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The Dirichlet problem by variational methods

Wolfgang Arendt and Daniel Daners

Abstract

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $\varphi \in C(\partial\Omega)$. Assume that φ has an extension $\Phi \in C(\bar{\Omega})$ such that $\Delta\Phi \in H^{-1}(\Omega)$. Then by the Riesz representation theorem there exists a unique

$$u \in H_0^1(\Omega) \quad \text{such that} \quad -\Delta u = \Delta\Phi \quad \text{in } \mathcal{D}(\Omega)'.$$

We show that $u + \Phi$ coincides with the Perron solution of the Dirichlet problem

$$\Delta h = 0, \quad h|_{\partial\Omega} = \varphi.$$

This extends recent results by Hildebrandt [Math. Nachr. 278 (2005), 141–144] and Simader [Math. Nachr. 279 (2006), 415–430], and also gives a possible answer to Hadamard's objection against Dirichlet's principle.

1. *The main result and its consequences*

Let Ω be a bounded open set in $\mathbb{R}^N$ with boundary $\partial\Omega$. Let $\varphi \in C(\partial\Omega)$. We consider the Dirichlet problem

$$D(\varphi,\Omega) \qquad\qquad h \in \mathcal{H}(\Omega) \cap C(\bar{\Omega}),\ h|_{\partial\Omega} = \varphi,$$

where $\mathcal{H}(\Omega) := \{u \in C^2(\Omega) \colon \Delta u = 0\}$ denotes the space of all harmonic functions. It follows from the maximum principle that $D(\varphi,\Omega)$ has at most one solution. We say that Ω is *Dirichlet regular* if for each $\varphi \in C(\partial\Omega)$ there exists a solution h of $D(\varphi,\Omega)$. Such a solution will be called a *classical solution* in what follows.

If Ω is not Dirichlet regular, then there always exists a generalised solution of $D(\varphi,\Omega)$ namely the *Perron solution* h_φ (see Section 2 for the definition). Moreover, if $D(\varphi,\Omega)$ has a classical solution h, then $h = h_\varphi$. There is an elaborate theory describing the points $z \in \partial\Omega$ for which $\lim_{x\to z} h_\varphi(x) = \varphi(z)$ for all $\varphi \in C(\partial\Omega)$, those are called the *regular points* (this is equivalent to the existence of a barrier at z, see e.g. Kellogg [**12**, Section XI.17]).

Our aim is to express that $h_\varphi = \varphi$ on $\partial\Omega$ in a weak sense instead by pointwise convergence. We denote by $H^1(\Omega) := \{u \in L^2(\Omega) \colon D_j u \in L^2(\Omega), j = 1,\dots,N\}$ the first Sobolev space and by $H_0^1(\Omega)$ the closure of the test functions $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. Finally, denote by $\mathcal{D}(\Omega)'$ the space of all distributions on Ω. Given $h \in \mathcal{H}(\Omega)$ we say that

$$h = \varphi \quad \text{on} \quad \partial\Omega \quad \text{weakly}$$

if φ has an extension $\Phi \in C(\bar{\Omega})$ such that $u := h - \Phi \in H_0^1(\Omega)$. This implies that $-\Delta u = \Delta\Phi$ in the sense of distributions, that is,

$$\langle \Delta\Phi, v\rangle := \int_\Omega \Phi\Delta v\,dx = \int_\Omega \nabla u\nabla v\,dx \tag{1.1}$$

for all $v \in \mathcal{D}(\Omega)$. As a consequence

$$|\langle \Delta\Phi, v\rangle| \leqslant c\Big(\int_\Omega |\nabla v|^2\,dx\Big)^{1/2} \tag{1.2}$$

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for all $v \in \mathcal{D}(\Omega)$ where $c = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$. By virtue of Poincaré's inequality, $(\int_\Omega |\nabla v|^2\,dx)^{1/2}$ defines an equivalent norm on $H_0^1(\Omega)$. Thus (1.2) means that $\Delta\Phi$ has a continuous extension from $\mathcal{D}(\Omega)$ to $H_0^1(\Omega)$. We keep the notation for the extension $\Delta\Phi \in H^{-1}(\Omega) := H_0^1(\Omega)'$. Our main result is the following.

THEOREM 1.1. *Let $\varphi \in C(\partial\Omega)$ and assume that φ has an extension $\Phi \in C(\bar{\Omega})$ such that $\Delta\Phi \in H^{-1}(\Omega)$. Let $u \in H_0^1(\Omega)$ be the unique solution of Poisson's equation*

$$-\Delta u = \Delta\Phi \quad \text{in } \mathcal{D}(\Omega)'. \tag{1.3}$$

Then $u + \Phi = h_\varphi$ is the Perron solution of the Dirichlet problem.

As seen before (1.3) has a unique solution $u \in H_0^1(\Omega)$. It follows that $\Delta(u+\Phi) = 0$ in the sense of distributions and hence $u + \Phi \in \mathcal{H}(\Omega)$, see [**7**, Chapter II § 3, Proposition 1] or [**13**, Appendix 34, Theorem 14]. Our main point is to prove that $u + \Phi = h_\varphi$ which will be done in Section 2. The Riesz representation theorem also says that $u \in H_0^1(\Omega)$ is the unique minimiser of

$$\frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \langle \Delta\Phi, u\rangle = \min\Big\{\frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \langle \Delta\Phi, v\rangle \colon v \in H_0^1(\Omega)\Big\}$$

(see [**5**, Théorème V.6]). Hence if $\Phi \in H^1(\Omega)$, then h_φ is actually the solution of *Dirichlet's principle*, which can now be formulated as follows.

COROLLARY 1.2. *Assume that φ has an extension $\Phi \in C(\bar{\Omega}) \cap H^1(\Omega)$. Then h_φ is the unique minimiser of*

$$\min\Big\{\int_\Omega |\nabla w|^2\,dx \colon w \in H^1(\Omega),\ w - \Phi \in H_0^1(\Omega)\Big\}. \tag{1.4}$$

Proof. Substitute $v \in H_0^1(\Omega)$ in (1.4) by $w = v + \Phi$. □

Hildebrandt [**10**, Theorem 1] shows that the minimiser h of (1.4) satisfies

$$\lim_{x\to z} h(x) = \varphi(z)$$

for all regular points $z \in \partial\Omega$. Thus, if Ω is Dirichlet regular, it follows that $h = h_\varphi$, which is also proved by Simader [**15**, Theorem 1.6] or [**7**, Proposition II.7.10]. However, even if Ω is Dirichlet regular, not every $\varphi \in C(\partial\Omega)$ has an extension $\Phi \in C(\bar{\Omega}) \cap H^1(\Omega)$. This follows from Hadamard's famous example [**9**] on the unit disc $\mathbb{D}$ of $\mathbb{R}^2$. Let

$$\varphi(e^{i\theta}) = \sum_{n=1}^\infty 2^{-n}\cos(2^{2n}\theta).$$

Then the classical solution of $D(\varphi, \mathbb{D})$ is given by

$$h_\varphi(re^{i\theta}) = \sum_{n=1}^\infty r^{2^{2n}} 2^{-n}\cos(2^{2n}\theta)$$

(see e.g [**6**, page 179–180]), and the energy of h_φ is

$$\int_{\mathbb{D}} |\nabla h_\varphi|^2\,dx = \infty,$$

hence $h_\varphi \notin H^1(\Omega)$. As a consequence of Theorem 1.1, for this φ there exists no extension $\Phi \in C(\bar{\Omega}) \cap H^1(\Omega)$ such that $\Phi|_{\partial\mathbb{D}} = \varphi$. Indeed, then (1.3) would imply that also

$h_\varphi = \Phi + u \in H^1(\Omega)$. We refer to Maz'ya and Shaposhnikova [**14**, § 123] for the interesting history of Hadamard's example.

On the other hand, the condition that φ has an extension $\Phi \in C(\bar\Omega)$ such that $\Delta\Phi \in H^{-1}(\Omega)$ is weaker than $\Phi \in H^1(\Omega)$. Indeed, if $D(\varphi, \Omega)$ has a classical solution $h \in \mathcal{H}(\Omega) \cap C(\bar\Omega)$, then $\Delta\Phi = \Delta h \in H^{-1}(\Omega)$ since $\Delta h = 0$.

Remark 1.3. Let $\Phi \in C(\bar\Omega)$. The following assertions are equivalent.

(i) $\Delta\Phi \in H^{-1}(\Omega)$;

(ii) $\Phi \in H^1_{\mathrm{loc}}(\Omega)$ and there exists $c > 0$ such that

$$\Big|\int_\Omega \nabla\Phi\nabla v\, dx\Big| \leqslant c\Big(\int_\Omega |\nabla v|^2\, dx\Big)^{1/2}$$

for all $v \in \mathcal{D}(\Omega)$.

In fact, if $\Delta\Phi \in H^{-1}(\Omega)$, then $u + \Phi = h_\varphi \in C^\infty(\Omega)$ where $\varphi = \Phi|_{\partial\Omega}$ and $u \in H^1_0(\Omega)$ solves (1.3). Thus $\Phi \in H^1_{\mathrm{loc}}(\Omega)$. Now (1.2) implies the estimate. Conversely, (ii) implies (i) since $(\int |\nabla v|^2\, dx)^{1/2}$ is an equivalent norm on $H^1_0(\Omega)$. However, as we saw above (ii) does not imply that $\Phi \in H^1(\Omega)$.

We note two further consequences for Poisson's equation. Let

$$C_0(\Omega) := \{u \in C(\bar\Omega) \colon u|_{\partial\Omega} = 0\}.$$

Corollary 1.4. *Let $v \in C_0(\Omega)$ such that $\Delta v \in H^{-1}(\Omega)$. Then $v \in H^1_0(\Omega)$.*

Proof. Let $\Phi := v$ so that $\varphi = \Phi|_{\partial\Omega} = 0$. Let $u \in H^1_0(\Omega)$ be the solution of (1.3). Then by Theorem 1.1 we get $u + v = h_\varphi = 0$. □

The following result extends [**2**, Lemma 2.2].

Corollary 1.5. *Assume that Ω is Dirichlet regular. Let $f \in L^p(\Omega)$ with $N/2 < p \leqslant \infty$ if $N \geqslant 2$, and let f be a bounded Borel measure on Ω if $N = 1$. Then there exists a unique solution $u \in C_0(\Omega)$ of the Poisson equation*

$$-\Delta u = f \quad \text{in} \quad \mathcal{D}(\Omega)'.$$

Proof. Since $H^1_0(\Omega) \subset L^q(\Omega)$ whenever $q < \frac{N}{N-2}$ if $N \geqslant 2$ and $H^1_0(\Omega) \subset L^\infty(\Omega)$ in the case $N = 1$ we have $L^p(\Omega) \subset H^{-1}(\Omega)$ if $p > N/2$ and $M(\Omega) \subset H^{-1}(\Omega)$ if $N = 1$, where $M(\Omega)$ denotes the space of all bounded signed Borel measures on Ω. It follows from the Riesz representation theorem that there exists a unique $u \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u\nabla v\, dx = \int_\Omega fv\, dx$$

for all $v \in H^1_0(\Omega)$, that is,

$$-\Delta u = f \quad \text{in} \quad \mathcal{D}(\Omega)'.$$

Let $\Phi(x) = \int_\Omega f(y)E(x-y)\, dy$ if $N \geqslant 2$ and $\Phi(x) = \int_\Omega E(x-y) df(y)$ if $N = 1$, where E is the Newtonian potential. Then $\Phi \in C(\bar\Omega)$. This follows from the fact that $E \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^N)$ if $N \geqslant 2$ and $E \in C(\mathbb{R})$ if $N = 1$. Moreover $\Delta\Phi = f$ in $\mathcal{D}(\Omega)'$. Let $\varphi = \Phi|_{\partial\Omega}$. It follows from Theorem 1.1 that $h_\varphi = \Phi + u$. Since Ω is Dirichlet regular, $h_\varphi \in C(\bar\Omega)$ and $h_\varphi|_\partial\Omega = \varphi$. Thus $u \in C_0(\Omega)$. In order to prove uniqueness, let $u \in C_0(\Omega)$ such that $-\Delta u = f$ in $\mathcal{D}(\Omega)'$. Then $h = u + \Phi \in C(\bar{\Omega})$ is a classical solution of $D(\varphi, \Omega)$. So uniqueness follows from the uniqueness of the classical solution of the Dirichlet problem $D(\varphi, \Omega)$. □

We conclude this section commenting on *weak solutions* of the Dirichlet problem.

REMARK 1.6. Let $\varphi \in C(\partial\Omega)$. We call $h \in \mathcal{H}(\Omega)$ a weak solution of $D(\varphi, \Omega)$ if φ has an extension $\Phi \in C(\bar{\Omega})$ such that $\Delta\Phi \in H^{-1}(\Omega)$ and $h - \Phi \in H_0^1(\Omega)$.

(a) It is not obvious that weak solutions are unique. Theorem 1.1 gives a positive answer: since $h = h_\varphi$ and since the Perron solution h_φ is unique there is at most one solution.

(b) If Ω is Dirichlet regular, then $\varphi \in C(\partial\Omega)$ has an extension $\Phi \in C(\bar{\Omega})$ with $\Delta\Phi \in H^{-1}(\Omega)$, namely the Perron solution h_φ. We do not know whether this is true in general. Here is a class of examples where it is true.

(c) Let $G \subset \mathbb{R}^N$ be a bounded open set which is Dirichlet regular and assume that $N \geqslant 2$. Let $F \subset G$ be a finite non-empty set and $\Omega = G \setminus F$. Then Ω is not Dirichlet regular. Let $\varphi \in C(\partial\Omega)$. Let $h \in \mathcal{H}(G) \cap C(\bar{G})$ such that $h(z) = \varphi(z)$ for all $z \in \partial G$. Let $\psi \in C^1(\mathbb{R}^N)$ such that $\psi = 0$ on $\mathbb{R}^N \setminus G$ and $\psi(z) = \varphi(z) - h(z)$ for all $z \in F$. Then $\Delta\Phi = \Delta(h + \psi) \in H^{-1}(\Omega)$ and $\Phi(z) = \varphi(z)$ for all $z \in \partial\Omega = \partial G \cup F$.

2. *Proof of Theorem* 1.1

We start this section by giving a definition of the Perron solution. A function $u \in C(\Omega)$ is called *subharmonic* if $\Delta u \geqslant 0$ in $\mathcal{D}(\Omega)'$ (that is, if $\int_\Omega u\Delta v\,dx \geqslant 0 \quad$ for all $\quad 0 \leqslant v \in \mathcal{D}(\Omega)$) and u is called *superharmonic* if $-u$ is subharmonic. We write

$$u \leqslant \varphi \text{ on } \partial\Omega \text{ if } \limsup_{x\to z} u(x) \leqslant \varphi(z) \text{ for all } z \in \partial\Omega$$

and

$$u \geqslant \varphi \text{ on } \partial\Omega \text{ if } \liminf_{x\to z} u(x) \geqslant \varphi(z) \text{ for all } z \in \partial\Omega.$$

Then by Perron's method

$$\begin{aligned} h_\varphi(x) &:= \sup\{u(x) \colon u \in C(\Omega) \text{ is subharmonic and } u \leqslant \varphi \text{ on } \partial\Omega\} \\ &= \inf\{u(x) \colon u \in C(\Omega) \text{ is superharmonic and } u \geqslant \varphi \text{ on } \partial\Omega\} \end{aligned}$$

exists for all $x \in \Omega$ and defines a bounded harmonic function h_φ. The mapping $\varphi \mapsto h_\varphi$ from $C(\partial\Omega)$ into $\mathcal{H}(\Omega) \cap L^\infty(\Omega)$ is linear, positive (that is, $\varphi \geqslant 0$ implies $h_\varphi \geqslant 0$) and contractive (that is, $\sup_{x\in\Omega} |h_\varphi(x)| \leqslant \sup_{z\in\partial\Omega} |\varphi(z)|$). We refer to [**7**, Chapter II § 4] for proofs of these classical results.

Let Ω be a bounded open set and let $\varphi \in C(\partial\Omega)$. For the proof we use the following alternative description of the Perron solution h_φ. Let $\Omega_n \subset \Omega$ be open, Dirichlet regular such that $\Omega_n \subset \Omega_{n+1}$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. Such Ω_n can always be constructed even of class C^∞ (see [**7**, Chapter II § 4 Lemma 1]). Extend φ to a function $\Phi \in C(\bar{\Omega})$. Let $h_n \in C(\bar{\Omega}_n) \cap \mathcal{H}(\Omega_n)$ such that $h_n = \Phi$ on $\partial\Omega_n$. Then

$$h_\varphi(x) = \lim_{n\to\infty} h_n(x) \tag{2.1}$$

uniformly on compact subsets of Ω. We refer to [**11**, Theorem II] or [**3**, Theorem 3.4] for this result. Now we assume in addition that

$$\Omega_n \subset \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > 1/n\}.$$

This can always be arranged by re-indexing the Ω_n. Let ϱ_n be a mollifier, that is, $0 \leqslant \varrho_n \in \mathcal{D}(\mathbb{R}^N)$, $\operatorname{supp} \varrho_n \subset B(0, 1/n)$ and $\int \varrho_n\,dx = 1$. We also assume that $\varrho_n(x) = \varrho_n(-x)$ for all $x \in \mathbb{R}^N$. Extend Φ to a uniformly continuous function on $\mathbb{R}^N$, which we still denote by Φ, and let $\Phi_n := \varrho_n * \Phi$. Then $\Phi_n \to \Phi$ uniformly on $\mathbb{R}^N$. Let $k_n \in \mathcal{H}(\Omega_n) \cap C(\bar{\Omega}_n)$ such that $k_n = \Phi_n$

on $\partial\Omega_n$, that is, k_n is the solution of $D(\Omega_n, \Phi_n|_{\partial\Omega_n})$. We show that also

$$k_n(x) \to h_\varphi(x) \tag{2.2}$$

uniformly on compact subsets of Ω. In fact, let $K \subset \Omega$ be compact. There exists n_0 such that $K \subset \Omega_n$ for all $n \geqslant n_0$. By the maximum principle we have

$$\|k_n - h_n\|_{C(K)} \leqslant \|k_n - h_n\|_{C(\bar{\Omega}_n)} \leqslant \|k_n - h_n\|_{C(\partial\Omega_n)} = \|\Phi_n - \Phi\|_{C(\partial\Omega_n)} \to 0$$

as $n \to \infty$. Now (2.2) follows from (2.1).

Consider the function

$$u_n = k_n - \Phi_n \in C_0(\Omega_n).$$

Then $-\Delta u_n = \Delta\Phi_n$ in $\mathcal{D}(\Omega_n)'$. It follows from [**2**, Lemma 2.2], (see also [**3**]) that $u_n \in H_0^1(\Omega_n)$. Now we assume in addition that $\Delta\Phi \in H^{-1}(\Omega)$, that is, there exists a constant $c > 0$ such that

$$\Big|\int_\Omega \Phi\Delta v\,dx\Big| \leqslant c\Big(\int_\Omega |\nabla v|^2\Big)^{1/2} \tag{2.3}$$

for all $v \in \mathcal{D}(\Omega)$. This will allow us to prove that

$$\Big(\int_\Omega |\nabla u_n|^2\,dx\Big)^{1/2} \leqslant c \tag{2.4}$$

for all $n \in \mathbb{N}$. In order to prove (2.4) fix $n \in \mathbb{N}$. Let $v \in \mathcal{D}(\Omega_n)$. Then

$$\int_{\Omega_n} \nabla u_n \nabla v\,dx = -\int_{\Omega_n} u_n\Delta v\,dx = \int_{\Omega_n} (\Phi_n - k_n)\Delta v\,dx = \int_{\Omega_n} \Phi_n\Delta v\,dx$$
$$= \int_{\mathbb{R}^N} (\varrho_n * \Phi)\Delta v\,dx = \int_{\mathbb{R}^N} \Phi(\varrho_n * \Delta v)\,dx = \int_{\mathbb{R}^N} \Phi\Delta v_n\,dx$$

where $v_n = \varrho_n * v \in C^\infty(\mathbb{R}^N)$ with $\operatorname{supp} v_n \subset B(0, 1/n) + \operatorname{supp} v \subset \Omega$. Thus $v_n \in \mathcal{D}(\Omega)$ and it follows from (2.3) that

$$\Big|\int_{\mathbb{R}^N} \nabla u_n \nabla v\,dx\Big| \leqslant c\Big(\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx\Big)^{1/2} \leqslant c\Big(\int_{\mathbb{R}^N} |\nabla v|^2\,dx\Big)^{1/2}$$

for all $v \in \mathcal{D}(\Omega_n)$ since $D_j v_n = \varrho_n * D_j v$ and $\|\varrho_n\|_{L^1} = 1$. As $\mathcal{D}(\Omega_n)$ is dense in $H_0^1(\Omega_n)$ and $\int_\Omega \nabla w_1 \nabla w_2\,dx$ defines an equivalent scalar product on $H_0^1(\Omega)$, the claim (2.4) follows. Now we define $\tilde{u}_n(x) = u_n(x)$ if $x \in \Omega_n$ and $\tilde{u}_n(x) = 0$ if $x \notin \Omega_n$. Then $\tilde{u}_n \in H_0^1(\Omega)$ and $\nabla\tilde{u}_n = \widetilde{\nabla u_n}$ (see e.g. [**5**, Proposition IX.18]). We identify $\tilde{u}_n$ and u_n to simplify the notation. By (2.4) the sequence (u_n) is bounded in $H_0^1(\Omega)$. Hence there exists a subsequence (u_{n_m}) converging weakly to a function $u \in H_0^1(\Omega)$ as $m \to \infty$. Since

$$u_{n_m} + \Phi_{n_m} = k_{n_m} \tag{2.5}$$

we have

$$-\int_{\Omega_{n_m}} \nabla u_{n_m} \nabla v + \int_{\Omega_{n_m}} \Phi_{n_m}\Delta v = \int_{\Omega_{n_m}} k_{n_m}\Delta v = 0$$

for all $v \in \mathcal{D}(\Omega_{n_m})$. Letting $m \to \infty$ we conclude that

$$-\int_\Omega \nabla u \nabla v + \int_\Omega \Phi\Delta v = 0$$

for all $v \in \mathcal{D}(\Omega) = \bigcup_{m\in\mathbb{N}} \mathcal{D}(\Omega_{n_m})$. Thus u is the solution of

$$u \in H_0^1(\Omega), \quad -\Delta u = \Delta\Phi \quad \text{in } \mathcal{D}(\Omega)'.$$

On the other hand, it follows from (2.5) and (2.2) that

$$u + \Phi = h_\varphi.$$

This completes the proof of Theorem 1.1.

3. *Further comments*

1. Our proof of Theorem 1.1 is based on exhausting Ω by Dirichlet regular sets. It allows us actually to identify the weak solution with Perron's solution. Hildebrandt [**10**] and Simader [**15**], in the case where φ has an extension $\Phi \in C(\bar{\Omega}) \cap H^1(\Omega)$, use barriers to show that the weak solution has the same regularity properties as the Perron solution. Simader's proof [**15**, Theorem 1.6] depends on the notion of H^1-barriers which are introduced in [**2**, Definition 3.1] (cf. [**15**, Definition 3.1]). By [**2**, Lemma 3.1] Ω is Dirichlet regular if and only if at every point $z \in \partial\Omega$ an H^1-barrier exists (cf. [**15**, Theorem 1.7]).

2. A further consequence of Corollary 1.4 is that

$$H_0^1(\Omega) \cap C(\bar{\Omega}) \subset C_0(\Omega) \tag{3.1}$$

whenever Ω is Dirichlet regular [**15**, Corollary 5.3]. We mention that Biegert and Warma [**4**] actually showed that (3.1) holds if and only if $\operatorname{cap}(B(z,r) \setminus \Omega) > 0$ for each $z \in \partial\Omega, r > 0$, that is, if Ω is *regular in capacity* [**1**, Definition 3.12]. By Wiener's criterion [**8**, (2.37)] regularity in capacity is weaker than Dirichlet regularity.

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Wolfgang Arendt
Institute of Applied Analysis
University of Ulm
D–89069 Ulm
Germany

wolfgang.arendt@uni-ulm.de

Daniel Daners
School of Mathematics and Statistics
The University of Sydney
NSW 2006
Australia

D.Daners@maths.usyd.edu.au