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# Uniform convergence for elliptic problems on varying domains

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*Dedicated to Professor Herbert Amann on the occasion of his 65th birthday*

Let $\Omega \subset \mathbb{R}^N$ be (Wiener) regular. For $\lambda > 0$ and $f \in L^\infty(\mathbb{R}^N)$ there is a unique bounded, continuous function $u\colon \mathbb{R}^N \to \mathbb{R}$ solving

$$\lambda u - \Delta u = f \quad \text{in} \quad \mathcal{D}(\Omega)', \qquad u = 0 \quad \text{on} \quad \mathbb{R}^N \setminus \Omega. \tag{$P_\Omega$}$$

Given open sets $\Omega_n$ we introduce the notion of *regular convergence* of $\Omega_n$ to $\Omega$ as $n \to \infty$. It implies that the solutions $u_n$ of $(P_{\Omega_n})$ converge (locally) uniformly to $u$ on $\mathbb{R}^N$. Whereas $L_2$-convergence has been treated in the literature, our criteria for uniform convergence are new. The notion of regular convergence is very general. For instance the sequence of open sets obtained by cutting into a ball converges regularly. Other examples show that uniform convergence is possible even if the measure of $\Omega_n \setminus \Omega$ stays larger than a positive constant for all $n \in \mathbb{N}$. Applications to spectral theory, parabolic equations and nonlinear equations are given.



## 0 Introduction

In this article we consider the Poisson equation

$$\begin{cases} \lambda u - \Delta u = f \quad \text{in} \quad \mathcal{D}(\Omega)', \\ u|_{\partial\Omega} = 0, \end{cases} \tag{$P_\Omega$}$$

where $\Omega$ is an open set in $\mathbb{R}^N$ and $\lambda \geq 0$. If $\Omega$ is Dirichlet regular, then for each $f \in L^\infty(\Omega)$ there exists a unique solution $u \in C(\overline{\Omega})$ solving the problem $(P_\Omega)$. Now let $\Omega_n$ be further open sets in $\mathbb{R}^N$. Consider the solutions $u_n$ of $(P_{\Omega_n})$. Extending $u_n$ and $u$ by zero to $\mathbb{R}^N$ we obtain uniformly bounded functions defined on $\mathbb{R}^N$. The purpose of this article is to study when $u_n$ converges to $u$ locally uniformly on $\mathbb{R}^N$. There is quite an extensive theory on $L^2$-convergence; see for instance [2, 10, 13, 15, 17, 27, 34, 37–39] and [18] with a final characterization.

Uniform convergence seems not been treated much in context of the Poisson problem and the corresponding parabolic problem (see [13, Remark 3, p. 129] and [17, Remark 4.6] for some results). The most comprehensive treatment seems to be in [9]. However, our results are complementary to those in [9]. The main subject of that paper is to study completeness properties of a metric space of open sets, where a sequence of open sets converges if the solutions of the corresponding Dirichlet problems converge uniformly. Our emphasis is on finding simple sufficient conditions for (local) uniform convergence of solutions, and then to discuss applications to semi-linear elliptic equations and the heat semigroup. We also emphasize that the perturbations we allow are rather singular perturbations. For smooth perturbations of the domain there are for instance results in [35].

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Of course we have to assume that $\Omega_n$ converges to $\Omega$ in a certain sense. If $\Omega_n \subset \Omega$ for all $n \in \mathbb{N}$, then we are able to characterize when $u_n$ converges to $u$ uniformly (see Section 3). Convergence in general is more complicated. We introduce the notion of *regular convergence* of $\Omega_n$ to $\Omega$ as $n \to \infty$ which involves somehow Dirichlet regularity. Our main result, Theorem 5.5, shows that regular convergence of $\Omega_n$ to $\Omega$ implies that the solutions $u_n$ of $(P_{\Omega_n})$ converge locally uniformly on $\mathbb{R}^N$ to the solution of $(P_\Omega)$.

There are many interesting examples. For instance, we may consider the sequence of open sets in $\mathbb{R}^2$ which are obtained by cutting into the unit disc. Also in the classical example of a dumbbell with shrinking handle we obtain uniform convergence to the solution corresponding to two disjoint balls. We obtain uniform convergence in some examples even if $\Omega_n \setminus \Omega$ has Lebesgue measure one for all $n \in \mathbb{N}$. All these examples go beyond the types of convergence considered in the literature. For instance Keldysh [30] and Hedberg [26] assume that $\bigcap \overline{\Omega}_n = \overline{\Omega}$ and they suppose throughout that $\Omega$ is *topologically regular*, that is, $\mathring{\overline{\Omega}} = \Omega$.

There is a sophisticated theory of approximation for the Dirichlet problem (see [26, 30] or [32, V §5] and the references therein). Even though the Dirichlet problem and the Poisson equation are closely related (see Section 4 and [8]) we do not use this theory. Our results are complementary to the above mentioned theory.

There are several reasons why we consider convergence of the solutions of the Poisson equation. First of all, the problem of non-uniqueness of certain nonlinear equations can be treated by varying domains as is shown in the pioneering work of Dancer [13, 15] (see also [16]). As an application of our theory for linear equations, we complement these results by a systematic theory showing that the convergence in many examples in [13, 15] is uniform, and not just in $L_p(\mathbb{R}^N)$ for all $p \in [1, \infty)$ (see Section 8). Another reason is that our results on elliptic equations yield results on convergence for the solutions of the heat equation (Section 9).

We allow arbitrary open sets, not necessarily bounded or connected. Hence, some preliminary results in Section 1 and Section 2 are needed in order to establish well-posedness in $L^\infty(\Omega)$. Section 3 then contains our first main result, namely a characterization of uniform convergence from the interior. To treat approximations from the outside we need more preparation concerning the local continuity at the boundary. This is given in Section 4. Our main approximation results are then given in Section 5, where we introduce the notion of regular convergence as a sufficient condition for (local) uniform convergence. Various examples showing the generality of our conditions are given in Section 6. The next three sections deal with consequences of the main results. First, in Section 7, we discuss consequences for spectral properties. Then, we look at applications to nonlinear problems in Section 8, and finally we study the heat equation in Section 9.

## 1 The elliptic equation in $L^\infty(\Omega)$

In this section we collect some properties of elliptic equations in $L^\infty(\Omega)$. In particular we provide a characterization of distributional solutions and a compactness lemma essential for our treatment of varying domains. Let $\Omega$ be an open set in $\mathbb{R}^N$. We identify the space $L^p(\Omega)$ with the subspace of $L^p(\mathbb{R}^N)$ consisting of those functions which vanish a.e. on $\Omega^c$. Also $H_0^1(\Omega)$ is identified with a subspace of $H^1(\mathbb{R}^n)$ extending functions by zero. This is consistent with derivation. In fact, let $u \in H_0^1(\Omega)$ and define

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \notin \Omega. \end{cases}$$

Then $\widetilde{u} \in H^1(\mathbb{R}^N)$ and $D_j\widetilde{u} = \widetilde{D_j u}$. Thus in the sequel we will omit the $\sim$ throughout. Let $\lambda \geq 0$ and let $f \in L^\infty(\mathbb{R}^N)$. If $\Omega$ is bounded, then by the Lax–Milgram Lemma (or simply the Theorem of Riesz–Fréchet) there exists a unique solution of the problem

$$\begin{cases} \lambda u - \Delta u = f & \text{in} \quad \mathcal{D}(\Omega)', \\ u \in H_0^1(\Omega). \end{cases} \tag{1.1}$$

We write $u = R_\Omega(\lambda)f$. Then $u$ is bounded, measurable on $\mathbb{R}^N$ and continuous on $\Omega$. We consider $R_\Omega(\lambda)$ as a bounded operator on $L^\infty(\mathbb{R}^N)$. If $\Omega$ is unbounded and $\lambda > 0$, then we define $R_\Omega(\lambda)$ by extrapolation, in the following way. First let $f \in L^2(\mathbb{R}^N)$. By the Lax–Milgram Lemma (1.1) has a unique solution $u$ which we denote by $R_{\Omega,2}(\lambda)f$. Then $R_{\Omega,2}(\lambda)$ is a self-adjoint bounded operator on $L^2(\mathbb{R}^N)$. It follows from the Beurling–Deny criterion (see [20, 1.3]), or a direct argument, that $\lambda R_{\Omega,2}(\lambda)$ is *submarkovian*, that is,

$$0 \leq f \leq 1 \quad \text{implies} \quad 0 \leq \lambda R_{\Omega,2}(\lambda) f \leq 1. \tag{1.2}$$

Consequently, there exists a unique operator $R_\Omega(\lambda) \in \mathcal{L}\big(L^\infty(\mathbb{R}^N)\big)$ such that

$$R_\Omega(\lambda) f = R_{\Omega,2}(\lambda) f \quad \text{for all} \quad f \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \tag{1.3}$$

and such that

$$R_\Omega(\lambda) \text{ is } \sigma^*\text{-continuous,}$$

that is, $f, f_n \in L^\infty(\mathbb{R}^N)$, $f_n \overset{*}{\rightharpoonup} f$ implies $R_\Omega(\lambda) f_n \overset{*}{\rightharpoonup} R_\Omega(\lambda) f$. Here $f_n \overset{*}{\rightharpoonup} f$ means that

$$\int_{\mathbb{R}^N} f_n g \longrightarrow \int_{\mathbb{R}^N} fg \quad \text{for all} \quad g \in L^1(\mathbb{R}^N). \tag{1.4}$$

**Remark 1.1** Note that $\sigma^*$-convergence is equivalent to $(f_n)$ being bounded in $L_\infty(\mathbb{R}^N)$ and (1.4) holding for $g$ in a dense subset of $L^1(\Omega)$, so for instance $g \in \mathcal{D}(\mathbb{R}^N)$ (see [40, Section V.1, Theorem 10]).

We collect some properties of the operators $R_\Omega(\lambda)$. They are positive linear operators on $L^\infty(\mathbb{R}^N)$, that is,

$$R_\Omega(\lambda) f \geq 0 \quad \text{for all} \quad f \in L^\infty(\mathbb{R}^N) \quad \text{nonnegative.}$$

Moreover, by (1.2),

$$\|R_\Omega(\lambda)\|_{\mathcal{L}(L^\infty(\mathbb{R}^N))} \leq \frac{1}{\lambda} \quad \text{for all} \quad \lambda > 0. \tag{1.5}$$

Furthermore, since

$$R_{\Omega,2}(\lambda) - R_{\Omega,2}(\mu) = (\mu - \lambda) R_{\Omega,2}(\lambda) R_{\Omega,2}(\mu) \geq 0$$

we have

$$0 \leq R_\Omega(\mu) \leq R_\Omega(\lambda) \quad \text{whenever} \quad 0 < \lambda \leq \mu.$$

Moreover, if $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ are open, then

$$\Omega_1 \subset \Omega_2 \quad \text{implies that} \quad 0 \leq R_{\Omega_1}(\lambda) \leq R_{\Omega_2}(\lambda) \tag{1.6}$$

(see for instance [8, Lemma 1.2]). In particular,

$$0 \leq R_\Omega(\lambda) \leq (\lambda - \Delta_\infty)^{-1}$$

where $\Delta_\infty$ is the Laplacian in $L^\infty(\mathbb{R}^N)$ defined on the domain

$$D(\Delta_\infty) = \big\{u \in L^\infty(\mathbb{R}^N) \colon \Delta u \in L^\infty(\mathbb{R}^N)\big\},$$

that is, $\Delta_\infty$ is the adjoint of the generator of the Gaussian semigroup on $L^1(\mathbb{R}^N)$. If $\Omega_2$ is bounded then (1.6) also holds for $\lambda = 0$.

If $\Omega$ is bounded, then by definition $R_\Omega(\lambda) L^\infty(\mathbb{R}^N) \subset H_0^1(\Omega)$. For unbounded sets and $f \in L^\infty(\mathbb{R}^N)$ it is not true that $R_\Omega(\lambda) f \in H_0^1(\mathbb{R}^N)$ in general, but only locally. More precisely, we let

$$H^1_{\mathrm{loc}}(\mathbb{R}^N) = \big\{u \in L^2_{\mathrm{loc}}(\mathbb{R}^N) \colon D_j u \in L^2_{\mathrm{loc}}(\mathbb{R}^N)\big\}.$$

 

For an open set $\Omega \subset \mathbb{R}^N$ we then define

$$H^1_{0,\text{loc}}(\Omega) := \bigl\{u \in H^1_{\text{loc}}\bigl(\mathbb{R}^N\bigr) : \psi u \in H^1_0(\Omega) \text{ for all } \psi \in \mathcal{D}\bigl(\mathbb{R}^N\bigr)\bigr\}.$$

In particular, for $u \in H^1_{0,\text{loc}}(\Omega)$ we have $u(x) = 0$ a.e. on $\Omega^c$. Our next theorem will characterize $R_\Omega(\lambda)f$ as the unique solution of $-\Delta u + \lambda u = f$ in $\mathcal{D}(\Omega)'$ with $u \in H_{0,\text{loc}}(\Omega) \cap L^\infty(\Omega)$. For the proof we need some preparation.

First we recall *Kato's inequality*

$$1_{\{u>0\}}\Delta u \le \Delta u^+ \quad \text{in} \quad \mathcal{D}(\Omega)' \tag{1.7}$$

which holds for all $u \in L^1_{\text{loc}}(\Omega)$ such that $\Delta u \in L^1_{\text{loc}}(\Omega)$ (see [22, V.II.21]). From this we deduce the following maximum principle (which is well-known if $u \in H^1_0(\Omega)$, see [25, Theorem 8.1]).

**Proposition 1.2** *Suppose that $\lambda \ge 0$, $u \in L^1_{\text{loc}}(\Omega)$ and $\Delta u \in L^1_{\text{loc}}(\Omega)$. If $u^+ \in H^1_0(\Omega)$ and $\lambda u - \Delta u \le 0$, then $u \le 0$.*

Proof. It follows from (1.7) that $\lambda u^+ - \Delta u^+ \le 1_{\{u>0\}}(\lambda u - \Delta u) \le 0$. Since $u^+ \in H^1_0(\Omega)$ we conclude that $\int_\Omega \lambda\, |u^+|^2 + \int_\Omega |\nabla u^+|^2 \le 0$, so $u^+ = 0$. □

We also need the elementary identity

$$|\nabla(\psi u)|^2 = u^2\, |\nabla\psi|^2 + \nabla u \nabla\bigl(\psi^2 u\bigr) \tag{1.8}$$

valid for all $\psi \in \mathcal{D}\bigl(\mathbb{R}^N\bigr)$ and $u \in H^1_{\text{loc}}\bigl(\mathbb{R}^N\bigr)$. We finally write shortly

$$\omega \subset\subset \Omega$$

for saying that $\omega$ is a bounded open set such that $\overline{\omega} \subset \Omega$.

**Theorem 1.3** *Suppose that $\Omega \subset \mathbb{R}^N$ is open, $f \in L^\infty(\Omega)$ and $\lambda > 0$. Then the following assertions are equivalent.*

(i) $u = R_\Omega(\lambda)f$;

(ii) $u \in H^1_{0,\text{loc}}(\Omega) \cap L^\infty(\Omega)$ *and* $\lambda u - \Delta u = f$ *in* $\mathcal{D}(\Omega)'$.

*If $\Omega$ is bounded the equivalence also holds for $\lambda = 0$.*

Proof. If $\Omega$ is bounded the assertion of the theorem is well-known, and follows from the discussion of (1.1). Hence assume that $\Omega$ is unbounded. We first prove (i) implies (ii). Let $f \in L^\infty(\Omega)$. Then there exist $f_n \in L^2\bigl(\mathbb{R}^N\bigr)$ such that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty\bigl(\mathbb{R}^N\bigr)$. Let $u_n := R_{\Omega_n}(\lambda)f_n$ and $u := R_\Omega(\lambda)f$. Note that $u_n \in H^1_0(\Omega)$ and $u \in L^\infty\bigl(\mathbb{R}^N\bigr)$. By definition of $R_\Omega(\lambda)$ we have $u_n \overset{*}{\rightharpoonup} u$ as $n \to \infty$. As $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty\bigl(\mathbb{R}^N\bigr)$ there exists $M \ge 0$ such that $\|f_n\|_\infty \le M$ for all $n \in \mathbb{N}$. Let now $B \subset\subset \mathbb{R}^N$ and $\psi \in \mathcal{D}\bigl(\mathbb{R}^N\bigr)$ such that $0 \le \psi \le 1$ and $\psi|_B \equiv 1$. Then by (1.8) and since $\psi^2 u_n \in H^1_0(\Omega_n)$,

$$\begin{aligned}
\|\nabla(\psi u_n)\|^2_{L^2} + \lambda\, \|\psi u_n\|^2_{L^2} &= \int_{\Omega_n} \nabla u_n \nabla\bigl(\psi^2 u_n\bigr) + \int_{\Omega_n} u_n^2\, |\nabla\psi|^2 + \int_{\Omega_n} \lambda u_n \psi^2 u_n \\
&= \int_{\Omega_n} f_n \psi^2 u_n + \int_{\Omega_n} u_n^2\, |\nabla\psi|^2 \quad \text{for all} \quad n \in \mathbb{N}.
\end{aligned}$$

Using (1.5) and $\|f_n\|_\infty \le M$ we get

$$\begin{aligned}
\|\nabla(\psi u_n)\|^2_{L^2} + \lambda\, \|\psi u_n\|^2_{L^2} &\le \frac{1}{\lambda}\, \|f_n\|^2_\infty\, \|\psi\|^2_{L_2} + \frac{1}{\lambda^2}\, \|f_n\|^2_\infty\, \|\nabla\psi\|^2_{L_2} \\
&\le \left(\frac{M}{\lambda}\right)^2 \bigl(\lambda\, \|\psi\|^2_{L_2} + \|\nabla\psi\|^2_{L_2}\bigr) \quad \text{for all} \quad n \in \mathbb{N}.
\end{aligned} \tag{1.9}$$

As $\bigl(\|\nabla(\psi u_n)\|^2_{L^2} + \lambda\, \|\psi u_n\|^2_{L^2}\bigr)^{1/2}$ is an equivalent norm on $H^1\bigl(\mathbb{R}^N\bigr)$ this shows that $u_n$ is bounded in $H^1(B)$ for all $B \subset\subset \mathbb{R}^N$. Hence there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $v \in H^1_{\text{loc}}\bigl(\mathbb{R}^N\bigr)$ such that $u_{n_k} \rightharpoonup v$ weakly in $H^1(B)$ for every $B \subset\subset \mathbb{R}^N$. By Rellich's Theorem $u_{n_k} \to v$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, and thus, by Remark 1.1, in $L^\infty(\mathbb{R}^N)$ with respect to the $\sigma^*$-topology.

We now show that $v = u$ and $u \in H^1_{0,\text{loc}}(\Omega)$. We know that $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\mathbb{R}^N)$. As $u_{n_k} \overset{*}{\rightharpoonup} v$ we conclude that $u = v$. Now, by what we proved, $u_n \rightharpoonup u$ weakly in $H^1(B)$ for all $B \subset\subset \mathbb{R}^N$. Thus, if we fix $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and choose $B \subset\subset \mathbb{R}^N$ such that $\operatorname{supp}\varphi \subset B$, then $\varphi u_n \in H^1_0(\Omega \cap B)$ for all $n \in \mathbb{N}$ and $\varphi u_n \to \varphi u$ weakly in $H^1(B)$. As $H^1_0(\Omega \cap B)$ is a weakly closed subspace of $H^1(B)$ we conclude that $\varphi u \in H^1_0(\Omega)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$, showing that $u \in H^1_{0,\text{loc}}(\Omega)$.

We finally prove that (ii) implies (i). In view of what we just proved it is sufficient to show uniqueness of solutions in (ii). Let $u \in H^1_{0,\text{loc}}(\Omega) \cap L^\infty(\Omega)$ such that $\lambda u - \Delta u = 0$ in $\mathcal{D}(\Omega)'$. We have to show that $u = 0$. Define $w(x) := 2N\lambda^{-1} + |x|^2$. Then $w \in C^\infty(\mathbb{R}^N)$ and $\Delta w = 2N \leq \lambda w$. Now fix $\varepsilon > 0$ and choose $R > 0$ such that $\varepsilon w(R) > \|u\|_\infty$. Setting $v := u - \varepsilon w$ we claim that $v^+ \in H^1_0(\Omega \cap B)$, where $B := B(0, R+1)$. To prove that, let $\psi \in \mathcal{D}(\Omega \cap B)$ such that $0 \leq \psi \leq 1$ and $\psi = 1$ on $B$. Since $\psi u \in H^1_0(\Omega)$ there exist $u_n \in \mathcal{D}(\Omega \cap B)$ such that $\|u_n\|_\infty \leq \|u\|_\infty$ and $u_n \to \psi u$ in $H^1(\Omega \cap B)$. Hence $u_n - \varepsilon w \to u - \varepsilon w$ in $H^1(\Omega \cap B)$ and thus also $(u_n - \varepsilon w)^+ \to (u - \varepsilon w)^+$ in $H^1(\Omega \cap B)$. But $\operatorname{supp}(u_n - \varepsilon w)^+ \subset \operatorname{supp} u_n \cap \overline{B}(0, R) \subset\subset \Omega \cap B$ for all $n \in \mathbb{N}$, implying that $(u - \varepsilon w)^+ \in H^1_0(\Omega \cap B)$ as claimed. Now $\lambda v - \Delta v = -\varepsilon(\lambda w - \Delta w) \leq 0$. It follows from Proposition 1.2 that $v \leq 0$. Thus $u \leq \varepsilon w$, and as $\varepsilon > 0$ was arbitrary we conclude that $u \leq 0$. Replacing $u$ by $-u$ we deduce that $u = 0$, completing the proof of the theorem. □

Next we will prove a compactness property of solutions of (1.1) when $f$ and $\Omega$ vary. We first recall the following simple regularity property of the Laplacian [19, II §3, Proposition 6].

**Lemma 1.4** *Let $u \in L^1_{\text{loc}}(\Omega)$. If $\Delta u \in L^\infty(\Omega)$, then $u \in C^1(\Omega)$. In particular, we note that $R_\Omega(\lambda)f \in C^1(\Omega)$ for all $f \in L^\infty(\mathbb{R}^N)$.*

An application of the closed graph theorem allows us to deduce the following interior estimate from Lemma 1.4.

**Lemma 1.5** *For every $\omega \subset\subset \Omega$ there exists a constant $c > 0$ such that*

$$\|u\|_{C^1(\overline{\omega})} \leq c\big(\|u\|_{L^\infty(\Omega)} + \|\Delta u\|_{L^\infty(\Omega)}\big) \tag{1.10}$$

*for all $u \in L^\infty(\Omega)$ such that $\Delta u \in L^\infty(\Omega)$.*

Given $u_n, u \in C(\Omega)$ we say that $u_n$ *converges to $u$ in $C(\Omega)$* if $u_n(x) \to u(x)$ as $n \to \infty$ uniformly on compact subsets of $\Omega$. We are now ready to prove the announced compactness result.

**Proposition 1.6** *Let $\Omega_n, \Omega \subset \mathbb{R}^N$ be open sets. Assume that for every compact set $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n \geq n_0$. Finally suppose that $f_n, f \in L^\infty(\mathbb{R}^N)$. If $\lambda > 0$ and $u_n := R_{\Omega_n}(\lambda)f_n$ then the following assertions hold.*

(i) *If $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\mathbb{R}^N)$ then there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $v \in H^1_{\text{loc}}(\mathbb{R}^N)$ such that $u_{n_k} \to v$ in $C(\overline{\omega})$ for all $\omega \subset\subset \Omega$, in $L^2_{\text{loc}}(\mathbb{R}^N)$, weakly in $H^1(B)$ for all $B \subset\subset \mathbb{R}^N$ and in $L^\infty(\mathbb{R}^N)$ for the $\sigma^*$-topology.*

(ii) *The limit points $v$ above satisfy the equation $-\Delta v + \lambda v = f$ in $\mathcal{D}(\Omega)'$.*

(iii) *If there exists a ball $B \subset \mathbb{R}^N$ such that $\Omega_n \subset B$ for all $n \in \mathbb{N}$ then then* (i) *and* (ii) *hold for $\lambda \geq 0$.*

Proof. (i) Let $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\mathbb{R}^N)$. Set $u_n := R_{\Omega_n}(\lambda)f_n$ and $u := R_\Omega(\lambda)f$. By Theorem 1.3 $u_n \in H^1_{0,\text{loc}}(\Omega)$ for all $n \in \mathbb{N}$. Moreover there exists $M \geq 0$ such that $\|f_n\|_\infty \leq M$ for all $n \in \mathbb{N}$. Now we can proceed as in the first part of the proof of Theorem 1.3 to find a subsequence $(u_{n_k})$ and $v$ such that $u_{n_k} \to v$ weakly in $H^1(B)$ for all $B \subset\subset \mathbb{R}^N$ and in $L^\infty(\mathbb{R}^N)$ for the $\sigma^*$-topology. By Rellich's Theorem convergence is also in $L^2_{\text{loc}}(\mathbb{R}^N)$. To show convergence in $C(\Omega)$, let $\omega \subset\subset U \subset\subset \Omega$. Then there exists $n_0 \in \mathbb{N}$ such that $\overline{U} \subset \Omega_n$ for all $n \geq n_0$. It follows from Lemma 1.5 (with $\Omega = U$) that the sequence $(u_n)$ is bounded and equi-continuous on $\overline{\omega}$. Thus $u_{n_k} \to v$ in $C(\overline{\omega})$ by the Arzelá–Ascoli Theorem.

(ii) We now show that $-\Delta v + \lambda v = f$ in $\mathcal{D}(\Omega)'$. To do so fix $\varphi \in \mathcal{D}(\Omega)$. By assumption on $\Omega_n$ there exists $n_0 \in \mathbb{N}$ such that $\varphi \in \mathcal{D}(\Omega_n)$ for all $n \geq n_0$. As $u_{n_k} \in H^1_{\text{loc}}(\mathbb{R}^N)$ we have

$$\langle -\Delta u_{n_k} + \lambda u_{n_k}, \varphi\rangle = \int_{\mathbb{R}^N} \nabla u_{n_k} \nabla\varphi + \lambda \int_{\mathbb{R}^N} u_{n_k}\varphi = \langle f_{n_k}, \varphi\rangle$$

 

for all $k$ large enough. As $u_{n_k} \rightharpoonup v$ in $H^1(B)$ for every $B \subset\subset \mathbb{R}^N$ and $f_{n_k} \overset{*}{\rightharpoonup} f$ in $L^\infty(\mathbb{R}^N)$ we can let $k \to \infty$ to get

$$\langle -\Delta v + \lambda v, \varphi\rangle = \int_{\mathbb{R}^N} \nabla v \nabla \varphi + \lambda \int_{\mathbb{R}^N} v\varphi = \langle f, \varphi\rangle \quad \text{for all} \quad \varphi \in \mathcal{D}(\Omega).$$

Hence (ii) follows.

(iii) Suppose now that there exists a ball $B$ with $\Omega_n \subset B$ for all $n \in \mathbb{N}$. Then $R_{\Omega_n}(0)$ exists, and by (1.6) we have $\|u_n\|_\infty = \|R_{\Omega_n}(0) f_n\|_\infty \leq M R_B(0)1$ for all $n \in \mathbb{N}$. Hence, in (1.9) we can substitute $M\lambda^{-1}$ by $M\, R_B(0)1$ and get a domain independent a priori estimate. We also replace $H^1(\mathbb{R}^N)$ by $H_0^1(B)$ and note that $\|\nabla u\|_2$ defines an equivalent norm on that space. The rest of the proof works similarly as before. □

## 2 Continuity on the boundary

The purpose of this section is to review and improve known results on continuity on the boundary for solutions to (1.1). Let $\Omega \subset \mathbb{R}^N$ be open and $\lambda > 0$. If $f \in L^\infty(\mathbb{R}^N)$ then $u = R_\Omega(\lambda) f$ is a bounded function on $\mathbb{R}^N$ which is continuous on $\Omega$ and vanishes on $\Omega^c$. We want to characterize when $u$ is continuous on $\mathbb{R}^N$.

**Definition 2.1** A bounded open set $\Omega$ is called *Dirichlet regular* or simply *regular*, if for each $\varphi \in C(\partial\Omega)$ there exists $h \in C(\overline{\Omega}) \cap \boldsymbol{H}(\Omega)$ such that $h|_{\partial\Omega} = \varphi$.

Here $\boldsymbol{H}(\Omega)$ denotes the space of all harmonic functions on $\Omega$. If $\Omega$ is bounded we let

$$C_0(\Omega) := \{u \in C(\mathbb{R}^N) \colon u|_{\Omega^c} = 0\}.$$

Let $\lambda > 0$. Recall from [8, Lemma 2.2] that

$$\text{for } u \in C_0(\Omega),\ \lambda u - \Delta u \in L^\infty(\Omega) \text{ implies } u \in H_0^1(\Omega).$$

Conversely, we have the following characterization of regularity [8, Theorem 2.4].

**Proposition 2.2** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $\lambda > 0$. Then $\Omega$ is regular if and only if $R_\Omega(\lambda) L^\infty(\mathbb{R}^N) \subset C_0(\Omega)$.*

If $\Omega$ has a Lipschitz boundary, then $\Omega$ is regular. In fact, many more general geometric sufficient conditions are known (see [19, 25, 31, 32]). Moreover, a bounded open set $\Omega$ is regular if and only if

$$\text{for all } z \in \partial\Omega \text{ there exists } r > 0 \text{ such that } \Omega \cap B(z, r) \text{ is regular.} \tag{2.1}$$

We take (2.1) as a definition of a regular domain if $\Omega$ is unbounded.

**Definition 2.3** An open subset $\Omega \subset \mathbb{R}^N$ is called *regular* if (2.1) holds.

If $\Omega \subset \mathbb{R}^N$ is a (possibly unbounded) set we let

$$BC_0(\Omega) := \{u \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \colon u = 0 \text{ on } \Omega^c\},$$
$$C_0(\Omega) := \left\{u \in BC_0(\Omega) \colon \lim_{|x|\to\infty} u(x) = 0\right\}.$$

Note that $BC_0(\Omega) = C_0(\Omega)$ if $\Omega$ is bounded.

**Proposition 2.4** *Let $\Omega \subset \mathbb{R}^N$ be an open, regular set and let $\lambda > 0$. Then*

$$R_\Omega(\lambda) L^\infty(\Omega) \subset BC_0(\Omega). \tag{2.2}$$

For the proof we refer to Proposition 4.2, where a more general local version will be established. Thus on a regular set we have the following characterization of $R_\Omega(\lambda)$.

**Theorem 2.5** *Let $\Omega \subset \mathbb{R}^N$ be an open regular set. If $\lambda > 0$ and $f \in L^\infty(\mathbb{R}^N)$ then the following assertions are equivalent.*

(i) $u = R_\Omega(\lambda) f$;
(ii) $u \in H^1_{0,\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ *and* $\lambda u - \Delta u = f$ *in* $\mathcal{D}(\Omega)'$;
(iii) $u \in BC_0(\Omega)$ *and* $\lambda u - \Delta u = f$ *in* $\mathcal{D}(\Omega)'$.

*If $\Omega$ is bounded the above statements are also equivalent for $\lambda = 0$.*

 

P r o o f. The equivalence of (i) and (ii) follows from Theorem 1.3 and (ii) $\Rightarrow$ (iii) from Proposition 2.4, so it remains to prove (iii) $\Rightarrow$ (i). By assumption $u = R_\Omega(\lambda)f$ satisfies (iii). Thus it suffices to show that (iii) has at most one solution. Let $v \in BC_0(\Omega)$ such that $\lambda v - \Delta v = 0$ in $\mathcal{D}(\Omega)'$. We have to show that $v = 0$. Fix $\varepsilon > 0$ arbitrary. Then $\lambda(v-\varepsilon) - \Delta(v-\varepsilon) = -\lambda\varepsilon \leq 0$ in $\mathcal{D}(\Omega)'$. Since $(v-\varepsilon)^+ = 1_{\{v>\varepsilon\}}(v-\varepsilon)$, Kato's inequality (1.7) implies that

$$\lambda(v-\varepsilon)^+ - \Delta(v-\varepsilon)^+ \leq 0 \quad \text{in} \quad \mathcal{D}(\Omega)'. \tag{2.3}$$

We claim that (2.3) also holds in $\mathcal{D}\big(\mathbb{R}^N\big)'$. Let $0 \leq \varphi \in \mathcal{D}\big(\mathbb{R}^N\big)$, $K := \operatorname{supp}\varphi$ and $A := \big\{x \in \mathbb{R}^N : v(x) \geq \varepsilon\big\}$. Then $\Omega_1 = A^c$ is open in $\mathbb{R}^N$. Since $A \subset \Omega$ we have $\Omega^c \subset \Omega_1$. Thus $K \subset \Omega_1 \cup \Omega$. Using a partition of unity we find $\varphi_1, \varphi_2 \in \mathcal{D}\big(\mathbb{R}^N\big)_+$ such that $\operatorname{supp}\varphi_1 \subset \Omega_1$, $\operatorname{supp}\varphi_2 \subset \Omega$ and $\varphi = \varphi_1 + \varphi_2$. By (2.3), $\big\langle (v-\varepsilon)^+, \lambda\varphi_2 - \Delta\varphi_2\big\rangle \leq 0$. On the other hand, $\big\langle (v-\varepsilon)^+, \lambda\varphi_1 - \Delta\varphi_1\big\rangle = 0$. Thus, $\big\langle (v-\varepsilon)^+, \lambda\varphi - \Delta\varphi\big\rangle \leq 0$, showing that

$$\lambda(v-\varepsilon)^+ - \Delta(v-\varepsilon)^+ \leq 0 \quad \text{in} \quad \mathcal{D}\big(\mathbb{R}^N\big)'. \tag{2.4}$$

Now recall that $\lambda - \Delta$ is a bijective operator from $\boldsymbol{S}'\big(\mathbb{R}^N\big)$ onto $\boldsymbol{S}'\big(\mathbb{R}^N\big)$ with positive inverse. Thus it follows from (2.4) that $(v-\varepsilon)^+ = 0$. Hence $v \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $v \leq 0$. Replacing $v$ by $-v$, we get $v = 0$.

Finally we look at the case $\lambda = 0$ and $\Omega$ bounded. By Theorem 1.3 we know that (i) and (ii) are equivalent. If (ii) holds then for arbitrary $\lambda > 0$ we have $\lambda u - \Delta u = f + \lambda u$ in $\mathcal{D}(\Omega)'$ and $f + \lambda u \in L^\infty(\Omega)$. Hence by what we proved $u \in BC_0(\Omega)$ and thus (ii) implies (iii). Interchanging the roles of $H_0^1$ and $BC_0$ we get that (iii) implies (ii), completing the proof of the theorem. ☐

The above shows that if $\Omega$ is a regular open set, then $R_\Omega(\lambda)$ is a positive, linear operator from $L^\infty\big(\mathbb{R}^N\big)$ into $BC_0(\Omega)$. We introduce the space

$$L_0^\infty\big(\mathbb{R}^N\big) := \big\{u \in L^\infty\big(\mathbb{R}^N\big) : \exists h \in C_0\big(\mathbb{R}^N\big)\ |u(x)| \leq h(x) \text{ a.e.}\big\}. \tag{2.5}$$

It is easy to see that $L_0^\infty\big(\mathbb{R}^N\big)$ is a closed subspace of $L^\infty\big(\mathbb{R}^N\big)$.

**Proposition 2.6** *Let $\Omega$ be a regular, open subset of $\mathbb{R}^N$. Then for $\lambda > 0$ (or $\lambda \geq 0$ if $\Omega$ is bounded),*

$$R_\Omega(\lambda)L_0^\infty\big(\mathbb{R}^N\big) \subset C_0(\Omega). \tag{2.6}$$

P r o o f. By [8, Theorem 3.3] we have $R_\Omega(\lambda)C_0(\Omega) \subset C_0(\Omega)$. Thus (2.6) follows by applying (2.5) and Theorem 2.5. ☐

## 3 Uniform convergence from the interior

We now present our main results concerning convergence from the interior. Let $\Omega, \Omega_n \subset \mathbb{R}^N$ be open sets and assume that $\Omega_n$ converges to $\Omega$ from the interior in the following sense.

**Definition 3.1** Let $\Omega_n, \Omega \subset \mathbb{R}^N$ be open sets. We say that $\Omega_n$ *converges to $\Omega$ from the interior as $n \to \infty$* if

(a) $\Omega_n \subset \Omega$ for all $n \in \mathbb{N}$ and

(b) for each compact subset $K$ of $\Omega$ there exists $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n \geq n_0$.

We write $\Omega_n \uparrow \Omega$ if $\Omega_n$ converges to $\Omega$ from the interior and $\Omega_n \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$.

The main result of this section is the following theorem.

**Theorem 3.2** *Suppose that $\Omega_n, \Omega$ are open sets, $\Omega$ is regular and $\Omega_n \to \Omega$ from the interior. Let $\lambda > 0$ and let $f_n, f \in L^\infty\big(\mathbb{R}^N\big)$. Then the following assertions hold.*

(a) *If $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$, then $R_{\Omega_n}(\lambda)f_n \to R_\Omega(\lambda)f$ locally uniformly on $\mathbb{R}^N$;*

(b) *If $f \in L_0^\infty(\Omega)$, then $R_{\Omega_n}(\lambda)f \to R_\Omega(\lambda)f$ uniformly on $\mathbb{R}^N$.*

*If $\Omega$ is bounded, then* (a) *and* (b) *also hold for $\lambda \geq 0$ and $R_{\Omega_n}(\lambda) \to R_\Omega(\lambda)$ in $\mathcal{L}\big(L^\infty(\Omega)\big)$ for all $\lambda \geq 0$.*

 

P r o o f. We assume throughout that $\lambda > 0$ if $\Omega$ is unbounded and that $\lambda \geq 0$ if $\Omega$ is bounded.

(a) Suppose that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$ and set $u_n := R_{\Omega_n}(\lambda) f_n$ and $u := R_\Omega(\lambda) f$. By Proposition 1.6 there exists a subsequence $(u_{n_k})$ converging to some $v \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ weakly in $H^1(B)$ for all $B \subset\subset \mathbb{R}^N$ and in $L^\infty(\mathbb{R}^N)$ with the $\sigma^*$-topology. Moreover, $\lambda v - \Delta v = f$ in $\mathcal{D}(\Omega)'$. As $\Omega_n \subset \Omega$ for all $n \in \mathbb{N}$, Theorem 1.3 implies that $u_n \in H^1_{0,\mathrm{loc}}(\Omega)$ for all $n \in \mathbb{N}$. Hence $v \in H^1_{0,\mathrm{loc}}(\Omega)$ and by Theorem 1.3 $v = u = R_\Omega(\lambda) f$. Having a unique $\sigma^*$-limit point, the whole sequence $(u_n)_{n \in \mathbb{N}}$ must converge to $u$. By Proposition 1.6 we also know that $u_n \to u$ in $C(\Omega)$. Let now $M \geq \|f\|_\infty$ be such that $\|f_n\|_\infty \leq M$ for all $n \in \mathbb{N}$. By Theorem 2.5, $w := R_\Omega(\lambda) M \in BC_0(\Omega)$. Furthermore, by (1.6) we have $-w \leq u_n \leq w$, that is, $|u_n| \leq w$ for all $n \in \mathbb{N}$. Similarly, $|u| \leq w$. Fix now $\varepsilon > 0$ arbitrary. Then for every $B \subset\subset \mathbb{R}^N$ the set

$$K := \{x \in \mathbb{R}^N : w(x) \geq \varepsilon/2\} \cap \overline{B}$$

is compact and $K \subset \Omega$. Also, by the above, $|u_n(x) - u(x)| \leq 2w(x) \leq \varepsilon$ for all $x \in B \setminus K$ and $n \in \mathbb{N}$. As $\varepsilon > 0$ was arbitrary and $u_n \to u$ in $C(\Omega)$ it follows that $u_n \to u$ uniformly in $B \subset\subset \mathbb{R}^N$. Hence $u_n \to u$ locally uniformly on $\mathbb{R}^N$.

(b) We may assume that $f \geq 0$. By Proposition 2.6, $u = R_\Omega(\lambda) f \in C_0(\Omega)$. Hence, for given $\varepsilon > 0$, the set $K := \{x \in \mathbb{R}^N : u(x) \geq \varepsilon\}$ is compact and $K \subset \Omega$. Now we can complete the proof as in part (a) taking $w = u$.

We finally look at the case of bounded $\Omega$. Since $R_\Omega(\lambda) 1_{\Omega^c} = 0$, we have

$$R_\Omega(\lambda) 1 = R_\Omega(\lambda) 1_\Omega \in C_0(\Omega)$$

by Proposition 2.6. Since $R_\Omega(\lambda) - R_{\Omega_n}(\lambda) \geq 0$ we have

$$\|R_\Omega(\lambda) - R_{\Omega_n}(\lambda)\|_{\mathcal{L}(L^\infty)} = \big\|\big(R_\Omega(\lambda) - R_{\Omega_n}(\lambda)\big) 1\big\|_{L^\infty(\Omega)}$$

which converges to zero as $n \to \infty$ by (a). □

Next we characterize convergence from the interior.

**Proposition 3.3** *Let $\Omega_n, \Omega$ be open sets such that $\Omega_n \subset \Omega$ and let $\Omega_n$ be regular for all $n \in \mathbb{N}$. Let $\lambda > 0$ and let $0 \leq f \in L^\infty(\mathbb{R}^N)$ such that $f \neq 0$ on every component of $\Omega$. Then $R_{\Omega_n}(\lambda) f \to R_\Omega(\lambda) f$ in $C(K)$ for every compact set $K \subset \Omega$ if and only if $\Omega_n \to \Omega$ from the interior.*

P r o o f. If $\Omega_n \to \Omega$ from the interior, then the first part of the proof of Theorem 3.2 shows that $u_n \to u := R_\Omega(\lambda) f$ in $C(K)$ for every compact set $K \subset \Omega$. To prove the converse assume that $\Omega_n \not\to \Omega$ from the interior. Then there exists $K \subset \Omega$ compact such that $K \cap \Omega_n^c \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. Let $x_k \in K \cap \Omega_{n_k}^c$ such that $x_k \to x \in K$ and $n_k \to \infty$ as $k \to \infty$. Since $x_k \notin \Omega_{n_k}$ we have $u_{n_k}(x_k) = 0$ and $|u(x)| \leq |u(x) - u(x_k)| + |u(x_k) - u_{n_k}(x_k)|$ for all $k \in \mathbb{N}$. As $u \in C(K)$ and $u_n \to u$ in $C(K)$ by assumption, the right-hand side of the above inequality goes to zero as $k \to \infty$. Hence $u(x) = 0$ for some $x \in \Omega$. By the strong maximum principle [25, Theorem 8.19], applied to $-u$, this implies that $u \equiv 0$ in the component of $\Omega$ containing $x$. As $f \neq 0$ on that component by assumption, $u \neq 0$ on that component. As this is a contradiction, $\Omega_n \to \Omega$ from the interior. □

**Remark 3.4** (Necessity of Dirichlet regularity) Proposition 3.3 also implies that the condition that $\Omega$ be regular cannot be omitted in Theorem 3.2. In fact, given an open set $\Omega$, there exist $\Omega_n$ open, bounded of class $C^\infty$ such that $\Omega_n \uparrow \Omega$. Take $f \in C_0(\Omega)$ such that $f(x) > 0$ on $\Omega$. Assume that $u_n = R_{\Omega_n}(\lambda) f$ converges uniformly on $\Omega$ to $u = R_\Omega(\lambda) f$. Since $u_n \in C_0(\Omega_n)$, it follows that $u \in C_0(\Omega)$. By [8, Corollary 3.12] this implies that $\Omega$ is regular.

## 4 Regular points

For our main result on uniform approximation (Section 5), it will be natural to consider regular points of the Dirichlet problem. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\varphi \in C(\partial\Omega)$. Then the Perron solution $h_\varphi$ yields a weak solution of the *Dirichlet problem*

$$h \in C(\overline{\Omega}) \cap \boldsymbol{H}(\Omega), \tag{4.1}$$

 

where $\boldsymbol{H}(\Omega)$ denotes the set of harmonic functions on $\Omega$. The *Perron solution* is defined with the help of subharmonic functions (see [25, Chapter 1], [19] or [31]). For our purposes it is more convenient to use another approach. Let $\omega_n \subset\subset \Omega$ be regular such that $\omega_n \uparrow \Omega$. Such sets always exist, they can even be chosen of class $C^\infty$, [22, V.4.8]. Let $\Phi \in C(\mathbb{R}^N)$ such that $\Phi|_{\partial\Omega} = \varphi$. Let $h_n \in C(\overline{\omega}_n) \cap \boldsymbol{H}(\omega_n)$ such that $h_n(x) = \Phi(x)$ on $\partial\omega_n$. Then $h_n$ converges in $C(\Omega)$ to a function $h \in \boldsymbol{H}(\Omega)$. This function $h$ does not depend on the choice of the sequence $\omega_n$ and not on the extension $\Phi$ of $\varphi$. The function $h$ coincides with the Perron solution. We refer to [30] for a proof of these assertions.

If the Dirichlet problem (4.1) has a solution $h$, then $h = h_\varphi$. The function $h_\varphi$ is bounded by $\|\varphi\|_{C(\partial\Omega)}$, but in general not continuous up to the boundary.

A point $z \in \partial\Omega$ is called *regular*, if

$$\lim_{\substack{x\to z\\ x\in\Omega}} h_\varphi(x) = \varphi(z) \quad \text{for all} \quad \varphi \in C(\partial\Omega).$$

Thus $\Omega$ is regular if and only if each point $z \in \partial\Omega$ is regular.

Recall that for $\lambda > 0$ and $f \in L^\infty(\mathbb{R}^N)$ the function $u = R_\Omega(\lambda)f$ is continuous on $\Omega$ and 0 on $\Omega^c$. Our aim is to show that $u$ is continuous at each regular point $z \in \partial\Omega$.

For this we need to establish a relation between the Perron solution and the solution of Poisson's equation. Let $\varphi \in C(\partial\Omega)$. Assume that there exists $\Phi \in C^2(\mathbb{R}^N)$ such that $\Phi|_{\partial\Omega} = \varphi$. Let $u \in H_0^1(\Omega)$ such that $\Delta u = \Delta\Phi$ in $\mathcal{D}(\Omega)'$. Then $h := \Phi - u \in \boldsymbol{H}(\Omega)$ and $h = \varphi$ on $\partial\Omega$ in a weak sense (namely, $h - \Phi \in H_0^1(\Omega)$). If $\Omega$ is regular, then by [8, Lemma 2.2] $h \in C(\overline{\Omega})$, that is, $h = h_\varphi$. This is always true as we show now.

**Proposition 4.1** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $\Phi \in C(\mathbb{R}^N)$ such that $\Delta\Phi \in C(\mathbb{R}^N)$. Let $\varphi = \Phi|_{\partial\Omega}$. Let $u \in H_0^1(\Omega)$ such that $\Delta u = \Delta\Phi$ in $\mathcal{D}(\Omega)'$. Then*

$$h_\varphi = \Phi - u.$$

Proof. Let $\omega_n \subset\subset \Omega$ be Dirichlet regular sets such that $\omega_n \uparrow \Omega$. Let $u_n \in H_0^1(\Omega_n)$ such that $\Delta u_n = \Delta\Phi$ in $\mathcal{D}(\omega_n)'$. Then $h_n = \Phi - u_n \in C(\overline{\omega}_n) \cap \boldsymbol{H}(\omega_n)$ and $h_n = \Phi$ on $\partial\omega_n$. Moreover, $h_\varphi = \lim_{n\to\infty} h_n$ in $C(\Omega)$ by the introductory remarks. On the other hand, by Theorem 3.2 we see that $u_n \to u$ in $C(\Omega)$. Thus $h_\varphi = \Phi - u$. □

Denote by

$$\operatorname{cap}(A) := \inf\bigl\{\|u\|^2_{H^1(\mathbb{R}^N)} : u \in H^1(\mathbb{R}^N),\ u \geq 1 \text{ a.e. on a neighborhood of } A\bigr\}$$

the *capacity* of a subset $A$ of $\mathbb{R}^N$. Then *Wiener's criterion* states that a point $z \in \partial\Omega$ is regular if and only if

$$\sum_{j=1}^{\infty} 2^{j(N-2)} \operatorname{cap}(B(z, 2^{-j}) \setminus \Omega) = \infty \tag{4.2}$$

if $N > 2$ and

$$\sum_{j=1}^{\infty} 2^{j} \operatorname{cap}(A_j) = \infty \tag{4.3}$$

where $A_j = \{x \in \Omega^c \colon 2^j < \ln|x - z|^{-1} \leq 2^{j+1}\}$ if $N = 2$ (see [32, p. 299] for the last case, and [32, V §1.3 Theorem 5.2 and (5.1.7)] for the case $N \geq 3$). Now, if $\Omega$ is an arbitrary open set, we call a point $z \in \partial\Omega$ *regular* if (4.2) (or (4.3) if $N = 2$) holds. Clearly, $z \in \partial\Omega$ is a regular point of $\Omega$ if and only if $z$ is a regular point of $\Omega \cap B(z, r)$ where $r > 0$. Thus, by Definition 2.3, an open set $\Omega$ is regular if and only if each point of $\partial\Omega$ is regular. With help of Proposition 4.1 we can now prove the following extension of Proposition 2.4. The argument is a modification of part one of the proof of [8, Theorem 3.5].

**Proposition 4.2** *Let $\Omega$ be an open set, $z \in \partial\Omega$ and $\lambda > 0$. Then $z$ is a regular point of $\Omega$ if and only if $u := R_\Omega(\lambda)f$ is continuous at $z$ for all $f \in L^\infty(\mathbb{R}^N)$.*

 

P r o o f. Suppose first that $z$ is a regular point of $\Omega$. Let $\Phi(x) = |x - z|^2$ and $x \in \mathbb{R}^N$. Then $\Phi \in C^2(\mathbb{R}^N)$ and $\Delta\Phi \equiv 2N$. Let $r_1 > 0$ and $B_1 = B(z, r_1)$. Let $w_1 \in H_0^1(\Omega \cap B_1)$ such that $\Delta w_1 = \Delta\Phi = 2N$ in $\mathcal{D}(\Omega \cap B_1)$. Let $\varphi = \Phi|_{\partial(\Omega\cap B_1)}$. Then $h_\varphi = \Phi - w_1$ by Proposition 4.1. It follows from the maximum principle (Proposition 1.2) that $w_1 \le 0$. Thus

$$h_\varphi(x) \ge \Phi(x) = |x - z|^2 \quad \text{on} \quad \Omega \cap B_1.$$

Now consider $u = R_\Omega(\lambda)f$ where $f \in L^\infty(\mathbb{R}^N)$. We can assume that $f \ge 0$. Thus $u \ge 0$. We have to show that $\lim_{x\to z} u(x) = 0$. Let $\varepsilon > 0$. Choose $w \in C^1(\mathbb{R}^N)$ such that $-\Delta w = -\lambda u + f$ in $\mathcal{D}(B_1)'$ and $w(z) = \varepsilon$. We may choose $w = E_N * (\eta(\lambda u - f)) + c$, where $\eta \in \mathcal{D}(\mathbb{R}^N)$, $\eta \equiv 1$ on $B_1$ and $E_N$ denotes the Newtonian potential and $c$ is a suitable constant, chosen in such a way that $w(z) = \varepsilon$. Choose a ball $B = B(z, r) \subset B_1$ such that $w \ge 0$ on $B$. Choose $c > 0$ so large that $c\,|x - z|^2 > \|u\|_\infty$ on $\partial B \cap \overline{\Omega}$ and let $v = c\,h_\varphi \in H^1(\Omega \cap B)$. Then $u - w - v \in H^1(\Omega \cap B)$ and $(u - w - v)^+ \in H_0^1(\Omega \cap B)$. In fact, let $\psi \in \mathcal{D}(\mathbb{R}^N)_+$ such that $\psi \equiv 1$ on $B$. Since $\psi u \in H_0^1(\Omega)$ there exist test functions $u_n \in \mathcal{D}(\Omega)$ such that $0 \le u_n \le \|u\|_\infty$ and $u_n \to \psi u$ in $H^1(\Omega)$. Thus $u_n \to u$ and $(u_n - w - v)^+ \to (u - w - v)^+$ in $H^1(\Omega \cap B)$. But $(u_n - w - v)^+ \le (u_n - w - c\,\Phi)^+ \in C_0(\Omega \cap B) \cap H^1(\Omega \cap B) \subset H_0^1(\Omega \cap B)$. Hence also $(u_n - w - v)^+ \in H_0^1(\Omega \cap B)$. This proves that $(u - w - v)^+ \in H_0^1(\Omega \cap B)$. Since by the choice of $w$, $\Delta(u - w - v) = -\Delta v \le 0$, it follows from the maximum principle Proposition 1.2 that $u - w - v \le 0$ on $\Omega \cap B$. Since $z$ is a regular point, we conclude that

$$\overline{\lim_{x\to z}}\, u(x) \le \overline{\lim_{x\to z}}\big(w(x) + v(x)\big) \le \varepsilon + \overline{\lim_{x\to z}}\, c\,h_\varphi(x) = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\overline{\lim}_{x\to z} u(x) \le \varepsilon$. Replacing $u$ by $-u$ we obtain $\overline{\lim}_{x\to z} u(x) = 0$.

Suppose now that $R(\lambda)f$ is continuous at $z$ for all $f \in L^\infty(\Omega)$. Fix $r > 0$ and let $\Omega_0 := \Omega \cap B(z, r)$. Since $R_{\Omega_0}(\lambda) \le R_\Omega(\lambda)$ it follows that $R_{\Omega_0}(\lambda)f$ is continuous at $z$ for all $f \in L^\infty(\mathbb{R}^N)$. By the resolvent identity

$$\lambda R_{\Omega_0}(\lambda) R_{\Omega_0}(0) f = R_{\Omega_0}(0) f - R_{\Omega_0}(\lambda) f,$$

so $R_{\Omega_0}(0)$ is continuous at $z$ for all $f \in L^\infty(\mathbb{R}^N)$. Consider

$$F := \big\{\varphi \in C(\partial\Omega) \colon \exists \Phi \in C^2(\mathbb{R}^N),\ \varphi = \Phi|_{\partial\Omega}\big\}.$$

Let $\varphi \in F$, $\Delta\Phi = f$ and $u = R_{\Omega_0} f$. Then $h_\varphi := \Phi - u \in \boldsymbol{H}(\Omega)$. As $u$ is continuous at $z$ and $u(z) = 0$,

$$\lim_{\substack{x\to z\\ x\in\Omega}} h_\varphi(x) = \varphi(x).$$

Since $F$ is dense in $C(\partial\Omega)$, $h_\varphi$ is continuous for all $\varphi \in C(\partial\Omega)$, that is, $z$ is a regular point of $\Omega$. □

## 5 Regular convergence of domains and uniform convergence

This section contains our main results concerning domain convergence. We introduce the following definition of domain convergence. It turns out to be most convenient for concrete geometric descriptions as we will see in Section 6. We call it regular convergence since it implies that the limit set is regular.

**Definition 5.1** Let $\Omega_n, \Omega \subset \mathbb{R}^N$ be open sets for all $n \in \mathbb{N}$. We say that $\Omega_n$ *converges regularly* to $\Omega$ as $n \to \infty$ if the following conditions hold:

(1) For every $\omega \subset\subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that $\omega \subset \Omega_n$ for all $n \ge n_0$.

(2) For each $z \in \Omega^c$ there exist a compact set $K_z$ and sequences $(z_n)$ and orthogonal transformations $T_n \in O_N(\mathbb{R})$ such that
  (a) $z_n \to z$ in $\mathbb{R}^N$ and $T_n \to I$ in $O_N(\mathbb{R})$;
  (b) there exists $n_0 \in \mathbb{N}$ such that $\Omega_n \cap \big(z_n + T_n(K_z)\big) = \emptyset$ for all $n \ge n_0$,
  (c) $0 \in \partial K_z$ and 0 is a regular point of $K_z^c$.

**Remark 5.2** (a) We could require that $T_n \to T$ in $O_N(\mathbb{R})$, but replacing $T_n$ by $T_nT^{-1}$ we can assume without loss of generality that $T_n \to I$.

(b) Note that by Wiener's criterion condition (c) in Definition 5.1 can be reformulated by saying that

$$\sum_{j=1}^{\infty} 2^j \operatorname{cap}(B(z, 2^{-j}) \cap K_z) = \infty, \tag{5.1}$$

if $N \geq 3$ (and by a similar condition if $N = 2$). Thus, the set $K$ may be replaced by $K_z \cap \overline{B}(z, \varepsilon)$ for each $\varepsilon > 0$. In other words, $K_z$ may be chosen arbitrarily small.

**Remark 5.3** If $\Omega_n \to \Omega$ regularly, then $\Omega$ is regular. To see this we first show that $\Omega \cap (K_z + z) = \emptyset$ for all $z \in \Omega^c$. Suppose to the contrary that there exists $y \in K_z$ such that $x = z + y \in \Omega$. Then there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset\subset \Omega$. By (1) there exists $n_0 \in \mathbb{N}$ such that $\Omega_n \supset B(x, \varepsilon)$ for all $n \geq n_0$. It follows from (b) that $\big\|x - \big(T_n(y) + z_n\big)\big\| \geq \varepsilon$ for all $n \geq n_0$, contradicting the fact that $z_n + T_n(y) \to x$. Hence the claim is proved and thus $\big(B(0, 2^{-j}) \cap K_z\big) + z \subset B(z, 2^{-j}) \setminus \Omega$ for $n$ large. Now (5.1) implies that $z$ is a regular point of $\Omega$.

**Remark 5.4** If $\Omega$ is a regular open set and $\Omega_n \to \Omega$ from the interior, then $\Omega_n \to \Omega$ regularly as $n \to \infty$. Indeed, let $z \in \Omega^c$. Then part (2) of Definition 5.1 holds if we set $K_z := \big(\Omega^c \cap \overline{B(z, r)}\big) - z$ for some arbitrary $r > 0$, $z_n = z$ and $T_n = I$ for all $n \in \mathbb{N}$. Part (1) is satisfied by Definition 3.1.

Based on Definition 5.1 the following result on uniform convergence holds. We also allow $\Omega = \emptyset$ as a regular open set, setting

$$R_\emptyset(\lambda) = 0 \quad (\lambda \geq 0).$$

**Theorem 5.5** *Let $\Omega, \Omega_n \subset \mathbb{R}^N$ be open. Assume that $\Omega_n \to \Omega$ regularly as $n \to \infty$. Let $f_n, f \in L^\infty\big(\mathbb{R}^N\big)$ and let $\lambda > 0$. Then $\Omega$ is regular and the following assertions hold.*

(a) *If $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty\big(\mathbb{R}^N\big)$, then $R_{\Omega_n}(\lambda) f_n \to R_\Omega(\lambda) f$ locally uniformly on $\mathbb{R}^N$;*

(b) *If $f \in L_0^\infty\big(\mathbb{R}^N\big)$, then $R_{\Omega_n}(\lambda) f \to R_\Omega(\lambda) f$ uniformly on $\mathbb{R}^N$.*

*If $\Omega, \Omega_n \subset B$ $(n \in \mathbb{N})$ for some bounded set $B \subset \mathbb{R}^N$, then* (a) *and* (b) *also hold for $\lambda \geq 0$. Moreover, $R_{\Omega_n}(\lambda) \to R_\Omega(\lambda)$ in $\mathcal{L}\big(L^\infty\big(\mathbb{R}^N\big)\big)$ for all $\lambda \geq 0$.*

Proof. 1. Suppose that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty\big(\mathbb{R}^N\big)$. Then $M = \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. We set $u_n := R_{\Omega_n}(\lambda) f_n$ and $u := R_\Omega(\lambda) f$. Moreover, for every $z \in \Omega^c$ let $K_z \subset \mathbb{R}^N$, $z_n$ and $T_n$ be as in Definition 5.1. Setting $w_z := M\, R_{(z+K_z^c)}(\lambda)1$ and $w_{z,n} := M\, R_{(z_n+T_n(K_z)^c)}(\lambda)1$ we have

$$w_{z,n}(x) = w_z\big(z + T_n^{-1}(x - z_n)\big) \quad \text{and} \quad w_z(z) = 0. \tag{5.2}$$

Since $T_n^{-1} \to I$ in $\mathcal{L}\big(\mathbb{R}^N\big)$ and $w_z \in C\big((z + K_z^c) \cup \{z\}\big)$ we conclude that

$$\lim_{n\to\infty} w_{z,n}(x_n) = w_z(x) \tag{5.3}$$

whenever $x_n \to x$ and $x \in (K_z^c + z) \cup \{z\}$. By choice of $M$, (1.6) and the fact that $\big(z_n + T_n(K_z)\big) \cap \Omega_n = \emptyset$ we conclude that $-w_{z,n}(x) \leq u_n(x) \leq w_{z,n}(x)$, that is,

$$|u_n(x)| \leq w_{z,n}(x) \text{ for all } n \in \mathbb{N} \text{ large, } x \in \mathbb{R}^N \text{ and } z \in \Omega^c. \tag{5.4}$$

2. Applying Proposition 1.6 there exists a subsequence $(u_{n_k})$ converging to some $v \in H^1_{\mathrm{loc}}\big(\mathbb{R}^N\big) \cap L_\infty\big(\mathbb{R}^N\big)$ in $C(\Omega)$ and in $L^2_{\mathrm{loc}}\big(\mathbb{R}^N\big)$. If $z \in \Omega^c$ then by (5.3) (for $x_n = z$) and (5.4) we get

$$\varlimsup_{n\to\infty} |u_n(z)| \leq \varlimsup_{n\to\infty} w_{z,n}(z) = w_z(z) = 0 \quad \text{for all} \quad z \in \Omega^c.$$

Hence $u_n \to 0$ on $\Omega^c$, showing that $v = 0$ a.e. on $\Omega^c$. Modifying $v$ on a set of measure zero we may assume that $v = 0$ on $\Omega^c$. As $u_{n_k} \to v$ in $C(\Omega)$ we therefore conclude that $u_{n_k} \to v$ point-wise on $\mathbb{R}^N$. Now from (5.3) and (5.4) we get $|v(x)| \leq w_z(x)$ for all $x \in \Omega$. The latter inequality is obvious for $x \in \Omega^c$, so

$$|v(x)| \leq w_z(x) \quad \text{for all} \quad x \in \mathbb{R}^N \quad \text{and} \quad z \in \Omega^c. \tag{5.5}$$

 

We next show that $v$ is continuous on $\overline{\Omega}$. As $u_{n_k} \to v$ in $C(\Omega)$ we know already that $v$ is continuous on $\Omega$. Hence we only need to show continuity on $\partial\Omega$. To do so let $x_n \in \Omega$ and $x_n \to z$ with $z \in \partial\Omega$. But then, by (5.5) and continuity of $w_z$ at $z$

$$\varlimsup_{n\to\infty} |v(x_n)| \le \varlimsup_{n\to\infty} w_z(x_n) = w_z(z) = 0,$$

showing that $v(x_n) \to 0$. As $v \equiv 0$ on $\Omega^c$ we conclude that $v \in BC_0(\Omega)$. We know from Proposition 1.6 that $-\Delta v + \lambda v = f$ in $\mathcal{D}(\Omega)'$. Moreover, $\Omega$ is regular by Remark 5.3, so by Theorem 2.5 $v = u = R_\Omega(\lambda)f$. As $u$ is the only possible limit point of $u_n$ the whole sequence $(u_n)$ must converge. Hence we have shown that $u_n \to u$ point-wise on $\mathbb{R}^N$.

3. We next show that $u_n \to u$ uniformly on bounded subsets of $\mathbb{R}^N$. Assume to the contrary that there exists $B \subset\subset \mathbb{R}^N$ such that $(u_n)$ does not converge uniformly on $\overline{B}$. Then there exist $\varepsilon > 0$ and a sequence $(x_n)$ in $\overline{B}$ such that

$$\varlimsup_{n\to\infty} |u_n(x_n) - u(x_n)| \ge 3\varepsilon.$$

As $\overline{B}$ is compact we can select a subsequence such that $x_{n_k} \to z$ in $\overline{B}$ and

$$|u_{n_k}(x_{n_k}) - u(x_{n_k})| \ge 2\varepsilon \quad \text{for all} \quad k \in \mathbb{N}.$$

As $u_n \to u$ in $C(\Omega)$ it is impossible that $z \in \Omega$, so $z \in \Omega^c$. But then by continuity $u(x_{n_k}) \to u(z) = 0$. Hence, for large enough $k \in \mathbb{N}$

$$|u_{n_k}(x_{n_k})| \ge \varepsilon.$$

However, by (5.3) and (5.4)

$$\varlimsup_{k\to\infty} |u_{n_k}(x_{n_k})| \le \varlimsup_{k\to\infty} w_{z,n_k}(x_{n_k}) = w_z(z) = 0.$$

As this is a contradiction $u_n$ must converge uniformly to $u$ on $\overline{B}$. This completes the proof of (a).

4. Assertion (b) follows from (a) as in the proof of Theorem 3.2.

5. We finally suppose there exists a ball $B \subset \mathbb{R}^N$ such that $\Omega_n, \Omega \subset B$ for all $n \in \mathbb{N}$. Note that all arguments in (a) and (b) then also work for $\lambda = 0$. Suppose now that $\lambda \ge 0$ but $R_{\Omega_n}(\lambda)$ does not converge in $\mathcal{L}\big(L_\infty(\mathbb{R}^N)\big)$. Then there exist $\varepsilon > 0$ and $f_n \in L^\infty\big(\mathbb{R}^N\big)$ with $\|f_n\|_\infty = 1$ such that

$$\varlimsup_{n\to\infty} \|R_{\Omega_n}(\lambda)f_n - R_\Omega(\lambda)f_n\|_\infty \ge 2\varepsilon.$$

Hence we can choose a subsequence $(f_{n_k})$ such that $f_{n_k} \overset{*}{\rightharpoonup} f$ in $L^\infty\big(\mathbb{R}^N\big)$ and

$$\|R_{\Omega_{n_k}}(\lambda)f_{n_k} - R_\Omega(\lambda)f_{n_k}\|_\infty \ge \varepsilon \quad \text{for all} \quad k \in \mathbb{N}. \tag{5.6}$$

As the support of $R_{\Omega_k}(\lambda)f_{n_k}$ lies in the fixed bounded set $B$ we get from part (a) of the theorem that $R_{\Omega_{n_k}}(\lambda)f_{n_k} \to R_\Omega(\lambda)f$ uniformly on $\mathbb{R}^N$. Similarly, $R_\Omega(\lambda)f_{n_k} \to R_\Omega(\lambda)f$ uniformly on $\mathbb{R}^N$ as $k \to \infty$. Hence

$$\|R_{\Omega_{n_k}}(\lambda)f_{n_k} - R_\Omega(\lambda)f_{n_k}\|_\infty \longrightarrow 0.$$

As this is a contradiction to (5.6), $R_{\Omega_n}(\lambda)$ must converge in the operator norm, completing the proof of the theorem. □

**Remark 5.6** In Definition 5.1 we could replace the orthogonal transformations by more general transformations. We just need to make sure that the family $R_{(z_n+T_n(K_z))^c}1$ is equi-continuous at zero. Then all results concerning regular convergence remain valid.

**Remark 5.7** Condition (1) in Definition 5.1 is necessary for $R_{\Omega_n}(\lambda)f$ to converge in $C(\Omega)$, so it is a necessary condition for the above theorem. To see this we just do some obvious modifications in the proof of Proposition 3.3

 

In the case of decreasing sequences we may use a slightly weaker hypothesis.

**Corollary 5.8** *Let $\Omega, \Omega_n \subset \mathbb{R}^N$ be open such that $\Omega \subset \Omega_{n+1} \subset \Omega_n$ for all $n \in \mathbb{N}$. Assume that for each $z \in \Omega^c$ there exist a compact set $K_z \subset \mathbb{R}^N$, $z_k \in \mathbb{R}^N$, $T_k \in O_N(\mathbb{R})$ and $n_k \in \mathbb{N}$ such that*

(a) $n_k < n_{k+1}$;

(b) $z_k \to z$ *and* $T_k \to I$ *as* $k \to \infty$;

(c) $\Omega_{n_k} \cap \big(z_k + T_k(K_z)\big) = \emptyset$ *for all* $k \in \mathbb{N}$;

(d) $0 \in \partial K_z$ *and* $0$ *is a regular point of* $K_z^c$.

*Then $\Omega_n \to \Omega$ regularly and the assertions of Theorem* 5.5 *hold.*

Proof. We may assume that there exists $\widetilde{z}_1 \in \mathbb{R}^N$ such that $(\widetilde{z}_1 + K_z) \cap \Omega_1 = \emptyset$. Choose $\widetilde{z}_m = \widetilde{z}_1$ and $\widetilde{T}_n := I$ for $1 \le m < n_1$. Let $\widetilde{z}_m = z_k$ and $\widetilde{T}_m := T_k$ for $n_k \le m < n_{k+1}$. Then $\big(\widetilde{z}_m + \widetilde{T}_m(K_z)\big) \cap \Omega_m = \emptyset$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} \widetilde{z}_m = z$. □

Regular convergence of $\Omega_n$ to $\Omega$ does not imply that $\bigcap_{n\in\mathbb{N}} \Omega_n \subset \overline{\Omega}$, in general. This is interesting for many examples (see Section 6). However, as a special case of Theorem 5.5 we may consider a more conventional notion of convergence, if we assume stronger regularity of $\Omega$.

**Definition 5.9** We call an open set $\Omega \subset \mathbb{R}^N$ *strongly regular*, if for each $z \in \Omega^c$ there exist a compact set $K_z \subset \mathbb{R}^N$, $z_k \in \mathbb{R}^N$ and $T_k \in O_N\big(\mathbb{R}^N\big)$ such that

(a) $z_k \to z$ and $T_k \to I$ as $k \to \infty$;

(b) $\overline{\Omega} \cap \big(z_k + T_k(K_z)\big) = \emptyset$ for all $k \in \mathbb{N}$;

(c) $0 \in \partial K_z$ and $0$ is a regular point of $K_z^c$.

Every strongly regular open set is regular (see the discussion in Remark 5.3). But the set $\Omega = (0,1) \cup (1,2) \subset \mathbb{R}$ is regular (as every open subset of $\mathbb{R}$) but not strongly regular. And indeed, we will see in Example 6.2 that the assertion of Corollary 5.11 below is not true for this set. Note however, that $\Omega$ is not topologically regular, that is, $\mathring{\overline{\Omega}} \neq \Omega$.

Keldysh studied convergence of solutions of the Dirichlet problem for approximations from the exterior, that is, $\Omega \subset \Omega_{n+1} \subset \Omega_n$ for all $n \in \mathbb{N}$ and $\overline{\Omega} = \bigcap_{n\in\mathbb{N}} \overline{\Omega}_n$. His example [30, V.3 p. 55] yields a topologically regular open set $\Omega \subset \mathbb{R}^3$ which is regular but not strongly regular.

If $\Omega \subset \mathbb{R}^2$ is bounded and has continuous boundary in the sense of [22, V.4.1], then it follows from [22, Theorem 4.4, p. 246] that $\Omega$ is strongly regular. However, Lebesgue's cusp [32, V. §1.3 Example p. 287] yields a bounded open set in $\mathbb{R}^3$ which has continuous boundary but is not strongly regular (in fact, it is not even regular). But if $\Omega \subset \mathbb{R}^N$ is regular and has continuous boundary then it is strongly regular.

We now look at a more conventional notion of convergence of $\Omega_n$, called *metric convergence* in [37, p. 29].

**Theorem 5.10** *Let $\Omega$ be a strongly regular open set in $\mathbb{R}^N$. Let $\Omega_n \subset \mathbb{R}^N$ be open sets such that for all compact sets $K_1 \subset \Omega$ and $K_2 \subset \overline{\Omega}^c$ there exists $n_0 \in \mathbb{N}$ such that $K_1 \subset \Omega_n$ and $K_2 \subset \overline{\Omega}_n^c$ for all $n \ge n_0$. Then $\Omega_n \to \Omega$ regularly and all assertions of Theorem* 5.5 *hold.*

Proof. We want to show that under the assumptions of the theorem $\Omega_n \to \Omega$ regularly, and thus Theorem 5.5 applies. Part (1) of Definition 5.1 is obviously satisfied by definition of metric convergence. Hence we need to show part (2) is satisfied. For $z \in \Omega^c$ let $K_z$, $(z_n)$ and $(T_n)$ be as in Definition 5.9. By assumption on $\Omega_n$ and the compactness of $K_z$, for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\big(z_k + T_k(K_z)\big) \cap \Omega_n = \emptyset$ for all $n \ge n_k$. Moreover, we can choose $n_k$ strictly increasing. If we set $\widetilde{z}_n := z_k$ and $\widetilde{T}_n := T_k$ for $n_k \le n < n_{k+1}$ then $\big(\widetilde{z}_k + \widetilde{T}_n(K_z)\big) \cap \Omega_n = \emptyset$ for all $n \ge n_k$. Hence part (2) of Definition 5.9 is satisfied, and thus $\Omega_n \to \Omega$ regularly. □

Finally we want to look at monotone convergence of $\Omega_n$ to $\Omega$ from the exterior. The following corollary is an immediate consequence of the above theorem.

**Corollary 5.11** *Let $\Omega \subset \mathbb{R}^N$ be a strongly regular open set. Let $\Omega_n$ be regular open sets such that $\Omega \subset \Omega_{n+1} \subset \Omega_n$ and $\overline{\Omega} = \bigcap_{n\in\mathbb{N}} \overline{\Omega}_n$. Then $\Omega_n \to \Omega$ regularly and all assertions of Theorem* 5.5 *hold.*

 

## 6 Examples

In the one-dimensional case we are able to characterize convergence from the exterior.

**Example 6.1** (Dimension $N = 1$) Let $\Omega_n \subset \mathbb{R}$ be open and bounded such that $\Omega_{n+1} \subset \Omega_n$. Let $\Omega = (\bigcap_{n\in\mathbb{N}} \Omega_n)^\circ$ and $\lambda \geq 0$. Then

$$R_{\Omega_n}(\lambda) \longrightarrow R_\Omega(\lambda) \quad \text{in} \quad \mathcal{L}\big(L^\infty(\mathbb{R})\big) \quad \text{as} \quad n \longrightarrow \infty.$$

P r o o f. We may consider $K = \{0\}$ since $\mathbb{R} \setminus \{0\}$ is regular. Let $z \in \Omega^c$. Then there exists $z_k \in (\bigcap_{n\in\mathbb{N}} \Omega_n)^c$ such that $\lim_{k\to\infty} z_k = z$. Thus, we find inductively $n_k < n_{k+1}$ such that $z_k \notin \Omega_{n_k}$, that is, $(K+z_k)\cap\Omega_{n_k} = \emptyset$. Thus $\Omega_n$ converges regularly to $\Omega$ by Corollary 5.8. The claim follows from Theorem 5.5. □

In the next example we produce a regularly converging sequence by cutting into the unit disc. Similar examples are possible in higher dimension.

**Example 6.2** (Cutting into a disc) Let $N = 2$ and

$$\begin{aligned} \Omega\ &= \big\{x \in \mathbb{R}^2 \colon |x| < 1\big\} \setminus \big\{(x_1, 0) \colon 0 \leq x_1 < 1\big\}, \\ \Omega_n &= \big\{x \in \mathbb{R}^2 \colon |x| < 1\big\} \setminus \big\{(x_1, 0) \colon \delta_n \leq x_1 < 1\big\}, \end{aligned}$$

where $\delta_n \downarrow 0$ as $n \to \infty$ as shown in Figure 1. Then $R_{\Omega_n}(\lambda) \to R_\Omega(\lambda)$ in $\mathcal{L}\big(L^\infty\big(\mathbb{R}^N\big)\big)$ for all $\lambda \geq 0$ as $n \to \infty$. We can obviously give examples for $N \geq 3$ by replacing the cutting line by part of a hyper-plane.

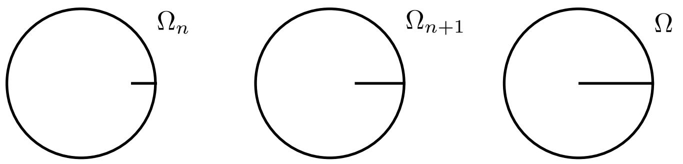


**Fig. 1** Cutting a disc

P r o o f. Let $z = (x_1, 0) \in \partial\Omega$ where $0 \leq x_1 \leq 1$. Let $K = \{(x_1, 0) \colon 0 \leq x_1 \leq 1\}$. Then $K^c$ is regular, [19, II §4 Sec. 1 Example 8, p. 337]. Let $z_n = z + (\delta_n, 0)$. Then $\lim_{n\to\infty} z_n = z$ and $(K+z_n)\cap\Omega_n = \emptyset$. This shows that $\Omega_n$ converges regularly to $\Omega$. The claim follows from Theorem 5.5. □

**Example 6.3** Let $N = 2$ and let $\Omega_n$ be a domain obtained by cutting a circular arc into an open set as shown in Figure 2. Then $R_{\Omega_n}(\lambda) \to R_\Omega(\lambda)$ in $\mathcal{L}\big(L^\infty\big(\mathbb{R}^N\big)\big)$ for all $\lambda > 0$ as $n \to \infty$. As the set $K$ in Definition 5.1 we take a small circular arc and argue similarly as in Example 6.2. We cannot just move that arc by translation but also need a rotations.

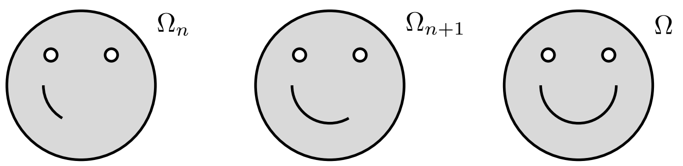


**Fig. 2** Cutting a domain with an arc

In the example given $\Omega \subset \Omega_n$ for all $n \in \mathbb{N}$. We can modify the above example by for instance cutting along a straight line segment and then turn that segment about a point as shown in Figure 3.

The next example shows that $\Omega_n$ may converge regularly to $\Omega$ even though $\Omega_n \setminus \Omega$ has Lebesgue measure one for all $n \in \mathbb{N}$.

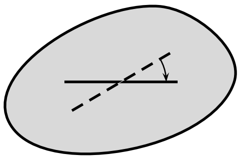

**Fig. 3** Rotate a line segment

**Example 6.4** (Converging combs) Let $R := (-1,1) \times (0,1)$, $\Omega := (-1,0) \times (0,1)$ and $\Omega_n = R \setminus T_n$ where

$$T_n = \big\{(x_1,x_2) \in R\colon x_1 \geq 0,\, (2k-1)/n \leq x_2 \leq 2k/n \text{ for } k = 1,2,\dots,[n/2]\big\}.$$

Here $[n/2]$ is the integer part of $n/2$. Then $\Omega_n$ is a comb as shown in Figure 4. The sequence $(\Omega_n)$ converges regularly to $\Omega$ as is easy to see. Just take for $K$ a horizontal line segment and note that the rational numbers are dense in $[0,1]$. Thus by Theorem 5.5, $R_{\Omega_n}(\lambda) \to R_\Omega(\lambda)$ as $n \to \infty$ in $\mathcal{L}\big(L^\infty\big(\mathbb{R}^2\big)\big)$ even though the measure of $\Omega_n \setminus \Omega$ does not go to zero as $n \to \infty$.

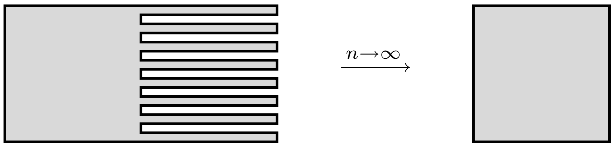


**Fig. 4** Combs converging to a square

**Example 6.5** (Closing up a sector) Let $\Omega_n$ be a disc with a sector of angle $\beta_n$ attached as shown in Figure 5. If $\beta_n \to 0$ then $\Omega_n \to \Omega$ regularly with $\Omega$ being the disc only. Hence Theorem 5.5 applies and $R_{\Omega_n}(\lambda)f \to R_\Omega(\lambda)f$ in $C\big(\mathbb{R}^N\big)$ for all $f \in L_\infty\big(\mathbb{R}^N\big)$ and uniformly on $\mathbb{R}^N$ if $f \in L_0^\infty\big(\mathbb{R}^N\big)$.

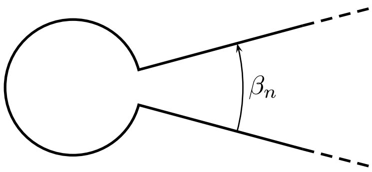


**Fig. 5** A sector closing up

**Example 6.6** (Dumbbells) Suppose that $\Omega_n$ is a dumbbell as shown in Figure 6. Suppose $B_0$ and $B_1$ are two fixed balls and that $C_n$ shrinks to a line. Then $\Omega_n \to \Omega$ regularly, where $\Omega$ is the union of the two balls. Again Theorem 5.5 applies.

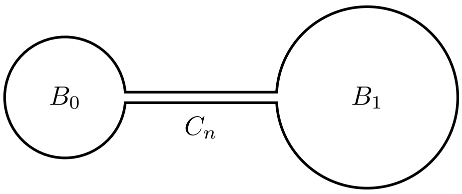


**Fig. 6** A dumbbell

Instead of looking at a fixed ball $B_1$ we now shift $B_1$ to infinity in the direction of $C_n$. At the same time we shrink $C_n$ to a line. Then $\Omega_n \to \Omega$ regularly, where $\Omega$ is just the ball $B_0$. Hence the assertions of Theorem 5.5 hold.

## 7 Application: Spectral theory

Let $\Omega \subset \mathbb{R}^N$ be an open regular set. We define the *Dirichlet Laplacian* $\Delta_\Omega$ on $L^\infty(\Omega)$ by

$$\begin{aligned} D(\Delta_\Omega) &= \big\{u|_\Omega \colon u \in BC_0(\Omega), \exists f \in BC_0(\Omega) \text{ such that } \Delta u = f \text{ in } \mathcal{D}(\Omega)'\big\}, \\ \Delta_\Omega u &= f. \end{aligned}$$

Then by Theorem 2.5, $(0,\infty) \subset \varrho(\Delta_\Omega)$ and

$$(\lambda - \Delta_\Omega)^{-1} = R_\Omega(\lambda)|_{L^\infty(\Omega)}.$$

For $\lambda \in \varrho(\Delta_\Omega)$ we set for $f \in L^\infty(\mathbb{R}^N)$

$$\big(R_\Omega(\lambda)\big) f(x) := \begin{cases} \big((\lambda - \Delta_\Omega)^{-1} f|_\Omega\big)(x), & x \in \Omega, \\ 0, & x \in \Omega^c. \end{cases}$$

Then $R_\Omega(\lambda) \in \mathcal{L}\big(L^\infty(\mathbb{R}^N)\big)$ and $R_\Omega(\lambda) L^\infty(\mathbb{R}^N) \subset BC_0(\Omega)$. It is clear that

$$R_\Omega \colon \varrho(\Delta_\Omega) \to \mathcal{L}\big(L^\infty(\Omega)\big)$$

is a pseudo resolvent which is maximal in the sense of [2, Definition 3.1.] (see [2, Proposition 3.5]).

**Remark 7.1** Alternatively, we may define the multi-valued operator $A_\Omega$ on $L^\infty(\mathbb{R}^N)$ by $D(A_\Omega) = D(\Delta_\Omega)$, $A_\Omega u = \big\{f \in L^\infty(\mathbb{R}^N) \colon \Delta u = f \text{ in } \mathcal{D}(\Omega)'\big\}$. Then $\varrho(\Delta_\Omega) = \varrho(A_\Omega)$ and $R_\Omega(\lambda) = (\lambda - A_\Omega)^{-1}$ for all $\lambda \in \varrho(A_\Omega)$.

It follows from [4] that $\varrho(\Delta_{\Omega,2}) = \varrho(\Delta_\Omega)$. Next we establish compactness of the resolvent.

**Theorem 7.2** *Assume that $\Omega$ has finite measure. Then $R_\Omega(\lambda)$ is compact for all $\lambda \in \varrho(\Delta_\Omega)$. Moreover, $\lambda \neq \mu \in \varrho(\Delta_\Omega)$ if and only if $(\mu - \lambda)^{-1} \in \varrho\big(R_\Omega(\lambda)\big)$.*

Proof. Let $\lambda \in \varrho(\Delta_\Omega)$ and $\mu > 0$. Consider the Laplacian $\Delta_2$ on $L^2(\mathbb{R}^N)$. Then $(\mu - \Delta_2)^{-k} L^2(\mathbb{R}^N) = H^{2k}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ if $k > N/2$. Since $0 \le R_\Omega(\mu) \le (\mu - \Delta_2)^{-1}$, it follows that $\|R_\Omega(\mu)^k f\|_\infty \le c\,\|f\|_{L^2}$ $\big(f \in L^2(\Omega)\big)$ for some $c \ge 0$. Thus $R_\Omega(\mu)^k$ extends to a compact operator on $L^2(\Omega)$. Since $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$, it follows that $R_\Omega(\mu)^k$ is compact. Observe that

$$\mu R_\Omega(\mu) R_\Omega(\lambda_0) = \frac{\mu}{\mu - \lambda_0}\big(R_\Omega(\lambda_0) - R_\Omega(\mu)\big)$$

converges to $R_\Omega(\lambda_0)$ in $\mathcal{L}\big(L^\infty(\mathbb{R}^N)\big)$ as $\mu \to \infty$. By induction we deduce that $\big(\mu R_\Omega(\mu)\big)^k R_\Omega(\lambda_0) \to R_\Omega(\lambda_0)$ as $\mu \to \infty$. Thus $R_\Omega(\lambda_0)$ is compact. The last assertion follows from [18, Appendix A] by setting $E = L^\infty(\mathbb{R}^N)$ and $F = L^\infty(\Omega)$ with the natural restriction and extension. □

Let $\Omega_n, \Omega$ be open sets all contained in a large ball such that $\Omega_n \to \Omega$ regularly. Then by Theorem 5.5 we know that

$$R_{\Omega_n}(\lambda) \to R_\Omega(\lambda) \text{ in } \mathcal{L}\big(L^\infty(\mathbb{R}^N)\big) \text{ as } n \longrightarrow \infty \text{ for all } \lambda \ge 0. \tag{7.1}$$

For the remainder of this section we assume that (7.1) holds. The following is then a consequence of Theorem 7.2 and [29, IV §3.5].

**Theorem 7.3** *Assume hat* (7.1) *holds for some $\lambda > 0$ and let $\mu \in \varrho(\Delta_\Omega)$. Then there exists $n_0 \in \mathbb{N}$ such that $\mu \in \varrho(\Delta_{\Omega_n})$ for all $n \ge n_0$ and*

$$\lim_{n\to\infty} R_{\Omega_n}(\mu) = R_\Omega(\mu) \quad \text{in} \quad \mathcal{L}\big(L^\infty(\mathbb{R}^N)\big).$$

*If $\lambda \in \sigma(\Delta_\Omega)$ then $\lambda$ is an isolated eigenvalue of finite algebraic multiplicity.*

*Let $U \subset \mathbb{C}$ be an open neighbourhood of $\lambda$ not containing any other eigenvalue of $\Delta_\Omega$ and let $m$ be the algebraic multiplicity of $\lambda$. Then there exists $n_0 \in \mathbb{N}$ such that, counting multiplicity, $\Delta_{\Omega_n}$ has precisely $m$ eigenvalues in $U$ for all $n \geq n_0$.*

*Denote these eigenvalues by $\lambda_{nk}$ ($k = 1, \dots, m$, $n \geq n_0$). Moreover denote by $P_n$ the spectral projection of $\Delta_{\Omega_n}$ corresponding to $\lambda_{n1}, \dots, \lambda_{nm}$, and by $P$ the spectral projection of $\Delta_\Omega$ corresponding to $\lambda$. Then*

$$\lim_{n\to\infty} \lambda_{nk} = \lambda \quad \textit{for all} \quad k = 1, \dots, m$$

*and*

$$\lim_{n\to\infty} P_n = P \quad \textit{in} \quad \mathcal{L}\big(L^\infty(\mathbb{R}^N)\big).$$

## 8 Application: Nonlinear problems

We now look at nonlinear elliptic problems of the form

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in} \quad \Omega, \\ u &= 0 \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{8.1}$$

and determine how solutions persist under perturbations of $\Omega$. Throughout we assume that $f \in C^1(\mathbb{R})$ and that $\Omega$ is a bounded open set, having possibly several components. We now want to rewrite (8.1) as a fixed point problem. Let $B \subset \mathbb{R}^N$ be an open ball containing $\Omega$. Then the substitution operator

$$F\colon L^\infty(B) \longrightarrow L^\infty(B), \quad u \longmapsto f \circ u$$

is Lipschitz continuous on every bounded set of $L^\infty(B)$. Hence if $u \in L^\infty(B)$ is a (weak) solution of (8.1) then $F(u) \in L^\infty(B)$ and thus $R_\Omega(0)F(u) = u \in L^\infty(B)$. Hence, $u$ is a fixed point of

$$G_\Omega := R_\Omega(0) \circ F$$

in $L^\infty(B)$. Also, by Theorem 1.3 every fixed point of $G_\Omega$ is a (weak) solution of (8.1). Hence there is a one-to-one correspondence between fixed points of $G_\Omega$ and solutions of (8.1).

We now state some properties of $G_\Omega$. Recall that a map between Banach spaces is called *completely continuous* if it is continuous and maps bounded sets onto relatively compact sets. Moreover, a map is called *compact* if it is continuous and its image is relatively compact (see [21, Definition 2.8.1]).

**Lemma 8.1** *The map $G_\Omega\colon L^\infty(B) \to L^\infty(B)$ is completely continuous.*

Proof. We know already that $F\colon L^\infty(B) \to L^\infty(B)$ is Lipschitz continuous on bounded sets. In particular, if $U$ is a bounded subset of $L^\infty(B)$ then $F(U)$ is bounded in $L^\infty(B)$. Hence by Theorem 7.2 the set $G_\Omega(U) = R_\Omega(0)F(U)$ is relatively compact in $L^\infty(B)$. As the image of $R_\Omega(0)$ lies in $L^\infty(B)$ the assertion of the lemma follows. □

One common technique to prove existence of solutions to (8.1) is by means of the *Leray–Schauder degree* (see [21, Chapter 2.8] or [33, Chapter 4]). The Leray–Schauder degree (or simply degree) is defined for compact perturbations of the identity. By Lemma 8.1 the map $G_\Omega$ is compact when restricted to a bounded set. Hence, if $U \subset L^\infty(B)$ is an open bounded set such that $u \neq G_\Omega(u)$ for all $u \in \partial U$, then the Leray–Schauder degree, $\deg(I - G_\Omega, U, 0) \in \mathbb{Z}$, is well-defined.

Now we look at a sequence of bounded open sets $\Omega_n$. As before, bounded weak solutions of

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in} \quad \Omega_n, \\ u &= 0 \qquad \text{on} \quad \partial\Omega_n \end{aligned} \tag{8.2}$$

correspond to fixed points of $G_{\Omega_n} := R_{\Omega_n}(0) \circ F$. The following is the main result of this section. The basic idea of the proof goes back to [13]. The proof given here is similar to the one of [17, Theorem 7.1] (see also [16, Theorem 6.1]). The advantage of working in $L^\infty$ is that there are no problems with growth conditions on the nonlinearity $f \in C^1(\mathbb{R})$, so the proofs partly simplify.

 

**Theorem 8.2** *Suppose $\Omega_n, \Omega$ are open sets contained in a ball $B \subset \mathbb{R}^N$ such that $\Omega_n \to \Omega$ regularly. Moreover, let $U \subset L^\infty(B)$ be an open bounded set such that $G_\Omega(u) \neq u$ for all $u \in \partial U$. Then there exists $n_0 \in \mathbb{N}$ such that $G_{\Omega_n}(u) \neq u$ for all $u \in \partial U$ and*

$$\deg(I - G_\Omega, U, 0) = \deg(I - G_{\Omega_n}, U, 0) \quad \textit{for all} \quad n \geq n_0. \tag{8.3}$$

P r o o f. We use the homotopy invariance of the degree (see [21, Section 2.8.3] or [33, Theorem 4.3.4]) to prove (8.3). We set $G := G_\Omega$ and $G_n := G_{\Omega_n}$ and define the homotopies $H_n(t, u) := tG_n(u) + (1 - t)G(u)$ for $t \in [0, 1]$, $u \in L^\infty(B)$ and $n \in \mathbb{N}$. To prove (8.3) it is sufficient to show that there exists $n_0 \in \mathbb{N}$ such that

$$u \neq H_n(t, u) \tag{8.4}$$

for all $n \geq n_0$, $t \in [0, T]$ and $u \in \partial U$. Assume to the contrary that there exist $n_k \to \infty$, $t_{n_k} \in [0, 1]$ and $u_{n_k} \in \partial U$ such that $u_{n_k} = H_{n_k}(t_{n_k}, u_{n_k})$ for all $k \in \mathbb{N}$. As $U$ is bounded in $L^\infty(B)$ and $F$ Lipschitz we can assume that

$$\begin{aligned} &t_{n_k} \longrightarrow t_0 \quad \text{in} \quad [0, 1], \\ &g_{n_k} := F(u_{n_k}) \overset{*}{\rightharpoonup} g \quad \text{in} \quad L^\infty(B) \end{aligned}$$

if we select a further subsequence. By Theorem 5.5

$$\begin{aligned} &G_{n_k}(u_{n_k}) = R_{\Omega_{n_k}}(0)g_{n_k} \longrightarrow R_\Omega(0)g \\ &G(u_{n_k}) = R_\Omega(0)g_{n_k} \longrightarrow R_\Omega(0)g \end{aligned}$$

in $L^\infty(B)$ as $k \to \infty$. Hence

$$\begin{aligned} u_{n_k} &= H_{n_k}(t_{n_k}, u_{n_k}) \\ &= t_{n_k} G_{n_k}(u_{n_k}) + (1 - t_{n_k})G(u_{n_k}) \xrightarrow{k\to\infty} t_0 R_\Omega(0)g + (1 - t_0)R_\Omega(0)g = R_\Omega(0)g \end{aligned}$$

in $L^\infty(B)$. Setting $u := R_\Omega(0)g$ we conclude that $u \in \partial U$ and $u_{n_k} \to u$ in $L^\infty(B)$. Moreover, $g_{n_k} = F(u_{n_k}) \to F(u) = g$ by continuity of $F$. Hence $u = G(u)$ for some $u \in \partial U$, contradicting our assumptions. Thus (8.4) must be true for some $n \geq n_0$, completing the proof of the theorem. □

Of course, we are most interested in the case $\deg(I - G_\Omega, U, 0) \neq 0$. Then, by the solution property of the degree (see [33, Theorem 4.3.2]), (8.1) has a solution in $U$. As a corollary to Theorem 8.2 we therefore have the following result.

**Corollary 8.3** *Suppose that $\Omega_n \to \Omega$ regularly and that $U \subset L^\infty(B)$ is open bounded and $u \neq G_\Omega(u)$ for all $u \in \partial U$. If $\deg(I - G_\Omega, U, 0) \neq 0$, then there exists $n_0 \in \mathbb{N}$ such that* (8.2) *has a solution in U for all $n \geq n_0$.*

Now we consider an isolated solution $u_0$ of (8.1) and recall the definition of its *index*. Denote by $B_\varepsilon(u_0)$ the open ball of radius $\varepsilon > 0$ and centre $u_0$ in $L^\infty(B)$. Then $\deg(I - G_\Omega, B_\varepsilon(u_0), 0)$ is defined for small enough $\varepsilon > 0$. Moreover, by the excision property of the degree $\deg(I - G_\Omega, B_\varepsilon(u_0), 0)$ stays constant for small enough $\varepsilon > 0$. Hence the index of $u_0$,

$$\mathrm{i}_0(G_\Omega, u_0) := \lim_{\varepsilon\to 0} \deg(I - G_\Omega, B_\varepsilon(u_0), 0)$$

is well-defined.

**Theorem 8.4** *Suppose that $u_0$ is an isolated solution of* (8.1) *with $\mathrm{i}_0(G_\Omega, u_0) \neq 0$. If $\Omega_n \to \Omega$ regularly then, for n large enough, there exist solutions $u_n$ of* (8.2) *such that $u_n \to u_0$ in $L^\infty(B)$ as $n \to \infty$.*

P r o o f. By assumption there exists $\varepsilon_0 > 0$ such that

$$\mathrm{i}_0(G_\Omega, u_0) = \deg(I - G_\Omega, B_\varepsilon(u_0), 0) \neq 0 \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_0).$$

Hence by Corollary 8.3 problem (8.2) has a solution in $B_\varepsilon(u_0)$ for all $\varepsilon \in (0, \varepsilon_0)$ if only $n$ large enough. Suppose now that there exists no sequence of solutions as claimed in the theorem. Then there exist $\varepsilon \in (0, \varepsilon_0)$ and a subsequence $n_k \to \infty$ such that (8.2) (for $n = n_k$) has no solution in $B_\varepsilon(u_0)$ for all $k \in \mathbb{N}$. However, this contradicts what we have just proved. □

Without additional assumptions it is possible that there are several different sequences of solutions of (8.2) converging to $u_0$. However, if $u_0$ is *non-degenerate*, that is, the linearized problem

$$\begin{aligned} -\Delta w &= f'(u_0)w \quad \text{in} \quad \Omega, \\ w &= 0 \qquad\quad\ \text{on} \quad \partial\Omega \end{aligned} \tag{8.5}$$

has only the trivial solution, then $u_n$ is unique for large $n \in \mathbb{N}$.

**Theorem 8.5** *Suppose that $f \in C^2(\mathbb{R})$, that $u_0$ is a non-degenerate solution of* (8.1) *and that $\Omega_n \to \Omega$ regularly. Then there exists $\varepsilon > 0$ such that* (8.2) *has a unique solution in $B_\varepsilon(u_0)$ for all $n$ large enough. Moreover these solutions are non-degenerate.*

Proof. As $f \in C^2(\mathbb{R})$ it follows that the substitution operator induced by $f'$ is also locally Lipschitz on $L_\infty(B)$ and $F$ is continuously differentiable. Hence also $G_\Omega$ is continuously differentiable. That the solution is non-degenerate means that $I - DG_\Omega(u_0)$ is invertible with bounded inverse. By [33, Theorem 5.2.3 and Theorem 4.3.14] $\mathrm{i}_0(G_\Omega, u_0) = \pm 1$, so by Theorem 8.4 there exists a sequence of solutions $u_n$ of (8.2) with $u_n \to u$ as $n \to \infty$. We now show uniqueness. Suppose to the contrary that there exist solutions $u_n$ and $v_n$ of (8.2) converging to $u_0$ and $u_n \neq v_n$ for all $n \in \mathbb{N}$ large enough. As $f' \in C^1(\mathbb{R})$ we get

$$f(u_n) - f(v_n) = \int_0^1 f'\big(\tau u_n + (1-\tau)v_n\big)\, d\tau (u_n - v_n)$$

and since $u_n$ and $v_n$ converge to $u_0$ in $L^\infty(B)$

$$a_n := \int_0^1 f'\big(\tau u_n + (1-\tau)v_n\big)\, d\tau \longrightarrow f'(u_0) \quad \text{in} \quad L^\infty(B) \quad \text{as} \quad n \longrightarrow \infty.$$

Passing to a subsequence we can assume that

$$w_n := \frac{u_n - v_n}{\|u_n - v_n\|_\infty} \overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty(B).$$

As $u_n$, $v_n$ solve (8.2) we conclude from the above that

$$w_n = \frac{G_{\Omega_n}(u_n) - G_{\Omega_n}(v_n)}{\|u_n - v_n\|_\infty} = R_{\Omega_n}(0)(a_n w_n).$$

Since $a_n \to f'(u_0)$ uniformly and $w_n \overset{*}{\rightharpoonup} w$ in $L^\infty(B)$ we have that $a_n w_n \overset{*}{\rightharpoonup} f'(u_0)w$ in $L^\infty(B)$. Hence by Theorem 5.5

$$w_n = R_{\Omega_n}(0)(a_n w_n) \xrightarrow{n\to\infty} R_\Omega(0)(f'(u_0)w)$$

Therefore, $w_n \to w$ in $L^\infty(B)$, $\|w\|_\infty = 1$ and $w = R_\Omega(0)(f'(u_0)w)$, showing that (8.5) has a nontrivial solution. However, by assumption no such nontrivial solution exists, proving uniqueness of $(u_n)$. It remains to show that $u_n$ is non-degenerate for large $n \in \mathbb{N}$. If we suppose not, then there exists a subsequence $(u_{n_k})$ and $w_{n_k} \in L^\infty(B)$ such that $\|w_{n_k}\|_\infty = 1$ such that $w_{n_k} = R_{\Omega_{n_k}}(0)(f'(u_{n_k})w_{n_k})$ for all $k \in \mathbb{N}$. By possibly passing to another subsequence we may assume that $w_{n_k} \overset{*}{\rightharpoonup} w$ in $L^\infty$. As $u_n \to u_0$ in $L^\infty(B)$ we have $f'(u_n) \to f'(u_0)$ in $L_\infty(B)$ and thus $f'(u_{n_k})w_{n_k} \overset{*}{\rightharpoonup} f'(u_0)w$ in $L^\infty(B)$. Hence by Theorem 5.5

$$w_{n_k} = R_{\Omega_{n_k}}(0)(f'(u_{n_k})w_{n_k}) \xrightarrow{k\to\infty} R_\Omega(0)(f'(u_0)w).$$

Thus $w_{n_k} \to w$ in $C_0(\Omega)$, $\|w\|_\infty = 1$ and $w$ satisfies (8.5), showing that (8.5) has a nontrivial solution. As this is a contradiction, $u_n$ must be non-degenerate for all $n$ large enough. □

To illustrate the use of the above results we finally give an example to the Gelfand equation from combustion theory (see [23, §15]), similar to those in [13]. The idea in [13] was to show that simple equations can have many solutions depending on the geometry (and not the topology) of the domain.

 

**Example 8.6** Consider the Gelfand equation

$$\begin{aligned} -\Delta u &= \mu e^u \quad \text{in} \quad \Omega, \\ u &= 0 \quad\ \ \text{on} \quad \partial\Omega, \end{aligned} \tag{8.6}$$

on a bounded domain of class $C^2$. If $\mu > 0$, then $\mu e^u > 0$ for all $u$ and thus by the maximum principle every solution of (8.6) is positive. By [24, Theorem 1] positivity implies that all solutions are radially symmetric if $\Omega$ is a ball.

It is well-known that there exists $\mu_0 > 0$ such that (8.6) has a minimal positive solution for $\mu \in [0, \mu_0]$ and no solution for $\mu > \mu_0$ (see [1, 11]). Moreover, for $\mu \in (0, \mu_0)$ this minimal solution is non-degenerate (see [11, Lemma 3]). Let now $\Omega = B_0 \cup B_1$ be the union of two balls $B_0$ and $B_1$ of the same radius and $\Omega_n$ the dumbbell-like domains as shown in Figure 6. If $\mu \in (0, \mu_0)$ and $N = 2$ there exists a second, solution for the problems on $B_0$ and $B_1$. By [28, p. 242] or [23, §15, p. 359] the equation has in fact *precisely two* solutions on the ball if $N = 2$ and $\mu \in (0, \mu_0)$. (This is not true for $N \geq 3$, see [28].) Equation (8) on page 415 together with the results in [12, Section 2] imply that there is bifurcation from every degenerate solution. Since we know that there is no bifurcation in the interval $(0, \mu_0)$, it follows that the second solution is also non-degenerate.

We now show that there are possibly more than two solutions on (simply connected) domains other than balls. Looking at $\Omega = B_0 \cup B_1$ we have four nontrivial non-degenerate solutions of (8.6). Hence by Theorem 8.5 there exist at least four non-degenerate solutions of (8.6) on $\Omega_n$ for $n$ large. These solutions are close to the original four solutions in the $L_\infty$-norm. The equation possibly has more than four solutions on $\Omega_n$. These additional solutions, if they exist, essentially live on the handle connecting the balls, and their $L_\infty$-norm goes to infinity as the handle of the dumbbell shrinks. These solutions are often called "large solutions." More on the existence and non-existence of such large solutions can be found for instance in [14]. Connecting $n$ balls with strips we can get domains where (8.6) has at least $2^n$ different solutions.

## 9 Application: Parabolic equations

Let $\Omega \subset \mathbb{R}^N$ be open and regular.

**Theorem 9.1** *There exists a unique bounded continuous function*

$$T_\Omega\colon (0,\infty) \to \mathcal{L}\big(L^\infty(\mathbb{R}^N)\big)$$

*such that*

$$R_\Omega(\lambda) = \int_0^\infty e^{-\lambda t} T_\Omega(t)\, dt \quad (\lambda > 0). \tag{9.1}$$

*Moreover, $T_\Omega$ has the following properties.*

(a) *There exists an angle $0 < \theta < \frac{\pi}{2}$ such that $T_\Omega$ has a unique holomorphic extension to*

$$\Sigma_\theta := \{re^{i\alpha} : r > 0,\ |\alpha| < \theta\}$$

*with values in $\mathcal{L}\big(L^\infty(\mathbb{R}^N)\big)$;*

(b) $T_\Omega(z_1 + z_2) = T_\Omega(z_1)T_\Omega(z_2)$ *for all $z_1, z_2 \in \Sigma_\theta$,*

(c) $T_\Omega(z)L^\infty\big(\mathbb{R}^N\big) \subset BC_0(\Omega)$ *for all $z \in \Sigma_\theta$.*

Proof. Because of (1.2) there exists the generator $\Delta_{\Omega,1}$ of a positive, contractive $C_0$-semigroup $T$ on $L^1(\Omega)$ such that $R(\lambda, \Delta_{\Omega,1})f = R_\Omega(\lambda)f$ for all $f \in L^1(\Omega) \cap L^\infty(\Omega)$ and $\lambda > 0$. This semigroup is bounded and holomorphic by [6] or [3] (see also [36]). Define $T_\Omega(z) = T(z)'$. Then $T_\Omega$ restricted to $L^\infty(\Omega)$ is a bounded holomorphic semigroup on $L^\infty(\Omega)$ (in the sense of [7, Definition 3.7.1]). Since $R(\lambda, \Delta_{\Omega,1})' = R_\Omega(\lambda)'_{L^\infty(\Omega)}$ its generator is $\Delta_\Omega$. Since $R_\Omega(\lambda)L^\infty(\Omega) \subset BC_0(\Omega)$ by Proposition 2.4, it follows from [7, Theorem 3.7.19] that $T_\Omega(t)L^\infty\big(\mathbb{R}^N\big) \subset BC_0(\Omega)$. Hence (c) follows from the uniqueness of holomorphic extensions. □

We call $T_\Omega$ the *semigroup on $L^\infty(\Omega)$ generated by* $\Delta_\Omega$. The following theorem is a simple consequence of [5, Theorem 5.2].

 

**Theorem 9.2** *Let $\Omega_n, \Omega$ be regular open sets and assume that*

$$\lim_{n\to\infty} R_{\Omega_n}(\lambda) = R_\Omega(\lambda)$$

*in $\mathcal{L}\big(L^\infty(\mathbb{R}^N)\big)$ for some $\lambda > 0$. Then*

$$\lim_{n\to\infty} T_{\Omega_n}(t) = T_\Omega(t) \quad \text{in} \quad \mathcal{L}\big(L^\infty(\mathbb{R}^N)\big)$$

*uniformly on $[\varepsilon, \varepsilon^{-1}]$ for all $0 < \varepsilon < 1$.*

In Section 6 diverse situations implying the assumptions of the above theorem are discussed.

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