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Positive Semigroups of Kernel Operators

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Dedicated to the memory of H.H. Schaefer

Abstract. Extending results of Davies and of Keicher on ℓ^p we show that the peripheral point spectrum of the generator of a positive bounded C_0-semigroup of kernel operators on L^p is reduced to 0. It is shown that this implies convergence to an equilibrium if the semigroup is also irreducible and the fixed space non-trivial. The results are applied to elliptic operators.

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0. Introduction

Irreducibility is a fundamental notion in Perron-Frobenius Theory. It had been introduced in a direct way by Perron and Frobenius for matrices, but it was H. H. Schaefer who gave the definition via closed ideals. This turned out to be most fruitful and led to a wealth of deep and important results. For applications Ouhabaz' very simple criterion for irreduciblity of semigroups defined by forms (see [Ouh05, Sec. 4.2] or [Are06]) is most useful. It shows that for practically all boundary conditions, a second order differential operator in divergence form generates a positive irreducible C_0-semigroup on $L^2(\Omega)$ where Ω is an open, connected subset of $\mathbb{R}^N$.

The main question in Perron-Frobenius Theory, is to determine the asymptotic behaviour of the semigroup. If the semigroup is bounded (in fact Abel bounded suffices), and if the fixed space is non-zero, then irreducibility is equivalent to convergence of the Cesàro means to a strictly positive rank-one operator, i.e. to an equilibrium of the system (see Theorem 1.1 below). However, one would like to prove strong convergence instead of just in-mean-convergence. For this a necessary condition is that the peripheral point spectrum is reduced to 0. Recently, Davies [Dav05] showed that this condition is automatically satisfied for positive contraction semigroups on ℓ^p and also for contraction semigroups on L^p enjoying the Feller property. Keicher [Kei05] generalized Davies' result to bounded positive C_0-semigroups on an arbitrary order continuous atomic Banach lattice and Wolff [Wol07] to more general atomic Banach lattices. In the main result of Section 3, Theorem 3.5, we prove that the peripheral point spectrum is reduced to 0 for any Abel bounded positive C_0-semigroup of kernel operators on L^p. Notice that the Feller property automatically implies that a semigroup consists of kernel operators and on ℓ^p every positive operator is a kernel operator.

Once we know that the peripheral point spectrum is trivial, one possible sufficient condition for convergence of a bounded mean ergodic C_0-semigroup is countability of the entire peripheral spectrum (cf. Theorem 4.1). This settles the problem in the case when the resolvent is compact. A natural example of convergence to an equilibrium is the semigroup generated by the Neumann Laplacian in $L^1(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded open domain. In fact, Neumann boundary conditions physically signify an isolated boundary and the heat flow should converge to an equilibrium. However, as Kunstmann [Kun02] showed, it may happen that the spectrum of the Neumann Laplacian on $L^1(\Omega)$ is the entire left half plane.

The main convergence result presented in this article, (Theorem 4.2), says that a bounded, positive, irreducible C_0-semigroup of kernel operators converges to a strictly positive equilibrium if a non-zero fixed vector exists. Thus the countability assumption can be omitted in the case of kernel operators and non-zero fixed space. This result can be applied to the Neumann Laplacian on L^1 and yields the desired convergence result. Theorem 4.2 is due to Greiner [Gre82, Korollar 3.11]. The reader might appreciate that we include a proof of this result (which is not complete in the monograph [Nag86]). In particular, we give a proof of an essential tool, namely non-disjointness of the powers of an irreducible kernel operator (which is a version of a result of Axmann [Axm80]).

Kernels of semigroups generated by elliptic operators, i.e. heat kernels, are intensely studied (see e.g. [Dav90], [Dan00], [Ouh05], [Are04], [Are06]). In the case of Neumann boundary conditions, on domains having the extension property, very precise information on the kernel is known: it satisfies Gaussian estimates [AtE97], [Are04], [Are06], [Ouh05], [Dan00] and hence the semigroup is holomorphic even on L^1. But even if the domain is irregular, it is easy to prove that some sort of kernel exists. This can be done via Bukhvalov's characterization of kernel operators (see Corollary 3.5) and using the de Giorgi-Nash result on local regularity of weak solutions (cf. [AB94]). A more serious problem is to show that the semigroup is bounded. If the spectral bound is 0 and the resolvent is compact, then by Perron-Frobenius Theory, it is again irreducibility which implies that the pole is simple and hence that the semigroup is at least Abel bounded. Our Theorem 3.5 ensures triviality of the peripherical point spectrum also in that case. Thus strong convergence of the semigroup is equivalent to its boundedness which may be violated, though. But in any case we can prove strong convergence in a weighted L^1-space, thus showing that in a quite general situation, asymptotic stability holds in a reasonable sense.

1. Ergodicity

Let T be a C_0-semigroup T on a Banach space X. We say that T is **mean ergodic** if

$$Pf = \lim_{t\to\infty} \frac{1}{t} \int_0^t T(s)f\,ds$$

exists for all $f \in X$. This implies that

$$X = \ker A \oplus \overline{R(A)}\ ,$$

where A is the generator of T and $R(A) := \{Af : f \in D(A)\}$ its range. In that case P is the projection onto $\ker A$ along this decomposition. We call P the **ergodic projection** of T. If T is bounded and X is reflexive, then T is ergodic.

Now let $X = L^p = L^p(\Omega)$ where (Ω, Σ, μ) is a σ-finite measure space and $1 \le p < \infty$. Assume that T is positive. We say that T is **irreducible** if the only invariant closed ideals of L^p are $\{0\}$ and L^p. A **closed ideal** in L^p is a space $\mathcal{J}$ of the form $\mathcal{J} = L^p(\omega)$ where $\omega \in \Sigma$ and $L^p(\omega) := \{f \in L^p : f = 0 \text{ a.e. on } \Omega \setminus \omega\}$. For $f \in L^p$ we write $f > 0$ if $f \ge 0$ and $\mu(\{x \in \Omega : f(x) > 0\}) > 0$ and we write $f \gg 0$ if $f(x) > 0$ a.e. Note that $\ker A = \{f \in L^p : T(t)f = f \text{ for all } t \ge 0\}$ is the fixed space of T. Thus, if T is irreducible and $0 < e \in \ker A$, then $e \gg 0$. Similarly, if $0 < \varphi \in \ker A'$, then $\varphi \gg 0$. We denote by

$$s(A) = \sup\{\Re\lambda : \lambda \in \sigma(A)\}$$

the **spectral bound** of T. Then $(s(A), \infty) \subset \varrho(A)$ and $R(\lambda, A) := (\lambda - A)^{-1} \ge 0$ for all $\lambda > s(A)$. Moreover,

$$\lambda R(\lambda, A) = \int_0^\infty \lambda e^{-\lambda t} T(t)\,dt$$

strongly for $\lambda > s(A)$, i.e. $\lambda R(\lambda, A)$ is the Abel mean of T. We say that T is **Abel bounded** if $s(A) \le 0$ and $\sup_{\lambda>0} \|\lambda R(\lambda, A)\| < \infty$. If T is bounded, then T is also Abel bounded. It will be important for us to consider this weaker notion of boundedness when discussing the important case where T is irreducible and A has compact resolvent. The Gaussian semigroup on $L^1(\mathbb{R})$ is positive, irreducible and contractive, but not mean ergodic. However, if $\ker A \ne 0$, then irreducibility implies mean ergodicity as the following theorem shows.

Theorem 1.1. *Let T be an Abel bounded positive C_0-semigroup on $L^p, 1 \le p < \infty$, with generator A. The following assertions are equivalent.*

(i) *T is irreducible and* $\ker A \ne \{0\}$*;*

(ii) *there exist* $0 \ll e \in \ker A,\ \ 0 \ll \varphi \in \ker A'$ *and* $\dim \ker A = 1$*;*

(iii) *T is mean ergodic with ergodic projection P given by*

$$Pf = \langle f, \varphi\rangle e \quad (f \in L^p) \quad \text{where} \quad 0 \ll \varphi \in \ker A'\ , \qquad 0 \ll e \in \ker A \quad \text{satisfy} \quad \langle e, \varphi\rangle = 1\ . \tag{1.1}$$

We denote the projection of rank 1 defined in (1.1) by $P = \varphi \otimes e$.

Proof. $(i) \Rightarrow (ii)$. Let $0 \neq f \in \ker A$. Then $|f| = |T(t)f| \leq T(t)|f|$. Let $0 < \psi \in X'$ such that $\langle |f|, \psi \rangle > 0$. Let φ be a weak* limit point of $\lambda R(\lambda, A)'\varphi$ as $\lambda \downarrow 0$. Then $0 < \varphi \in \ker A'$ by the proof of [ABHN01, Proposition 4.3.6]. Hence $\varphi \gg 0$ by irreducibility. Since $\langle T(t)|f| - |f|, \varphi \rangle = \langle |f|, T(t)'\varphi - \varphi \rangle = 0$ and $T(t)|f| - |f| \geq 0$, it follows that $T(t)|f| = |f|$. Thus $e := |f| \in \ker A$. Since T is irreducible it follows that $e \gg 0$ and that $\ker A$ is 1-dimensional (see [Nag86, C-III Prop. 3.5, p. 310] and its proof).

$(ii) \Rightarrow (iii)$. Since the order interval is weakly compact and invariant, it follows from [ABHN01, Proposition 4.3.1a)] that T is Abel ergodic. Since T is positive, T is also mean ergodic by [ABHN01, Theorem 4.3.7]. Normalizing e and φ such that $\langle e, \varphi \rangle = 1, Pf = \langle f, \varphi \rangle e$ is a projection onto $\ker A$ vanishing on $R(A) \subset \ker \varphi$. Thus P is the ergodic projection.

$(iii) \Rightarrow (i)$. Let $\omega \in \Sigma$, $\mathcal{J} = L^p(\omega)$. Assume that $\mu(\omega) > 0$ and $\mu(\Omega \setminus \omega) > 0$. Let $0 < f \in \mathcal{J}$, $0 < \psi \in X' = L^{p'}$ such that $\operatorname{supp} \psi \subset \Omega \setminus \omega$. Then $\frac{1}{t} \int_0^t \langle T(s)f, \psi \rangle \, ds \to \langle Pf, \psi \rangle = \langle f, \varphi \rangle \langle e, \psi \rangle$ which is strictly positive. Hence there exists $s > 0$ such that $\langle T(s)f, \psi \rangle > 0$ and so $T(s)f \notin \mathcal{J}$. Thus $\mathcal{J}$ is not invariant. We have shown that T is irreducible. □

The following simple example shows that in condition (ii) we have to assume that e and φ are both strictly positive.

Example 1.2. Let $X = \mathbb{R}^2, A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$. Then $T(t) = \begin{pmatrix} e^{-t} & 1 - e^{-t} \\ 0 & 0 \end{pmatrix}$. Thus $\lim_{t \to \infty} T(t) = P, Px = \langle x, \varphi \rangle e,\ e = (1,1)^\top, \varphi = (0,1)$. But T is not irreducible.

An important case where the assumption $\ker A \neq \{0\}$ can be verified occurs when T is an irreducible, positive C_0-emigroup whose generator A has compact resolvent. Then by de Pagter's Theorem $s(A) > -\infty$. We assume that $s(A) = 0$. This is just a normalization which can be obtained by considering $A - s(A)$ instead of A. Thus 0 is a pole of the resolvent and $\ker A \neq 0$. Now by [Nag86, C-III. Prop. 3.5, p. 310], the order of the pole is 1 (because T is irreducible). Hence T is Abel bounded. Thus we can apply Theorem 1.1 and obtain the following.

Corollary 1.3. *Let T be a positive, irreducible C_0-semigroup on $L^p, 1 \leq p < \infty$, whose generator A has compact resolvent. Assume that $s(A) = 0$. Then T is ergodic with ergodic projection P given by*

$$Pf = \langle f, \varphi \rangle e \quad (f \in L^p)$$

where $0 \ll e \in \ker A$ and $0 \ll \varphi \in \ker A'$.

Theorem 1.1 shows that under the assumption that T is Abel bounded and $\ker A \neq 0$, irreducibility is equivalent to saying that T converges in the Cesàro sense to an equilibrium P given by

$$Pf = \langle f, \varphi \rangle e \quad (f \in L^p)$$

where $0 \ll \varphi \in \ker A', 0 \ll e \in \ker A$. The aim of this article is to give conditions which imply strong convergence instead of merely Cesàro convergence. Such results can be seen as Tauberian theorems (see [ABHN01, Part 3]). A well-known Tauberian condition is that T is bounded and that the peripheral spectrum $\sigma(A) \cap i\mathbb{R}$ (and not just the peripheral point spectrum $\sigma_p(A) \cap i\mathbb{R}$) is countable ([EN00, Thm. V.2.21], [ABHN01, Sec. 5.5]). Here we want to consider a different condition, which is not of spectral type, namely that the semigroup consists of kernel operators.

2. Kernel operators

Let $K : X \to X$ be a positive operator where $X = L^p(\Omega), 1 \leq p < \infty$, as before. We say that K is a **kernel operator** if there exists a measurable function $k : \Omega \times \Omega \to \mathbb{R}_+$ such that

$$Kf(x) = \int_\Omega k(x,y)f(y)\,dy$$

$x - a.e.$ for all $f \in X$. Kernel operators can be characterized in an abstract way. By [Sch74, Proposition IV.9.8, p. 290], K is a kernel operator if and only if $K \in (X' \otimes X)^{\perp\perp}$, i.e. if there exist operators K_n, R_n such that $0 \leq K_n \leq K_{n+1}, \lim_{n\to\infty} K_n f = Kf$ for all $f \in X$ and $K_n \leq R_n, R_n$ an operator of finite rank. As a consequence we obtain the following permanence property. Recall that a closed sublattice of an L^p-space is an L^p-space again (see e.g. [Sch74, Exercise 23, p. 149]).

Proposition 2.1. *Let $0 \leq K \in \mathcal{L}(X)$ be a kernel operator.*

a) *If $0 \leq S \in \mathcal{L}(X)$, then also SK and KS are kernel operators.*
b) *If $Y \subset X$ is a closed sublattice of X such that $Y \subset K$, then also $K_{|_Y}$ is a kernel operator.*

Proof. a) One has $\lim SK_n = SK$ and $SK_n \leq SR_n$. Similarly $\lim K_n S = KS$ and $K_n S \leq R_n S$. Since $R_n S$ and SR_n are of finite rank, the claim follows.

b) There exists a positive projection P from X onto Y (see [Sch74, III.11.4]). Thus $\tilde{K}_n = PK_{n_{|_Y}} \in \mathcal{L}(Y), \lim_{n\to\infty} \tilde{K}_n f = Kf$ $(f \in Y)$ and $\tilde{K}_n \leq PR_{n_{|_F}}$ which is of finite rank.

□

We also mention the following characterization [MN91, Thm. 3.3.11] which is most useful for applications in Section 5.

Theorem 2.2 (Bukhvalov). *Let $0 \leq K \in \mathcal{L}(X)$. The following assertions are equivalent.*

(i) *K is a kernel operator;*
(ii) *if $f_n \to f$ in $X, |f_n| \leq g$ for some $g \in X$, then*

$$\lim_{n\to\infty} Kf_n(x) = Kf(x) \quad a.e.$$

Remark 2.3. In [MN91] condition(ii) in Theorem 2.2 is replaced by the condition (iii)if $f_n \xrightarrow{*} f$ and $|f_n| \le g$ for some $g \in X$, then $\lim\limits_{n\to\infty} (Kf_n)(x) = (Kf)(x)$ a.e.

Here $f_n \xrightarrow{*} f$ means that each subsequence of $(f_n)_{n\in\mathbb{N}}$ has a subsequence which converges a.e. to f. But if $|f_n| \le g \in X$, then by the dominated convergence theorem $f_n \xrightarrow{*} f$ if and only if $f_n \to f$ in X. Thus (iii) is equivalent to (ii). The equivalence of $*$-convergence and convergence in X for a dominated sequence also shows that (i) implies (iii).

Corollary 2.4. *Assume that* Ω *is an open set in* $\mathbb{R}^N$ *and* μ *the Lebesgue measure. If* $0 \le K \in \mathcal{L}(X)$ *satisfies* $KX \subset L^\infty_{\mathrm{loc}}(\Omega)$, *then* K *is a kernel operator.*

Proof. Let $\omega_n \subset\subset \Omega$ (i.e. ω_n is bounded and $\overline{\omega_n} \subset \Omega$) such that $\bigcup\limits_{n\in\mathbb{N}} \omega_n = \Omega$. By the Closed Graph Theorem there exist constants $c_n \ge 0$ such that

$$\|Kf\|_{L^\infty(\omega_n)} \le c_n \|f\|_X \ .$$

Now let $f_m \to f$ in X as $m \to \infty$. Then $Kf_m \to Kf$ in $L^\infty(\omega_n)$ as $m \to \infty$ for each $n \in \mathbb{N}$. Hence $(Kf_n)(x) \to (Kf)(x)$ x- a.e. in Ω as $m \to \infty$. Thus (ii) of the previous theorem is satisfied. □

3. Trivial peripheral point spectrum

Let (Ω, Σ, μ) be a σ-finite measure space, $1 \le p < \infty$, $L^p = L^p(\Omega)$. We first prove a special case of the main result Theorem 3.5.

Theorem 3.1. *Let* T *be a positive contractive* C_0*-semigroup on* L^p *with generator* A*. Assume that* $T(t)$ *is a kernel operator for some* $t > 0$*. Then* $\sigma_p(A) \cap i\mathbb{R} \subset \{0\}$.

Here $\sigma_p(A) = \{\lambda \in \mathbb{C} : \exists\, 0 \ne f \in D(A), Af = \lambda f\}$ denotes the **point spectrum** of A. Theorem 3.1 is due to Davies [Dav05] in the case where $L^p = \ell^p$, i.e., if (Ω, Σ, μ) is atomic. Keicher [Kei05] generalized the result to arbitrary bounded, positive C_0-semigroups on an atomic Banach lattice with order continuous norm and Wolff [Wol07] to more general atomic Banach lattices. Moreover Wolff merely assumes that the semigroup is (ω)-solvable, a notion of boundedness which is more general than Abel bounded. Note that on ℓ^p each positive operator is a kernel operator.

Proof of Theorem 3.1. Let $\alpha \in \mathbb{R}, 0 \ne f \in L^p$ such that $T(t)f = e^{i\alpha t} f$ for all $t \ge 0$. We want to show that $\alpha = 0$.

We have $|f| = |T(t)f| \le T(t)|f|$. Since T is contractive and the norm is strictly monotonic on the positive cone of L^p, it follows that $e := |f| = T(t)|f|$ for all $t \ge 0$. We may assume that $|f| \gg 0$. Otherwise we replace Ω by $\{x \in \Omega : f(x) \ne 0\}$. Since the order interval $[0, e]$ is weakly compact and total in L^p and also invariant under T, the semigroup T is relatively weakly compact. Hence there exists a projection $P \in \{T(t) : t \ge 0\}^{-\mathcal{L}_\sigma}$ commuting with T onto the space $X_r := \overline{\mathrm{span}}\{g \in D(A) : Ag = i\beta g \text{ for some } \beta \in \mathbb{R}\}$, see [EN00, Theorem V.2.8] (cf.

[KN07] in this volume). Since $P \in \{T(t) : t \geq 0\}^{-\mathcal{L}_\sigma}$, it follows that P is positive and contractive. Note that for $f \in X_r, |f| = |Pf| \leq P|f|$. Since the norm is strictly monotonic on the positive cone it follows that $|f| = P|f|$. Thus X_r is a closed sublattice of L^p. The restriction T_r of T to X_r acts as a group of lattice isomorphisms on X_r (see [KN07, Proposition 2.2]). By assumption and Proposition 2.1.b) there exists $t_0 > 0$ such that $T_r(t_0)$ is a kernel operator. The ideal property Proposition 2.1.a) implies that the identity on X_r is a kernel operator. By [MN91, Proposition 3.3.13, p. 189] this implies that X_r is atomic, i.e., $X_r = \ell^p(J)$ for some index set J. Now T_r is a bounded C_0-group of lattice isomorphisms on $\ell^p(J)$. This implies that $T_r(t) = I$ for all $t \in \mathbb{R}$ (see [Kei05, Proposition 3.5] for arbitrary atomic spaces). In fact, since $T_r(t)$ maps atoms to atoms, one has $T_r(t)e_j = h(t,j)e_{\varphi(t,j)}$ for some $h(t,j) > 0, \varphi(t,j) \in J$. Strong continuity implies that $\varphi(t,j) = j$. Hence $T_r(t)e_j = h(t,j)e_j$. This implies that $h(t,j) = e^{\lambda_j t}$, and hence $h(t,j) = 1$ since T_r is bounded. Since $f \in X_r$, it follows that $\alpha = 0$. □

Remark 3.2. One can also use the direct argument of Davies [Dav05], i.e. the space $\mathcal{M}$ of [Dav05, Theorem 4] instead of the Glicksberg-deLeeuw space X_r in the proof of Theorem 3.1.

Next we will show that the peripheral point spectrum is trivial for each positive C_0-semigroup of kernel operators which may even mildly grow as $t \to \infty$. We will use a construction which will be useful later again. Let T be a positive C_0-semigroup on $L^p(\Omega,\mu)$. Let $0 \ll \varphi \in L^{p'}(\Omega,\mu)$ (the dual space of $L^p(\Omega,\mu)$) such that

$$T(t)'\varphi \leq \varphi \ , \ \text{for all } t \geq 0 \ . \tag{3.1}$$

Consider the space $L^1(\Omega,\varphi\mu)$. Since for $f \in L^p(\Omega,\mu)$,

$$\|f\|_{L^1(\Omega,\varphi\mu)} = \int_\Omega |f|\varphi d\mu \leq \|f\|_{L^p(\Omega,\mu)}\|\varphi\|_{L^{p'}(\Omega,\mu)} \tag{3.2}$$

the space $L^p(\Omega,\mu)$ is continuously embedded into $L^1(\Omega,\varphi\mu)$. Moreover, $L^p(\Omega,\mu)$ is a sublattice of $L^1(\Omega,\varphi\mu)$.

Proposition 3.3. *Assume* (3.1). *Then there exists a unique C_0-semigroup T_φ on $L^1(\Omega,\varphi\mu)$ such that $T_\varphi(t)f = T(t)f$ for all $f \in L^p(\Omega,\mu), t \geq 0$. The semigroup T_φ is positive and contractive. Moreover, $T_\varphi(t)$ is a kernel operator, if $T(t)$ is one, $t > 0$. If T is irreducible, then T_φ is so too.*

Proof. Let $f \in L^p(\Omega,\mu)$. Then

$$\begin{aligned}\|T(t)f\|_{L^1(\Omega,\varphi\mu)} &= \int_\Omega |T(t)f|\varphi d\mu \leq \int_\Omega T(t)|f|\varphi d\mu \\ &= \int |f| \ T(t)'\varphi d\mu \leq \int_\Omega |f|\varphi d\mu \\ &= \|f\|_{L^1(\Omega,\varphi\mu)} \ .\end{aligned}$$

Since $L^p(\Omega,\mu)$ is dense in $L^1(\Omega,\varphi\mu)$, the operator $T(t)$ has a unique continuous extension $T_\varphi(t)\in\mathcal{L}(L^1(\Omega,\varphi\mu))$. Clearly, $T_\varphi(t)$ is positive and contractive. Moreover, $T_\varphi(t+s)=T_\varphi(t)T_\varphi(s), s,t\geq 0$. In order to show strong continuity, let $f\in L^p(\Omega,\mu)$. Then by (3.2),

$$\|T_\varphi(t)f-f\|_{L^1(\Omega,\varphi\mu)}\leq\|T(t)f-f\|_{L^p(\Omega,\mu)}\cdot\|\varphi\|_{L^{p'}(\Omega,\mu)}\to 0$$

as $t\to 0$. Since $L^p(\Omega,\mu)$ is dense in $L^1(\Omega,\varphi\mu)$ and $T_\varphi(t)$ is contractive, the claim follows. If $T(t)$ is given by a kernel $k_t(x,y)$, then $T_\varphi(t)$ is given by the kernel $\frac{1}{\varphi(y)}k_t(x,y)$. Each closed ideal $\mathcal{J}$ of $L^1(\Omega,\varphi\mu)$ is of the form $\mathcal{J}=L^1(\omega,\varphi\mu)$ for some $\omega\in\Sigma$. Thus $\mathcal{J}\cap L^p(\Omega,\mu)=L^p(\omega,\mu)$. If $T_\varphi(t)\mathcal{J}\subset\mathcal{J}$, then $T(t)L^p(\omega,\mu)\subset L^p(\omega,\mu)$. Thus T_φ is irreducible if T is irreducible. □

We need the following lemma whose easy proof is omitted.

Lemma 3.4. *Let S be a C_0-semigroup on a Banach space X which is the direct sum of two closed subspaces X_1 and X_2. Assume that $S(t)X_1\subset X_1$. Denote by P_2 the projection onto X_2 according to the decomposition $X=X_1\oplus X_2$. Then $T_2(t)x=P_2T(t)x$ defines a C_0-semigroup on X_2. If $y\in X, T(t)y=e^{i\alpha t}y$, then*

$$T_2(t)(P_2y)=e^{i\alpha t}(P_2y)$$

for all $t\geq 0$.

Now we can prove a fairly general result on the triviality of the peripheral point spectrum. Recall from Section 1 that each bounded C_0-semigroup is Abel bounded.

Theorem 3.5. *Let T be a positive, Abel bounded C_0-semigroup on $L^p(\Omega,\mu)$, where $1\leq p<\infty$, with generator A. Assume that $T(t)$ is a kernel operator for some $t>0$. Then $\sigma_p(A)\cap i\mathbb{R}\subset\{0\}$.*

Proof. Let $0\neq f\in L^p(\Omega,\mu)$ such that $T(t)f=e^{i\alpha t}f\quad(t\geq 0)$. Then $|f|\leq T(t)|f|\quad(t\geq 0)$. Let $0<\psi\in L^{p'}(\Omega,\mu)$ such that $\langle|f|,\psi\rangle>0$. Let φ be a ω^*-limit point of $\lambda R(\lambda,A)'\psi$ as $\lambda\downarrow 0$. Then $\varphi\in\ker A'$ and $\langle|f|,\varphi\rangle\geq\langle|f|,\psi\rangle>0$ [ABHN01, proof of Prop. 4.3.6]. Moreover, $\varphi\geq 0$. Let $\omega:=\{x\in\Omega:\varphi(x)>0\}$. Then $T(t)L^p(\Omega\backslash\omega,\mu)\subset L^p(\Omega\backslash\omega,\mu)$. In fact, let $f\in L^p(\Omega\backslash\omega,\mu)$, i.e., $\int_\Omega|f|\varphi d\mu=0$. Then

$$\int_\Omega|T(t)f|\varphi d\mu\leq\int_\Omega(T(t)|f|)\varphi d\mu=\int_\Omega|f|\cdot T(t)'\varphi d\mu=\int_\Omega|f|\varphi d\mu=0\ .$$

Thus $T(t)f\in L^p(\Omega\setminus\omega,\mu)$. Define the C_0-semigroup T_2 on $L^p(\omega,\mu)$ by $T_2(t)f=1_\omega T(t)f\quad(f\in L^p(\omega,\mu))$, see the previous lemma. Then $f_1:=1_\omega f\neq 0$ since $\langle\varphi,|f|\rangle>0$ and $T_2(t)f_1=e^{i\alpha t}f_1\quad(t\geq 0)$. Moreover, $T_2(t)'\varphi=\varphi$, since for $g\in L^p(\omega,\mu)$,

$$\int_\omega T_2(t)g\varphi d\mu=\int_\Omega T(t)g\varphi d\mu=\int_\Omega gT(t)'\varphi d\mu=\int_\Omega g\varphi d\mu=\int_\omega g\varphi d\mu\ .$$

Finally, $T_2(t)$ is a kernel operator for some $t > 0$. Now the semigroup $T_{2\varphi}$ defined on $L^1(\omega, \varphi d\mu)$ as in Proposition 3.3 is contractive. Moreover $T_{2\varphi}(t)f_1 = e^{i\alpha t}f_1 \quad (t \geq 0)$ and $f_1 \neq 0$. It follows from Theorem 3.1 that $\alpha = 0$. ☐

Corollary 3.6. *Let T be a positive, irreducible C_0-semigroup on L^p, where $1 \leq p < \infty$. Assume that*

(a) *$T(t)$ is a kernel operator for some $t > 0$, and that*
(b) *$s(A) = 0$ is a pole of the resolvent.*

Then $\sigma_p(A) \cap i\mathbb{R} = \{0\}$.

Proof. By [Nag86, C-III. Prop. 3.5, p. 310] the pole is simple and hence T is Abel bounded. Now Theorem 3.5 proves the claim. ☐

Assume now that Ω is an open subset of $\mathbb{R}^N$ and μ the Lebesgue measure dx. As in [Dav05] we say that a positive C_0-semigroup on $L^p(\Omega, dx)$ has the **Feller property** if

$$T(t)L^p(\Omega, dx) \subset C(\Omega)$$

(the space of all continuous functions on Ω). By Corollary 2.3 this implies that $T(t)$ is a kernel operator for each $t > 0$. Thus we obtain the following corollary which is due to Davies [Dav05, Theorem 12] in the case where $T(t)$ is contractive.

Corollary 3.7. *Let T be a positive, Abel bounded C_0-semigroup on $L^p(\Omega, dx)$ where $1 \leq p < \infty$. Assume that T has the Feller property. Then*

$$\sigma_p(A) \cap i\mathbb{R} \subset \{0\} \ ,$$

where A denotes the generator of T.

4. Convergence of the semigroup

In this section we want to establish strong convergence of $T(t)$ itself as $t \to \infty$, and not merely convergence of the Cesaro means. It is clear that triviality of the peripheral point spectrum is a necessary condition. In fact, if $i\alpha \in \sigma_p(A)$, then $T(t)f = e^{i\alpha t}f \quad (t \geq 0)$ for some $f \neq 0$. If $T(t)f$ converges as $t \to \infty$, then $\alpha = 0$.

Throughout this section, (Ω, Σ, μ) is a σ-finite measure space, $1 \leq p < \infty$, $X = L^p(\Omega, \mu)$. We first give a result based on the hypothesis that the peripheral spectrum (and not just the peripheral point spectrum) is countable.

Theorem 4.1. *Let T be a positive, bounded, ergodic C_0-semigroup on X, where $1 \leq p < \infty$, with generator A. Assume that $T(t)$ is a kernel operator for some $t > 0$. If $\sigma(A) \cap i\mathbb{R}$ is countable, then*

$$Pf = \lim_{t\to\infty} T(t)f$$

exists for all $f \in X$.

Proof. By [ABHN01, Prop. 4.3.13] the semigroup T is totally ergodic. This implies that $\sigma_p(A') \cap i\mathbb{R} = \sigma_p(A) \cap i\mathbb{R}$. Thus it follows from Theorem 3.5 that $\sigma_p(A') \cap i\mathbb{R} \subset \{0\}$. Denote by P the ergodic projection and let $X_0 = (I - P)X$. Applying [ABHN01, Theorem 5.5.6] (or [EN00, Thm. V.2.21]) to the restriction of T to X_0 proves the claim. □

It is remarkable that the assumption that the peripheral spectrum be countable can be omitted if we assume that $\ker A \neq \{0\}$. The following result is due to Greiner [Gre82, Korollar 3.11] and is stated in [Nag86, C-IV].

Theorem 4.2. *Let T be a positive, irreducible, contractive C_0-semigroup on L^p with generator A. Assume that*

(a) *$T(t_0)$ is a kernel operator for some $t_0 > 0$ and that*
(b) *$\ker A \neq \{0\}$.*

Then there exist $0 \ll e \in \ker A, 0 \ll \varphi \in \ker A'$ such that

$$\lim_{t\to\infty} T(t)f = \langle f, \varphi \rangle e$$

for all $f \in L^p$.

Proof. We know from Theorem 1.1 that T is ergodic with ergodic projection $P = \varphi \otimes e$ for some $0 \ll e \in \ker A, 0 \ll \varphi \in \ker A'$. Observe that $T(\tau)$ is mean ergodic for $\tau > 0$. In fact, the weakly compact order interval $[0, e]$ is invariant under $T(\tau)$ and total in X. This implies mean ergodicity (cf. [ABHN01, Proposition 4.3.1]). Moreover,

$$\ker(I - T(\tau)) = \mathbb{C} \cdot e \ . \tag{4.1}$$

This follows from the fact that

$$\sigma_p(A) \cap i\mathbb{R} = \{0\} \ , \tag{4.2}$$

which is a consequence of Theorem 3.1. In fact, if $f \in \ker(I - T(\tau))$, then

$$0 = T(\tau)f - f = (A - 2\pi ik/\tau) \int_0^\tau e^{-2\pi ikt/\tau} T(t)f \, dt \ .$$

Since $2\pi ik/\tau \notin \sigma_p(A)$ for $k \neq 0$ it follows that the $k - th$ Fourier coefficient of the funtion $T(\cdot)f \in C([, \tau]; X)$ is 0 for $k \neq 0$. Hence $T(\cdot)f$ is constant. Thus $f \in \ker A = \mathbb{C} \cdot e$.

As a consequence of mean ergodicity and (4.1) we have

$$X = \mathbb{C} \cdot e + \overline{R(I - T(\tau))} \ . \tag{4.3}$$

Note that for $g \in X$,

$$P(g - T(\tau)g) = P\left(A \int_0^\tau T(s)g \, ds\right) = \left\langle A \int_0^\tau T(s)g \, ds \ , \ \varphi \right\rangle \cdot e = 0$$

and $Pe = e$. Thus P is the projection according to the decomposition (4.3), i.e.,

$$P = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T(\tau)^k$$

strongly. Now it follows as in the proof of Theorem 1.1 that $T(\tau)$ is irreducible. By Greiner's 0-2-law [Gre82, 3.7] or [Nag86, C-IV.Thm. 2.6, p. 346] for every $\tau > 0$ one has the following alternative

$$|T(t+\tau) - T(t)|e \to 0 \quad \text{as} \quad t \to \infty \ , \tag{4.4}$$

or

$$|T(t+\tau) - T(t)|e = 2e \quad \text{for all} \quad t > 0 \ , \tag{4.5}$$

We show that there exists at least one $\tau \geq t_0$ such that (4.4) holds. In fact, otherwise (4.5) holds for all $\tau \geq t_0$. Let $S = T(t_0)$. Then S is a positive, irreducible kernel operator. Letting $\tau = (n-1)t_0, t = t_0$ in (4.5), it follows that

$$|S^n - S|e = 2e \tag{4.6}$$

for all $n \in \mathbb{N}, n \geq 2$. Denote by k_n the kernel of $S^n, n = 1, 2 \cdots$. Then $|k_n - k_1|$ is the kernel of $|S^n - S|$. Hence there exists a null set $N \subset \Omega$ such that for all $x \in \Omega \setminus N, n \geq 2$

$$\begin{aligned} \int_\Omega |k_n(x,y) - k_1(x,y)|e(y)\,dy &= 2e(x) \\ &= (S^n e)(x) + (Se)(x) \\ &= \int_\Omega (k_n(x,y) + k_1(x,y))e(y)\,dy \ . \end{aligned}$$

Let $g(y) = (k_n(x,y) + k_1(x,y) - |k_n(x,y) - k_1(x,y)|) \cdot e(y)$. Then $g \geq 0$ a.e. and $\int_\Omega g(y)\,dy = 0$. This implies that $g(y) = 0$ a.e. Since for $a, b \geq 0$, $|a - b| = a + b$ if and only if $a \cdot b = 0$, we conclude that for all $x \in \Omega \setminus N, n \geq 2$

$$k_n(x,y)k_1(x,y) = 0 \quad y - \text{a.e.}$$

Hence

$$S^n \wedge S = 0$$

for all $n \geq 2$. Since S is an irreducible kernel operator this contradicts Theorem 6.1 in the appendix. We have shown that (4.4) holds for some $\tau \geq t_0$. For this τ and $|g| \leq e$ we have

$$|T(t)(I - T(\tau))g| = |(T(t+\tau) - T(t)g| \leq |T(t+\tau) - T(t)|e \to 0 \quad \text{as} \quad t \to \infty.$$

Since the order interval $[0, e]$ is total, it follows that $\lim_{t\to\infty} T(t)f = 0$ for all $f \in \overline{R(I - T(\tau))}$. Now the claim follows from (4.3). □

The rotation semigroup on $L^p(\mathbb{T}), \mathbb{T}$ the torus, $1 \le p < \infty$, is positive, irreducible and contractive. Its generator A has a compact resolvent and $1_{\mathbb{T}} \in \ker A$. But $T(t)$ does not converge strongly as $t \to \infty$. In fact, T is an isometric group. Moreover, $2\pi i\mathbb{Z} \subset \sigma_p(A)$. In this example condition (a) of Theorem 4.2 is violated, i.e., $T(t)$ is not a kernel operator for any $t > 0$ (even though $R(\lambda, A)$ is a kernel operator for all $\lambda > 0$).

Remark 4.3. The proof of Corollary 2.11 in [Nag86, p. 350], which would imply Theorem 4.2, seems not to be complete. In fact, [Nag86, C-III Cor. 3.2] cannot be applied to A_j since T_j is not known to be periodic. Moreover, Example 1.2 in Section 1 shows that irreducibility of T_j needs a proof as does the irreducibility of $T_j(t_0)$.

One problem in applying Theorem 4.2 is that it might not be known or not true that T is bounded. In fact, let T be a positive, irreducible C_0-semigroup of kernel operators. Then by Jentzsch's Theorem [Sch74, V. Thm. 6.6. p. 337] $s(A) > -\infty$. Replacing A be $A - s(A)$ we may assume that $s(A) = 0$. Now the problem is that in general T will not be bounded. Assume in addition that 0 is a pole of the resolvent. Then it follows from [Nag86, C-III Thm. 3.12, p. 315] that the pole is simple. Thus T is Abel bounded. It follows from Theorem 1.1 that T is ergodic with ergodic projection $P = \varphi \otimes e$ where $0 \ll \varphi \in \ker A', 0 \ll e \in \ker e$. Now by Theorem 4.3, T converges strongly to P if and only if T is bounded. But even if T is not bounded the extended semigroup T_φ is contractive on $L^1(\Omega, \varphi d\mu)$. In general, the peripheral spectrum of this semigroup is no longer countable. But $T_\varphi(t)$ is still a kernel operator. So Theorem 4.2 implies that

$$\lim_{t\to\infty} T_\varphi(t)f = \int f\varphi d\mu \cdot e$$

in $L^1(\Omega, \varphi\mu)$ for all $f \in L^1(\Omega, \varphi\mu)$. In particular, we obtain the following.

Theorem 4.4. *Let T be a positive, irreducible C_0-semigroup on $L^p(\Omega, \mu)$ where $1 \le p < \infty$. Denote by A its generator. Assume that*

(a) *$T(t)$ is a kernel operator for some $t > 0$, and that*
(b) *$s(A) = 0$ is a pole of the resolvent.*

Then there exist $0 \ll e \in \ker A, 0 \ll \varphi \in \ker A'$ such that

$$\lim_{t\to\infty} \int_\Omega |T(t)f - c(f)e|\varphi d\mu = 0$$

where $c(f) = \int f\varphi d\mu$ for all $f \in L^P(\Omega, \mu)$.

We add another result of Perron Frobenius Theory which will be useful for the applications given in Section 5.

Theorem 4.5. *Let T be a positive, irreducible C_0-semigroup on X. Assume that there exist*

(a) *$0 < e \in \ker A$ and*
(b) *$\tau > 0$ such that $T(\tau)$ is compact.*

Then there exist $0 \ll \varphi \in \ker A'$, $M \geq 0, \varepsilon > 0$ *such that*

$$\|T(t) - P\|_{\mathcal{L}(X)} \leq Me^{-\varepsilon t} \quad (t \geq 0)$$

where $Pf = \langle f, \varphi \rangle e \quad (f \in X)$.

Proof. It follows from irreducibility that $e \gg 0$ and that $s(A) = 0$. Since $T(\tau)$ is compact, the semigroup is quasicompact [Nag86, B-IV.2.8 p. 214]. Now the result follows from [Nag86, C-IV.2.1 p. 343 and C-III.3.5(d) p. 310]. □

5. Applications

We will consider elliptic operators with measurable coefficients. In the first two examples we consider a domain $\Omega \subset \mathbb{R}^N$ with finite volume. If Ω is sufficiently regular, we can apply Theorem 4.5 and show exponential convergence to an equilibrium. For arbitrary Ω exponential convergence is not true, in general. Heat propagation might be too slow if the boundary is complicated. Analytically, this means in particular that the semigroup consists no longer of compact operators. It is interesting that they are still kernel operators. This follows from the de Giorgi-Nash Theorem. So we can apply Theorem 4.2 and prove strong convergence to an equilibrium. Finally, we consider operators with unbounded drift. Here the semigroup is not known to be bounded and we apply Theorem 4.4 to deduce convergence to an equilibrium in a weighted norm.

5.1. The Neumann Laplacian

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set. Consider the Neumann Laplacian Δ_2^N on $L^2(\Omega)$. It generates a selfadjoint submarkovian C_0-semigroup $T_2(t)$ on $L^2(\Omega)$. By Ouhabaz' simple criterion [Ouh05, Sec. 4.2] it follows that T_2 is irreducible. It had been proved in [AB94, Theorem 5.2] that $T_2(t)$ is a kernel operator for all $t > 0$. Let T_1 be the extrapolation semigroup on $L^1(\Omega)$. Then T_1 is contractive, positive, irreducible and $T_1(t)$ is a kernel operator for all $t > 0$. Note that $T_1(t)1_\Omega = 1_\Omega$ for all $t \geq 0$. Thus it follows from Theorem 4.2 that

$$\lim_{t\to\infty} T_1(t)f = \frac{1}{|\Omega|} \int_\Omega f(x)\,dx \cdot 1_\Omega \tag{5.1}$$

in $L^1(\Omega)$. If Ω is irregular, it can happen that the generator Δ_1^N of T_1 has the entire left half-plane as spectrum (see Kunstmann [Kun02] for this surprising phenomenon).

Remark. However, in the case of Example 5.1, we can also argue in a different way. Since T_2 is holomorphic, it follows that for $f \in L^2(\Omega)$

$$\lim_{t\to\infty} T_2(t)f = \frac{1}{|\Omega|} \int f(x)\,dx 1_\Omega$$

in $L^2(\Omega)$. By standard arguments this implies (5.1).

In fact, let $0 \leq f \leq 1$. Since $T(t)f \to c(f)1_\Omega$ in $L^2(\Omega)$, for each sequence $t_n \to \infty$ there is a subsequence such that $T(t_{n_k})f \to c(f)1_\Omega$ a.e. Thus $T(t_{n_k})f \to c(f)1_\Omega$ in $L^1(\Omega)$ by the dominated convergence theorem. Here $c(f) = \frac{1}{|\Omega|}\int\limits_\Omega f(x)\,dx$. Since $[0, 1_\Omega]$ is total in $L^1(\Omega)$, the claim follows.

5.2. Elliptic operators

Let $\Omega \subset \mathbb{R}^N$ be open, connected of finite volume. Let $a_{ij} \in L^\infty(\Omega), b_j \in L^\infty(\Omega)$ such that

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \quad (\xi \in \mathbb{R}^N, x \in \Omega)\ ,$$

where $\nu > 0$. Define the form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ given by

$$a(u,v) = \int\limits_\Omega \left\{ \sum_{i,j=1}^N a_{ij}(x) D_i u \overline{D_j v} + \sum_{j=1}^N b_j D_j u \bar{v} \right\} dx\ .$$

Then a is continuous and $L^2(\Omega)$-elliptic, i.e.,

$$\Re a(u,u) + \omega\|u\|^2_{L^2} \geq \beta\|u\|^2_{H^1}$$

for all $u \in H^1(\Omega)$ and some $\omega \in \mathbb{R}, \beta > 0$. Denote by $-A_2$ the operator associated with a and by T_2 the semigroup generated by A_2. The semigroup T_2 is positive and irreducible [Ouh05, 4.2 p. 103f]. Since $a(1,v) = 0$ for all $v \in H^1(\Omega)$ one has $A_2 1_\Omega = 0$, hence $T_2(t)1_\Omega = 1_\Omega \quad (t \geq 0)$. Consequently, the restriction $T_p(t)$ of $T_2(t)$ to $L^p(\Omega)$ defines a positive C_0-semigroup T_p on $L^p(\Omega), 2 < p < \infty$, whose generator we denote by A_p. It is shown by Ouhabaz [Ouh05, Theorem 4.28 p. 135] that there also exist C_0-semigroups T_p on $L^p(\Omega)$ for $1 < p < 2$ such that $\|T_p(t)\| \leq e^{\omega_p t} \quad (t \geq 0)$ where $\omega_p \in \mathbb{R}$ and $T_p(t)_{|_{L^2(\Omega)}} = T_2(t) \quad (t \geq 0)$, see also Daners [Dan00]. However, if the b_j are not differentiable, then it may happen that $\lim\limits_{p\downarrow 1} \omega_p = \infty$, and it is not clear that a C_0-semigroup extending T_2 exists on $L^1(\Omega)$ without further assumptions.

Nonetheless, since the semigroup governs a heat equation, the space L^1 is of particular interest. Indeed, given an initial heat distribution $0 \leq f \in L^p(\Omega), u(t,x) = (T_p(t)f)(x)$ is the density of the heat at time t at the point $x \in \Omega$. For a Borel set $B \subset \Omega$ the integral

$$\int\limits_B u(t,x)\,dx$$

is the heat amount in B at time t. We want to discuss in which sense $u(t,\cdot)$ converges to an equilibrium as $t \to \infty$ depending on various hypotheses on Ω and on the coefficients. Before doing so we collect some properties which hold without further assumptions. The semigroup T_2 is holomorphic. Consequently, T_p is holomorphic for $1 < p < \infty$; cf. [Are04, 7.2.2]. Next we show that $T_p(t)$ is a kernel operator for $1 < p < \infty$, $t > 0$. In fact, it is obvious that the property of being a kernel operator extrapolates from one L^p space to all others, on which the semigroup operates, even to L^1 (in contrast to properties as holomorphy and compactness as Example 5.1 shows). Thus we may consider $p > N/2$. Then for $\lambda > 0$ large, $R(\lambda, A_p)L^p(\Omega) \subset L^\infty_{\text{loc}}(\Omega)$. In fact, let $f \in L^p(\Omega), u = R(\lambda, A_p)f$. Then $\lambda u - A_p u = f$. It follows from the de Giorgi-Nash Theorem [GT77, Theorem 8.22] that u is continuous. Hence $D(A_p) \subset L^\infty_{\text{loc}}(\Omega)$. Since T_p is holomorphic, $T_p(t)L^p(\Omega) \subset D(A_p) \subset L^\infty_{\text{loc}}(\Omega)$. It follows from Corollary 2.3 that $T_p(t)$ is a kernel operator for all $t > 0$.

In order to obtain a semigroup on $L^1(\Omega)$ and kernels in $L^\infty(\Omega\times\Omega)$ we discuss several assumptions on the domain and the coefficients.

First case: Assume that Ω has the **Sobolev injection property**, i.e. $H^1(\Omega) \subset L^{2^*}(\Omega)$ where $L^{2^*}(\Omega) = L^{2N/N-2}(\Omega)$ if $N \geq 3, L^{2^*}(\Omega) = \bigcup_{2\leq q<\infty} L^q(\Omega)$ if $N = 2$ and $L^{2^*}(\Omega) = L^\infty(\Omega)$ if $N = 1$. This condition is satisfied if Ω is bounded and has Lipschitz boundary, or more generally the extension property. But the **Sobolev injection property** is also true for any $\Omega = \tilde{\Omega} \setminus L$ where $\tilde{\Omega}$ has the extension property and L is closed subset of a hyperplane. Such Ω has not longer the extension property if $\text{int}(L) \cap \Omega \neq \emptyset$, where $\text{int}(L)$ denotes the interior of L in the relative topology of the hyperplane (see also [Dan00]). The assumption that Ω has the Sobolev injection property implies that T_p has Gaussian bounds [Ouh05]. Consequently, $T_p(t)$ is given by a bounded kernel and the semigroup extends to a C_0-semigroup T_1 on $L^1(\Omega)$. Moreover, each $T_1(t)$ is a compact operator. Now we can apply Theorem 4.5 and find $0 \ll \varphi \in \ker A_1'$, $\varepsilon > 0, M \geq 0$ such that

$$\|T_1(t) - P\|_{\mathcal{L}(L^1)} \leq Me^{-\varepsilon t} \quad (t \geq 0)$$

where $Pf = \int_\Omega f\varphi\, dx \cdot e$. Thus $T_1(t)$ converges in operator norm exponentially fast to the equilibrium P. This implies in particular that $\sigma(A_1) \subset \{\lambda \in \mathbb{C} : \Re e\lambda \leq -\varepsilon\}\cup\{0\}$. Thus, the Neumann Laplacian considered in 5.1 shows that such a result cannot hold for irregular Ω.

Second case: The domain Ω is arbitrary, of finite volume. In order to make sure that the semigroup extends to $L^1(\Omega)$ we impose further conditions on the drift terms, namely that $b_j \in W_0^{1,\infty}(\Omega)$ and

$$\sum_{j=1}^N D_j b_j \leq 0 \ .$$

Then by the Beurling-Deny criterion, $\|T_2(t)\|_{L^1(\Omega)} \leq \|f\|_{L^1}$ (see [Ouh05], [Are06, Example 9.3.3]) and we obtain a contractive C_0-semigroup T_1 on $L^1(\Omega)$ such that $T_1(t)_{|_{L^2(\Omega)}} = T_2(t)$ for all $t \geq 0$. Since $T_2(t)$ is a positive kernel operator on $L^2(\Omega)$, also $T_1(t)$ is a positive kernel operator on $L^1(\Omega)$. Moreover, T_1 is irreducible since T_2 is so. Finally, $T_1(t)1_\Omega = T_2(t)1_\Omega = 1_\Omega$ for all $t \geq 0$. Now Theorem 4.2 implies that there exists $0 \ll \varphi \in \ker A_1' \subset L^\infty(\Omega)$ such that

$$\lim_{t\to\infty} T_1(t)f = \int_\Omega f\varphi\, dx 1_\Omega$$

in $L^1(\Omega)$ for all $f \in L^1(\Omega)$.

5.3. Elliptic operators with unbounded drift

Let $a_{ij} \in C^1(\mathbb{R}^N)$ be bounded with bounded derivatives and assume that

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2$$

for all $x \in \mathbb{R}^N, \xi \in \mathbb{R}^N$. Let $b \in C^1(\mathbb{R}^N, \mathbb{R}^N), V \in C(\mathbb{R}^N)$ such that

$$\frac{\text{div} b}{p} \le V$$

where $1 \le p < \infty$, is fixed. Consider the operator

$$\mathcal{A} : W^{11}_{\text{loc}}(\mathbb{R}^N) \to \mathcal{D}'(\mathbb{R}^N)$$

given by

$$\mathcal{A}u = \sum_{i,j=1}^N D_i(a_{ij}D_ju) - \sum_{j=1}^N b_jD_ju - Vu\ .$$

Let $A_{\max}$ be the part of $\mathcal{A}$ in $L^p(\mathbb{R}^N)$; i.e.

$$D(A_{\max}) = \{u \in L^p(\mathbb{R}^N) \cap W^{11}_{\text{loc}}(\mathbb{R}^N) : \mathcal{A}u \in L^p(\mathbb{R}^N)\}$$

$A_{\max}u = \mathcal{A}u$. Then there exists a unique operator $A \subset A_{\max}$ on $L^p(\mathbb{R}^N)$ which generates a minimal positive C_0-semigroup T on L^p (see [AMP06]). It follows from the construction that this semigroup is irreducible and that $T(t)$ is a kernel operator for all $t > 0$. Moreover, $\|T(t)\| \le 1$ for all $t \ge 0$. Now assume in addition that

$$\lim_{|x|\to\infty} (V(x) - p^{-1}\text{div} b(x)) = \infty\ . \tag{5.2}$$

Then the operator A has compact reslovent [AMP06].

Now we consider two cases.

First case: $s(A) = 0$. Then by Theorem 4.2 there exist $0 \ll e \in \ker A, 0 \ll \varphi \in \ker A'$ such that

$$\lim_{t\to\infty} T(t)f = \int_\Omega f(x)\varphi(x)\, dx \cdot e$$

in $L^p(\mathbb{R}^N)$ for all $f \in L^p(\mathbb{R}^N)$.

Second case: $s(A) < \infty$. It follows from de Pagter's or from Jentzsch's Theorem [MN91, Thm. 4.3.3., Cor. 4.2.14], [Sch74, Thm. V.6.6. p. 337] that $s(A) > -\infty$.

Since $s(A)$ is a pole, the rescaled semigroup $(e^{-s(A)t}T(t))_{t\geq 0}$ is ergodic with ergodic projection $P = \varphi \otimes e$ for some

$$0 \ll e \in \ker(A - s(A)),\ 0 \ll \varphi \in \ker(A' - s(A)) \ .$$

By Theorem 4.1 the rescaled semigroup converges strongly to P if and only if it is bounded. However, this will not be the case in general. So we use Theorem 4.4 which gives us the following asymptotic behaviour. For all $f \in L^p(\mathbb{R}^N)$ one has

$$\lim_{t\to\infty} \int\limits_{\mathbb{R}^N} |e^{-s(A)t}T(t)f - c(f)e|\varphi(x)\, dx = 0 \tag{5.3}$$

where $c(f) = \int\limits_{\mathbb{R}^N} f(x)\varphi(x)\, dx$.

6. Appendix

The aim of this section is to prove a result which is a version of [Axm80, Satz 3.5] with the same arguments as given by Axmann.

Let (Ω, Σ, μ) be a σ-finite measure space, $1 \leq p < \infty$, $X = L^p(\Omega, \mu)$. We assume that $\dim X \geq 2$.

Theorem 6.1. *Let $0 \leq T \in \mathcal{L}(X)$ be a kernel operator such that $T^n \wedge T = 0$ for all $n \geq 2$. Then T is not irreducible.*

Denote by $k_n : \Omega\times\Omega \to \mathbb{R}_+$ the kernel of T^n. By [Sch74, Prop. IV. 9.8 p. 290], the mapping which associates to each kernel operator its kernel is a lattice homomorphism into the space of all measurable functions on $\Omega\times\Omega$. Thus, the hypothesis $T^n \wedge T = 0 \quad (n \geq 2)$ means that

$$k_n(x, y) \cdot k_1(x, y) = 0 \quad x, y -\ \text{a.e.}$$

if $n \geq 2$, or equivalently (by Fubini's Theorem, cf. [AB94]) that for all $n > 2$ there exists a null set $N \subset \Omega$ such that for $x \in \Omega \setminus N, k_n(x, y) \cdot k_1(x, y) = 0$ for μ-almost all $y \in \Omega$.

Proof of Theorem 6.1. First step: We show that we may assume that $p = 1$ and $\mu(\Omega) < \infty$. In fact, let $\lambda > r(T)$ (the spectral radius of T), and let $\psi \in L^{p'}(\Omega, \mu)$ (the dual space of $L^p(\Omega, \mu)$), $\varphi := R(\lambda, T')\psi$. Then $\varphi \geq 0, \lambda\varphi - T'\varphi = \psi$. Hence $T'\varphi \leq \lambda\varphi$. Consider the space $E = L^1(\Omega, \varphi d\mu)$. Then $L^p(\Omega, \mu) \subset E$ by Hölder's inequality and

$$\begin{aligned} \|Tf\|_E &= \int\limits_{\omega} |Tf|\varphi d\mu \\ &\leq \int\limits_{\Omega} T|f|\varphi d\mu \\ &= \int\limits_{\Omega} |f|T'\varphi d\mu \leq \lambda\|f\|_E \ , \end{aligned}$$

for all $f \in L^p(\Omega,\mu)$. Thus T has a unique continuous extension $\tilde{T} \in \mathcal{L}(E)$. Observe that $\varphi \gg 0$. In fact, if $0 < f \in L^p(\Omega,\mu)$, then $\int_\Omega f\varphi d\mu = \int_\Omega R(\lambda,T)f\psi d\mu > 0$ since $R(\lambda,T)f > 0$ and $\psi \gg 0$. Denoting the kernel of T by k_1 as before, one easily sees that $\tilde{T}$ has the kernel $\tilde{k}$ given by $\tilde{k}(x,y) = \frac{1}{\varphi(y)}k(x,y)$. Moreover, since $T \wedge T^n = 0$ also $(T \wedge T^n)^\sim = \tilde{T} \wedge \tilde{T}^n = 0$ for $n \geq 2$. Finally, $\tilde{T}$ is irreducible if and only if T is so. Thus, replacing T by $\tilde{T}$ we may assume that $p = 1$. Now let $0 \ll \eta \in L^1(\Omega,\mu)$. Then

$$V : L^1(\Omega,\mu) \to L^1(\Omega,\eta\mu)\ ,\ Vf = \frac{f}{\eta}$$

defines a lattice isomorphism. Replacing T by VTV^{-1} and μ by $\eta\mu$ we may assume that the measure μ is finite. Thus in the following we assume throughout that $p = 1$ and $\mu(\Omega) < \infty$.

Second step: There exists $0 < f \in L^1(\Omega,\mu)$ such that

$$f \wedge T^n f = 0 \quad \text{for all} \quad n \in \mathbb{N}\ . \tag{6.1}$$

Since the operator T'^n on $L^\infty(\Omega)$ is given by the kernel k'_n where $k'_n(x,y) = k_n(y,x)$, it follows that $T'^n \wedge T' = 0$ for $n = 2,3,\dots$. Now we identify $L^\infty(\Omega)$ with $C(K)$, where K is a compact space (using Kakutani's Theorem). Then the operator T' corresponds to an operator $0 \leq S$ on $C(K)$ satisfying $S \wedge S^n = 0$ for all $n \geq 2$. Hence for $n \geq 2$,

$$0 = (S \wedge S^n)1_K = \inf_{0\leq h\leq 1_K}(Sh + S^n(1_K - h))\ , \tag{6.2}$$

where the infinum is taken in the space $C(K)$. For $m,n \in \mathbb{N}, n \geq 2$, let $O^n_m := \{t \in K : (Sh)(t) + (S^n(1_K - h))(t) < \frac{1}{m}$ for some $h \in C(K)$ satisfying $0 \leq h \leq 1_K\}$.

Then O^n_m is open, and it follows from (6.2) that O^n_m is dense in K. It follows from Baire's Theorem that $G := \bigcap_{\substack{n,m\in\mathbb{N}\\ n\geq 2}} O^n_m$ is dense in K. Note that for $t \in K, \langle S'\delta_t \wedge S'^n\delta_t, 1_K\rangle = \inf_{0\leq h\leq 1_K}(Sh(t) + (S^n(1_K - h))(t))$. Thus

$$S'\delta_t \wedge S'^n\delta_t = 0 \quad \text{for all} \quad t \in G \tag{6.3}$$

and all $n \geq 2$. For $t \in K, S'\delta_t$ is a positive linear form on $X' = C(K)$. We identify $X = L^1(\Omega,\mu)$ with a band in $X'' = C(K)'$. The assumption that T is a kernel operator implies that there exists $t \in G$ such that $S'\delta_t \in X^\perp$. In fact, we may assume that $S \neq 0$. Then $k_0 = k \wedge 1_{\Omega\times\Omega} > 0$. Consider the kernel operator R_0 on $L^1(\Omega,\mu)$ defined by the kernel k_0. Then $R_0 = T \wedge R_1 > 0$ where $R_1 f = \int_\Omega f d\mu 1_\Omega$ for all $f \in L^1(\Omega,\mu)$. The adjoint S_1 of R_1 considered as an operator on $C(K)$ is given by $S_1 g = \langle g, \tilde{\mu}\rangle 1_K$ where $\tilde{\mu} \in L^\infty(\Omega)'$ is given by $\langle f, \tilde{\mu}\rangle = \int_\Omega f d\mu$ for all $f \in L^\infty(\Omega)$. Since $S_1 \wedge S = (R_1 \wedge T)' > 0$, it follows that

$$\varrho := (S_1 \wedge S)1_K \in C(K)$$

is a positive function different from 0. Thus $U := \{t \in K : \varrho(t) > 0\}$ is open and non-empty. Consequently, $U \cap G \neq \emptyset$. For $0 \leq h \leq 1_K$ one has

$$\begin{aligned}\varrho = (S_1 \wedge S)1_K &= (S_1 \wedge S)h + (S_1 \wedge S)(1_K - h) \\ &\leq S_1 h + S(1_K - h)\ .\end{aligned}$$

Hence for $t \in U, 0 < \varrho(t) \leq \langle h, \tilde{\mu}\rangle + \langle 1_K - h, S'\delta_t\rangle$. Thus for $t \in U, f := \tilde{\mu} \wedge S'\delta_t > 0$. Now observe that $\tilde{\mu} \in X \subset X''$. Since $0 < f \leq \tilde{\mu}$, it follows that $f \in X = L^1(\Omega, \mu)$. If we choose $t \in U \cap G$, then $0 < f \leq S'\delta_t$ and for $n \in \mathbb{N}, f \wedge T^n f \leq S'\delta_t \wedge (S')^{n+1}\delta_t = 0$ by (6.3). Thus (6.1) is proved.

Third step: We prove that T is not irreducible. Since $\dim X \geq 2$, we may assume that $T \neq 0$. Let $u = Tf$. Then $u \geq 0$. If $u = 0$, then $TJ = 0$ where $J = f^{\perp\perp}$. Thus J is an invariant closed ideal, $J \neq 0$. One has $J \neq X$, since $T \neq 0$. Thus J is a non-trivial closed invariant ideal. If $u \neq 0$, let

$$J_0 = \{g \in X :\ \exists\ c \geq 0, m \in \mathbb{N}, |g| \leq c(u + Tu + \cdots + T^m u)\}\ .$$

Then J_0 is an ideal, $J_0 \neq \{0\}, TJ_0 \subset J_0$. Thus, $J := \bar{J}_0$ is a closed ideal and $TJ \subset J$. Since by (6.1) $J \subset f^\perp$, it follows that $J \neq X$.

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