DECOMPOSING AND TWISTING OF SECTORIAL OPERATORS

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Abstract. Bisectorial operators play an important role since exactly these operators lead to a well-posed equation on the entire line. The most simple example of a bisectorial operator is obtained by taking the direct sum of an invertible generator of a bounded holomorphic semigroup and the negative part of such an operator. Our main result in this paper shows that each bisectorial operator A is of this form if we allow a more general notion of direct sum defined by an unbounded closed projection. As a consequence we can express the solution of the evolution equation on the line by an integral operator involving two semigroups associated with A.

1. Introduction

Let us first explain the motivation for investigating bisectorial operators. An invertible operator A on a Banach space is called **bisectorial** if the imaginary line is in the resolvent set of A and $\lambda(\lambda-A)^{-1}$ is bounded on the imaginary line. Such bisectorial operators were considered by McIntosh and Yagi [8] in the framework of spectral calculus. Mielke [9] showed in 1987 that, on Hilbert spaces, an operator A is bisectorial if and only if for some (equivalently all) $f \in L^p(\mathbb{R}; X)$, $1 < p < \infty$, there is a unique solution $u \in W^{1,p}(\mathbb{R}; X)$ such that

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}. \tag{1.1}$$

In other words, Mielke proved a result on maximal regularity for the evolution equation on the line for bisectorial operators on Hilbert space. He applied such results to non-linear equations and in particular to prove the existence of central manifolds. Mielke's result on maximal regularity was generalized to Banach space by Schweiker [10], and in [4] with the help of the operator-valued Fourier multiplier theorem due to Weis. Maximal regularity in Hölder spaces was considered in [1].

Most interesting is the spectral theory of bisectorial operators. A striking problem is the question whether it is possible to decompose the Banach space with the help of a spectral projection commuting with A such that the operator is the direct sum of an invertible generator of a bounded holomorphic semigroup and the negative invertible generator of a bounded holomorphic semigroup. There is always a natural spectral projection (see Section 3) defined by a contour integral, but this projection is unbounded in general as was shown by McIntosh and Yagi [8], see also Dore and Venni [6].

However the spectral projection P is always closed. This means that its kernel and its image are closed subspaces of X whose sum is dense in X, but possibly different from the entire space. The part of A in these subspaces is the generator or the negative generator of a holomorphic semigroup. In our main result we show that twisting A by its spectral projection, we obtain the generator of a holomorphic semigroup on the entire space X. This is surprising since it shows that each bisectorial operator is, in fact, the twisted version of a sectorial operator. Another corollary of the main result shows that the square of a bisectorial operator A is always sectorial (i.e. $-A^2$ always generates a holomorphic semigroup). These results clarify somehow the nature of bisectorial operators.

The spectral projection had been investigated before by Sybille Schweiker [10]. In particular, Schweiker associated two semigroups with a bisectorial operator which operate on the entire space X. These semigroups are holomorphic but singular as the time goes to 0. However the singularity can never be worse than logarithmic, as Schweiker showed. We now obtain these semigroups very simply

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from the semigroup generated by the twisted version of A. They allow one to give a representation formula of the solutions of (1.1) which will be exploited further in [12].

Our result holds also for non densely defined operators. For simplicity we do not consider more general operators which are merely bisectorial outside a compact set as in [3], where a spectral theory for these operators is developed.

2. Twisting bisectorial operators by unbounded projections

Let X be a Banach space. We start defining unbounded projections.

Definition 1. *A **projection** P on X is a linear operator P on X with domain $D(P)$ such that $P^2 = P$, i.e. such that $Px \in D(P)$ and $P^2x = Px$ for all $x \in D(P)$.*

Let P be a projection. Then $X_2 = \ker P$ and

$$X_1 = \mathrm{im}P := \{Px : x \in D(P)\} = \{x \in D(P) : Px = x\}$$

are subspaces of X such that $X_1 \cap X_2 = \{0\}$.

Lemma 2. *The projection P is closed if and only if $\ker P$ and $\mathrm{im}P$ are closed.*

This is easy to see. Conversely, the following holds. If X_1 and X_2 are subspaces of X such that $X_1 \cap X_2 = \{0\}$, then letting

$$\begin{aligned} D(P) &= X_1 + X_2 \\ P(x_1 + x_2) &= x_1 \end{aligned}$$

defines a projection on X with $\mathrm{im}P = X_1$ and $\ker P = X_2$. This projection is closed if and only if X_1 and X_2 are closed.

Remark 3. *Closability of projections.*

(i) *If P is closable, then $\overline{P}$ is a projection.*
(ii) *Let $X_1, X_2 \subset X$ be subspaces such that $X_1 \cap X_2 = \{0\}$. Then the projection onto X_1 defined above is closable if and only if $\overline{X_1} \cap \overline{X_2} = \{0\}$.*
(iii) *Let X_1 be a dense subspace of X which is different from X. Let X_2 be an algebraic complement. Then the projection onto X_1 with domain X is not closable.*
(iv) *Let A be a densely defined invertible operator which is not bounded. Let $x' \in X' \setminus D(A')$, and let $u \in D(A)$ such that $\langle x', Au\rangle = 1$. Then $Px = \langle Ax, x'\rangle u$, with domain $D(P) = D(A)$, defines an unbounded non-closable projection.*

Next we consider an operator A on X with non-empty resolvent set $\rho(A)$.

Proposition 4. *Let P be a projection on X such that $D(A) \subset D(P)$. Then the following statements are equivalent.*

(i) $PR(\mu, A)x = R(\mu, A)Px$, *for all $x \in D(P)$ and for some $\mu \in \rho(A)$;*
(ii) *if $y \in D(A)$ is such that $Ay \in D(P)$, then $Py \in D(A)$ and $PAy = APy$;*
(iii) $PR(\mu, A)x = R(\mu, A)Px$, *for all $x \in D(P)$ and for all $\mu \in \rho(A)$.*

Proof. (*i*) $\Rightarrow$ (*ii*). Let $y \in D(A)$ such that $Ay \in D(P)$. Then $x = \mu y - Ay \in D(P)$ and, by (*i*), $Py = PR(\mu, A)x = R(\mu, A)Px \in D(A)$. Moreover $(\mu - A)Py = Px = \mu Py - PAy$. Thus $APy = PAy$.

(*ii*) $\Rightarrow$ (*iii*). Let $\mu \in \rho(A)$, $x \in D(P)$, $y = R(\mu, A)x$. Then $y \in D(A)$ and $\mu y - Ay = x$. Hence $Ay \in D(P)$. By hypothesis it follows that $Py \in D(A)$ and $(\mu - A)Py = P(\mu - A)y = Px$. Hence $R(\mu, A)Px = Py = PR(\mu, A)x$. ∎

Definition 5. *We say that* ***a projection*** P ***commutes with*** A *if* $D(A) \subset D(P)$ *and the equivalent conditions* $(i) - (iii)$ *of Proposition 4 are satisfied.*

Now we introduce the basic notion of this paper.

Definition 6. *An operator* A *is called* ***bisectorial*** *if*
- (i) $i\mathbb{R} \subset \rho(A)$;
- (ii) $\sup_{s\in\mathbb{R}} |s|\|R(is, A)\| < \infty$.

For $0 < \theta < \frac{\pi}{2}$ we consider the open horizontal sector

$$\Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}$$

and the open vertical bisector

$$\Sigma'_\theta := \mathbb{C} \setminus \{\overline{\Sigma}_\theta \cup -\overline{\Sigma}_\theta\}.$$

Let A be bisectorial. Then by the usual geometric series expansion one obtains $\omega \in (0, \frac{\pi}{2})$ such that

$$\Sigma'_\omega \subset \rho(A) \tag{2.1}$$

and

$$\sup_{\lambda\in\Sigma'_\omega} \|\lambda R(\lambda, A)\| < \infty. \tag{2.2}$$

We say that an operator A **generates a bounded holomorphic semigroup** if $\lambda \in \rho(A)$ for $\mathrm{Re}\lambda > 0$ and

$$\sup_{\mathrm{Re}\lambda>0} \|\lambda R(\lambda, A)\| < \infty.$$

In fact, then we may construct a semigroup $\left(e^{tA}\right)_{t>0} \subset \mathcal{L}(X)$, which has a bounded and holomorphic extension to a sector Σ_θ for some $0 < \theta < \frac{\pi}{2}$. This semigroup is a C_0-semigroup if and only if A is densely defined. Moreover, A is invertible if and only if the semigroup is **exponentially stable**, i.e. if

$$\|e^{tA}\| \le Me^{-\varepsilon t}, \qquad t > 0$$

for some $\varepsilon > 0$, $M > 0$. We refer to the monographs [7] and [2] for these properties. Thus, if A generates a bounded holomorphic semigroup, then A is in particular bisectorial.

In the following let A be a bisectorial operator on X. Let P be a closed projection on X commuting with A. Then $X_+ = \ker P$ and $X_- = \mathrm{im}P$ are closed subspaces on X. Consider the **parts** A_+ and A_- on X_+ and X_- respectively, i.e.

$$\begin{aligned} D(A_\pm) &= D(A) \cap X_\pm \\ A_\pm x &= Ax, \qquad x \in D(A_\pm). \end{aligned}$$

Then it follows from Proposition 4 that A_+ and A_- are both bisectorial.

Next we define the **twisted operator** $\tilde{A}$ which formally is given by $\tilde{A} = -A$ on X_+ and $\tilde{A} = A$ on X_-. Let $Z := X_+ \oplus X_-$ with norm $\|x_1 + x_2\|_Z := \|x_1\|_X + \|x_2\|_X$ where $x_1 \in X_+$ and $x_2 \in X_-$. Then Z is a Banach space such that

$$D(A) \subset Z \hookrightarrow X. \tag{2.3}$$

Moreover, the projections $P_+ = P_{|Z}$ and $P_- = (I - P^+)_{|Z}$ are bounded as operators on Z.

Definition 7 (The twisted operator)**.** *Define the operator $\tilde{A}$ on X by*

$$\begin{aligned} D(\tilde{A}) &:= \{x \in Z : -P_+x + P_-x \in D(A)\} \\ \tilde{A}x &:= A(-P_+x + P_-x). \end{aligned}$$

We call $\tilde{A}$ ***the operator*** A ***twisted by*** P_+*.*

The part $\tilde{A}_{|Z}$ of $\tilde{A}$ in Z is just the direct sum of $-A_+$ and A_-. Thus $\tilde{A}_{|Z}$ is a bisectorial operator on Z. For A itself we can show the following.

Proposition 8. *Let $\lambda \in \tilde{\rho} := \rho(A) \cap \rho(-A)$. Then $\lambda \in \rho(\tilde{A})$ and*

$$R(\lambda, \tilde{A}) = P_+R(-\lambda, A) + P_-R(\lambda, A).$$

In particular, $i\mathbb{R} \subset \rho(\tilde{A})$. Moreover,

$$\sup_{s\in\mathbb{R}} \|R(is, \tilde{A})\| < \infty. \tag{2.4}$$

Finally, $\sigma(\tilde{A}) = -\sigma(A_+) \cup \sigma(A_-)$.

Proof. Let $\lambda \in \tilde{\rho}$. Define

$$\tilde{R}(\lambda) := P_+R(\lambda, -A) + P_-R(\lambda, A).$$

Then

$$\begin{aligned} -P_+\tilde{R}(\lambda) + P_-\tilde{R}(\lambda) &= -P_+R(\lambda, -A) + P_-R(\lambda, A) \\ &= -P_+R(\lambda, -A) - P_+R(\lambda, A) + R(\lambda, A) \\ &= P_+(R(-\lambda, A) - R(\lambda, A)) + R(\lambda, A) \\ &= 2\lambda P_+R(-\lambda, A)R(\lambda, A) + R(\lambda, A) \\ &= 2\lambda R(-\lambda, A)P_+R(\lambda, A) + R(\lambda, A) \end{aligned}$$

which maps X into $D(A)$. Thus $\tilde{R}(\lambda)$ maps X into $D(\tilde{A})$ and

$$\begin{aligned} (\lambda - \tilde{A})\tilde{R}(\lambda) &= \lambda\tilde{R}(\lambda) - A(-P_+\tilde{R}(\lambda) + P_-\tilde{R}(\lambda)) \\ &= \lambda\tilde{R}(\lambda) + AP_+(R(\lambda, -A) + R(\lambda, A)) - AR(\lambda, A) \\ &= \lambda P_+(R(\lambda, -A) - R(\lambda, A)) + \lambda R(\lambda, A) + \\ &\quad +AP_+(R(\lambda, -A) + R(\lambda, A)) - AR(\lambda, A) \\ &= P_+\{\lambda R(\lambda, -A) - \lambda R(\lambda, A) + AR(\lambda, -A) + AR(\lambda, A)\} + I \\ &= I. \end{aligned} \tag{2.5}$$

Now let $y \in D(\tilde{A})$, i.e. $y \in Z$ and $-P_+y + P_-y \in D(A)$. Then

$$\begin{aligned} \tilde{R}(\lambda)\tilde{A}y &= (P_+R(\lambda, -A) + P_-R(\lambda, A))A(-P_+y + P_-y) \\ &= AR(\lambda, -A)(-P_+y) + AR(\lambda, A)P_-y \\ &= A(-P_+R(\lambda, -A)y + P_-R(\lambda, A)y) \\ &= \tilde{A}\tilde{R}(\lambda)y. \end{aligned}$$

This shows, by (2.5) that

$$\tilde{R}(\lambda)(\lambda - \tilde{A})y = (\lambda - \tilde{A})\tilde{R}(\lambda)y = y, \qquad y \in D(\tilde{A}).$$

It follows from [2, Proposition 3.10.3] that $\sigma(\tilde{A}) = \sigma(\tilde{A}_{|Z})$. But $\tilde{A}_{|Z}$ is the direct sum of $-A_+$ and A_-. Hence $\sigma(\tilde{A}_{|Z}) = -\sigma(A_+) \cup \sigma(A_-)$.

Finally, in order to prove (2.4) observe first that $isR(is, \tilde{A}_{|Z}) = \tilde{A}_{|Z}R(is, \tilde{A}_{|Z}) + I_{|Z}$. Thus $\|\tilde{A}_{|Z}R(is, \tilde{A}_{|Z})\|_{\mathcal{L}(Z)}$ is bounded since $\tilde{A}_{|Z}$ is bisectorial. Let $x \in X$. We consider $D(A)$ as a Banach space for $\|x\|_A := \|Ax\|_X$ (remind that $0 \in \rho(A)$). Then $D(A) \hookrightarrow Z \hookrightarrow X$. Hence there exists constants $C, C_1, C_2 > 0$ such that

$$\begin{aligned} \|R(is,\tilde{A})x\|_X &= C\|R(is,\tilde{A})x\|_Z \\ &= C\|\tilde{A}R(is,\tilde{A})A^{-1}x\|_Z \\ &\leq C\sup_{s\in\mathbb{R}}\|\tilde{A}R(is,\tilde{A})\|_{\mathcal{L}(Z)}\|A^{-1}x\|_Z \\ &\leq C_1\|A^{-1}x\|_Z \\ &\leq C_2\|A^{-1}x\|_{D(A)} = C_2\|x\|_X. \end{aligned}$$

□

We conclude this section by an example which shows that the estimate (2.4) cannot be essentially improved and that $\tilde{A}$ is not bisectorial in general.

Example 9. *Let*

$$X = \left\{ x = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty}\left\{\frac{1}{n^2}(|x_{2n}|^2 + |x_{2n+1}|^2) + |x_{2n} - x_{2n+1}|^2\right\} < \infty \right\}.$$

Then X is a Hilbert space for the scalar product

$$(x|y) = \sum_{n=1}^{\infty}\left\{\frac{1}{n^2}(x_{2n}\overline{y}_{2n} + x_{2n+1}\overline{y}_{2n+1}) + (x_{2n} - x_{2n+1})(\overline{y}_{2n} - \overline{y}_{2n+1})\right\}.$$

Define the operator A on X by

$$(Ax)_{2n} = -nx_{2n}, \qquad (Ax)_{2n+1} = -nx_{2n+1},$$

with domain $D(A) = \{x \in X : Ax \in X\}$. Then A is invertible and generates a bounded holomorphic C_0-semigroup on X. In fact, for $\mathrm{Re}\lambda \geq 0$, the resolvent of A is given by $R(\lambda, A)y = \tilde{x}$, with

$$\tilde{x}_{2n} = \frac{1}{\lambda + n}y_{2n}, \qquad \tilde{x}_{2n+1} = \frac{1}{\lambda + n}y_{2n+1}.$$

Hence

$$\begin{aligned} |\lambda|\|\tilde{x}\| &= |\lambda|\left(\sum_{n=1}^{\infty}\left\{\frac{1}{|\lambda+n|^2}(|y_{2n}|^2 + |y_{2n+1}|^2)\frac{1}{n^2} + \frac{1}{|\lambda+n|^2}|y_{2n} - y_{2n+1}|^2\right\}\right)^{\frac{1}{2}} \\ &\leq \sup_{\mathrm{Re}\lambda\geq 0, n\in\mathbb{N}}\frac{|\lambda|}{|\lambda+n|}\|y\|. \end{aligned}$$

*Let $X_+ = \{x \in X : x_{2n+1} = 0, \quad n \in \mathbb{N}\}$, $X_- = \{x \in X : x_{2n} = 0, \quad n \in \mathbb{N}\}$. Then X_+, X_- are closed subspaces of X (which are isomorphic to ℓ^2), such that $X_+ \cap X_- = \{0\}$. The constant-*1 *sequence is in X but not in $X_+ + X_-$. Let P be the projection given by $D(P) = X_+ + X_-$,*

$$(Px)_{2n} = x_{2n}, \qquad (Px)_{2n+1} = 0.$$

Then P is closed, $D(A) \subset D(P)$ and P commutes with A in the sense of Definition 5. Let $\tilde{A}$ be the operator A twisted by P. Then $D(\tilde{A}) = D(A)$ and

$$(\tilde{A}x)_{2n} = nx_{2n}, \qquad (\tilde{A}x)_{2n+1} = -nx_{2n+1}.$$

It is not difficult to see that $\sigma(\tilde{A}) = \mathbb{N} \cup (-\mathbb{N})$ and that

$$(R(\lambda,\tilde{A})y)_{2n} = \frac{1}{\lambda - n}y_{2n}, \qquad (R(\lambda,\tilde{A})y)_{2n+1} = \frac{1}{\lambda + n}y_{2n+1},$$

for $\lambda \notin \mathbb{N} \cup (-\mathbb{N})$. By (2.4) we know that $\sup_{s\in\mathbb{R}}\|R(is,\tilde{A})\| < \infty$. However, in this example,

$$\sup_{s\in\mathbb{R}}|s|\|R(is,\tilde{A})\| = \infty.$$

In fact, let $e_n = (0, \ldots, 0, 1, 0, \ldots)$ *be the n-th unit vector and* $v_n = \frac{n}{\sqrt{2}}(e_{2n} - e_{2n+1})$. *Then* $\|v_n\| = 1$. *Let*

$$\begin{aligned} u_n = R(in, \tilde{A})v_n &= \frac{n}{\sqrt{2}}\left(\frac{1}{in-n}e_{2n} - \frac{1}{in+n}e_{2n+1}\right) \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{i-1}e_{2n} - \frac{1}{i+1}e_{2n+1}\right). \end{aligned}$$

Then

$$\begin{aligned} \|u_n\|^2 &\geq \frac{1}{2}\left|\frac{1}{i-1} + \frac{1}{i+1}\right|^2 \\ &= 2\frac{1}{|(i-1)(i+1)|^2} \\ &= \frac{1}{2} \end{aligned}$$

Hence $\|R(in, \tilde{A})\| \geq \frac{1}{2}$ *for all* $n \in \mathbb{N}$.

In the following section we will see that for each bisectorial operator A there exists a special projection, namely the **spectral projection** P, such that the operator $\tilde{A}$ obtained by twisting A by P, is sectorial. Since $\tilde{\tilde{A}} = A$, we may reformulate this in the following way. Given a bisectorial operator B there exists a sectorial operator A , a (in general unbounded) projection P which commutes with A such that $B = \tilde{A}$, where $\tilde{A}$ is the operator A twisted by P.

3. Twisting by the spectral projection

The simplest way to obtain a bisectorial operator is the following. Assume that $X = X_+ \oplus X_-$ is the direct sum of two closed subspaces. Let $-A_+$ and A_- be invertible generators of bounded and holomorphic semigroups, and let $A = A_+ \oplus A_-$. Then A is bisectorial. Moreover, A_+ is the part of A in X_+ and A_- is the part of A in X_-. We want to give this simple situation a name.

Definition 10. *A bisectorial operator* A *on* X *is called* ***decomposable*** *if* X *is the direct sum* $X = X_+ \oplus X_-$ *of closed subspaces such that* $R(is, A)X_+ \subset X_+$ *and* $R(is, A)X_- \subset X_-$ *for all* $s \in \mathbb{R} \setminus \{0\}$ *and*

$$\sigma(A_+) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq 0\},$$
$$\sigma(A_-) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\},$$

where A_+ *is the part of* A *in* X_+ *and* A_- *is the part of* A *in* X_-.

It is not difficult to see the following (see e.g. the appendix of [7]).

Proposition 11. *Let* A *be the generator of a holomorphic* C_0*-semigroup such that* $i\mathbb{R} \subset \rho(A)$. *Then* A *is bisectorial and decomposable.*

Even on Hilbert spaces there exist undecomposable invertible bisectorial operators. This was shown by McIntosh and Yagi (see [8]).

Theorem 12 (McIntosh-Yagi)**.** *Let* X *be a separable Hilbert Space. Then there exists an invertible bisectorial operator* A *which is not decomposable.*

Our aim is to prove the following.

Theorem 13. *Let A be an invertible, bisectorial operator. Then there exists a (possibly unbounded) projection P commuting with A such that the operator $\tilde{A}$ obtained by twisting A by P generates a bounded holomorphic semigroup.*

We start defining the projection P which will fulfill the requirement. Let A be a bisectorial, invertible operator. Let $0 < \omega < \frac{\pi}{2}$ such that $\Sigma'_\omega \subset \rho(A)$ and $\sup_{\lambda \in \Sigma'_\omega} \|\lambda R(\lambda, A)\| < \infty$ (see (2.1) and (2.2)).

Let $\varepsilon > 0$ such that $\{z \in \mathbb{C} : |z| \leq \varepsilon\} \subset \rho(A)$. For $\omega < \theta < \frac{\pi}{2}$ we consider the contour $\Gamma^+_{\theta,\varepsilon}$ which consists of the line $\{re^{-i\theta} : r > \varepsilon\}$, the arc $\{\varepsilon e^{i\alpha} : -\theta \leq \alpha \leq \theta\}$ and the line $\{re^{i\theta} : r > \varepsilon\}$ oriented downwards. Let

$$Q_+ := \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} R(\lambda, A) \frac{d\lambda}{\lambda}.$$

Then $Q_+ \in \mathcal{L}(X)$ does not depend on the choice of θ and $\varepsilon > 0$ satisfying the requirement above (by Cauchy's Theorem).

Proposition 14. *Let $P_+ = AQ_+$ with domain $D(P_+) = \{x \in X : Q_+x \in D(A)\}$. Then P_+ is a closed projection commuting with A.*

Proof. Let $\omega < \theta' < \theta < \frac{\pi}{2}$, and $0 < \varepsilon < \varepsilon'$ such that $\{z \in \mathbb{C} : |z| \leq \varepsilon'\} \subset \rho(A)$. Then by Cauchy's Theorem and the resolvent identity,

$$\begin{aligned} Q_+^2 &= \frac{1}{(2\pi i)^2} \int_{\Gamma^+_{\theta,\varepsilon}} R(\lambda, A) \int_{\Gamma^+_{\theta',\varepsilon'}} \frac{1}{\lambda' - \lambda} \frac{d\lambda'}{\lambda'} \frac{d\lambda}{\lambda} + \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma^+_{\theta',\varepsilon'}} \frac{R(\lambda', A)}{\lambda'} \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{1}{\lambda' - \lambda} \frac{d\lambda}{\lambda} d\lambda' \\ &= \frac{1}{2\pi i} \int_{\Gamma^+_{\theta',\varepsilon'}} \frac{R(\lambda', A)}{(\lambda')^2} d\lambda' \\ &= \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\lambda^2} d\lambda. \end{aligned}$$

Hence $Q_+^2 X \subset D(A)$ and

$$\begin{aligned} AQ_+^2 &= \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{\lambda R(\lambda, A) - I}{\lambda^2} d\lambda \\ &= Q_+. \end{aligned}$$

Let $x \in D(P_+)$, i.e. $Q_+x \in D(A)$. Then

$$Q_+P_+x = Q_+AQ_+x = AQ_+^2x = Q_+x.$$

Hence $P_+x \in D(P_+)$ and

$$P_+P_+x = AQ_+P_+x = AQ_+x = P_+x.$$

□

Let $X_+ := \text{im} P_+$ and let A_+ be the part of A in X_+ (cf. Section 2). Then the following holds.

Proposition 15. $\sigma(A_+) \subset \{\mu \in \mathbb{C} : \text{Re}\mu > 0\}$.

Proof. It follows from Proposition 4 that $\rho(A) \subset \rho(A_+)$ and $R(\mu, A)_{|X_+} = R(\mu, A_+)$ for all $\mu \in \rho(A)$. Let $\text{Re}\mu < 0$. We have to show that $\mu \in \rho(A_+)$. Define

$$R_+ = \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\mu - \lambda} d\lambda.$$

Then $R_+ \in \mathcal{L}(X)$. We first show that $(\mu - A)R_+ = P_+$. In fact, let $\omega < \theta' < \theta < \frac{\pi}{2}$, and $0 < \varepsilon < \varepsilon'$. We have

$$\begin{aligned} R_+Q_+ &= \frac{1}{(2\pi i)^2}\int_{\Gamma^+_{\theta,\varepsilon}}\left(\int_{\Gamma^+_{\theta',\varepsilon'}}\frac{R(\lambda',A)}{\mu-\lambda'}d\lambda'\right)\frac{R(\lambda,A)}{\lambda}d\lambda \\ &= \frac{1}{(2\pi i)^2}\int_{\Gamma^+_{\theta,\varepsilon}}\int_{\Gamma^+_{\theta',\varepsilon'}}\frac{R(\lambda',A)-R(\lambda,A)}{\lambda-\lambda'}\frac{1}{\mu-\lambda'}\frac{1}{\lambda}d\lambda' d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta',\varepsilon'}}\frac{R(\lambda',A)}{\mu-\lambda'}\left(\frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}}\frac{d\lambda}{\lambda(\lambda-\lambda')}\right)d\lambda' + \\ &\quad -\frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}}\frac{R(\lambda,A)}{\lambda}\left(\frac{1}{2\pi i}\int_{\Gamma^+_{\theta',\varepsilon'}}\frac{d\lambda'}{(\mu-\lambda')(\lambda-\lambda')}\right)d\lambda \\ &=: I_1 - I_2. \end{aligned}$$

The function $f(\lambda') := [(\mu-\lambda')(\lambda-\lambda')]^{-1}$ is holomorphic in the set $\Sigma_{\theta'} \cap \{z \in \mathbb{C} : |z| > \varepsilon'\}$ so that $I_2 = 0$. Moreover by Cauchy's Formula we have that

$$\frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}}\frac{d\lambda}{\lambda(\lambda-\lambda')} = \frac{1}{\lambda'}.$$

Thus

$$R_+Q_+ = \frac{1}{2\pi i}\int_{\Gamma^+_{\theta',\varepsilon'}}\frac{R(\lambda',A)}{(\mu-\lambda')\lambda'}d\lambda'.$$

Consequently $R_+Q_+ \in \mathcal{L}(X; D(A))$ and

$$\begin{aligned} (\mu-A)R_+Q_+ &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta',\varepsilon'}}\frac{((\mu-\lambda')+(\lambda'-A))R(\lambda',A)}{(\mu-\lambda')\lambda'}d\lambda' \\ &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta',\varepsilon'}}\frac{R(\lambda',A)}{\lambda'}d\lambda' + \frac{1}{2\pi i}\int_{\Gamma^+_{\theta',\varepsilon'}}\frac{d\lambda'}{(\mu-\lambda')\lambda'} \\ &= Q_+. \end{aligned}$$

Observe that for $x \in D(A)$, $R_+x \in D(A)$ and $AR_+x = R_+Ax$. It follows that

$$(\mu - A)R_+P_+x = P_+x$$

for all $x \in D(P_+)$ and

$$R_+(\mu - A)x = x$$

for all $x \in D(A) \cap X_+$. This shows that $\mu \in \rho(A_+)$ and

$$R(\mu, A_+) = R_{+|X_+}.$$

□

Since the spectrum of A_+ is included in the right half-plane, we call P_+ the **positive spectral projection** associated with A.

Similarly, we let $\Gamma^-_{\theta,\varepsilon} = -\Gamma^+_{\theta,\varepsilon}$ oriented from down to up,

$$Q_- = \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,\varepsilon}} R(\lambda, A)\frac{d\lambda}{\lambda}$$

and $P_- = AQ_-$.

Observe that, for integrable $f : \Gamma^+_{\theta,\varepsilon} \to X$,

$$-\int_{\Gamma^-_{\theta,\varepsilon}} f(-\lambda)d\lambda = \int_{\Gamma^+_{\theta,\varepsilon}} f(\lambda)d\lambda.$$

Since $R(\lambda, -A) = -R(-\lambda, A)$, P_- is the positive spectral projection associated with $-A$. It follows from the residuum theorem that $Q_+ + Q_- = A^{-1}$. Hence $D(P_+) = D(P_-)$ and $P_+ = I - P_-$. Defining $X_- := \ker P_+$, and letting A_- being the part of A in X_- we deduce from Proposition 15 that $\sigma(A_-)$ is in the left half-plane. Now consider the operator $\tilde{A}$ obtained by twisting A by P_+. Then by Proposition 8 one has

$$\sigma(\tilde{A}) \subset -\sigma(A_+) \cup \sigma(A_-) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}. \tag{3.1}$$

Moreover

$$\sup_{s\in\mathbb{R}} \|R(is, \tilde{A}\| < \infty. \tag{3.2}$$

Finally we show that

$$\sup_{s\in\mathbb{R}} \|isR(is, \tilde{A})\| < +\infty. \tag{3.3}$$

For $\mu \in \rho(A) \cap \rho(-A)$ we have that

$$\begin{aligned} R(\mu, \tilde{A}) &= P_+R(\mu, -A) + P_-R(\mu, A) \\ &= P_+(R(\mu, -A) - R(\mu, A)) + R(\mu, A) \\ &= S(\mu) + R(\mu, A), \end{aligned}$$

where $S(\mu) := P_+(R(\mu, -A) - R(\mu, A))$.

Lemma 16. $\sup_{s\in\mathbb{R}} \|sS(is)\| < +\infty$.

Proof. We compute $S(\mu)$. Let μ be to the left of $\Gamma^+_{\theta,\varepsilon}$. Then

$$\begin{aligned} Q_+R(\mu, A) &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)R(\mu, A)}{\lambda} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A) - R(\mu, A)}{(\mu - \lambda)\lambda} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{(\mu - \lambda)\lambda} d\lambda. \end{aligned}$$

Since $P_+R(\mu, A) = AQ_+R(\mu, A)$ and $AR(\lambda, A) = \lambda R(\lambda, A) - I$ for each $\lambda \in \rho(A)$ we have

$$\begin{aligned} P_+R(\mu, A) &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} \frac{AR(\lambda, A)}{(\mu - \lambda)\lambda} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\mu - \lambda} d\lambda - \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} \frac{1}{\lambda(\mu - \lambda)} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\mu - \lambda} d\lambda. \end{aligned}$$

Let μ be to the left of the curve Γ' consisting of the lines $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ oriented upwards.

Then we have, by Cauchy's Theorem,

$$P_+R(\mu, A) = \frac{1}{2\pi i}\int_{\Gamma'} \frac{R(\lambda, A)}{\mu - \lambda} d\lambda,$$

and, if $-\mu$ is to the left of Γ' then

$$\begin{aligned} P_+R(\mu,-A) &= -P_+R(-\mu,A) \\ &= -\frac{1}{2\pi i}\int_{\Gamma'}\frac{R(\lambda,A)}{-\mu-\lambda}d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma'}\frac{R(\lambda,A)}{\lambda+\mu}d\lambda \end{aligned}$$

In particular, for $\mu = is$, $(s \neq 0)$ we have

$$\begin{aligned} S(is) &= P_+(R(is,-A)-R(is,A)) \\ &= \frac{1}{2\pi i}\int_{\Gamma'}\left(\frac{1}{is+\lambda}-\frac{1}{is-\lambda}\right)R(\lambda,A)d\lambda \\ &= \frac{1}{i\pi}\int_{\Gamma'}\frac{\lambda}{s^2+\lambda^2}R(\lambda,A)d\lambda. \end{aligned}$$

Observe that

$$|a+be^{2i\theta}| \geq \sqrt{\frac{1+\cos(2\theta)}{2}}(a+b), \qquad a,b \geq 0.$$

Thus

$$\begin{aligned} \|sS(is)\| &\leq \left\|\frac{1}{i\pi}\int_{\Gamma'}\frac{\lambda s}{s^2+\lambda^2}R(\lambda,A)d\lambda\right\| \\ &\leq \frac{2}{\pi}\int_0^{+\infty}\frac{M|s|}{|s^2+r^2e^{2i\theta}|}dr \\ &\leq \frac{2M}{\pi}\sqrt{\frac{2}{1+\cos(2\theta)}}\int_0^{+\infty}\frac{|s|}{s^2+r^2}dr \\ &= \frac{2M}{\pi}\sqrt{\frac{2}{1+\cos(2\theta)}}\int_0^{+\infty}\frac{|s|}{s^2(1+(\frac{r}{|s|})^2)}dr \\ &= \frac{2M}{\pi}\sqrt{\frac{2}{1+\cos(2\theta)}}\int_0^{+\infty}\frac{1}{1+t^2}dt \\ &= \frac{2M}{\pi}\sqrt{\frac{2}{1+\cos(2\theta)}}\frac{\pi}{2} \\ &= M\sqrt{\frac{2}{1+\cos(2\theta)}}, \end{aligned}$$

where $M = \sup_{s\in\mathbb{R}} \|sR(is,A)\|$. This proves the claim (3.3). $\square$

Next we use the following theorem

Theorem 17 (Phragmen-Lindelöff, [5, Corollary 6.4.4])**.** *Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re} z > 0\}$, and let $h : \overline{\mathbb{C}_+} \to X$ be continuous and holomorphic on $\mathbb{C}_+$. Assume that for each $\delta > 0$ there exists $c > 0$ such that*

$$\|h(z)\| \leq Ce^{\delta|z|}, \qquad z \in \mathbb{C}_+.$$

Assume that

$$\|h(is)\| \leq M, \qquad s \in \mathbb{R}.$$

Then $\|h(z)\| \leq M$ for all $z \in \mathbb{C}_+$.

Now let $h(\mu) = \mu R(\mu, \tilde{A})$ for $\mathrm{Re}\mu \geq 0$. Then $M_1 := \sup_{s\in\mathbb{R}} \|h(is)\| < +\infty$, and, by (3.2),

$$\|h(\mu)\| \leq |\mu| M_2, \qquad \mathrm{Re}\mu \geq 0,$$

for some $M_2 > 0$. It follows from the Phragmen-Lindelöff Theorem that $\|h(\mu)\| \leq M$ for $\mathrm{Re}\mu \geq 0$. We have proved that $\tilde{A}$ generates a bounded holomorphic semigroup. Thus Theorem 13 is proved.

4. The semigroups associated with a bisectorial operator

Let A be an invertible, bisectorial operator on X. We consider the operators Q_+ and Q_- defined in the previous section, and the spectral projections $P_+ = AQ_+$, $P_- = AQ_-$. Let $\tilde{A}$ be the operator A twisted by P_+ and let $\tilde{T}$ be the holomorphic semigroup generated by $\tilde{A}$.

Proposition 18. *Define, for $t > 0$,*

$$T^+(t) := P_+\tilde{T}(t), \qquad T^-(t) := P_-\tilde{T}(t).$$

Then $T^+(t), T^-(t) \in \mathcal{L}(X)$ for all $t > 0$ and

$$T^+(t+s) = T^+(t)T^+(s), \quad T^-(t+s) = T^-(t)T^-(s), \qquad t, s > 0.$$

Moreover $T^+(t)T^-(s) = T^-(t)T^+(s) = 0$ for all $t, s > 0$.

Proof. Since $\tilde{T}(t)X \subset D(\tilde{A}) \subset Z$, the operators $T^+(t), T^-(t)$ are bounded. Since Q_+, Q_- commute with the resolvent of A, they also commute with $R(\mu, \tilde{A}) = P_+R(\mu, -A) + P_+R(\mu, A)$ (see Proposition 8). Consequently, Q_+, Q_- also commute with $\tilde{T}(t)$. Hence also P_+, P_- commute with $\tilde{T}$. This implies the semigroup property. Since

$$P_-x = x - P_+x, \qquad x \in D(P_+) = D(P_-)$$

we have $P_+P_-x = P_-P_+x = 0$. This implies that $T^+(t)T^-(s) = T^-(s)T^+(t) = 0$. □

It follows from the definition that

$$\tilde{T}(t) = T^+(t) + T^-(t), \qquad t > 0. \tag{4.1}$$

Moreover $T^+, T^- \in C^\infty((0, +\infty), X)$ and

$$\frac{d}{dt}T^\pm(t) = \mp AT^\pm(t), \qquad t > 0. \tag{4.2}$$

It follows that, for $x \in Z$

$$\mp A \int_0^t T^\pm(s)x\,ds = T^\pm(t)x - x. \tag{4.3}$$

It is possible to express the semigroups T^+, T^- directly by a contour integral, without using $\tilde{T}$. Let $\omega < \theta < \frac{\pi}{2}$ as in Section 3.

Proposition 19. *One has, for $t > 0$,*

$$T^+(t) = \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} e^{-\lambda t}R(\lambda, A)\,d\lambda, \tag{4.4}$$

$$T^-(t) = \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,\varepsilon}} e^{\lambda t}R(\lambda, A)\,d\lambda. \tag{4.5}$$

Proof. For $t > 0$ let

$$S(t) := \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} e^{-\lambda t}R(\lambda, A)\,d\lambda \in \mathcal{L}(X).$$

If $x \in X_+$, then $R(\lambda, A)x$ has a holomorphic extension to $\mathbb{C}_+$ (by Proposition 15). Hence $S(t)x = 0$ by Cauchy's Theorem.

Let $x \in X_+$. Then substituting λ by $-\lambda$ we have

$$\begin{aligned} S(t)x &= -\frac{1}{2\pi i}\int_{\Gamma^-_{\theta,\varepsilon}} e^{\lambda t} R(-\lambda, A)x d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,\varepsilon}} e^{\lambda t} R(\lambda, -A)x d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,\varepsilon}} e^{\lambda t} R(\lambda, \tilde{A})x d\lambda \\ &= \tilde{T}(t)x = T^+(t)x \end{aligned}$$

by the usual exponential formula of the holomorphic semigroup $\tilde{T}$. Hence $S(t)x = T^+(t)x$ for all $x \in X_+ + X_-$ and hence for all $x \in X$ by density. This proves (4.4). The identity (4.5) is proved similarly. □

Even though $\tilde{T}(t) = T^+(t) + T^-(t)$ converges strongly to the identity as $t \to 0$, each of the semigroups T^+, T^- are singular at 0 as $t \to 0$ whenever $P_\pm$ is unbounded.

In fact, the following holds.

Proposition 20. *a) For $x \in X$ the following are equivalent:*

(i) $x \in D(P_+)$,
(ii) $\lim_{t\to 0} T^+(t)x$ *exists.*

In that case $P_+x = \lim_{t\to 0} T^+(t)x$.

b) If P_+ is unbounded, then $\lim_{t\to 0} \|T^+(t)\| = \infty$.

Proof. a) If $x \in D(P_+)$, then $\lim_{t\to 0} T^+(t)x = \lim_{t\to 0} \tilde{T}(t)P_+x = P_+x$. Conversely, assume that $\lim_{t\to 0} T^+(t)x = y$. Since $\lim_{t\to 0} \tilde{T}(t)x = x$ and since P_+ is closed it follows that $x \in D(P_+)$ and $P_+x = \lim_{t\to 0} P_+\tilde{T}(t)x = \lim_{t\to 0} T^+(t)x = y$.

b) Assume that there exists $t_n \to 0$ such that $\|T^+(t_n)\| \le C$. Then for $x \in D(P_+)$ one has $\|P_+x\| = \lim_{n\to+\infty} \|T^+(t_n)x\| \le C\|x\|$. Since $D(P_+)$ is dense, it follows that P_+ is bounded. □

However, the following corollary of Proposition 19 shows that the singularity of $T^\pm$ at 0 is mild.

Corollary 21. *There exists a constant $c > 0$ such that*

$$\|T^\pm(t)\|_{\mathcal{L}(X)} \le c|\log t|, \qquad 0 < t \le 1/2.$$

Proof. Since $0 \in \rho(A)$, there exists a constant $c > 0$ such that

$$\|R(\lambda, A)\| \le \frac{M}{1 + |\lambda|}$$

for all $\lambda = re^{\pm i\theta}$, $r \ge 0$. Let Γ' consist of the two rays $\{re^{\pm i\theta} : r \ge 0\}$ where θ is chosen as in Section 3, directed upwards. Then by Cauchy's Theorem

$$T^+(t) = \frac{1}{2\pi i}\int_{\Gamma'} e^{-\lambda t} R(\lambda, A) d\lambda.$$

Hence, for $0 < t \leq 1/2$,

$$\begin{aligned}
\|T^+(t)\|_{\mathcal{L}(X)} &\leq \frac{1}{2\pi} 2M \int_0^{+\infty} e^{-rt\cos(\theta)} \frac{1}{1+r} dr \\
&= \frac{M}{\pi} \int_1^{+\infty} e^{-(s-1)t\cos(\theta)} \frac{ds}{s} \\
&= \frac{M}{\pi} e^{t\cos(\theta)} \int_1^{\infty} e^{-st\cos(\theta)} \frac{ds}{s} \\
&\leq \frac{M}{\pi} e^{\cos(\theta)/2} \int_t^{+\infty} e^{-r\cos(\theta)} \frac{dr}{r} \\
&\leq \frac{M}{\pi} e^{\cos(\theta)/2} \left(\int_1^{+\infty} e^{-r\cos(\theta)} \frac{dr}{r} + \int_t^1 \frac{dr}{r} \right) \\
&\leq c_1(c_2 - \log t),
\end{aligned}$$

where $c_1, c_2 > 0$ are constants. □

Schweiker [10] defined the semigroups T^+, T^- directly, by the expressions given in Proposition 19, and also proved the estimate of Corollary 21

5. Squares and roots

Let A be an invertible bisectorial operator on X and let $\tilde{A}$ be its twisted version.

Proposition 22. *The operator $\tilde{A}$ is invertible and $\tilde{A}^2 = A^2$.*

Proof. It follows from Proposition 8 that $\tilde{A}^{-1} = -P_+A^{-1} + P_-A^{-1}$. Since P_+ and P_- commute with A, it follows that

$$\tilde{A}^{-2} = (\tilde{A}^{-1})^2 = A^{-2}.$$

Consequently, $\tilde{A}^2 = A^2$. □

As a consequence of Proposition 22, the operator A^2 is sectorial. This is surprising since the spectral mapping theorem alone does not allow us to conclude that the spectrum of A^2 is contained in a sector.

Since A^2 is sectorial, we may consider its square root $(A^2)^{\frac{1}{2}}$ [2, Section 3.8], which is a sectorial operator again.

Theorem 23. *One has*

$$-\tilde{A} = (A^2)^{\frac{1}{2}}.$$

Proof. Let Q_+, Q_-, P_+, P_- be defined as in Section 3. A change of variable and the resolvent identity show that

$$\begin{aligned} Q_+ - Q_- &= \frac{1}{2\pi i}\left\{\int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda,A)}{\lambda}d\lambda - \int_{\Gamma^-_{\theta,\varepsilon}} \frac{R(\lambda,A)}{\lambda}d\lambda\right\} \\ &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda,A)-R(-\lambda,A)}{\lambda}d\lambda \\ &= -2\frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} R(\lambda,A)R(-\lambda,A)d\lambda \\ &= 2\frac{1}{2\pi i}\int_{\Gamma^+_{\theta,\varepsilon}} R(\lambda^2,A^2)d\lambda \\ &= \frac{1}{2\pi i}\int_{(\Gamma^+_{\theta,\varepsilon})^2} R(w,A^2)w^{-\frac{1}{2}}dw \\ &= -(A^2)^{-\frac{1}{2}}, \end{aligned}$$

where $(\Gamma^+_{\theta,\varepsilon})^2 = \{z^2 : z \in \Gamma^+_{\theta,\varepsilon}\}$, and the last identity is the well-known formula for the square root [2, (3.51) p.166]. It follows that

$$\begin{aligned} \tilde{A}^{-1} &= -P_+A^{-1} + P_-A^{-1} \\ &= -(Q_+ - Q_-) \\ &= -(A^2)^{-\frac{1}{2}}. \end{aligned}$$

Hence $\tilde{A} = -(A^2)^{-\frac{1}{2}}$. □

6. Mild solutions

Let A be a linear operator on X. Given $f \in L^1_{loc}(\mathbb{R};X)$ we consider the problem

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}. \tag{6.1}$$

A continuous function $u : \mathbb{R} \to X$ is called a **mild solution** of (6.1) if $\int_0^t u(s)ds \in D(A)$ and

$$u(t) = u(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds$$

for all $t \in \mathbb{R}$.

In order to prove uniqueness of the solution of (6.1) we need a spectral condition on A and a growth condition on u. Let $g \in L^1_{loc}(\mathbb{R};X)$. We say that g is **polynomially bounded** if

$$\|g(t)\| \le c(1+|t|)^k, \qquad t \in \mathbb{R},$$

for some $k \in \mathbb{N}$, $c > 0$. The function f is called **weakly polynomially bounded** if

$$\int_{-\infty}^{+\infty} \|g(t)\|(1+|t|)^{-k}dt < +\infty,$$

for some $k \in \mathbb{N}$. This last notion is clearly weaker than that of polynomially boundedness. Note that g is weakly polynomially bounded whenever $g \in L^p(\mathbb{R};X)$ for some $1 \le p \le +\infty$.

Proposition 24. *Assume that $i\mathbb{R} \subset \rho(A)$. Then there exists at most one weakly polynomially bounded solution u of (6.1).*

Proof. Let u be a weakly polynomially bounded solution of (6.1) for $f = 0$. Then the Carleman spectrum of u (as defined in [2, Section 4.6]) is empty. This is proved as the last 6 lines of [4, Theorem 2.7]. It follows from [2, Theorem 4.8.2] that $u(t) = 0$ for all $t \in \mathbb{R}$. □

Remark 25. *Conversely Schweiker* [11, Theorem 1.1] *showed the following. If for each $f \in \mathrm{BUC}(\mathbb{R}; X)$ there is a unique mild solution $u \in \mathrm{BUC}(\mathbb{R}; X)$ of (6.1), then $i\mathbb{R} \subset \rho(A)$ and $\sup_{s\in\mathbb{R}} \|R(is, A)\| < +\infty$. She also showed that on Hilbert spaces this condition is sufficient for this type of well-posedness.*

Now we assume that A is bisectorial and invertible and keep the notation of Sections 3 and 4. In particular, we consider the semigroups T^+ and T^- associated with A. Recall that there exist $\omega > 0$ and $c > 0$ such that

$$\|T^{\pm}(t)\| \leq c(1 + |\log t|)e^{-\omega t}, \qquad t > 0. \tag{6.2}$$

Let $f \in L^1_{loc}(\mathbb{R}; X)$ be weakly polynomially bounded. Then the function $u : \mathbb{R} \to X$ given by

$$u(t) := \int_{-\infty}^{t} T^-(t - s)f(s)ds - \int_{t}^{+\infty} T^+(s - t)f(s)ds \tag{6.3}$$

is continuous and polynomially bounded. In fact

$$\begin{aligned}
\left|\int_{-\infty}^{t} T^-(t-s)f(s)ds\right| &\leq \int_{-\infty}^{t} e^{-\omega(t-s)} f(s)(1+|s|)^{-k}(1+|s|)^k ds \\
&\leq C \sup_{s\leq t} e^{-\omega(t-s)}(1+|s|)^k \\
&= C \sup_{r\geq 0} e^{-\omega r}(1+|t-r|)^k \\
&\leq C\left(1 + \sup_{r\geq 0}\left|\sum_{n=0}^{k}(-1)^n\binom{k}{n}t^n r^{k-n}\right|\right).
\end{aligned}$$

Analogously for the second term in the right-hand side of (6.3).

Theorem 26. *The function u defined by (6.3) is the unique mild solution of (6.1).*

Proof. Let u be defined by (6.3). In order to show that u is a mild solution we consider the function v given by $v(t) = A^{-2}u(t)$. It suffices to show that

$$v(t) = v(0) + A\int_0^t v(s)ds + \int_0^t A^{-1}f(s)ds. \tag{6.4}$$

Note that, by definition,

$$v(s) = \int_{-\infty}^{s} T^-(s-r)A^{-1}f(r)dr - \int_{s}^{+\infty} T^+(r-s)A^{-1}f(r)dr.$$

Hence by Fubini's Theorem

$$\begin{aligned}
\int_0^t v(s)ds &= \int_{-\infty}^{0}\int_{0}^{t} T^-(s-r)A^{-1}f(r)dsdr + \int_{0}^{t}\int_{r}^{t} T^-(s-r)A^{-1}f(r)dsdr \\
&\quad - \int_{0}^{t}\int_{0}^{r} T^+(r-s)A^{-1}f(r)dsdr - \int_{t}^{\infty}\int_{0}^{t} T^+(r-s)A^{-1}f(r)dsdr.
\end{aligned}$$

Since A is closed we obtain by (4.3), for $t > 0$

$$\begin{aligned} A\int_0^t v(s)ds &= \int_{-\infty}^0 \big(T^-(t-r)A^{-1}f(r) - T^-(-r)A^{-1}f(r)\big)\,dr + \\ &\quad + \int_0^t \big(T^-(t-r)A^{-1}f(r) - P_- A^{-1}f(r)\big)\,dr + \\ &\quad - \int_0^t \big(P_+ A^{-1}f(r) - T^+(r)A^{-1}f(r)\big)\,dr + \\ &\quad - \int_t^{+\infty} \big(T^+(r-t)A^{-1}f(r) - T^+(r)A^{-1}f(r)\big)\,dr \\ &= v(t) - \int_{-\infty}^0 T^-(-r)A^{-1}f(r)dr + \\ &\quad - \int_0^t A^{-1}f(r)dr + \int_0^{+\infty} T^+(r)A^{-1}f(r)dr \\ &= v(t) - \int_0^t A^{-1}f(r)dr - v(0). \end{aligned}$$

□

Our point is the representation formula (6.3). In special cases it had been proved before. Lunardi [7, (4.4.26) p.164] gave a proof when A generates a holomorphic semigroup and Schweiker [10, Chapter 2] gave a proof if $f \in \mathrm{BUC}(\mathbb{R}; X)$ and A is densely defined. Here we do not address the question of maximal regularity. This had been done in previous work with the help of multiplier theorems. In fact, in [1] it is shown that for each $f \in C^\alpha(\mathbb{R}; X)$ there exists a unique classical solution $u \in C^{1+\alpha}(\mathbb{R}; X)$ of (6.1), where $0 < \alpha < 1$. Since a classical solution is also a weak solution we now have a representation formula for this solution. On the other hand, with the help of the representation formula (6.3) one may prove that $u \in C^{1+\alpha}(\mathbb{R}; X)$ for $f \in C^\alpha(\mathbb{R}; X)$ more directly as in [7, Theorem 4.3.1] without making use of Fourier multiplier theorems. This will be done in the forthcoming paper [12].

In the L^p-context the following is known. Let $1 < p < +\infty$. If X is a Hilbert space and $f \in L^p(\mathbb{R}; X)$, then there exists a unique strong solution $u \in W^{1,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$ of (6.1) (see [4] or [9]). Again we can deduce that u is given by (6.1). If X is a UMD-space this result remains true if A is R-bisectorial (instead of merely sectorial, see [4]).

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