# SPECTRAL PROPERTIES OF THE DIRICHLET-TO-NEUMANN OPERATOR ON LIPSCHITZ DOMAINS

WOLFGANG ARENDT AND RAFE MAZZEO

Abstract. The Dirichlet-to-Neumann operator $D_\lambda$ is defined on $L^2(\Gamma)$ where $\Gamma$ is the boundary of a Lipschitz domain $\Omega$ and $\lambda$ a real number which is not an eigenvalue of the Dirichlet Laplacian on $L^2(\Omega)$. We show that $D_\lambda$ is a selfadjoint lower bounded operator with compact resolvent. There is a close connection between its eigenvalues and those of the Laplacian $\Delta_\mu$ on $L^2(\Omega)$ with Robin boundary conditions $\frac{\partial u}{\partial \nu} = \mu u_{|\Gamma}$ where $\mu \in \mathbb{R}$. This connection is used to generalize L. Friedlander's result $\lambda_{k+1}^N \leq \lambda_k^D$ to Lipschitz domains (where $\lambda_k^D$ is the $k-th$ Dirichlet and $\lambda_k^N$ the $k-th$ Neumann eigenvalue). We show that this Euclidean result is false, though, if an arbitrary compact Riemannian manifold $M$ is considered instead of $\mathbb{R}^d$ and $\Omega$ is suitable domain in $M$.

## 0. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. There is a wealth of interesting results comparing the eigenvalues $\lambda_1^D < \lambda_2^D \leq \lambda_3^D \leq \cdots$ of the Dirichlet Laplacian and those of the Neumann Laplacian denoted by $\lambda_1^N < \lambda_2^N \leq \lambda_3^N \leq \cdots$, each time repeated according to multiplicity. A first result of this type is due to Polya [Pol52] who showed that

$$\lambda_2^N < \lambda_1^D \ .$$

Shortly after, in 1955, Payne [Pay55] showed that

$$\lambda_{k+d}^N \leq \lambda_k^D \quad (k = 1, 2, \cdots) \tag{0.1}$$

whenever $\Omega$ is a convex, planar domain with $C^2$-boundary. Levine and Weinberger [LW86] proved inequality (0.1) for arbitrary bounded convex domains in $\mathbb{R}^d$ without any regularity assumption. They also showed that (0.1) remains true if convexity is replaced by more general conditions on the mean curvature of the boundary (which is assumed to be $C^{2+\alpha}$). However, without any geometric condition, in dimension 2, it may happen that $\lambda_3^N > \lambda_1^D$ for $\Omega \subset \mathbb{R}^2$, (see [Avi86]). It was only in 1991 that L. Friedlander [Fri91] proved the inequality

$$\lambda_{k+1}^N \leq \lambda_k^D \quad (k = 1, 2, \cdots \tag{0.2}$$

for arbitray domains in $\mathbb{R}^d$ of class $C^1$ without any restriction on the geometry. However, his assumption on the $C^1$-regularity of the boundary is crucial for his arguments (which are actually given for $C^\infty$-domains, referring to a general approximation result of $C^1$-domains by $C^\infty$-domains with convergence of the corresponding eigenvalues in [CH89]). In view of the preceding diverse results involving geometric and regularity assumptions one may wonder whether the $C^1$-assumption is optimal in Friedlander's result, even though some hypothesis on $\Omega$ is needed to garantee that the Neumann Laplacian has compact resolvent.

In the present paper we show that (0.2) does hold for Lipschitz domains by very elementary arguments. As Friedlander we use the Dirichlet-to-Neumann operator $D_1$ on $L^2(\Gamma)$ where $\Gamma$ is the boundry of $\Omega$ and $\lambda \notin \{\lambda_k^D : k \in \mathbb{N}\}$ a real parameter. A major point is to define the operator $D_\lambda$ on Lipschitz domains using merely the trace operator $Tr : H^1(\Omega) \to L^2(\Gamma)$ and a weak definition of the normal derivative. Instead of investigating the spectrum of $D_\lambda$ directly we consider the Laplacian

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$\Delta_\mu$ on $L^2(\Omega)$ with Robin boundary conditions $\frac{\partial u}{\partial \nu} = \mu u_{|\Gamma}$. The crucial relation between the spectra of these operators is

$$\lambda \in \sigma(\Delta_\mu) \Leftrightarrow \mu \in \sigma(D_\lambda) \; . \tag{0.3}$$

We show that (0.2) is equivalent to the first eigenvalue of $D_\lambda$ being non-positive, which is always true for Lipschitz domains in $\mathbb{R}^d$. However, (0.2) fails, in general, if $\Omega$ is a Lipschitz domain in a compact manifold $M$. This had been already proved in [Maz91] if $M$ is the sphere. Here we show that in any compact manifold $M$ there are domains for which (0.2) fails. It suffices to consider as $\Omega$ the manifold with a small hole. In the paper we also show that the semigroup generated by $-D_\lambda$ is positive whenever $\lambda < \lambda_1^D$. This allows us to prove some assertions of Krein-Rutman type on the principal eigenvalue.

An extended version of the present article will be published elsewhere.

## 1. Preliminaries on forms

Let $H$ be a Hilbert space and $V$ a Hilbert space which is densely and continuously embedded in $H$. Let $a : V \times V \to \mathbb{R}$ be symmetric, continuous and **elliptic**, i.e.

$$a(u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \quad (u \in V)$$

for some $\omega \in \mathbb{R}, \alpha > 0$ where $a(u) = a(u,u)$. This is equivalent to saying that the form $a$ with domain $V$ is lower bounded and closed in $H$. Denote by $A$ the operator on $H$ associated with $a$. That is, for $x, y \in H$ one has $x \in D(A), Ax = y$ if and only if $x \in V$ and $a(x,v) = (y|v)_H$ for all $v \in V$. Then $A$ is selfadjoint. The form $a$ is **accretive** (i.e. $a(u) \geq 0$ for all $u \in V$) if and only if $(Au \mid u) \geq 0$ for all $u \in D(A)$, i.e. if and only if $A$ is **monotone**. $A$ has compact resolvent if and only if the injection $V \hookrightarrow H$ is compact. We assume throughout that $H$ is separable and infinite dimensional. If $A$ has compact resolvent, then $H$ has an orthogonal basis $\{e_n : n \in \mathbb{N}\}$ such that

$$Ae_n = \lambda_n e_n \quad (n \in \mathbb{N})$$

where $\lambda_1 \leq \lambda_2 \leq \cdots$ and $\lim\limits_{n\to\infty} \lambda_n = \infty$. We call this **the sequence of eigenvalues of** $A$ counting multiplicity. One has

$$\lambda_n = \sup\{\min_{u \in W_1} a(u) : W \subset V, \dim V/W = n-1\} \; . \tag{1.1}$$

Here $W \subset V$ is a subspace and $W_1 := \{u \in W : \|u\|_H = 1\}$.

## 2. The Laplacian on open sets

In this section we consider the Laplacian on $L^2(\Omega)$ with Robin boundary conditions, where $\Omega$ is an open bounded set in $\mathbb{R}^d$ with Lipschitz boundary (in §6 we will consider more generally a relatively compact, open subset of a Riemannian manifold). We define the Robin Laplacians as selfadjoint operators depending on a parameter. They all have compact resolvent and we study continuity properties of the $k-$th eigenvalue.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ which is connected and has Lipschitz boundary $\Gamma := \partial\Omega$. Such a subset will be called a **Lipschitz domain** in the sequel. We consider $L^2(\Omega)$ with respect to the Lebesgue measure. Denote by $H^1(\Omega)$ the first Sobolev space which is a Hilbert space for the norm

$$\|u\|_{H^1}^2 = \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \, dx \; .$$

We denote by $H_0^1(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, where $\mathcal{D}(\Omega)$ is the space of all test functions. There is a unique bounded operator

$$Tr : H^1(\Omega) \to L^2(\Gamma)$$

such that $Tr(u) = u_{|\Gamma}$ whenever $u \in H^1(\Omega) \cap C(\bar{\Omega})$. This operator is called the **trace operator**. We keep the symbol $u_{|\Gamma} := Tr(u)$ for $u \in H^1(\Omega)$ even if $u \notin C(\bar{\Omega})$. Here $L^2(\Gamma)$ is defined with respect to the surface measure on $\Gamma$. For $\mu \in \mathbb{R}$ we want to consider **Robin boundary conditions** $\frac{\partial u}{\partial \nu} = \mu u_{|\Gamma}$. In order to do so, we define the weak normal derivative in the following way.

**Definition 2.1.** a) Let $u \in H^1(\Omega)$. We say that $\Delta u \in L^2(\Omega)$ if there exists $f \in L^2(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v = \int_\Omega f v \tag{2.1}$$

for all $v \in \mathcal{D}(\Omega)$ (equivalently for all $v \in H_0^1(\Omega)$). In that case we let $\Delta u := f$. b) Let $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. We say that $\frac{\partial u}{\partial \nu} \in L^2(\Gamma)$ if there exists $b \in L^2(\Gamma)$ such that

$$\int_\Omega \nabla u \nabla v - \int_\Omega \Delta u v = \int_\Gamma b v \tag{2.2}$$

for all $v \in H^1(\Omega)$. In that case we let $\frac{\partial u}{\partial \nu} := b$.

Here and later on we let $\int_\Gamma bv = \int_\Gamma b u_{|\Gamma}$, i.e. we omit the trace signs under the integral over $\Gamma$. Since by the Stone-Weierstraß Theorem the space $\{v_{|\Gamma} : v \in C^\infty(\mathbb{R}^d)\}$ is dense in $L^2(\Gamma)$, the function $b \in L^2(\Gamma)$ is unique. The definition is such that Green's formula

$$\int_\Omega \nabla u \nabla v - \int_\Omega \Delta u v = \int_\Gamma \frac{\partial u}{\partial \nu} v \tag{2.3}$$

holds for all $v \in H^1(\Omega)$ whenever $u \in H^1(\Omega), \Delta u \in L^2(\Omega)$ and $\frac{\partial u}{\partial \nu} \in L^2(\Gamma)$. In the case of smooth domain and smooth functions $\frac{\partial u}{\partial \nu}$ is the outer normal derivative where $\nu$ denotes the outer normal. If $u \in C^2(\bar{\Omega})$, then $\Delta u \in L^2(\Omega)$ and

$$\Delta u = -\sum_{j=1}^{d} D_j^2 u \ .$$

We use the sign of the Laplacian which makes it a form-positive operator.

For $\mu \in \mathbb{R}$ we define the **Robin Laplacian** $\Delta_\mu$ on $L^2(\Omega)$ by

$$\begin{aligned} D(\Delta_\mu) &:= \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \frac{\partial u}{\partial \nu} = \mu u_{|\Gamma}\} \\ \Delta_\mu u &:= \Delta u \ . \end{aligned}$$

**Proposition 2.2.** *The operator $\Delta_\mu$ is associated with the symmetric, continuous and elliptic form*

$$b_\mu(u, v) = \int_\Omega \nabla u \nabla v - \mu \int_\Gamma uv$$

$$b_\mu : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R} \ .$$

*Consequently, $\Delta_\mu$ is selfadjoint, bounded below and has compact resolvent.*

In order to prove ellipticity, recall that the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ as well as the trace operator $H^1(\Omega) \to L^2(\Gamma)$ are compact (cf. [Nec67, Chap. 2 § 6, Theorem 6.2]). Thus if $u_n \rightharpoonup u$ in $H^1(\Omega)$ (weak convergence) then $u_n \to u$ in $L^2(\Omega)$ and $u_{n_{|\Gamma}} \to u_{|\Gamma}$ in $L^2(\Gamma)$.

We need the following standard estimate which will also be useful later.

**Lemma 2.3.** *Let $X_1, X_2, X_3$ be Banach spaces, $X_1$ reflexive. Let $T \in \mathcal{L}(X_1, X_3)$ be compact and $S \in \mathcal{L}(X_1, X_2)$ injective. Let $\varepsilon > 0$. Then there exists $c > 0$ such that for all $x \in X_1$,*

$$\|Tx\|_{X_3}^2 \le \varepsilon \|x\|_{X_1}^2 + c\|Sx\|_{X_2}^2 \ .$$

*Proof.* If not, there exist $x_n \in X_1$ such that $\|x_n\|_{X_1} = 1, \|Tx_n\|^2_{X_3} \geq \varepsilon + n\|Sx_n\|^2_{X_2}$. Since $X_1$ is reflexive, we may assume that $x_n \rightharpoonup x$ in $X_1$. Since $T$ is compact, it follows that $Tx_n \to Tx$ in $X_3$. Hence $\|Tx\|^2_{X_3} \geq \varepsilon$. On the other hand $\|Sx_n\|^2_{X_2} \leq \frac{1}{n}\|Tx_n\|^2_{X_3} \to 0$ as $n \to \infty$. Since $Sx_n \rightharpoonup Sx$, it follows that $Sx = 0$. Since $S$ is injective, it follows that $x = 0$. Hence $Tx = 0$, a contraction. □

*Proof of Proposition 2.2.* The form $b_\mu$ is clearly continuous. In order to prove ellipticity, let $X_1 = H^1(\Omega)$, $X_3 = L^2(\Gamma)$, $T \in \mathcal{L}(X_1, X_3)$ the trace operator, $X_2 = L^2(\Omega)$ and $S \in \mathcal{L}(H^1(\Omega), L^2(\Omega))$ the injection. By Lemma 2.3 there exists $c > 0$ such that for all $u \in H^1(\Omega)$,

$$\begin{aligned} \mu \int_\Gamma u^2 &\leq \tfrac{1}{2}\|u\|^2_{H^1} + c\int_\Omega u^2 \\ &= \tfrac{1}{2}\int_\Omega |\nabla u|^2 + (c + 1/2)\int_\Omega u^2 \ . \end{aligned}$$

Let $\omega = c + 1$. Then

$$\begin{aligned} b_\mu(u) + \omega \int_\Omega u^2 &= \int_\Omega |\nabla u|^2 - \mu \int_\Gamma u^2 + \omega \int_\Omega u^2 \\ &\geq \frac{1}{2}\Big(\int_\Omega |\nabla u|^2 + \int_\Omega u^2\Big) \ . \end{aligned}$$

Thus $b_\mu$ is elliptic. Let $B$ be the operator associated with $b_\mu$. Let $u \in H^1(\Omega), f \in L^2(\Omega)$. Then $u \in D(B)$ and $Bu = f$ if and only if

$$\int_\Omega \nabla u \nabla v - \mu \int_\Gamma uv = \int_\Omega fv \tag{2.4}$$

for all $v \in H^1(\Omega)$. Taking $v \in H^1_0(\Omega)$ we see that (2.4) implies that $\Delta u = f$. Hence inserting $f = \Delta u$ into (2.4) we deduce that $\frac{\partial u}{\partial \nu} = \mu u$. Thus $u \in D(\Delta_\mu)$ and $\Delta_\mu u = Bu$ for all $u \in D(B)$. Conversely, if $u \in D(\Delta_\mu)$, then

$$\int_\Omega \nabla u \nabla v - \int_\Omega \Delta u v = \int_\Gamma \mu u v \tag{2.5}$$

for all $v \in H^1(\Omega)$ by the definition of $\frac{\partial u}{\partial \nu} = \mu u$. Thus (2.4) holds for $f = \Delta u$. Hence $u \in D(B)$ and $Bu = \Delta u$. We have shown that $B = \Delta_\mu$. Since the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it follows that $\Delta_\mu$ has compact resolvent. □

If $\mu = 0$, then we find the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$. We also use the symbol $\Delta^N = \Delta_0$ in this case and call $\Delta^N$ the **Neumann Laplacian**. The **Dirichlet Laplacian** $\Delta^D$ on $L^2(\Omega)$ is definied by

$$\begin{aligned} D(\Delta^D) &= \{u \in H^1_0(\Omega) : \Delta u \in L^2(\Omega)\} \\ \Delta^D u &= \Delta u \ . \end{aligned}$$

The operator $\Delta^D$ is associated with the form

$$\begin{aligned} b_{-\infty} &: H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R} \\ b_{-\infty}(u, v) &= \int_\Omega \nabla u \nabla v \ . \end{aligned}$$

Thus $\Delta^D$ is selfadjoint and has compact resolvent. For $\mu \in \mathbb{R}$ we denote the $k-th$ eigenvalue of $\Delta_\mu$ by $\lambda_k(\mu)$. We let

$$\lambda^N_k = \lambda_k(0) \ ,$$

which is the $k-th$ eigenvalue of $\Delta^N$. By $\lambda^D_k = \lambda_k(-\infty)$ we denote the $k-th$ eigenvalue of $\Delta^D = \Delta_{-\infty}$.

**Theorem 2.4.** *The functions $\lambda_k : [-\infty, \infty) \to \mathbb{R}$ are continuous and nonincreasing. In particular,* $\lim\limits_{k \to -\infty} \lambda_k(\mu) = \lambda^D_k$.

Note also that by definition

$$\lambda_k(\mu) \leq \lambda_{k+1}(\mu) \quad \text{for all} \quad -\infty \leq \mu < \infty \ .$$

Since $\lambda_1(0) = \lambda_1^N = 0$, it follows that $\lambda_k(\mu) \geq 0$ if $-\infty \geq \mu \geq 0$ for all $k \in \mathbb{N}$. We will obtain further information on these functions in Section 4.

For the proof of Theorem 2.3 we need the continuity of the resolvents. This can be proved with the help of the following lemma (see [Dan03, Appendix B]).

**Lemma 2.5.** *Let $T_n, T \in \mathcal{L}(X,Y)$, where $X, Y$ are Banach spaces and $X$ is reflexive. The following assertions are equivalent:*

(i) $\lim\limits_{n\to\infty} \|T_n - T\| = 0$ *and $T$ is compact;*

(ii) $x_n \rightharpoonup x$ *in $X$ implies $T_n x_n \to Tx$ in $Y$.*

**Proposition 2.6.** *Let $-\infty \leq \mu_0 < \infty$. Then for $\lambda \in \mathbb{R}$ large enough*

$$\lim_{\mu\to\mu_0} (\lambda + \Delta_\mu)^{-1} = (\lambda + \Delta_{\mu_0})^{-1}$$

*in $\mathcal{L}(L^2(\Omega))$.*

*Proof.* Let $f_n \rightharpoonup f$ in $L^2(\Omega), u_n = (\lambda + \Delta_{\mu_n})^{-1} f_n$ where $\mu_n \to \mu_0 \in [-\infty, \infty)$. We have to show that $u_{n_k} \to u := (\lambda + \Delta_{\mu_0})^{-1} f$ in $L^2(\Omega)$ for some subsequence. By definition we have

$$\lambda \int_\Omega u_n v + \int_\Omega \nabla u_n \nabla v - \mu_n \int_\Gamma u_n v = \int_\Omega f_n v \quad (v \in H^1(\Omega)) \ . \tag{2.6}$$

In particular,

$$\lambda \int_\Omega u_n^2 + \int_\Omega |\nabla u_n|^2 - \mu_n \int_\Gamma u_n^2 = \int_\Omega f_n u_n \ . \tag{2.7}$$

Taking $\lambda > 0$, sufficiently large, by ellipticity in Proposition 2.3, the left hand side is larger or equal than $\alpha \|u_n\|_{H^1}^2$ for all $n$ and some $\alpha > 0$. Since $\int_\Omega f_n u_n \leq \|f_n\|_{L^2} \|u_n\|_{L^2}$, it follows $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. Taking a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$. Consequently, $u_n \to u$ in $L^2(\Omega)$ and $u_{n_{|\Gamma}} \to u_{|\Gamma}$ in $L^2(\Gamma)$. Now we consider the two cases $\mu_0 \neq -\infty$ and $\mu_0 = \infty$ separately.

a) Let $\mu_0 \neq -\infty$. Then it follows from (2.6), letting $n \to \infty$, that

$$\lambda \int_\Omega uv + \int_\Omega \nabla u \nabla v - \mu_0 \int_\Gamma uv = \int_\Omega fv$$

for all $v \in H^1(\Omega)$. Thus $u = (\lambda + \Delta_{\mu_0})^{-1} f$. Now (2.7) implies that

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 &= \int_\Omega fu - \lambda \int_\Omega u^2 + \mu_0 \int_\Gamma u^2 \\ &= \int_\Omega |\nabla u|^2 \ . \end{aligned}$$

We have shown that $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $\|u_n\|_{H^1} \to \|u\|_{H^1}$. This implies that $u_n \to u$ in $H^1(\Omega)$.

b) Let $\mu_0 = -\infty$. Thus $\lim\limits_{n\to\infty} (-\mu_n) = \infty$. Then (2.7) implies that $\lim\limits_{n\to\infty} \int_\Gamma u_n^2 = 0$. Hence $u_{|\Gamma} = 0$, i.e. $u \in H_0^1(\Omega)$. Letting $v \in H_0^1(\Omega)$ in (2.6) and $n \to \infty$ shows that

$$\lambda \int_\Omega uv + \int_\Omega \nabla u \nabla v = \int_\Omega fv$$

for all $v \in H_0^1(\Omega)$. Hence $u = (\lambda + \Delta^D)^{-1} f$. Since $u_n \to u$ in $L^2(\Omega)$ it follows from Lemma 2.6 that $(\lambda + \Delta_{\mu_n})^{-1} \to (\lambda + \Delta^D)^{-1}$ in $\mathcal{L}(L^2(\Omega))$. □

*Remark* 2.7. If $\mu_0 \neq -\infty$ we have shown that $(\lambda + \Delta_{\mu_n})^{-1} \to (\lambda + \Delta_\mu)^{-1}$ even in $\mathcal{L}(L^2(\Omega), H^1(\Omega))$ as $n \to \infty$.

Now Theorem 2.5 follows from the following result, cf. [Kat66, IV § 3.5].

**Proposition 2.8.** *Let $B_n, B$ selfadjoint operators with compact resolvent on a separable Hilbert space $H$ such that*

$$(B_n x \mid x) \geq \omega (x \mid x)$$

*for all $x \in D(B_n)$ and all $n \in \mathbb{N}$ where $\omega \in \mathbb{R}$. Assume that $(\lambda + B_n)^{-1} \to (\lambda + B)^{-1}$ in $\mathcal{L}(H)$ as $n \to \infty$ for some $\lambda > \omega$. Denote by $\lambda_k^n$ the $k-th$ eigenvalue of $B_n$ and by $\lambda_k$ the $k-th$ eigenvalue of $B$ repeating eigenvalues according to their multiplicity. Then $\lim_{n\to\infty} \lambda_k^n = \lambda_k$.*

## 3. The Dirichlet-to-Neumann Operator

In this section we define the Dirichlet-to-Neumann operator on a Lipschitz domain $\Omega$. Let $\lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$. Denote the boundary of $\Omega$ by $\Gamma$. The **Dirichlet-to-Neumann Operator** $D_\lambda$ is defined on $L^2(\Gamma)$ by

$$\begin{aligned} D(D_\lambda) \quad &:= \quad \{\varphi \in L^2(\Gamma) : \exists u \in H^1(\Omega) \text{ such that} \\ &\qquad u_{|\Gamma} = \varphi, \Delta u = \lambda u \text{ and } \tfrac{\partial u}{\partial \nu} \text{ exists in } L^2(\Gamma)\}\ , \\ D_\lambda \varphi \quad &= \quad \tfrac{\partial u}{\partial \nu}\ . \end{aligned}$$

Here we use the definition of $\frac{\partial u}{\partial \nu}$ given in the preceding section.

**Theorem 3.1.** *The operator $D_\lambda$ is selfadjoint, bounded below and has compact resolvent.*

The theorem will be proved by showing that $D_\lambda$ is associated with a symmetric form on $L^2(\Gamma)$. We need the following lemma.

**Lemma 3.2.** *For $\lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$ one has*

$$H^1(\Omega) = H_0^1(\Omega) \oplus H^1(\lambda)$$

*where $H^1(\lambda) = \{u \in H^1(\Omega) : \Delta u = \lambda u\}$.*

*Proof.* a) Consider the operator $\mathcal{A} : H_0^1(\Omega) \to H_0^1(\Omega)'$ given by $\langle \mathcal{A}u, v\rangle = \int_\Omega \nabla u \nabla v$. Thus $\Delta^D$ is the part of $\mathcal{A}$ in $L^2(\Omega)$ where we consider $L^2(\Omega) \hookrightarrow H_0^1(\Omega)'$ by letting $\langle f, v\rangle := \int_\Omega f v\, dx$ for $f \in L^2(\Omega), v \in H_0^1(\Omega)$. It follows from [ABHN01, Proposition 3.10.3] that $\sigma(\mathcal{A}) = \sigma(\Delta^D)$. Thus $\lambda - \mathcal{A}$ is invertible.
b) Let $u \in H^1(\Omega)$. Consider $F \in H_0^1(\Omega)'$ given by $F(v) = \int_\Omega \nabla u \nabla v - \lambda \int_\Omega uv$. Then by a) there exists $u_0 \in H_0^1(\Omega)$ such that $\mathcal{A}(u_0) = F$. Thus $u_1 := u - u_0 \in H^1(\lambda)$. Hence $u = u_0 + u_1 \in H_0^1(\Omega) + H^1(\lambda)$. We have shown that $H^1(\Omega) = H_0^1(\Omega) + H^1(\lambda)$. Since $\lambda \notin \sigma(\Delta^D)$ one has $H_0^1(\Omega) \cap H^1(\lambda) = \{0\}$. ☐

Let $V := \{u_{|\Gamma} : u \in H^1(\Omega)\}$ be the trace space which is a subspace of $L^2(\Gamma)$. If $\lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$, then the trace operator restricted to $H^1(\lambda)$, i.e. the mapping $u \in H^1(\lambda) \mapsto u_{|\Gamma} \in V$ is linear and bijective by Lemma 3.2. Defining $\|u_{|\Gamma}\|_V := \|u\|_{H^1(\lambda)}$, the space $V$ becomes a Hilbert space. It follows from the closed graph theorem that a different choice of $\lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$ leads to an equivalent norm on $V$. Since the trace is a compact operator from $H^1(\Omega)$ into $L^2(\Gamma)$, it follows that the embedding of $V$ into $L^2(\Gamma)$ is compact. The Stone-Weierstrass Theorem implies that $V$ is dense in $L^2(\Gamma)$.

Let $\lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$. We define the bilinear mapping $a_\lambda : V \times V \to \mathbb{R}$ by

$$a_\lambda(\varphi, \psi) := \int_\Omega \nabla u \nabla v - \lambda \int_\Omega uv$$

where $u, v \in H^1(\lambda)$ such that $\varphi = u_{|\gamma}, \psi = v_{|\Gamma}$. Then $a_\lambda$ is clearly continuous and symmetric. Now Theorem 3.1 is a consequence of the following.

**Proposition 3.3.** *The form $a_\lambda$ is elliptic and $D_\lambda$ is the operator on $L^2(\Gamma)$ associated with $a_\lambda$.*

*Proof.* 1. In order to show ellipticity we apply Lemma 2.3 to the compact embedding $T : H^1(\lambda) \to L^2(\Omega), u \mapsto u$, and the trace operator $S : H^1(\lambda) \to L^2(\Gamma), u \mapsto u_{|\Gamma}$ which is injective on $H^1(\lambda)$. Given $1 > \delta > 0$ we find $c > 0$ such that

$$\int_\Omega u^2 \leq \delta \|u\|^2_{H^1} + c \int_\Gamma u^2$$

for all $u \in H^1(\lambda)$. Since $\|u\|^2_{H^1} = \int_\Omega |\nabla u|^2 + \int_\Omega u^2$, it follows that

$$\int_\Omega u^2 \leq \frac{\delta}{1-\delta} \int_\Omega |\nabla u|^2 + \frac{c}{1-\delta} \int_\Gamma u^2 \ .$$

Thus, given $\varepsilon > 0$ there exists $c_1 \geq 0$ such that

$$\int_\Omega u^2 \leq \varepsilon \int_\Omega |\nabla u|^2 + c_1 \int_\Gamma u^2 \tag{3.1}$$

for all $u \in H^1(\lambda)$. Let $\varepsilon > 0$ such that $\varepsilon(|\lambda| + 1/2) = 1/2$ and let $\omega = c_1(|\lambda| + 1/2)$. Then by (3.1),

$$\begin{aligned}
&a_\lambda(u_{|\Gamma}) + \omega \int_\Gamma u^2 = \\
&\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2 + \omega \int_\Gamma u^2 \geq \\
&\int_\Omega |\nabla u|^2 + \tfrac{1}{2} \int_\Omega u^2 - (|\lambda| + 1/2) \int_\Omega u^2 + \omega \int_\Gamma u^2 \geq \\
&\int_\Omega |\nabla u|^2 + \tfrac{1}{2} \int_\Omega u^2 - \varepsilon(|\lambda| + 1/2) \int_\Omega |\nabla u|^2 - (|\lambda| + 1/2) c_1 \int_\Gamma u^2 + \omega \int_\Gamma u^2 = \\
&\tfrac{1}{2} \|u\|^2_{H^1} \quad \text{for all} \quad u \in H^1(\lambda) \ .
\end{aligned}$$

2. Let $B$ be the operator on $L^2(\Gamma)$ which is associated with $a_\lambda$. We want to show that $B = D_\lambda$. Let $u \in H^1(\lambda), b \in L^2(\Gamma)$. Then $u_{|\Gamma} \in D(B)$ and $Bu_{|\Gamma} = b$ if and only if

$$\int_\Omega \nabla u \nabla v - \lambda \int_\Omega uv = \int_\Gamma bv \tag{3.2}$$

for all $v \in H^1(\lambda)$.
a) Assume that $u_{|\Gamma} \in D(B)$ and $Bu_{|\Gamma} = b$. Notice that for $u \in H^1(\lambda)$ one has

$$\int_\Omega \nabla u \nabla v - \lambda \int_\Omega uv = 0 = \int_\Gamma vb$$

for all $v \in H^1_0(\Omega)$. Since $H^1_0(\Omega) \oplus H^1(\lambda) = H^1(\Omega)$, it follows that (3.2) holds for all $v \in H^1(\Omega)$. Now introducing $\lambda u = \Delta u$ into (3.2) one sees that $\frac{\partial u}{\partial \nu} = b$ in the sense of our definition. Hence $u \in D(D_\lambda)$ and $D_\lambda u = Bu$.
b) Conversely, let $\varphi \in D(D_\lambda)$ and $D_\lambda \varphi = b$. Then there exists $u \in H^1(\lambda)$ such that $u_{|\Gamma} = \varphi$ and $\frac{\partial u}{\partial \nu} = b$. Hence

$$\begin{aligned}
\int_\Omega \nabla u \nabla v \quad &- \quad \lambda \int_\Omega uv = \\
\int_\Omega \nabla u \nabla v \quad &- \quad \int_\Omega \Delta u v = \int_\Gamma \frac{\partial u}{\partial v} v = \int_\Gamma bv
\end{aligned}$$

for all $v \in H^1(\Omega)$. It follows that $\varphi \in D(B)$ and $B\varphi = b$. □

We retain from the proof of Proposition 3.2.

**Lemma 3.4.** *Let $u \in H^1(\Omega), b \in L^2(\Gamma)$. Then $u_{|\Gamma} \in D(D_\lambda)$ and $D_\lambda u_{|\Gamma} = b$ if and only if*

$$\int_\Omega \nabla u \nabla v - \lambda \int_\Omega uv = \int_\Gamma bv$$

*for all $v \in H^1(\Omega)$.*

We conclude this section showing that the first eigenvalue $\alpha_1(\lambda)$ of $D_\lambda$ is $\leq 0$ whenever $\lambda \geq 0$. This is done with the help of the same function used by Friedlander [Fri91, Lemma 1.3].

**Lemma 3.5.** *Let $0 \leq \lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$. Then the first eigenvalue $\alpha_1(\lambda)$ of $D_\lambda$ is $\leq 0$.*

*Proof.* We extend $a_\lambda$ to a symmetric sesquilinear form from $V_\mathbb{C} \times V_\mathbb{C} \to \mathbb{C}$, where $V_\mathbb{C} = \{u_{|\Gamma} : u \in H^1(\lambda, \mathbb{C})\}$, $H^1(\lambda, \mathbb{C}) = \{u \in H^1(\Omega, \mathbb{C}) : \Delta u = \lambda u\}$. Let $\omega \in \mathbb{R}^d$ such that $|\omega|^2 = \lambda$. Let $u(x) = e^{ix\omega}$. Then $D_j u(x) = i\omega_j e^{ix\omega}, \Delta u(x) = \lambda u(x)$. Thus $u \in H^1(\lambda, \mathbb{C})$. Let $\varphi = u_{|\Gamma}$. Then

$$\begin{aligned} a_\lambda(\varphi) &= \int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2 \\ &= |\omega|^2 \int_\Omega 1 - \lambda \int_\Omega 1 = 0 \ . \end{aligned}$$

Thus by (1.1), $\alpha_1(\lambda) \leq \inf\{a_\lambda(\psi) : \psi \in V_\mathbb{C}, \|\psi\|_{L^2(\Gamma)} = 1\} \leq 0$. ☐

We will see in Section 6 that Lemma 3.5 is no longer valid if $\Omega$ is a Lipschitz domain in a compact Riemannian manifold $M$.

## 4. Comparing eigenvalues

In this section we establish relations between the eigenvalue of the Robin Laplacian $\Delta_\mu$ and the Dirichlet-to-Neumann operator $D_\lambda$. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain as before.

**Theorem 4.1.** *Let $\lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$. Then for $\mu \in \mathbb{R}$,*

a) *$\mu \in \sigma(D_\lambda) \Leftrightarrow \lambda \in \sigma(\Delta_\mu)$, and*
b) *$\dim \ker(\mu - D_\lambda) = \dim \ker(\lambda - \Delta_\mu)$.*

*Proof.* We show that the mapping $S : u \mapsto u_{|\Gamma}$ is an isomorphism from $\ker(\Delta_\mu - \lambda)$ onto $\ker(D_\lambda - \mu)$. In fact, let $u \in \ker(\Delta_\mu - \lambda)$. Then $b_\mu(u, v) = \lambda \int_\Omega uv$ for all $v \in H^1(\Omega)$, i.e.

$$\int_\Omega \nabla u \nabla v - \lambda \int_\Omega uv = \mu \int_\Gamma uv \tag{4.1}$$

for all $v \in H^1(\Omega)$. By Lemma 3.4 this implies that
$u_{|\Gamma} \in D(D_\lambda)$ and $D_\lambda u_{|\Gamma} = \mu u_{|\Gamma}$. If $u_{|\Gamma} = 0$, then $u \in H^1_0(\Omega) \cap D(D_\lambda) \subset H^1_0(\Omega) \cap H^1(\lambda) = \{0\}$. We have shown that $S$ defines a $1-1$-mapping from $\ker(\Delta_\mu - \lambda)$ into $\ker(D_\lambda - \mu)$. In order to show surjectivity, let $\varphi \in \ker(D_\lambda - \mu)$. Then by Lemma 3.4 there exists $u \in H^1(\lambda)$ such that $\varphi = u_{|\Gamma}$ and (4.1) holds for all $v \in H^1(\Omega)$. Thus $b_\mu(u, v) = \int_\Omega \lambda uv$ for all $v \in H^1(\Omega)$. It follows that $u \in D(\Delta_\mu)$ and $\Delta_\mu u = \lambda u$. ☐

Next we prove Friedlander's result [Fri91] for Lipschitz domains in $\mathbb{R}^d$. Recall that $\lambda_k^N$ is the $k-th$ eigenvalue of $\Delta^N$ repeated according to multiplicity.

**Theorem 4.2.** *One has $\lambda_{k+1}^N \leq \lambda_k^D \quad (k = 1, 2, 3 \cdots)$.*

*Proof.* For $\mu \in \mathbb{R}, m \in \mathbb{N}$ denote by $\lambda_m(\mu)$ the $m-th$ eigenvalue of $\Delta_\mu$. Recall that $\lambda_m(\mu)$ is decreasing in $\mu, \lambda_m(0) = \lambda_m^N, \lim_{\mu \to -\infty} \lambda_m(\mu) = \lambda_m^D$. Now assume that there exists $k \in \mathbb{N}$ such that $\lambda_k^D < \lambda_{k+1}^N$. Choose $\lambda_k^D < \lambda < \lambda_{k+1}^N$. Then $\lambda_m(\mu) \leq \lambda_k(\mu) \leq \lambda_k^D$ whenever $m \leq k, \mu \in \mathbb{R}$, and for $m \geq k+1, \mu \leq 0, \lambda_m(\mu) \geq \lambda_{k+1}(\mu) \geq \lambda_{k+1}(0) = \lambda_{k+1}^N$. Hence $\lambda \neq \lambda_m(\mu)$ for all $\mu \leq 0, m \in \mathbb{N}$, i.e.

$\lambda \notin \sigma(\Delta_\mu)$ whenever $\mu \le 0$. By Theorem 4.1 this implies that $\sigma(D_\lambda) \cap (-\infty, 0] = \emptyset$. This contradicts Lemma 3.5. □

We continue to study the functions $\lambda_k : \mathbb{R} \to \mathbb{R}$, where $\lambda_k(\mu)$ is the $k-th$ eigenvalue of $\Delta_\mu$.

**Corollary 4.3.** *Let $k \in \mathbb{N}, \mu_1, \mu_2 \in \mathbb{R}$ such that $\lambda_k(\mu_1) = \lambda_k(\mu_2)$. If $\lambda_k(\mu_1) \notin \sigma(\Delta^D)$, then $\mu_1 \neq \mu_2$.*

*Proof.* Assume that $\mu_1 < \mu_2$. Then $\lambda_k(\mu) = \lambda := \lambda_k(\mu_1)$ for all $\mu \in [\mu_1, \mu_2]$ since $\lambda_k$ is nonincreasing. It follows from Theorem 4.1 that $[\mu_1, \mu_2] \subset \sigma(D_\lambda)$. This is impossible since the spectrum of $D_\lambda$ is discrete. □

**Corollary 4.4.** $\lim\limits_{\mu \to \infty} \lambda_k(\mu) = -\infty$ *for all $k \in \mathbb{N}$.*

*Proof.* Let $k \in \mathbb{N}$. Assume that there exists $\lambda < \inf\limits_{\mu \in \mathbb{R}} \lambda_k(\mu)$. We may assume that $\lambda < 0$. Then $\lambda \notin \sigma(\Delta^D)$. For $m \ge k$, one has $\lambda < \lambda_k(\mu) \le \lambda_m(\mu)$ for all $\mu \in \mathbb{R}$. It follows from Theorem 4.1 that $\sigma(D_\lambda) = \{\mu \in \mathbb{R} : \exists\ m < k, \lambda = \lambda_m(\mu)\}$. Corollary 4.3 implies that $\sigma(D_\lambda)$ has at most $k-1$ eigenvalues, which is impossible, since $D_\lambda$ has compact resolvent and $\dim L^2(\Gamma) = \infty$. □

We let $\lambda_0^D := -\infty, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$ we denote by $\alpha_k(\lambda)$ the $k-th$ eigenvalue of $D_\lambda$ repeating eigenvalues according to multiplicity.

**Proposition 4.5.** *Let $n \in \mathbb{N}_0, \lambda_n^D < \lambda < \lambda_{n+1}^D$. Then for each $k \ge n+1$ there exists a unique $\mu_k \in \mathbb{R}$ such that $\lambda_k(\mu_k) = \lambda$. Moreover, $\alpha_k(\lambda) = \mu_{n+k}$.*

*Proof.* Let $k \ge n+1$. Then $\lim\limits_{\mu \to \infty} \lambda_k(\mu) = \lambda_k^D > \lambda$ and $\lim\limits_{\mu \to -\infty} \lambda_k(\mu) = -\infty$. Hence there exists $\mu_k \in \mathbb{R}$ such that $\lambda_k(\mu_k) = \lambda$. Uniqueness follows from Corollary 4.3. It follows from Theorem 4.1 that $\mu_k \in \sigma(D_\lambda)$.
We show that $\mu_k \le \mu_{k+1}$. In fact, assume that $\mu_{k+1} < \mu_k$. Then $\lambda_{k+1}(\mu_{k+1}) = \lambda = \lambda_k(\mu_k) < \lambda_k(\mu_{k+1}) \le \lambda_{k+1}(\mu_{k+1})$, a contradiction. If $\mu \in \sigma(D_\lambda)$ then by Theorem 4.1, there exists $k \in \mathbb{N}$ such that $\lambda_k(\mu) = \lambda$. Hence $\mu = \mu_k$. Moreover, $k \ge n+1$ (since for $k \le n, \lambda_k(\mu) \le \lambda_n^D < \lambda$ for all $\mu \in \mathbb{R}$). We have shown that

$$\sigma(D_\lambda) = \{\mu_k : k \ge n+1\} \quad \text{and} \quad \mu_k \le \mu_{k+1} \quad \text{for all} \quad k \ge n+1\ .$$

It remains to show that the multiplicity is correctly expressed by the series $\mu_1, \mu_2, \cdots$. Let $k \in \mathbb{N}, p \in \mathbb{N}_0$ such that $\mu := \mu_k = \mu_{k+1} = \cdots = \mu_{k+p} < \mu_{k+p+1}$ and assume that $\mu_{k-1} < \mu_k$ if $k \neq 1$. Then $\lambda_k(\mu) = \lambda = \lambda_{k+1}(\mu_{k+1}) = \lambda_{k+1}(\mu) = \cdots = \lambda_{k+p}(\mu)$, but $\lambda_{k+p+1}(\mu) < \lambda_{k+p+1}(\mu_{k+p+1}) = \lambda$ and, if $k > 1, \lambda_{k-1}(\mu) > \lambda_{k-1}(\mu_{k-1}) = \lambda$. Thus by the definition of $\lambda_k(\mu), \dim\ker(\lambda + \Delta_\mu) = p+1$. Hence by Theorem 4.1, also $\dim(\mu - \Delta_\lambda) = p+1$, i.e. the multiplicity of the eigenvalue $\mu = \mu_k$ is $p+1$ as it was claimed. □

The following beautiful identity was proved by Friedlander (for smooth domains) as a tool for his proof of Theorem 4.2.

**Corollary 4.6.** *For $\lambda > 0$ let $N^D(\lambda)$ be the number of $k$ such that $\lambda_k^D < \lambda$ and $N^N(\lambda)$ the number of those $k$ satisfying $\lambda_k^N < \lambda$. If $\lambda \notin \sigma(\Delta^D)$, then $N^N(\lambda) - N^D(\lambda)$ is the number of all negative eigenvalues of $D_\lambda$.*

*Proof.* Let $\lambda_n^D < \lambda < \lambda_{n+1}^D$. Thus $N^D(\lambda) = n$. Assume that $\lambda_{n+1}^N \le \cdots \le \lambda_{n+p}^N < \lambda \le \lambda_{n+p+1}^N$ so that $N^N(\lambda) = p+n$ and $N^N(\lambda) - N^D(\lambda) = p$. We keep the notation of Proposition 4.5. Thus $\lambda_{n+\ell}(\mu_{n+\ell}) = \lambda$ for all $\ell \in \mathbb{N}$. Since for $\ell = 1, \cdots, p, \lambda_{n+\ell}(0) = \lambda_{n+\ell}^N < \lambda$ and $\lambda_{n+\ell}$ is nonincreasing, it follows that $\mu_{n+\ell} < 0$ for $\ell = 1, \cdots, p$. On the other hand $\mu_{n+p+1} \ge 0$. In fact, otherwise $\mu_{n+p+1} < 0$ and consequently, $\lambda = \lambda_{n+p+1}(\mu_{n+p+1}) > \lambda_{n+1+p}(0) = \lambda_{n+p+1}^N$ contradicting our assumption. Thus there are exactly $p$ negative eigenvalues of $D_\lambda$. □

Next we want to describe the spectrum of $D_\lambda$. We denote by $\alpha_k(\lambda)$ the $k-th$ eigenvalue of $D_\lambda$ repeating the eigenvalue according to its multiplicity ($\lambda \notin \sigma(-\Delta^D)$). Let us denote by $d_1 < d_2 < d_3 \cdots$ the eigenvalues of $-\Delta^D$ not counting multiplicity and denote by $m_k$ the multiplicity of $d_k$. We let $d_0 = -\infty$. Then the picture is the following.

**Corollary 4.7.** *The function $\alpha_k$ have the following properties.*

a) *Each function $\alpha_k$ is continuous on $\mathbb{R} \setminus \sigma(\Delta^D)$;*
b) *each $\alpha_k$ is strictly increasing on $(d_n, d_{n+1})$ for each $n \in \mathbb{N}_0$;*
c) $\lim_{\lambda \to -\infty} \alpha_k(\lambda) = -\infty$;
d) $\lim_{\lambda \uparrow d_n} \alpha_k(\lambda) = \infty$ *for $k = 1, \cdots, m_n$, but*
e) $\lim_{\lambda \uparrow d_n} \alpha_k(\lambda) < \infty$ *for $k > m_n$;*
f) $\lim_{\lambda \downarrow d_n} \alpha_k(\lambda) > -\infty$ *for $n = 1, 2, \cdots$,*
g) $\alpha_1(\lambda) > 0$ *if* $\lambda > 0$,
$\alpha_1(0) = 0$,
$\alpha_1(\lambda) < 0$ *if* $\lambda < 0$.

*Proof.* If $\lambda \in (\lambda_n^D, \lambda_{n+1}^D)$, where $n \in \mathbb{N}_0$, then $\alpha_k(\lambda) = \mu_{n+k}$ where $\mu_{n+k} \in \mathbb{R}$ is the unique number such that

$$\lambda_{n+k}(\mu_{n+k}) = \lambda \quad (k = 1, 2 \cdots) .$$

Thus $\alpha_k$ is the inverse function of $\lambda_{n+k}$ and the properties a) - e) follows from the preceedings results. □

Friedlander [Fri91] showed some of these properties directly for the operator $D_\lambda$ in the case where $\Omega$ has $C^\infty$ boundary.

*Remark* 4.8. The forms $a_\lambda$ have all the same form domain. From Corollary 4.7b), one might conjecture that

$$a_{\lambda_1}(\varphi) \geq a_{\lambda_2}(\varphi) \quad (\varphi \in V) \tag{4.2}$$

if $d_n < \lambda_1 < \lambda_2 < d_{n+1}, n \in \mathbb{N}_0$.

This seems also to be used in Friedlander's proof [Fri91, (2.5)] of Theorem 4.2 in the case of $C^\infty$-domain. We are able to verify (4.2) only for $n = 0$. Our proof depends on Poincaré's inequality

$$\int |\nabla u|^2 \geq d_1 \int u^2 \quad (u \in H_0^1(\Omega)) \tag{4.3}$$

which comes in naturally. So let $-\infty < \lambda_1 < \lambda_2 < d_1, \varphi \in V, u \in H^1(\lambda_1), u_{|\Gamma} = \varphi$. Then $u = u_0 + u_2 \in H_0^1(\Omega) \oplus H^1(\lambda_2)$ so that $a_{\lambda_2}(\varphi) = \int_\Omega |\nabla u_2|^2 - \lambda_2 \int_\Omega u_2^2$ and $a_{\lambda_1}(\varphi) = \int_\Omega |\nabla u|^2 - \lambda_1 \int_\Omega u^2$. Inserting $u = u_2 + u_0$ and using that $\int_\Omega \nabla u_2 \nabla u_0 = \lambda_2 \int_\Omega u_2 u_0$ (since $u_2 \in H^1(\lambda_2)$) we obtain

$$\begin{aligned}
a_{\lambda_1}(\varphi) &= \int_\Omega |\nabla u_2|^2 + 2\int_\Omega \nabla u_2 \nabla u_0 + \int_\Omega |\nabla u_0|^2 \\
&- \lambda_1 \int_\Omega u_2^2 - 2\lambda_1 \int_\Omega u_2 u_0 - \lambda_1 \int_\Omega u_0^2 \\
&= a_{\lambda_2}(\varphi) + (\lambda_2 - \lambda_1) \int_\Omega u_2^2 + (\lambda_2 - \lambda_1) \int_\Omega 2u_2 u_0 \\
&+ \int_\Omega |\nabla u_0|^2 - \lambda_1 \int_\Omega u_0^2 \\
&\geq a_{\lambda_2}(\varphi) + (\lambda_2 - \lambda_1) \int_\Omega (u_2^2 + 2u_2 u_0)(d_1 - \lambda_1) \int_\Omega u_0^2 \\
&\geq a_{\lambda_2}(\varphi) + (\lambda_2 - \lambda_1) \int_\Omega (u_2 + u_0)^2 \geq a_{\lambda_2}(\varphi) .
\end{aligned}$$

## 5. Positivity

Here we study the semigroup generated by $-A_\lambda$ on $L^2(\Gamma)$ for positivity properties. A $C_0$-semigroup $T = (T(t))_{t\geq 0}$ on a space $L^p$ is called **positive** if $0 \leq f \in L^p$ implies that $T(t)f \geq 0$ for all $t \geq 0$.

**Theorem 5.1.** *If $\lambda < \lambda_1^D$, then the semigroup generated by $-D_\lambda$ on $L^2(\Gamma)$ is positive.*

*Proof.* Let $\varphi \in V$. Then $\varphi^+, \varphi^- \in V$. In fact, let $u \in H^1(\Omega)$ such that $u_{|\Gamma} = \varphi$. Then $u^+, u^- \in H^1(\Omega)$ and $u^+_{|\Gamma} = \varphi^+, u^-_{|\Gamma} = \varphi^-$. By the Beurling-Deny criterion (see [Dav90] or [Ouh05, Theorem 2.6]) the semigroup is positive if and only if $a_\lambda(\varphi^+, \varphi^-) \leq 0$ for all $\varphi \in V$. Let $\varphi \in V, \varphi = u_{|\Gamma}$ where $u \in H^1(\lambda)$. Write $u^+ = u_0 + u_1 \in H^1_0 \oplus H^1(\lambda)$ and $u^- = \bar{u}_0 + u_2 \in H^1_0 \oplus H^1(\lambda)$. Since $u = (u_0 - \bar{u}_0) + (u_1 - u_2) \in H^1(\lambda)$ it follows that $u_0 = \bar{u}_0$. Now

$$
\begin{aligned}
a_\lambda(\varphi^+, \varphi^-) &= \int_\Omega \nabla u_1 \nabla u_2 - \lambda \int_\Omega u_1 u_2 \\
&= \int_\Omega \nabla(u_1 + u_0)\nabla(u_2 + u_0) - \int_\Omega \nabla u_1 \nabla u_0 - \int_\Omega \nabla u_0 \nabla u_2 - \int_\Omega \nabla u_0 \nabla u_0 \\
&- \lambda \int_\Omega (u_1 + u_0)(u_2 + u_0) + \lambda \int_\Omega u_1 u_0 + \lambda \int_\Omega u_0 u_2 + \lambda \int_\Omega u_0^2 \\
&= \int_\Omega \nabla u^+ \nabla u^- - \lambda \int_\Omega u^+ u^- - \int_\Omega |\nabla u_0|^2 + \lambda \int_\Omega u_0^2 \leq 0
\end{aligned}
$$

by Poincaré's inequality. In the last identity we used that fact that

$$\int_\Omega \nabla u_i \nabla u_0 = \lambda \int_\Omega u_i u_0, \quad \text{since} \quad u_i \in H^1(\lambda),\ i = 1, 2\ .$$

□

For $\lambda > \lambda_1^D$ the semigroup generated by $-D_\lambda$ is not positive, in general, as can be seen by the explicite computation in dimension 1, for example.

Let $(Y, \Sigma, \nu)$ be a measure space and $1 \leq p < \infty$. A holomorphic positive $C_0$-semigroup $T$ on $L^p(Y, \Sigma, \nu)$ is **irreducible** if and only if for $0 \leq f, f \neq 0, T(t)f(x) > 0$ a.e. for all $t > 0$ (see [Ouh05]). Denote by $-A$ the generator of $A$ and assume that $A$ is selfadjoint and has compact resolvent. Let $\lambda_1$ be the first eigenvalue of $A$. If $T$ is irreducible, then $\lambda_1$ is simple and there exists $u \in D(A), u(x) > 0$ a.e. such that $Au = \lambda_1 u$, i.e. there exists a strictly positive eigenvector corresponding to $\lambda_1$. Moreover $\lambda_1$ is the only eigenvalue with a positive eigenvector. It follows form [Ouh05, Theorem 2.9] that the semigroup generated by $-\Delta_\mu$ is irreducible. Hence $\lambda_1(\mu)$ is a simple eigenvalue with a strictly positive eigenfunction for each $\mu \in \mathbb{R}$. From this we conclude the following for the Dirichlet-to-Neumann operator.

**Theorem 5.2.** *Let $-\infty < \lambda < \lambda_1^D$. Denote by $\mu$ the first eigenvalue of $D_\lambda$. Then there exists a positive eigenfunction of $D_\lambda$ corresponding to $\mu$. Conversely, if $\mu$ is an eigenvalue with a positive eigenfunction $\varphi \in D(D_\lambda)$, then $\mu$ is the first eigenvalue of $D_\lambda$. Moreover, let $u \in H^1(\lambda)$ such that $u_{|\Gamma} = \varphi$. Then $u(x) > 0$ a.e. in $\Omega$.*

*Proof.* Since the semigroup generated by $-D_\lambda$ is positive, the first assertion follows from the Krein-Rutman-Theorem (cf. [Nag86]). In order to prove the second let $0 \leq \varphi \in D(D_\lambda)$ such that $\varphi \neq 0, D_\lambda \varphi = \mu\varphi$. Let $u \in H^1(\lambda)$ such that $u_{|\Gamma} = \varphi$. Then by Lemma 3.4,

$$\int_\Omega \nabla u \nabla v - \lambda \int_\Omega uv = \int_\Gamma \mu uv \tag{5.1}$$

for all $v \in H^1(\Omega)$. Since $\varphi \geq 0$ one has $u^- \in H^1_0(\Omega)$. Taking $v = u^-$ in (5.1) we obtain

$$-\int_\Omega |\nabla u^-|^2 + \lambda \int_\Omega u^{-2} = 0 \ .$$

By Poincaré's inequality,

$$\lambda \int_\Omega (u^-)^2 = \int_\Omega |\nabla u^-|^2 \geq \lambda_1^D \int_\Omega (u^-)^2 \ .$$

Since $\lambda < \lambda_1^D$, it follows that $u^- = 0$. Thus $u \geq 0$. Hence $u$ is a positive eigenvector of $\Delta_\mu$ corresponding to the eigenvalue $\lambda$. Hence $u \gg 0$ and $\lambda$ is the first eigenvalue of $\Delta_\mu$. Hence $\mu$ is the first eigenvalue of $D_\lambda$ by Theorem 4.1. $\square$

Even though we do not know whether the semigroup generated by $-D_\lambda$ is irreducible, Theorem 5.2 establishes the usual consequences of irreducibility. For further properties of the semigroup generated by the Dirichlet-to-Neumann operator we refer to Escher [Esc94].

## 6. Manifolds

The purpose of this section is to consider the previous analysis in a more general setting where $\mathbb{R}^d$ is replaced by a Riemannian manifold. All properties established so far will be valid in this more general context with one exception. Lemma 3.5 may fail, and in fact we will see that this lemma is equivalent to the validity of Friedlander's Theorem (Theorem 4.2).

Let $M$ be a Riemannian manifold which we assume to be connected and orientable. Let $\Omega \subset M$ be a **Lipschitz domain** in $M$, i.e., we assume that $\Omega$ is relatively compact, open, connected and has Lipschitz boundary $\Gamma := \partial\Omega$. The first Sobolev space $H^1(\Omega)$ is defined as the completion of

$$H^1(\Omega) \cap C^\infty(\Omega) := \{u \in C^\infty(\Omega) : \|u\|_{H^1(\Omega)} < \infty\}$$

for the norm

$$\|u\|^2_{H^1(\Omega)} := \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2$$

(cf. [Heb96], [Heb99], [MMT01], [MT99]). Here $\nabla u : \Omega \to T\Omega$ is defined as the vector field $\nabla u(x) := j_x^{-1} du(x)$, where $j_x : T_x\Omega \to T_x^*\Omega$ is the canonical map identifying functionals with vectors via the Riemannian metric. The space $H^1(\Omega)$ is compactly injected into $L^2(\Omega)$. On $\Gamma$ we consider the surface measure. As in the Euclidean case, the trace $u \in C(\bar{\Omega}) \cap H^1(\Omega) \to L^2(\Gamma), u \mapsto u_{|\Gamma}$ has a compact linear extension to $H^1(\Omega)$ with values in $L^2(\Gamma)$ (which we denote still by $u \mapsto u_{|\Gamma}$). We omit the trace sign $u_{|\Gamma}$ when writing $\int_\Gamma uv$ for $u, v \in H^1(\Omega)$. By $H^1_0(\Omega)$ we denote the closure of the test functions. Note that for $u \in H^1(\Omega)$ one has $u \in H^1_0(\Omega)$ if and only if $u_{|\Gamma} = 0$. For $u, v \in H^1(\Omega)$, there exists a unique function $\nabla u \cdot \nabla v \in L^1(\Omega)$ such that for each local coordinate $\varphi : V \to \mathbb{R}$,

$$\nabla u \cdot \nabla v = \Big(\sum_{i,j=1}^d g^{ij} \big(\frac{\partial}{\partial \varphi^i} u\big)\big(\frac{\partial}{\partial \varphi^j} v\big)\Big)^{1/2} \ .$$

If $u \in H^1(\Omega)$, we say that $\Delta u \in L^2(\Omega)$ if there exists $f \in L^2(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v = \int_\Omega f v$$

for all $v \in \mathcal{D}(\Omega) := C_c^\infty(\Omega)$. In that case we put $\Delta u := f$.

Next we define the outer normal. If $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, then we say that $\frac{\partial u}{\partial \nu} \in L^2(\Gamma)$ if there exists $b \in L^2(\Gamma)$ such that

$$\int_\Omega \nabla u \nabla v - \int_\Omega \Delta u v = \int_\Gamma b v$$

for all $v \in H^1(\Omega)$. Then we set $\frac{\partial u}{\partial \nu} = b$.

Assume that $\Omega \neq M$ (this is automatic if $M$ is non-compact). Then we can define as in the Euclidean case the operators $\Delta_\mu$ $(\mu \in \mathbb{R}), \Delta^N = \Delta_0$ and $\Delta^D = \Delta_{-\infty}$. They are selfadjoint and have compact resolvent. Denoting by $\lambda_k(\mu)$ the $k-th$ eigenvalue of $\Delta_\mu$ we obtain continuous, increasing functions $\lambda_k : [-\infty, \infty) \to \mathbb{R}$ as in Theorem 2.4. If $M$ is a compact manifold, and $\Omega = M$, then $H^1_0(M) = H^1(M)$ and $\Gamma = \emptyset$. Thus $\Delta^N = \Delta^D$ which is just the Laplace-Beltrami operator.

Next, for $\lambda \in \mathbb{R} \backslash \sigma(\Delta^D)$ and $\Omega \neq M$, we define the Dirichlet-to-Neumann operator $D_\lambda$ as in Section 3. It is a selfadjoint, lower bounded operator on $L^2(\Gamma)$ which has compact resolvent. Theorem 4.1 stays true in this general context. Now the point in the non-Euclidean case is that Lemma 3.5 fails in general (as we will show below). In fact, it is equivalent to the validity of Friedlander's result Theorem 4.2. More precisely, the following holds. By $\alpha_k(\lambda)$ we denote the $k-th$ eigenvalue of $D_\lambda, \lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$.

**Theorem 6.1.** *Let $\Omega$ be a Lipschitz domain in $M, \Omega \neq M$. Let $k \in \mathbb{N}$ such that $\lambda_k^D < \lambda_{k+1}^D$. The following assertions are equivalent:*

(i) *One has $\alpha_1(\lambda) \leq 0$ for all $\lambda \in (\lambda_k^D, \lambda_{k+1}^D)$;*
(ii) $\lambda_{k+1}^N \leq \lambda_k^D$.

*Proof.* $(i) \Rightarrow (ii)$ as in Section 4. $(ii) \Rightarrow (i)$ Assume that $\lambda_k^D < \lambda_{k+1}^N$.
Let $\lambda_k^D < \lambda < \lambda_{k+1}^N$. Since $\lambda_m(\mu) \leq \lambda_k^D$ for all $\mu \in \mathbb{R}, m \leq k$, and $\lambda_m(\mu) > \lambda_{k+1}^N$ for all $m \geq k+1, \mu \geq 0$, there exist no $\mu \geq 0$ and no $m \in \mathbb{N}$ such that $\lambda_m(\mu) = \lambda$. Hence by Theorem 4.1, $\mu > 0$ for all $\mu \in \sigma(D_\lambda)$. □

Next we show that it may happen that the two equivalent conditions of Theorem 6.1 fail in the non-Euclidean case. In fact, taking an arbitrary compact Riemannian manifold, we show that these assertions fail for $\Omega = M \setminus K$ if $K$ is a compact subset of $M$ which is small enough.

**Theorem 6.2.** *Let $M$ be a compact Riemannian manifold of dimension $d \geq 2$. Let $K_{n+1} \subset K_n \subset M$ be compact sets such that $\bigcap_{n\in\mathbb{N}} K_n = \{a\}$ where $a \in M$. Let $\Omega_n = M \setminus K_n$. Then for $\lambda > 0, (\lambda + \Delta^D_{\Omega_n})^{-1} \to (\lambda + \Delta_M)^{-1}$ and $(\lambda + \Delta^N_{\Omega_n})^{-1} \to (\lambda + \Delta_\mu)^{-1}$ in $\mathcal{L}(L^2(M))$, where $\Delta_M$ denotes the Laplace-Beltrami operator on $L^2(M)$ and $\Delta^D_{\Omega_n}$ the Dirichlet- and $\Delta^N_{\Omega_n}$ the Neumann Laplacian on $L^2(\Omega_n)$. Consequently, $\lim_{n\to\infty} \lambda_k^D(\Omega_n) = \lambda_k(M), \lim_{k\to\infty} \lambda_k^N(\Omega_n) = \lambda_k(M)$ for all $k \in \mathbb{N}$, where $\lambda_k(M), \lambda_k^D(\Omega_n), \lambda_k^N(\Omega_n)$ denotes the $k-th$ eigenvalue of the Laplace-Beltrami, Dirichlet- and Neumann Laplacian on $L^2(M)$ and $L^2(\Omega_n)$, respectively.*

*Proof.* We identify $L^2(\Omega_n)$ with a closed subspace of $L^2(M)$ extending functions in $L^2(\Omega_n)$ by 0 outside $\Omega_n$ and we also consider $\mathcal{L}(L^2(\Omega_n)) \subset \mathcal{L}(L^2(M))$ in a canonical way. Since dimension $d \geq 2$, the space $C_c^\infty(M \setminus \{a\})$ is dense in $H^1(M)$ (cf. [AB93, Remark 2.6], [Bré83, p. 171]). Let $\lambda > 0$.
a) We show that $(\lambda + \Delta^N_{\Omega_n})^{-1} \to (\lambda + \Delta_M)^{-1}$ in $\mathcal{L}(L^2(M))$ as $n \to \infty$. Let $f_n \rightharpoonup f$ in $L^2(M), u_n = (\lambda + \Delta^N_{\Omega_n})^{-1} f_n$. By Lemma 2.5 we have to show that $u_n \to u := (\lambda + \Delta_M)^{-1} f$ in $L^2(M)$. By definition

$$\lambda \int_{\Omega_n} u_n v + \int_{\Omega_n} \nabla u_n \nabla v = \int_{\Omega_n} f_n v \tag{6.1}$$

for all $v \in H^1(\Omega_n)$. In particular,

$$\lambda \int_{\Omega_n} u_n^2 + \int_{\Omega_n} |\nabla u_n|^2 = \int_{\Omega_n} f_n u_n \leq \|f_n\|_{L^2(M)} \|u_n\|_{L^2(M)} \,.$$

It follows that $\|u_n\|_{H^1(\Omega_n)}$ is bounded by a constant $c$. Let $(u_{n_k})$ be a subsequence. By the diagonal argument, there exist a $w \in L^2(M)$ and a subsequence $(u_{n_{k_\ell}})$ such that $w_{|\Omega_m} \in H^1(\Omega_m)$ and

$u_{n_{k_\ell}} \rightharpoonup w_{|\Omega_n}$ in $H^1(\Omega_m)$ as $\ell \to \infty$ for all $m \in \mathbb{N}$. Hence $w \in H^1_{\mathrm{loc}}(M \setminus \{a\})$ and $\int_M |\nabla w|^2 < \infty$. Letting $\ell \to \infty$ it follows from (6.1) that

$$\lambda \int_M wv + \int_M \nabla w \nabla v = \int_M fv$$

for all $v \in C_c^\infty(M \setminus \{a\})$ and hence for all $v \in H^1(M)$ by density. Thus $w = u$ and $u_{n_{k_\ell}} \to u$ in $L^2(\Omega_m)$ for all $m \in \mathbb{N}$. Recall that $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(M)$. Since $|M \setminus \Omega_m| \to 0$ as $m \to \infty$, it follows that $u_{n_{k_\ell}} \to u$ in $L^2(M)$. We have shown that each subsequence of $(u_n)_{n\in\mathbb{N}}$ has a subsequence converging to $u$ in $L^2(M)$. Hence the sequence itself converges to $u$ and a) is proved.
b) The proof that $(\lambda + \Delta^D_{\Omega_n})^{-1} \to (\lambda + \Delta_M)^{-1}$ in $\mathcal{L}(L^2(M))$ as $n \to \infty$ is similar to a) but easier since $H^1_0(\Omega_n) \subset H^1(M)$ (via the extension by 0). The remaining assertions follow from Proposition 2.8. □

Now we obtain in each compact Riemannian manifold a Lipschitz domain $\Omega$ such that the inequality $\lambda^N_{k+1} \leq \lambda^D_k$ fails. We keep the notations of the preceding theorem.

**Corollary 6.3.** *Let $k \in \mathbb{N}$ such thath $\lambda_k(M) < \lambda_{k+1}(M)$. Then for $n$ large enough $\lambda^N_{k+1}(\Omega_n) > \lambda^D_k(\Omega_n)$.*

*Proof.* Since $\lim_{n\to\infty} \lambda^N_{k+1}(\Omega_n) = \lambda_{k+1}(M)$ and $\lim_{n\to\infty} \lambda^D_k(\Omega_n) = \lambda_k(M)$ this is obvious. □

In fact, in special cases one can give more precise information. If $M = S^3$, then taking of $M$ a cap in the upper half sphere, we obtain a domain for which the inequality $\lambda^N_{k+1} \leq \lambda^D_k$ is violated, see [Maz91]. For other results on subdomains of the sphere see Ashbaugh and Levine [AL97].

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Institute of Applied Analysis, University of Ulm, D - 89069 Ulm, Germany
*E-mail address*: wolfgang.arendt@uni-ulm.de

Stanford University, Department of Mathemtics, Stanford, CA 94305-2125, USA
*E-mail address*: mazzeo@math.stanford.edu