let $v_s \in MR(0, \tau)$ be the unique solution of

$$\dot{v}_s + A(t)v_s = f_s \quad \text{a.e. } t \in [0, \tau], \qquad v_s(0) = 0,$$

and set $u_s(t) := v_s(t) + w_s(t)$ for $t \in [s, \tau]$. Observe that $v_s(s) = 0$ since $A \in \mathcal{MR}_p(0, s)$ and $f_s = 0$ on $(0, s)$. Thus, u_s solves (2.2).

Estimate: By definition,

$$\lim_{n\to\infty} \|w_{s_n} - w_s\|_{MR(s\vee s_n, \tau)} = 0$$

and thus also

$$\lim_{n\to\infty} \|f_{s_n} - f_s\|_{L^p(0, \tau; X)} = 0.$$

The latter estimate and the boundedness of L_A^{-1} implies

$$\lim_{n\to\infty} \|v_{s_n} - v_s\|_{MR(0, \tau; X)} = 0,$$

from where the estimate for u_s. $\square$

Let $\Delta := \{(t,s) \in [0,\tau] \times [0,\tau]\colon\ t \geqslant s\}$, and assume that $A \in \mathcal{MR}_p(0,\tau')$ for every $0 < \tau' \leqslant \tau$. By Lemma 2.2, for every $(t,s) \in \Delta$ and every $x \in Tr$ we can define

$$U(t,s)x := u(t),$$

where u is the unique solution of the initial value problem (2.2).

Proposition 2.3. *The family $(U(t,s))_{(t,s)\in\Delta}$ is a bounded, strongly continuous evolution family on Tr, i.e.*

(i) $U(t,s) \in \mathcal{L}(Tr)$ *for every* $(t,s) \in \Delta$ *and* $\sup_{(t,s)\in\Delta} \|U(t,s)\|_{\mathcal{L}(Tr)} \leqslant M$,
(ii) $U(t,t) = I$ *and* $U(t,s) = U(t,r)U(r,s)$ *for every* $0 \leqslant s \leqslant r \leqslant t \leqslant \tau$, *and*
(iii) *for every* $x \in Tr$ *the function* $\Delta \to Tr$, $(t,s) \mapsto U(t,s)x$ *is continuous.*

Proof. By the estimate from Lemma 2.2 and the boundedness of the embedding $MR(s,\tau) \hookrightarrow C([s,\tau],Tr)$, for every $x \in Tr$ the function $(t,s) \mapsto U(t,s)x$ is continuous with values in Tr. By the closed graph theorem, there exists $M \geqslant 0$ such that

$$\sup_{(t,s)\in\Delta} \left\|U(t,s)x\right\|_{Tr} \leqslant M\|x\|_{Tr}.$$

The property (ii) is an easy consequence of unique solvability of the initial value problem (2.2). □

Next we show that the solution of the inhomogeneous problem (2.1) with initial value 0 is given by convolution of the non-homogeneity f and the evolution family U.

Proposition 2.4. *Assume that $A \in \mathcal{MR}_p(0,\tau')$ for every $0 < \tau' \leqslant \tau$. For every $f \in L^p(0,\tau;Tr)$ the unique solution u of the inhomogeneous problem* (2.1) *is given by*

$$u(t) = \int_0^t U(t,s)f(s)\,ds.$$

Proof. By the estimate from Lemma 2.2, for every $x \in Tr$ the function $U(\cdot,\cdot)x$ belongs to $L^p(\Delta,D)$. Hence, for every simple function $f \in L^p(0,\tau;Tr)$, the function $(t,s) \mapsto U(t,s)f(s)$ belongs to $L^p(\Delta,D)$. Thus, if we put $v(t) = \int_0^t U(t,s)f(s)\,ds$, then v is well defined for almost every $t \in (0,\tau)$. Note that

$$U(t,s)f(s) = f(s) - \int_s^t A(r)U(r,s)f(s)\,dr$$

for $0 \leqslant s \leqslant t \leqslant \tau$, by the definition of $U(t,s)$. Thus, by Fubini's theorem, for almost every $t \in (0,\tau)$

$$\begin{aligned}v(t) &= \int_0^t f(s)\,ds - \int_0^t \int_s^t A(r)U(r,s)f(s)\,dr\,ds \\ &= \int_0^t f(s)\,ds - \int_0^t \int_0^r A(r)U(r,s)f(s)\,ds\,dr \\ &= \int_0^t f(s)\,ds - \int_0^t A(r)\int_0^r U(r,s)f(s)\,ds\,dr \\ &= \int_0^t f(s)\,ds - \int_0^t A(r)v(r)\,dr.\end{aligned}$$

Hence, v is a solution of (2.1), and by uniqueness, $u = v$. For general $f \in L^p(0,\tau;Tr)$ one argues by density. □

So far we described consequences of L^p-maximal regularity of the non-autonomous problem (2.1). Next we give a criterion which implies L^p-maximal regularity. It is based on the following definition.

Definition 2.5. A function $A:[0,\tau] \to \mathcal{L}(D,X)$ is called *relatively continuous* if for each $t \in [0,\tau]$ and all $\varepsilon > 0$ there exist $\delta > 0$, $\eta \geqslant 0$ such that for all $x \in D$, $s \in [0,\tau]$, $|s-t| \leqslant \delta$ implies that

$$\big\|A(t)x - A(s)x\big\|_X \leqslant \varepsilon\|x\|_D + \eta\|x\|_X.$$

Remark 2.6. If A is relatively continuous then by a compactness argument A is *uniformly relatively continuous*, by which we mean that for every $\varepsilon > 0$ there exist $\delta > 0$ and $b \geqslant 0$ such that for all $x \in D$ and all $s, t \in [0,\tau]$ one has

$$\big\|A(t)x - A(s)x\big\|_X \leqslant \varepsilon\|x\|_D + b\|x\|_X$$

whenever $|t-s| \leqslant \delta$. This implies in particular that each relatively continuous function is bounded.

Now the main result is the following.

Theorem 2.7. *Let $A:[0,\tau] \to \mathcal{L}(D,X)$ be strongly measurable and relatively continuous. Assume that $A(t) \in \mathcal{MR}$ for all $t \in [0,\tau]$. Then $A \in \mathcal{MR}_p(0,\tau')$ for every $0 < \tau' \leqslant \tau$ and every $p \in (1,\infty)$.*

In particular, for each $f \in L^p(0,\tau;X)$ and each $x \in (X,D)_{\frac{1}{p^*},p}$ there exists a unique $u \in W^{1,p}(0,\tau;X) \cap L^p(0,\tau;D)$ satisfying

$$\begin{cases} \dot{u} + A(t)u = f & \text{a.e. on } (0,\tau), \\ u(0) = x. \end{cases} \tag{2.3}$$

This result generalizes a result by Prüss and Schnaubelt [30] (see also Amann [3]) where it is supposed that A is norm continuous. If A is norm continuous then the semigroup generated by $-A(t)$ are uniformly exponentially bounded, i.e. $\|e^{-sA(t)}\|_{\mathcal{L}(X)} \leqslant Me^{\omega t}$ $(t \geqslant 0)$ for all $s \geqslant 0$, $t \in [0,\tau]$. Our more general hypothesis does not imply such a uniform bound. For the proof we need the following compactness property.

Lemma 2.8. *For each $t \in [0,\tau]$ let be given $\delta_t > 0$. Then there exist a partition $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = \tau$ and $t_i \in [0,\tau]$, $i = 0, 1, \ldots, n$, such that*

$$t_i \in [\tau_i, \tau_{i+1}] \subset [t_i - \delta_{t_i}, t_i + \delta_{t_i}]$$

for all $i = 0, 1, \ldots, n-1$.

Proof. By compactness, we find $t_i \in [0,\tau]$ such that $[0,\tau] \subset \bigcup_{i=0}^{n-1} [t_i - \delta_i, t_i + \delta_i]$ where $\delta_i = \delta_{t_i}$. We may assume that this covering is minimal. Then $t_i \neq t_j$ for $i \neq j$. Then we can arrange t_i in such a way that $0 \leqslant t_0 < t_1 < t_2 < \cdots < t_n \leqslant \tau$. It follows that for $i = 0, 1, \ldots, n-2$,

$$t_i - \delta_i \leqslant t_{i+1} - \delta_{i+1} \leqslant t_i + \delta_i \leqslant t_{i+1} + \delta_{i+1}.$$

Now let $\tau_0 = 0$ and

$$\tau_i = \max\{t_{i-1}, t_i - \delta_i\},$$

$i = 1, \ldots, n-1$ and $\tau_n = \tau$. □

Proof of Theorem 2.7. (a) Let $f \in L^p(0,\tau;X)$. By assumption on A, for every $t \in [0,\tau]$ there exist $\delta_t > 0$ and $\eta_t \geqslant 0$ such that for every $s \in [t - \delta_t, t + \delta_t]$ and every $x \in D$,

$$\|A(t)x - A(s)x\|_X \leqslant \frac{1}{2M(t)} \|x\|_D + \eta_t \|x\|_X,$$

where $M(t) \geqslant \|(\lambda + L_{A(t)})^{-1}\|_{\mathcal{L}(L^p(a,b;X), MR(a,b))}$ for all $\lambda \geqslant 0$ and all $(a,b) \subset (0,\tau)$ (cf. Lemma 1.2).

By Lemma 2.8, there exist a partition $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots < \tau_n = \tau$ and $t_i \in [\tau_i, \tau_{i+1}]$ such that $[\tau_i, \tau_{i+1}] \subset [t_i - \delta_i, t_i + \delta_i]$ $(\delta_i := \delta_{t_i})$. We consider the functions

$$B_i : [\tau_i, \tau_{i+1}] \to \mathcal{L}(D, X)$$

given by $B_i(s) = A(s) - A(t_i)$ $(i = 0, 1, \ldots, n-1)$. It follows from Proposition 1.3 that for each $x_i \in (X,D)_{\frac{1}{p^*},p}$ there exists a unique $u_i \in MR(\tau_i, \tau_{i+1})$ such

$$\dot{u}_i + A(t_i)u_i + B_i(t)u_i = f \quad \text{a.e. on } (\tau_i, \tau_{i+1}), \qquad u_i(\tau_i) = x.$$

Note that $A(t_i) + B_i(t) = A(t)$ on $[\tau_i, \tau_{i+1}]$.

Now let $x \in (X,D)_{\frac{1}{p^*},p}$. Then we find $u_0 \in MR(0,\tau_1)$ such that

$$\dot{u}_0 + A(t)u_0 = f \quad \text{a.e. on } (0,\tau_1), \qquad u_0(0) = x.$$

Let $x_1 = u_0(\tau_1)$. We find $u_1 \in MR(\tau_1, \tau_2)$ satisfying

$$\dot u_1 + A(t)u_1 = f \quad \text{a.e. on } (\tau_1, \tau_2), \qquad u_1(\tau_1) = x_1.$$

Continuing in this way we find functions $u_i \in MR(\tau_i, \tau_{i+1})$ such that

$$\dot u_i + A(t)u_i = f \quad \text{a.e. on } (\tau_i, \tau_{i+1}) \ (i = 0, 1, \ldots, n-1)$$

and such that $u_i(\tau_{i+1}) = u_{i+1}(\tau_{i+1})$. Thus, the function $u : [0, \tau] \to X$ given by

$$u(t) = u_i(t) \quad \text{for } t \in [\tau_i, \tau_{i+1}]$$

solves the problem (3.1).

(b) In order to show uniqueness, consider a function $u \in MR(0, \tau)$ such that $\dot u + A(t)u = 0$ a.e. on $(0, \tau)$ and $u(0) = 0$. It follows from (a) that $u = 0$ a.e. on $[0, \tau_1]$. Then we obtain successively $u = 0$ a.e. on $[\tau_i, \tau_{i+1}]$ for $i = 1, \ldots, n-1$. □

We conclude this section by establishing a perturbation result for relatively continuous functions. For this we consider an intermediate Banach space Y, i.e.,

$$D \hookrightarrow Y \hookrightarrow X.$$

For the purpose of this paper, we say that Y is *close to X compared with D* if for each $\varepsilon > 0$ there exists $\eta \geqslant 0$ such that

$$\|x\|_Y \leqslant \varepsilon \|x\|_D + \eta \|x\|_X, \quad x \in D.$$

Notice that $\beta \|x\|_X \leqslant \|x\|_Y$ for every $x \in Y$ and some constant $\beta > 0$. Thus the condition says that the norm of Y is equivalent to the norm of X up to perturbations by $\varepsilon \|x\|_D$.

There are several examples.

Example 2.9. (a) Assume that Y satisfies an interpolation inequality

$$\|x\|_Y \leqslant c \|x\|_D^{\alpha} \|x\|_X^{1-\alpha} \quad (x \in D)$$

where $0 < \alpha < 1$, $c \geqslant 0$. Then for $\delta > 0$,

$$\|x\|_Y \leqslant \delta^{\alpha} \|x\|_D^{\alpha} c \frac{1}{\delta^{\alpha}} \|x\|_X^{1-\alpha} \leqslant \alpha \delta \|x\|_D + (1-\alpha)\big(c/\delta^2\big)^{\frac{1}{1-\alpha}} \|x\|_X.$$

Thus Y is near X compared with D.

(b) Let $Y = (X, D)_{\alpha,p}$, $0 < \alpha < 1$, $1 \leqslant p \leqslant \infty$, be a real interpolation space or $Y = [X, D]_{\alpha}$ a complex interpolation space. Then the interpolation inequality (a) is valid.

(c) Let $Y = D(B^{\alpha})$ with graph norm $\|x\|_Y := \|B^{\alpha} x\|_X$ where B is an invertible sectorial operator and $0 < \alpha < 1$. Then the interpolation inequality (2.6) is valid [28, Chapter 2, Theorem 10.6].

(d) Let $D \hookrightarrow_c Y \hookrightarrow X$ where c indicates that the inclusion $D \hookrightarrow Y$ is compact. Then by Ehrling's Lemma [2, p. 334] Y is close to X compared with D.

Proposition 2.10. *Let $A : [0,\tau] \to \mathcal{L}(D,X)$ be relatively continuous and let $B : [0,\tau] \to \mathcal{L}(Y,X)$ be strongly measurable and bounded, where Y is close to X compared with D. Then $A+B$ is relatively continuous.*

Proof. Let $t \in [0,\tau]$, $\varepsilon > 0$. There exist $\delta > 0$, $\eta \geqslant 0$ such that $|t-s| \leqslant \delta$ implies

$$\bigl\| \bigl(A(t) - A(s)\bigr)x \bigr\|_X \leqslant \frac{\varepsilon}{3} \|x\|_D + \eta \|x\|_X.$$

Moreover, $\|B(s)x\|_X \leqslant c\|x\|_Y \leqslant \varepsilon/3\|x\|_D + \eta_1\|x\|_X$ for all $s \in [0,\tau]$ and some η_1. Hence

$$\bigl\| \bigl(\bigl(A(t) + B(t)\bigr) - \bigl(A(s) + B(s)\bigr)\bigr)x \bigr\|_X \leqslant \varepsilon\|x\|_D + (\eta + 2\eta_1)\|x\|_X$$

whenever $|s-t| \leqslant \delta$. □

Theorem 2.11. *Let $A : [0,\tau] \to \mathcal{L}(D,X)$ be relatively continuous and let $B : [0,\tau] \to \mathcal{L}(Y,X)$ be strongly measurable and bounded, where Y is close to X compared with D. Assume that $A(t) \in \mathcal{MR}$ for every $t \in [0,\tau]$. Then $A + B \in \mathcal{MR}_p$ for every $p \in (1,\infty)$.*

Proof. By Proposition 1.3 and by the assumption on $A(t)$, $A(t) + B(t) \in \mathcal{MR}$ for every $t \in [0,\tau]$. In fact, apply Proposition 1.3 to the operator $A = A(t)$ and to the constant function $B(t)$; use also that Y is close to X compared with D. By Proposition 2.10, $A + B$ is relatively continuous. The claim thus follows from Theorem 2.7. □

3. Accretive operators

In this section we consider the non-autonomous problem assuming that each operator $A(t)$ is accretive. We recall some facts concerning the notion of accretivity. Let X be a Banach space. By $N(x) = \|x\|$ we denote the norm on X which is a sublinear mapping. For $x \in X$, $y \in X$ we denote by

$$D_y N(x) := \lim_{h \downarrow 0} \frac{\|x + hy\| - \|x\|}{h}$$

the right Gâteaux derivative of N at x in the direction of y. Let

$$\partial N(x) := \bigl\{x' \in X' \colon \|x'\| \leqslant 1,\ \langle x', x\rangle = \|x\|\bigr\}$$

be the subdifferential of N at x. It follows from the Hahn–Banach Theorem that $\partial N(x) \neq \emptyset$ for all $x \in X$. From the definition it follows that

$$D_y N(x) \geqslant \operatorname{Re}\langle x', y\rangle \tag{3.1}$$

for all $x' \in \partial N(x)$. In fact,

$$\begin{aligned} \|x + hy\| - \|x\| &= \|x + hy\| - \langle x', x\rangle \\ &\geqslant \operatorname{Re}\langle x', x + hy\rangle - \langle x', x\rangle \\ &= h \operatorname{Re}\langle x', y\rangle. \end{aligned}$$

An operator B on X with domain $D(B)$ is called *accretive* if for every $x \in D(B)$ there exists $x' \in \partial N(x)$ such that $\mathrm{Re}\langle x', Ax\rangle \geqslant 0$. The operator B is called *strictly accretive* if $\mathrm{Re}\langle x', Ax\rangle \geqslant 0$ for all $x' \in \partial N(x)$. If $-B$ generates a contractive C_0-semigroup, then B is strictly accretive [19, Proposition 3.23]. Conversely, the Lumer–Phillips Theorem says that $-B$ generates a contractive C_0-semigroup whenever B is densely defined, accretive and $\lambda + B$ is surjective for some $\lambda > 0$. We need the following chain rule (see e.g. [27, B-II, Proposition 2.3]).

Lemma 3.1. *Let $u : [t, t+\delta) \to X$ be right-differentiable at t with right derivative $\dot{u}(t)$. Then*

$$\frac{d}{ds}\|u(s)\|\bigg|_{s=t} = D_{\dot{u}(t)}N\big(u(t)\big).$$

After these preparations we consider the non-homogeneous Cauchy problem. Let X and D be two Banach spaces such that $D \hookrightarrow_d X$. Let $A : (a,b) \to \mathcal{L}(D, X)$ be a strongly measurable function.

Proposition 3.2. *Assume that $A(t)$ is accretive for all $t \in (a,b)$. Let $u \in W^{1,p}(a,b;X) \cap L^p(a,b;D)$ be a solution of*

$$\dot{u} + A(t)u = 0 \quad \text{a.e. on } (a,b).$$

Then $\|u(t)\|$ is decreasing on $[a,b]$. In particular, if $u(a) = 0$, then $u \equiv 0$.

Proof. Let

$$J = \big\{t \in (0,\tau)\colon\ u \text{ is differentiable at } t,\ u(t) \in D,\ \dot{u} + A(t)u = 0\big\}.$$

Then, by assumption, $(0,\tau) \setminus J$ is a null set. Let $v(t) = u(\tau - t)$. We have to show that $\|v(t)\|$ is increasing. Let $t \in J$. Choose $x' \in \partial N(u(\tau - t))$ such that $\mathrm{Re}\langle x', A(\tau - t)u(\tau - t)\rangle \geqslant 0$. Then

$$\begin{aligned}
\frac{d}{ds}\|v(s)\|\bigg|_{s=t} &= D_{\dot{v}(t)}N\big(v(t)\big) \\
&\geqslant \mathrm{Re}\big\langle x', \dot{v}(t)\big\rangle \\
&= \mathrm{Re}\big\langle x', -\dot{u}(\tau - t)\big\rangle \\
&= \mathrm{Re}\big\langle x', -\dot{u}(\tau - t) - A(\tau - t)u(\tau - t)\big\rangle \\
&\quad + \mathrm{Re}\big\langle x', A(\tau - t)u(\tau - t)\big\rangle \\
&\geqslant 0.
\end{aligned}$$

Since v is absolutely continuous, also $\|v(\cdot)\|$ is absolutely continuous. Hence $\|v(t)\| = \|v(0)\| + \int_0^t \frac{d}{ds}\|v(s)\|\,ds$ is increasing. □

From Proposition 3.2 we deduce uniqueness of the non-autonomous Cauchy problem.

Theorem 3.3. *Let $A : [0,\tau] \to \mathcal{L}(D, X)$ be strongly measurable and relatively continuous. Assume that $A(t)$ is accretive for all $t \in [0,\tau]$. Then the following assertions are equivalent.*

(i) $A \in \mathcal{MR}_p(0, \tau)$ *for some* $p \in (1, \infty)$.
(ii) $A \in \mathcal{MR}_p(0, \tau)$ *for all* $p \in (1, \infty)$.
(iii) $A(t) \in \mathcal{MR}$ *for all* $t \in [0, \tau]$.

If one of the equivalent conditions (i), (ii) *or* (iii) *is satisfied then for every* $p \in (1, \infty)$ *the operator* L_A *given by*

$$D(L_A) = \{u \in W^{1,p}(0, \tau; X) \cap L^p(0, \tau; D) \colon\ u(0) = 0\},$$
$$L_A u = \dot{u} + A(\cdot)u$$

is the negative generator of a contractive C_0*-semigroup on* $L^p(0, \tau; X)$.

Proof. (i) $\Rightarrow$ (iii) We assume that $A \in \mathcal{MR}_p(0, \tau)$ for some $p \in (1, \infty)$.

(a) We first show that $A \in \mathcal{MR}_p(a, b)$ for all $(a, b) \subset (0, \tau)$. Let $f \in L^p(a, b; X)$. Extend f by 0 to the interval $(0, \tau)$ and consider the solution u of (2.1) on $(0, \tau)$. Then $u \equiv 0$ on $[0, a]$ by Proposition 3.2. Thus, $u|_{(a,b)}$ is a solution of

$$\dot{u} + A(t)u = f \quad \text{a.e. on } (a, b), \qquad u(a) = 0.$$

Uniqueness follows from Proposition 3.2.

(b) Consider the multiplication operator $\mathcal{A}$ on $L^p(0, \tau; X)$ given by

$$D(\mathcal{A}) = L^p(0, \tau; D),$$
$$\mathcal{A}u = A(\cdot)u.$$

Let $\lambda > 0$, $u \in D(\mathcal{A})$, $\lambda u + \mathcal{A}u = f$. Then $\lambda u + A(t)u = f$ a.e. on $(0, \tau)$. Since $A(t)$ is accretive, it follows that $\lambda\|u(t)\| \leqslant \|f(t)\|$ for almost all $t \in (0, \tau)$ and so $\lambda\|u\|_{L^p} \leqslant \|f\|_{L^p}$. We have shown that $\mathcal{A}$ is accretive.

Consider the negative shift-generator $\mathcal{B}$ on $L^p(0, \tau; X)$ defined in (1.3). Then $\mathcal{B}$ is strictly accretive. It follows that $L_A = \mathcal{A} + \mathcal{B}$ is accretive. Since L_A is invertible by the assumption of maximal regularity and since $\varrho(L_A)$ is open, it follows from the Lumer–Phillips Theorem that $-L_A$ generates a contractive C_0-semigroup.

(c) Choose $\varepsilon > 0$ such that

$$\varepsilon \cdot \|L_A^{-1}\|_{\mathcal{L}(L^p(0,\tau;X), MR(0,\tau))} =: q < 1/2.$$

Then $\varepsilon\|L_A^{-1}\|_{\mathcal{L}(L^p(a,b;X), MR(a,b))} \leqslant q$ whenever $(a, b) \subset (0, \tau)$.

Let $t_0 \in [0, \tau]$. Choose a nondegenerate interval $[a, b] \subset [0, \tau]$ such that $t_0 \in [a, b]$ and $\|(A(t) - A(t_0))x\|_X \leqslant \varepsilon\|x\|_D + \eta\|x\|_X$ for all $x \in D$ and all $t \in [a, b]$. Let $\mathcal{C} : L^p(a, b; D) \to L^p(a, b; X)$ be defined by $(\mathcal{C}u)(t) = (A(t_0) - A(t))u(t)$. Then

$$\begin{aligned}
\|\mathcal{C}u\|_{L^p(a,b;X)} &\leqslant \varepsilon\|u\|_{L^p(a,b;D)} + \eta\|u\|_{L^p(a,b;X)} \\
&= \varepsilon\|L_A^{-1} L_A u\|_{L^p(a,b;D)} + \eta\|u\|_{L^p(a,b;X)} \\
&\leqslant q\|L_A u\|_{L^p(a,b;X)} + \eta\|u\|_{L^p(a,b;X)}.
\end{aligned}$$

Since L_A generates a contractive C_0-semigroup and since $q < 1/2$, it follows that $\lambda + L_A + \mathcal{C} = (I + \mathcal{C}(\lambda + L_A)^{-1})(\lambda + L_A)$ is invertible for $\lambda > 0$ large enough. Thus, for all $f \in L^p(a, b; X)$ there exists a unique $u \in D(L_A)$ such that $\lambda u + L_A u + \mathcal{C}u = f$; i.e. $\dot{u} + \lambda u + A(t_0)u = f$ a.e. on (a, b), $u(a) = 0$. Thus $A(t_0) \in \mathcal{MR}$. We have shown that $A(t) \in \mathcal{MR}$ for all $t \in [0, \tau]$ and also the additional assertion concerning L_A.

The implication (iii) ⇒ (ii) follows from Theorem 2.7, while (ii) ⇒ (i) is trivial. □

For uniformly continuous $A : [0, \tau] \to \mathcal{L}(D, X)$ (but not necessarily accretive $A(t)$) Theorem 3.3 is proved in [3, Proposition 7.1] and [30, Theorem 2.5].

Next we want to establish the existence of the evolution family governing the non-autonomous problem. This can be done very easily in the accretive case. It can also be done without the accretivity assumption if one assumed that $A : [0, \tau] \to \mathcal{L}(D, X)$ is norm continuous. In fact, Prüss and Schnaubelt [30] use an approximation argument which is not easy to prove and they also use many results of the theory of evolution semigroups to do this. So the easy direct argument in the accretive case is of some interest.

Corollary 3.4. *Let $A : [0, \tau] \to \mathcal{L}(D, X)$ be strongly measurable and relatively continuous. Assume that $A(t)$ is accretive and that $A(t) \in \mathcal{MR}$ for every $t \in [0, \tau]$. Let $p \in (1, \infty)$. Then there exists a contractive evolution family $(U(t, s))_{(t,s)\in\Delta} \subset \mathcal{L}(X)$ such that for every $x \in X$ the function $u(t) := U(t, 0)x$ is the unique solution in*

$$C\big([0, \tau]; X\big) \cap L^p_{\mathrm{loc}}\big((0, \tau]; D\big) \cap W^{1,p}_{\mathrm{loc}}\big((0, \tau]; X\big)$$

of

$$\dot{u} + A(t)u = 0 \quad \text{a.e. on } (0, \tau), \qquad u(0) = x.$$

Moreover, there exists a constant $M \geqslant 0$, depending on p but independent of $x \in X$, such that

$$\big\| tu(t) \big\|_{MR(0,\tau)} \leqslant M \|x\|_X.$$

Proof. By Theorem 2.7, $A \in \mathcal{MR}_p(0, \tau')$ for every $\tau' \in (0, \tau]$ and every $p \in (1, \infty)$.

Fix $p \in (1, \infty)$, and let $(U(t, s))_{(t,s)\in\Delta}$ be the associated evolution family on the trace space (Lemma 2.3). By Proposition 3.2, for every $x \in Tr$ and every $(t, s) \in \Delta$,

$$\big\| U(t, s)x \big\|_X \leqslant \|x\|_X.$$

Hence, the evolution family U extends to a contractive, strongly continuous evolution family on X, which we will also denote by U.

For every $x \in Tr$ the function $v(t) := tU(t, 0)x$ is the unique solution of the non-homogeneous problem

$$\dot{v} + A(t)v = U(t, 0)x \quad \text{a.e. on } (0, \tau), \qquad v(0) = 0.$$

Hence,

$$\|v\|_{MR(0,\tau)} \leqslant \big\| L_A^{-1} \big\|_{\mathcal{L}(L^p(0,\tau;X), MR(0,\tau))} \big\| U(\cdot, 0)x \big\|_{L^p(0,\tau;X)} \leqslant M \|x\|_X.$$

By density, this estimate holds for every $x \in X$. In particular, for every $x \in X$ and every $p \in (1, \infty)$,

$$U(\cdot, 0)x \in L^p_{\text{loc}}\big((0, \tau]; D\big) \cap W^{1,p}_{\text{loc}}\big((0, \tau]; X\big).$$

The claim follows from the definition of U. □

Corollary 3.4 gives estimates for the homogeneous problem. As in Proposition 2.4 we can now represent the solution of the inhomogeneous problem by the evolution family U also for $f \in L^p(0, \tau; X)$ (and not only for functions with values in the trace space). Putting all together, we can formulate the following final result.

Corollary 3.5. *Let $A : [0, \tau] \to \mathcal{L}(D, X)$ be strongly measurable and relatively continuous. Assume that $A(t)$ is accretive and that $A(t) \in \mathcal{MR}$ for every $t \in [0, \tau]$. Let $p \in (1, \infty)$.*

Then for every $x \in X$ and every $f \in L^p(0, \tau; X)$ the function

$$u(t) := U(t, 0)x + \int_0^t U(t, s) f(s)\, ds$$

is the unique solution in $C([0, \tau]; X) \cap L^p_{\text{loc}}((0, \tau]; D) \cap W^{1,p}_{\text{loc}}((0, \tau]; X)$ of the problem (2.3).

4. An example

Let $\Omega \subset \mathbb{R}^n$ be an open set such that $\partial\Omega$ is bounded and of class C^2. Assume that

(H1) $a_{ij} \in C([0, \tau] \times \bar{\Omega})$ for $i, j = 1, \dots, n$ is uniformly continuous, bounded and *uniformly elliptic*, i.e.,

$$\sum_{i,j=1}^n a_{ij}(t, x)\xi_i \xi_j \geqslant \beta |\xi|^2$$

for some $\beta > 0$ and all $\xi \in \mathbb{R}^n$, $x \in \bar{\Omega}$, $t \in [0, \tau]$, and

(H2) $b_j \in L^\infty((0, \tau) \times \Omega)$ for $j = 0, 1, \dots, n$.

Define the partial differential operators $\mathcal{A}(t, x, D)$ by

$$\mathcal{A}(t, x, D)u(x) := \sum_{i,j=1}^n a_{ij}(t, x)\partial_i \partial_j u(x) + \sum_{j=1}^n b_j(t, x)\partial_j u(x) + b_0(t, x)u(x).$$

For the definition of the Besov spaces $B^s_{pq}(\Omega)$ and their properties we refer to [32]. By definition, $\mathring{B}^s_{pq}(\Omega)$ is the closure of the space of test functions on Ω in $B^s_{pq}(\Omega)$.

Theorem 4.1. *Let $p, q \in (1, \infty)$. Then for every $u_0 \in B_{pq}^{2/q'} \cap \mathring{B}_{pq}^{1/q'}(\Omega)$ and every $f \in L^q(0, \tau; L^p(\Omega))$ there exists a unique*

$$u \in C\big([0, \tau]; B_{pq}^{2/q'} \cap \mathring{B}_{pq}^{1/q'}(\Omega)\big) \cap W^{1,q}\big(0, \tau; L^p(\Omega)\big) \cap L^q\big(0, \tau; W^{2,p} \cap W_0^{1,p}(\Omega)\big)$$

solution of

$$\begin{cases} \partial_t u(t,x) - \mathcal{A}(t,x,D)u(t,x) = f(t,x) & \text{a.e. on } (0,\tau) \times \Omega, \\ u(t,x) = 0 & \text{on } (0,\tau) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{a.e. on } \Omega. \end{cases} \tag{4.1}$$

Here we let $u(t,x) = u(t)(x)$.

Proof. Let $D := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and define for every $t \in (0, \tau]$ the operator $A(t) \in \mathcal{L}(D, L^p(\Omega))$ by

$$A(t)u = -\sum_{i,j=1}^{n} a_{ij}(t,\cdot)\partial_i\partial_j u, \quad u \in D.$$

It follows from [15, Theorem 8.2] that $A(t) \in \mathcal{MR}$ for all $t \in [0, \tau]$. Moreover, A is continuous from $[0, \tau]$ into $\mathcal{L}(D, L^p(\Omega))$.

Let $Y := (L^p(\Omega), W^{2,p}(\Omega))_{\theta,s}$, where $\theta \in (\frac{1}{2}, 1)$ and $q \in (1, \infty)$. Then $Y = B_{pq}^{2\theta}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ by [32]. Hence, Y and *a fortiori* $W^{1,p}(\Omega)$ are close to $L^p(\Omega)$ compared with $W^{2,p}(\Omega)$.

Let $B : (0, \tau) \to \mathcal{L}(W^{1,p}(\Omega), L^p(\Omega))$ be given by

$$(Bu)(t) = -\sum_{j=1}^{n} b_j(t,\cdot)\partial_j u - b_0(t,\cdot)u.$$

Then B is weakly measurable. In fact, for every $g \in L^{p'}(\Omega)$,

$$\big\langle (Bu)(t), g\big\rangle = \sum_{j=1}^{n} \int_\Omega b_j(t,x)\partial_j u(x)g(x)\,dx + \int_\Omega b_0(t,x)u(x)g(x)\,dx$$

is measurable for all $u \in W^{1,p}(\Omega)$. It follows from Pettis' Theorem that B is strongly measurable. Moreover, B is clearly bounded.

Now the claim follows from Theorem 2.11. □

Theorem 4.2. *In addition to* (H1) *and* (H2), *assume that*

(H1)′ $a_{ij}(t,\cdot) \in W^{1,\infty}(\Omega)$ *for every* $t \in [0, \tau]$ *and* $\partial_i a_{ij} \in L^\infty((0,\tau) \times \Omega)$ *for* $i, j = 1, \ldots, n$.

Then for every $u_0 \in L^p(\Omega)$ and every $f \in L^q(0, \tau; L^p(\Omega))$ there exists a unique solution

$$u \in C\big([0,\tau]; L^p(\Omega)\big) \cap C\big((0,\tau]; B_{pq}^{2/q'} \cap \mathring{B}_{pq}^{1/q'}(\Omega)\big)$$
$$\cap W_{\mathrm{loc}}^{1,q}\big((0,\tau]; L^p(\Omega)\big) \cap L_{\mathrm{loc}}^q\big((0,\tau]; W^{2,p} \cap W_0^{1,p}(\Omega)\big)$$

of the problem (4.1).

Proof. Fix $p \in (1,\infty)$ and let A and B be defined as in the proof of Theorem 4.1. Then it was shown that $A+B$ is bounded and strongly measurable, relatively continuous and $A+B \in \mathcal{MR}_p$ for every $q \in (1,\infty)$.

By the additional regularity of the coefficients a_{ij} and by [14, Theorem 5.1], there exists $\omega_p \geqslant 0$ depending on p and also on the L^∞ norms of the coefficients such that the operators $A(t)+B(t)+\omega_p I$ are accretive on $L^p(\Omega)$, i.e. the $A(t)+B(t)$ are uniformly quasi-accretive. Hence, by Corollary 3.5, for every $u_0 \in L^p(\Omega)$ and every $f \in L^q(0,\tau; L^p(\Omega))$ there exists a unique function u with the regularity prescribed in the statement and which is a solution of (4.1) with b_0 replaced by $b_0+\omega_p$. The claim follows from this and a simple renormalization. □

Remark 4.3. In the proof of Theorem 4.2, instead of applying [15] in order to obtain maximal regularity for the operators $A(t)+B(t)$ one could also use that the semigroup generated by $-A(t)-B(t)$ has Gaussian estimates [14, Theorem 6.1], and the fact that Gaussian estimates imply maximal regularity [22].

Alternatively, one can use the quasicontractivity and positivity of the associated semigroups on $L^p(\Omega)$ and the fact that this also implies maximal regularity [24].

5. The non-autonomous second order problem

Let X, D_A and D_B be three Banach spaces such that D_A and D_B are densely and continuously embedded into X. Actually, in the following we assume that

$$D_A \underset{d}{\hookrightarrow} D_B \underset{d}{\hookrightarrow} X,$$

although the definition of L^p-maximal regularity makes sense in the general case, too.

Let $A \in \mathcal{L}(D_A, X)$ and $B \in \mathcal{L}(D_B, X)$.

Definition 5.1. Let $p \in (1,\infty)$. We say that the couple (A,B) has *L^p-maximal regularity* (and we write $(A,B) \in \mathcal{MR}_p$) if for some interval (a,b) and all $f \in L^p(a,b;X)$ there exists a unique $u \in W^{2,p}(a,b;X) \cap L^p(a,b;D_A)$ with $\dot u \in L^p(a,b;D_B)$ such that

$$\ddot u + B\dot u + Au = f \quad \text{a.e. on } (a,b), \qquad u(a)=\dot u(a)=0. \tag{5.1}$$

We recall that $W^{2,p}(a,b;X) \subset C^1([a,b];X)$ so that the condition $u(a)=\dot u(a)=0$ makes sense. It is known that L^p-maximal regularity is independent of the bounded interval (a,b) [9, Corollary 2.4], and it is independent of $p \in (1,\infty)$ [10].

By

$$MR(a,b) := \big\{u \in W^{2,p}(a,b;X) \cap L^p(a,b;D_A):\ \dot u \in L^p(a,b;D_B)\big\}$$

we denote the *maximal regularity space* which is a Banach space for the norm

$$\|u\|_{MR} = \|u\|_{W^{2,p}(a,b;X)} + \|u\|_{L^p(a,b;D_A)} + \|\dot u\|_{L^p(a,b;D_B)}.$$

Moreover, we consider the *trace space* $Tr := \{(u(a), \dot u(a)):\ u \in MR(a,b)\}$ with the norm

$$\big\|(x,y)\big\|_{Tr} := \inf\big\{\|u\|_{MR}:\ x = u(a),\ y = \dot u(a)\big\}.$$

For further properties of those spaces, we refer to [9].

By [9, Theorem 2.3], if $(A,B) \in \mathcal{MR}_p$ then for every $(x,y) \in Tr$ there exists a unique solution $u \in MR(a,b)$ of the homogeneous problem

$$\ddot u + B\dot u + Au = 0 \quad \text{a.e. on } (a,b), \qquad u(a) = x, \qquad \dot u(a) = y. \tag{5.2}$$

Clearly, the couple (A,B) has L^p-maximal regularity if and only if for some (for all) bounded intervals (a,b) the operator L on $L^p(a,b;X)$ given by

$$D(L) = \big\{u \in MR(a,b):\ u(a) = \dot u(a) = 0\big\},$$
$$Lu = \ddot u + B\dot u + Au$$

is invertible. Moreover, the operator L is invertible if and only if for some (for every) $\lambda \in \mathbb{C}$ the operator $L_\lambda : D(L) \to L^p(a,b;X)$ given by

$$L_\lambda u = \ddot u + (B+\lambda)\dot u + \big(\lambda^2 + \lambda B + A\big)u$$

is invertible. In fact, L and L_λ, and thus also their inverses, are similar:

$$L_\lambda^{-1} f = e^{-\lambda\cdot} L^{-1}\big(e^{\lambda\cdot} f\big). \tag{5.3}$$

Fix $\tau > 0$. For each subinterval $(a,b) \subset (0,\tau)$ we may consider the operator L on $L^p(a,b;X)$. We do not use different notations for these operators in order to keep notations simple.

Lemma 5.2. *Assume that $(A,B) \in \mathcal{MR}_p$. Then there exists a constant $M \geqslant 0$ such that*

$$\big\|L_\lambda^{-1}\big\|_{\mathcal{L}(L^p(a,b;X),L^p(a,b;D_A\cap D_B))} \leqslant M,$$
$$\big\|(1+\lambda)L_\lambda^{-1}\big\|_{\mathcal{L}(L^p(a,b;X))} \leqslant M,$$
$$\left\|\left(\frac{d}{dt}+\lambda\right)L_\lambda^{-1}\right\|_{\mathcal{L}(L^p(a,b;X),L^p(a,b;D_B))} \leqslant M, \quad \textit{and}$$
$$\left\|\big(1+\lambda^{\frac{p-1}{p}}\big)\left(\frac{d}{dt}+\lambda\right)L_\lambda^{-1}\right\|_{\mathcal{L}(L^p(a,b;X))} \leqslant M$$

for every $\lambda \geqslant 0$ and every interval $(a,b) \subset (0,\tau)$.

For the proof of Lemma 5.2 we need the following maximum principle.

Lemma 5.3. *Let X, Y be two Banach spaces such that Y is continuously embedded into X. Let $\mathbb{C}_+ := \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > 0\}$, and let $F : \mathbb{C}_+ \to Y$ be an analytic function which extends continuously to $\overline{\mathbb{C}_+}$. Assume that*

$$\sup_{\lambda \in \mathbb{C}_+} \|F(\lambda)\|_X < \infty \quad \text{and} \quad \sup_{s \in \mathbb{R}} \|F(is)\|_Y < \infty.$$

Then

$$\sup_{\lambda \in \mathbb{C}_+} \|F(\lambda)\|_Y < \infty.$$

Proof. Since the function F is bounded and analytic with values in X, we have the following Poisson representation

$$F(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}} F(is) \frac{\operatorname{Re}\lambda}{(\operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda - s)^2}\, ds$$

for every $\lambda \in \mathbb{C}_+$ [18]. Since F is bounded on the imaginary axis with values in Y, and since Y embeds continuously into X, this representation holds also in Y. The claim follows from a simple integral estimate. □

Proof of Lemma 5.2. It suffices to prove the estimate for the interval $(0, \tau)$. The same estimate then holds for arbitrary subintervals $(a, b) \subset (0, \tau)$ (cp. Lemma 1.2).

We first note that the function $\lambda \to L_\lambda^{-1}$, $\mathbb{C} \to \mathcal{L}(L^p(0, \tau; X), L^p(0, \tau; D_A \cap D_B))$ is entire.

By [9, Proposition 2.6] and the similarity (5.3), there exists a function

$$S \in C\big([0, \tau]; \mathcal{L}(X)\big) \cap C^\infty\big((0, \tau]; \mathcal{L}(X, D_A \cap D_B)\big)$$

such that

$$S(0) = 0 \quad \text{and} \quad L_\lambda^{-1} f = \big(e^{-\lambda \cdot} S\big) * f. \tag{5.4}$$

In fact, the case $\lambda = 0$ follows from [9] and the case of general $\lambda \in \mathbb{C}$ from (5.3). The regularity of S implies that

$$\sup_{\lambda \in \mathbb{C}_+} \big\|(1 + \operatorname{Re}\lambda) L_\lambda^{-1}\big\|_{\mathcal{L}(L^p(0,\tau;X))} < \infty,$$

which yields already the second estimate. By the similarity (5.3) and since the mapping $f \mapsto e^{-is\cdot} f$ is an isometric isomorphism both on $L^p(0, \tau; X)$ and on $L^p(0, \tau; D_A \cap D_B)$,

$$\sup_{s \in \mathbb{R}} \big\|L_{is}^{-1}\big\|_{\mathcal{L}(L^p(0,\tau;X), L^p(0,\tau;D_A \cap D_B))} < \infty.$$

Hence, by Lemma 5.3,

$$\sup_{\lambda \in \mathbb{C}_+} \big\|L_\lambda^{-1}\big\|_{\mathcal{L}(L^p(0,\tau;X), L^p(0,\tau;D_A \cap D_B))} < \infty,$$

and this is the first estimate.

In order to prove the third and the fourth estimates, note that $S(0) = 0$ and so

$$\left(\frac{d}{dt} + \lambda\right) L_\lambda^{-1} f = \left(\frac{d}{dt} + \lambda\right)\left(e^{-\lambda \cdot} S\right) * f = \left(e^{-\lambda \cdot} \dot{S}\right) * f.$$

Applying the representation (5.4) for $\lambda = 0$ to constant functions f and using L^p-maximal regularity, we obtain that for every $x \in X$,

$$\dot{S}(\cdot)x \in L^p(0, \tau; X)$$

and

$$\left\| \dot{S}(\cdot)x \right\|_{L^p(0,\tau;X)} \leqslant C\|x\|_X,$$

where C is a constant independent of x. By Hölder's inequality and Fubini's theorem, for every $f \in L^p(0, \tau; X)$,

$$\begin{aligned}
\left\| \left(e^{-\lambda \cdot} \dot{S}\right) * f \right\|_{L^p(0,\tau;X)}^p &\leqslant \int_0^\tau \left(\int_0^t \left\| e^{-\lambda(t-s)} \dot{S}(t-s) f(s) \right\|_X ds \right)^p dt \\
&\leqslant \int_0^\tau \left(\int_0^t e^{-\lambda s p'} ds \right)^{p-1} \left(\int_0^t \left\| \dot{S}(t-s) f(s) \right\|_X^p ds \right) dt \\
&\leqslant \frac{C}{1 + \lambda^{p-1}} \int_0^\tau \int_s^\tau \left\| \dot{S}(t-s) f(s) \right\|_X^p dt\, ds \\
&\leqslant \frac{C}{1 + \lambda^{p-1}} \int_0^\tau \left\| f(s) \right\|_X^p ds \\
&\leqslant \frac{C}{1 + \lambda^{p-1}} \|f\|_{L^p(0,\tau;X)},
\end{aligned}$$

so that we have proved the fourth estimate. By L^p-maximal regularity, the function

$$\begin{aligned}
&\mathbb{C} \to \mathcal{L}\big(L^p(0, \tau; X), L^p(0, \tau; D_B)\big), \\
&\lambda \mapsto \left(\frac{d}{dt} + \lambda\right) L_\lambda^{-1}
\end{aligned}$$

is entire. Moreover, for every $f \in L^p(0, \tau; X)$

$$\|\dot{S} * f\|_{L^p(0,\tau;D_B)} = \left\| L^{-1} f \right\|_{L^p(0,\tau;D_B)} \leqslant C\|f\|_{L^p(0,\tau;X)},$$

and by similarity, as above,

$$\left\| \left(e^{-is\cdot} \dot{S}\right) * f \right\|_{L^p(0,\tau;D_B)} \leqslant M \|f\|_{L^p(0,\tau;X)}$$

for all $s \in \mathbb{R}$ and some constant $C \geqslant 0$ independent of s. The third estimate thus follows from Lemma 5.3 again. □

As in the first order case, we prove a perturbation result for maximal regularity.

Proposition 5.4. *Assume that $(A, B) \in \mathcal{MR}_p$. Let $(a, b) \subset (0, \tau)$ and let $C : (a, b) \to \mathcal{L}(D_A, X)$, $D : (a, b) \to \mathcal{L}(D_B, X)$ be two strongly measurable functions. Suppose that there exists a constant $\eta \geqslant 0$ such that for every $x \in D_A$, $y \in D_B$, and every $t \in (a, b)$,*

$$\left\| C(t)x \right\|_X \leqslant \frac{1}{3M} \|x\|_{D_A} + \eta \|x\|_X, \quad \text{and} \tag{5.5}$$

$$\left\| D(t)y \right\|_X \leqslant \frac{1}{3M} \|y\|_{D_B} + \eta \|y\|_X, \tag{5.6}$$

where M is the constant from Lemma 5.2.

Then for all $f \in L^p(a, b; X)$, $(x, y) \in Tr$ there exists a unique $u \in MR(a, b)$ satisfying

$$\ddot{u} + B\dot{u} + D(t)\dot{u} + Au + C(t)u = f \quad \text{a.e. on } (a, b), \qquad u(a) = x, \qquad \dot{u}(a) = y. \tag{5.7}$$

Proof. (a) We define two operators $\tilde{C} \in \mathcal{L}(L^p(a, b; D_A), L^p(a, b; X))$ and $\tilde{D} \in \mathcal{L}(L^p(a, b; D_B), L^p(a, b; X))$ by

$$(\tilde{C}u)(t) := C(t)u(t) \quad \text{and} \quad (\tilde{D}u)(t) := D(t)u(t).$$

Then the problem

$$\ddot{u} + B\dot{u} + D(t)\dot{u} + Au + C(t)u = f \quad \text{a.e. on } (a, b), \qquad u(a) = \dot{u}(a) = 0, \tag{5.8}$$

admits for every $f \in L^p(a, b; X)$ a unique solution $u \in MR(a, b)$ if and only if the operator $\tilde{L} : D(L) \to L^p(a, b; X)$ given by

$$\tilde{L}u := Lu + \tilde{D}\dot{u} + \tilde{C}u$$

is boundedly invertible. However, the latter operator is invertible if and only if for some (for all) $\lambda \in \mathbb{C}$ the operator $\widetilde{L_\lambda} : D(L) \to L^p(a, b; X)$ given by

$$\widetilde{L_\lambda}u := L_\lambda u + \tilde{D}\dot{u} + \lambda\tilde{D}u + \tilde{C}u$$

is boundedly invertible, and in this case

$$\widetilde{L_\lambda}^{-1} f = e^{-\lambda\cdot} \tilde{L}^{-1}\left(e^{\lambda\cdot} f\right).$$

(b) By assumption on the functions C and D, we obtain

$$\begin{aligned}\|\tilde{C}u\|_{L^p(a,b;X)} &= \left(\int_a^b \big\|C(t)u(t)\big\|_X^p\,dt\right)^{\frac{1}{p}}\\ &\leqslant \left(\int_a^b \left(\frac{1}{3M}\big\|u(t)\big\|_{D_A} + \eta\big\|u(t)\big\|_X\right)^p dt\right)^{\frac{1}{p}}\\ &\leqslant \frac{1}{3M}\|u\|_{L^p(a,b;D_A)} + \eta\|u\|_{L^p(a,b;X)},\end{aligned}$$

and similarly

$$\|\tilde{D}u\|_{L^p(a,b;X)} \leqslant \frac{1}{3M}\|u\|_{L^p(a,b;D_B)} + \eta\|u\|_{L^p(a,b;X)}.$$

Hence, for every $\lambda \geqslant 0$, by Lemma 5.2,

$$\begin{aligned}&\left\|\left(\tilde{D}\frac{d}{dt} + \tilde{D}\lambda + \tilde{C}\right)L_\lambda^{-1}f\right\|_{L^p(a,b;X)}\\ &\quad\leqslant \frac{1}{3M}\left\|\left(\frac{d}{dt}+\lambda\right)L_\lambda^{-1}f\right\|_{L^p(a,b;D_B)} + \eta\left\|\left(\frac{d}{dt}+\lambda\right)L_\lambda^{-1}f\right\|_{L^p(a,b;X)}\\ &\qquad + \frac{1}{3M}\big\|L_\lambda^{-1}f\big\|_{L^p(a,b;D_A)} + \eta\big\|L_\lambda^{-1}f\big\|_{L^p(a,b;X)}\\ &\quad\leqslant \left(\frac{2}{3} + \frac{\eta M}{1+\lambda^{\frac{p-1}{p}}} + \frac{\eta M}{1+\lambda}\right)\|f\|_{L^p(a,b;X)}.\end{aligned}$$

Choosing $\lambda \geqslant 0$ large enough, we find that

$$\left\|\left(\tilde{D}\frac{d}{dt} + \tilde{D}\lambda + \tilde{C}\right)L_\lambda^{-1}\right\|_{\mathcal{L}(L^p(a,b;X))} \leqslant \frac{3}{4},$$

and hence the operator

$$\widetilde{L_\lambda} = \left(I + \left(\tilde{D}\frac{d}{dt} + \tilde{D}\lambda + \tilde{C}\right)L_\lambda^{-1}\right)L_\lambda$$

is invertible. In particular, by (a), for every $f \in L^p(a,b;X)$ the problem (5.8) admits a unique solution $u \in MR(a,b)$.

(c) Let $(x,y) \in Tr$. Then there exists $w \in MR(a,b)$ such that $w(a) = x$ and $\dot{w}(a) = y$. By (b), there exists a unique function $v \in MR(a,b)$ such that

$$\begin{aligned}&\ddot{v} + \big(B + D(t)\big)\dot{v} + \big(A + C(t)\big)v\\ &\quad = -\ddot{w} - \big(B + D(t)\big)\dot{w} - \big(A + C(t)\big)w + f \quad \text{a.e. on } (a,b),\\ &v(a) = \dot{v}(a) = 0.\end{aligned}$$

Putting $u := v + w$, we have proved existence for (5.7). Uniqueness follows from (b). □

By a natural modification of the proofs of Theorems 2.7 and 2.11, we deduce from the perturbation result Proposition 5.4 the following two theorems on the non-autonomous second order problem.

Theorem 5.5. *Let $A : [0,\tau] \to \mathcal{L}(D_A, X)$ and $B : [0,\tau] \to \mathcal{L}(D_B, X)$ be relatively continuous. Let $p \in (1,\infty)$, and assume that $(A(t), B(t)) \in \mathcal{MR}_p$ for all $t \in [0,\tau]$. Then for every $f \in L^p(0,\tau; X)$ and every $(x, y) \in Tr$ there exists a unique $u \in MR(0,\tau)$ satisfying*

$$\ddot{u} + B(t)\dot{u} + A(t)u = f \quad \text{a.e. on } (0,\tau), \qquad u(0) = x, \qquad \dot{u}(0) = y. \tag{5.9}$$

Theorem 5.6. *Let $A : [0,\tau] \to \mathcal{L}(D_A, X)$ and $B : [0,\tau] \to \mathcal{L}(D_B, X)$ be relatively continuous. Let $p \in (1,\infty)$, and assume that $(A(t), B(t)) \in \mathcal{MR}_p$ for all $t \in [0,\tau]$. Let $C : [0,\tau] \to \mathcal{L}(Y_A, X)$ and $D : [0,\tau] \to \mathcal{L}(Y_B, X)$ be strongly measurable and bounded, where Y_A respectively Y_B are close to X compared with D_A respectively D_B. Then $(A + C, B + D) \in \mathcal{MR}_p$. In particular, for every $f \in L^p(0,\tau; X)$ and every $(x, y) \in Tr$ there exists a unique $u \in MR(0,\tau)$ satisfying*

$$\begin{cases} \ddot{u} + \big(B(t) + C(t)\big)\dot{u} + \big(A(t) + D(t)\big)u = f \quad \text{a.e. on } (0,\tau), \\ u(0) = x, \quad \dot{u}(0) = y. \end{cases} \tag{5.10}$$

6. An example

Let $\Omega \subset \mathbb{R}^n$ be an open set such that $\partial\Omega$ is bounded and of class C^2. Assume the conditions (H1) and (H2) from Section 4 and

(H3) $c_j \in L^\infty((0,\tau) \times \Omega)$ for $j = 0, 1, \ldots, n$.

We define partial differential operators $\mathcal{A}(t, x, D)$ and $\mathcal{B}(t, x, D)$ by

$$\mathcal{A}(t, x, D)u(x) := \sum_{i,j=1}^{n} a_{ij}(t, x)\partial_i\partial_j u(x) + \sum_{j=1}^{n} b_j(t, x)\partial_j u(x) + b_0(t, x)u(x)$$

and

$$\mathcal{B}(t, x, D)u(x) := \sum_{i,j=1}^{n} a_{ij}(t, x)\partial_i\partial_j u(x) + \sum_{j=1}^{n} c_j(t, x)\partial_j u(x) + c_0(t, x)u(x).$$

Theorem 6.1. *Let $p, q \in (1,\infty)$. Then for every $u_0 \in W^{2,p} \cap W_0^{1,p}(\Omega)$, every $u_1 \in B_{pq}^{2/q'} \cap \mathring{B}_{pq}^{1/q'}(\Omega)$ and every $f \in L^q(0,\tau; L^p(\Omega))$ there exists a unique*

$$u \in W^{1,q}\big(0,\tau; W^{2,p} \cap W_0^{1,p}(\Omega)\big) \cap C^1\big([0,\tau]; B_{pq}^{2/q'} \cap \mathring{B}_{pq}^{1/q'}(\Omega)\big) \cap W^{2,q}\big(0,\tau; L^p(\Omega)\big)$$

solution of

$$\begin{cases} \partial_t^2 u(t,x) - \mathcal{B}(t,x,D)\partial_t u(t,x) - \mathcal{A}(t,x,D)u(t,x) = f(t,x) \quad \text{a.e. on } (0,\tau)\times\Omega, \\ u(t,x) = 0 \quad \text{on } (0,\tau)\times\partial\Omega, \\ u(0,x) = u_0(x) \quad \text{a.e. on } \Omega, \\ \partial_t u(0,x) = u_1(x) \quad \text{a.e. on } \Omega. \end{cases} \tag{6.1}$$

Here we let $u(t,x) = u(t)(x)$.

Proof. Let $D_A = D_B := D := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and define for every $t \in [0,\tau]$ the operator $A(t) \in \mathcal{L}(D, L^p(\Omega))$ by

$$A(t)u = -\sum_{i,j=1}^{n} a_{ij}(t,\cdot)\partial_i\partial_j u, \quad u \in D.$$

It follows from [15, Theorem 8.2] that $A(t)$ has a bounded H^∞ functional calculus on some sector of angle $\beta_t \in (0, \frac{\pi}{2})$ for all $t \in [0,\tau]$ (the sector may depend on t). Since $L^p(\Omega)$ has property (α), $A(t)$ has in fact a bounded RH^∞ functional calculus on the same sector. By [9, Theorem 4.1], the couple $(A(t), A(t))$ has L^q-maximal regularity for every $q \in (1,\infty)$. Moreover, A is continuous from $[0,\tau]$ into $\mathcal{L}(D, L^p(\Omega))$.

Let $Y_A = Y_B := Y := (L^p(\Omega), W^{2,p}(\Omega))_{\theta,s}$, where $\theta \in (\frac{1}{2}, 1)$ and $s \in (1,\infty)$. Then $Y = B_{ps}^{2\theta}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ by [32]. Hence, Y and *a fortiori* $W^{1,p}(\Omega)$ are close to $L^p(\Omega)$ compared with $W^{2,p}(\Omega)$.

Let $B, C : (0,\tau) \to \mathcal{L}(W^{1,p}(\Omega), L^p(\Omega))$ be given by

$$(Bu)(t) := -\sum_{j=1}^{n} b_j(t,\cdot)\partial_j u - b_0(t,\cdot)u \quad \text{and}$$

$$(Cu)(t) := -\sum_{j=1}^{n} c_j(t,\cdot)\partial_j u - c_0(t,\cdot)u.$$

Then B and C are strongly measurable; compare the proof of Theorem 4.1. Moreover, B and C are clearly bounded.

Now the claim follows from Theorem 5.6. □

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