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VARYING DOMAINS: STABILITY OF THE DIRICHLET AND THE POISSON PROBLEM

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Dedicated to Professor Edward Norman Dancer
on the occasion of his 60th birthday

Abstract. For Ω a bounded open set in $\mathbb{R}^N$ we consider the space $H_0^1(\bar{\Omega}) = \{u_{|\Omega}\colon u \in H^1(\mathbb{R}^N)\colon u(x) = 0 \text{ a.e. outside } \bar{\Omega}\}$. The set Ω is called *stable* if $H_0^1(\Omega) = H_0^1(\bar{\Omega})$. Stability of Ω can be characterised by the convergence of the solutions of the Poisson equation

$$-\Delta u_n = f \quad \text{in } \mathcal{D}(\Omega_n)', \qquad u_n \in H_0^1(\Omega_n)$$

and also the Dirichlet Problem with respect to Ω_n if Ω_n converges to Ω in a sense to be made precise. We give diverse results in this direction, all with purely analytical tools not referring to abstract potential theory as in Hedberg's survey article [Expo. Math. **11** (1993), 193–259]. The most complete picture is obtained when Ω is supposed to be Dirichlet regular. However, stability does not imply Dirichlet regularity as Lebesgue's cusp shows.

1. Introduction.

There are two natural ways to define Dirichlet boundary conditions on a bounded open set $\Omega \subset \mathbb{R}^N$ for functions in the Sobolev space $H^1(\Omega) := \{u \in L^2(\Omega)\colon D_j u \in L^2(\Omega), j = 1, \dots N\}$. The most common one consists in considering the space $H_0^1(\Omega)$ defined as the closure of the test functions $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. The other natural space is

$$H_0^1(\bar{\Omega}) := \{u_{|\Omega}\colon u \in H^1(\mathbb{R}^N),\, u(x) = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega\}.$$

The last one always contains $H_0^1(\Omega)$ but the inclusion is strict in general.

The purpose of this article is to characterise when both spaces coincide. We then say that Ω is *stable*. We may associate two realizations Δ_Ω and $\Delta_{\bar{\Omega}}$ of the Laplacian on $L^2(\Omega)$ associated with the form domains $H_0^1(\Omega)$ and $H_0^1(\bar{\Omega})$ corresponding to a "minimal" and a "maximal" realisation of the Laplacian with Dirichlet boundary conditions. Both can be obtained by approximation, the first from the inside, the second from the outside. If $\Omega_n \uparrow \Omega$, that is, if $\Omega_n \subset \Omega_{n+1}$ are open and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$, then

$$e^{t\Delta_{\Omega_n}} \to e^{t\Delta_\Omega}$$

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strongly in $L^2(\Omega)$. On the other hand, if $\Omega_n \downarrow \Omega$, that is, if $\Omega_{n+1} \subset \Omega_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} \Omega_n = \bar{\Omega}$, then

$$e^{t\Delta_{\Omega_n}} \to e^{t\Delta_{\bar{\Omega}}}$$

as $n \to \infty$. Thus stability of Ω is equivalent to the fact that $e^{t\Delta_{\Omega_n}}$ converges to $e^{t\Delta_{\Omega}}$ if $\Omega_n \downarrow \Omega$. The convergence of the heat semigroups is equivalent to the convergence of the solutions of the Poisson equation

$$-\Delta u = f$$

where $u \in H_0^1(\Omega)$ or $u \in H_0^1(\bar{\Omega})$. Much more interesting is the relation to convergence of the solutions of the Dirichlet problem

$$D(\varphi, \Omega) \qquad h \in \mathcal{H}(\Omega) \cap C(\bar{\Omega}), \quad h = \varphi \quad \text{on} \quad \partial\Omega$$

where $\mathcal{H}(\Omega)$ denotes the space of all harmonic functions. As usual we call Ω *Dirichlet regular* if the above problem has a unique solution for every $\varphi \in C(\partial\Omega)$. We will show by rather elementary means that for a Dirichlet regular set Ω, stability is equivalent to the fact that

$$h_n \to h \quad \text{uniformly on} \quad \bar{\Omega}$$

whenever $\Omega_n \downarrow \Omega, \Phi \in C(\bar{\Omega}_1), h_n$ is the solution of $D(\Phi_{|\partial\Omega_n}, \Omega_n)$ and h the solution of $D(\Phi_{|\partial\Omega}, \Omega)$. Indeed we will study the close connections of Poisson's equation and the Dirichlet problem. For questions of uniform convergence, harmonic functions are easier to treat because of the mean-value theorem. Indeed, we exploit and generalise a clever argument by Keldyš (see [22, Lemma 4.1]) and prove a very general convergence result (Theorem 5.2). Our approach is based on elementary arguments in analysis. There is a different approach by abstract potential theory which is exposed in the survey article of Hedberg [21] and also subject of the books by Landkof [23, Chapter V.5] and Adams and Hedberg [1, Chapter 11]. We obtain several of the results of Hedberg [21] concerning stability by elementary arguments. Such a direct approach might be welcome by analysts who encounter the stability problem treating linear and non-linear equations such as, for instance, Dancer [13, 14, 15, 16], or [17], [11, 10], [12], [7, 8]. Our paper in particular complements results contained in Dancer's paper [16].

The last section is devoted to a detailed analysis of Lebesgue's cusp which is an example of a stable open set which is not Dirichlet regular. In this example, the Dirichlet problem is not solvable for a function $\varphi \in C(\partial\Omega)$ which is the restriction of an extremely regular function defined on $\mathbb{R}^N$ (C^∞ and even locally harmonic).

2. **The Poisson problem with Dirichlet and pseudo Dirichlet boundary conditions.** The purpose of this section is to discuss the two realisations of the Laplace operator with Dirichlet boundary conditions, a minimal and a maximal realisation. Let Ω be a bounded open set in $\mathbb{R}^N$. For $u \in L^2(\Omega)$ we denote by $\tilde{u}$ the extension of u by 0. If $u \in H_0^1(\Omega)$, then $\tilde{u} \in H^1(\mathbb{R}^N)$ and $D_j\tilde{u} = \widetilde{D_j u}$ (see e.g. [9, Proposition IX.18]). We let

$$H_0^1(\bar{\Omega}) := \{u_{|\Omega} \colon u \in H^1(\mathbb{R}^N),\, u = 0 \text{ a.e. on } \bar{\Omega}^c\}.$$

Thus $H_0^1(\Omega) \subset H_0^1(\bar{\Omega})$. Note that for $u \in H^1(\Omega), u^+ \in H^1(\Omega)$ and $D_j u^+ = 1_{\{u>0\}} D_j u$ (see [20, §7.4]). Moreover, the mapping $u \mapsto u^+$ is continuous. It follows that $u \in H_0^1(\Omega)$ implies $u^+ \in H_0^1(\Omega)$ and $u \in H_0^1(\bar{\Omega})$ implies $u^+ \in H_0^1(\bar{\Omega})$. Moreover, if $u \in H_0^1(\Omega), v \in H_0^1(\bar{\Omega})$ and $0 \le v \le u$, then $v \in H_0^1(\Omega)$. This means that $H_0^1(\Omega)$ is a *closed ideal* in $H_0^1(\bar\Omega)$. We shall use this property without further comment in the sequel.

We also identify $H_0^1(\bar\Omega)$ with a subspace of $H^1(\mathbb{R}^N)$ by identifying u and $\tilde u$. If $u \in H^1(\Omega)$ vanishes outside a compact subset K of Ω, then $u \in H_0^1(\Omega)$ (see [9, Lemma IX.5]). Thus, if Ω_1, Ω_2 are open sets such that $\bar\Omega_1 \subset \Omega_2$, and if Ω_1 is bounded then

$$H_0^1(\bar\Omega_1) \subset H_0^1(\Omega_2). \tag{2.1}$$

We consider the bilinear form

$$a\colon H_0^1(\bar\Omega) \times H_0^1(\bar\Omega) \to \mathbb{R}$$

given by

$$a(u,v) = \int_\Omega \nabla u \nabla v \, dx.$$

Then a is continuous and coercive, that is,

$$a(u,u) \geq \alpha \|u\|^2_{H_0^1(\bar\Omega)}$$

for all $u \in H_0^1(\bar\Omega)$. In fact, let B be a ball such that $\bar\Omega \subset B$. Then $u \in H_0^1(B)$ by (2.1) and the claim follows from Poincaré's inequality. We denote by $-\Delta_{\bar\Omega}$ the operator associated with a and call $\Delta_{\bar\Omega}$ the *pseudo-Dirichlet Laplacian*. The operator associated with the restriction a_0 of a to $H_0^1(\Omega) \times H_0^1(\Omega)$ is denoted by $-\Delta_\Omega$. We call Δ_Ω the *Dirichlet Laplacian*. The operators Δ_Ω and $\Delta_{\bar\Omega}$ are both self-adjoint and invertible. Thus for $f, u, v \in L^2(\Omega)$ we have

$$u \in D(\Delta_{\bar\Omega}), \quad -\Delta_{\bar\Omega} u = f \tag{2.2}$$

if and only if $u \in H_0^1(\bar\Omega)$ and $\int_\Omega \nabla u \nabla w \, dx = \int_\Omega f w \, dx$ for all $w \in H_0^1(\bar\Omega)$. On the other hand,

$$u \in D(\Delta_\Omega), \quad -\Delta_\Omega u = f \tag{2.3}$$

if and only if $u \in H_0^1(\Omega)$ and $\int_\Omega \nabla u \nabla w \, dx = \int_\Omega f w \, dx$ for all $w \in H_0^1(\Omega)$. Since the space of all test functions $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, we have

$$\begin{aligned} D(\Delta_\Omega) &:= \{u \in H_0^1(\Omega)\colon \Delta u \in L^2(\Omega)\}, \\ \Delta_\Omega u &:= \Delta u, \end{aligned} \tag{2.4}$$

where $\Delta u \in \mathcal{D}(\Omega)'$ is understood in the sense of distributions. Note that $\Delta_\Omega = \Delta_{\bar\Omega}$ if and only if $H_0^1(\Omega) = H_0^1(\bar\Omega)$. More precisely, if $H_0^1(\Omega) \neq H_0^1(\bar\Omega)$ then $D(\Delta_\Omega) \not\subset D(\Delta_{\bar\Omega})$ and $D(\Delta_{\bar\Omega}) \not\subset D(\Delta_\Omega)$. This means in particular, if $H_0^1(\Omega) \neq H_0^1(\bar\Omega)$, then there exists $u \in H_0^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$ but there exists $w \in H_0^1(\bar\Omega)$ such that

$$\int_\Omega \nabla u \nabla w \, dx \neq -\int_\Omega w \Delta u \, dx.$$

Example 2.1. Let $N = 1$, $\Omega = (-1,0) \cup (0,1)$. Observe that $H^1(\mathbb{R}) \subset C^b(\mathbb{R})$ (the bounded continuous function on $\mathbb{R}$). Then $H_0^1(\Omega) = \{u \in H^1(-1,1)\colon u(0) = u(1) = u(-1) = 0\}$ and

$$H_0^1(\bar\Omega) = \{u \in H^1(-1,1)\colon u(-1) = u(1) = 0\} = H_0^1(-1,1).$$

If Ω is reasonably smooth, then

$$H_0^1(\Omega) = H_0^1(\bar\Omega). \tag{2.5}$$

A proof of the following result can be found in [25, pages 24–26].

Proposition 2.2. *If Ω is bounded and has continuous boundary, then (2.5) is true.*

The purpose of this article is to characterise when (2.5) holds in terms of the Dirichlet and the Poisson problems. This section is devoted to the Poisson equation. For $\lambda \geq 0$ we denote by $R(\lambda, \Delta_\Omega)$ and by $R(\lambda, \Delta_{\bar\Omega})$ the resolvents of Δ_Ω and $\Delta_{\bar\Omega}$ at λ. Since we will mainly be interested in $\lambda = 0$ we let shortly

$$R_\Omega := R(0, \Delta_\Omega) \quad \text{and} \quad R_{\bar\Omega} := R(0, \Delta_{\bar\Omega}).$$

Thus $R_\Omega = (-\Delta_\Omega)^{-1}$ and $R_{\bar\Omega} = (-\Delta_{\bar\Omega})^{-1}$. For $f, u, v \in L^2(\Omega)$ we have $R_\Omega f = u$ if and only if $u \in H^1_0(\Omega)$ and $-\Delta u = f$ in $\mathcal{D}(\Omega)'$ if and only if $u \in H^1_0(\Omega)$ and $\int_\Omega \nabla u \nabla w \, dx = \int_\Omega f w \, dx$ for all $w \in H^1_0(\Omega)$. Similarly we have $R_{\bar\Omega} f = v$ if and only if $v \in H^1_0(\bar\Omega)$ and $\int_\Omega \nabla v \nabla w \, dx = \int_\Omega f w \, dx$ for all $w \in H^1_0(\bar\Omega)$. Thus $R_\Omega f$ and $R_{\bar\Omega} f$ are the solutions of the Poisson problem with Dirichlet and pseudo-Dirichlet boundary conditions.

Next we establish some monotonicity properties.

Proposition 2.3. *Let Ω_1, Ω_2 be bounded, open sets, such that $\bar\Omega_1 \subset \Omega_2$, and let $\lambda \geq 0$. Then*

$$0 \leq R(\lambda, \Delta_{\Omega_1}) \leq R(\lambda, \Delta_{\bar\Omega_1}) \leq R(\lambda, \Delta_{\Omega_2}).$$

Proof. Let $f \in L^2(\mathbb{R}^n)$, $u_1 := R_{\Omega_1}(\lambda) f$, $\bar u_1 := R_{\bar\Omega_1}(\lambda) f$ and $u_2 := R_{\Omega_2}(\lambda) f$. Then

$$\lambda \int u_j v \, dx + \int \nabla u_j \nabla v \, dx = \int f v \, dx$$

for all $v \in H^1_0(\Omega_j)$, $i = 1, 2$, and

$$\lambda \int \bar u_1 v \, dx + \int \nabla \bar u_1 \nabla v \, dx = \int f v \, dx.$$

for all $v \in H^1_0(\bar\Omega_1)$. First assume that $f \leq 0$ and set $v = u_1^+$. Then

$$\lambda \int |u_1^+|^2 \, dx + \int |\nabla u_1^+|^2 \, dx = \lambda \int u_1 v \, dx + \int \nabla u_1 \nabla v \, dx = \int f v \, dx \leq 0.$$

Hence $u_1^+ = 0$ and so $u_1 \leq 0$. This shows that $R_{\Omega_1}(\lambda) \geq 0$. Since also $\bar u_1^+ \in H^1_0(\bar\Omega_1)$, it follows similarly that $R_{\bar\Omega_1}(\lambda) \geq 0$.

Next let $f \geq 0$. Then $u_1, \bar u_1, u_2 \geq 0$ by what we have just shown, and $\lambda \int (u_1 - \bar u_1) v + \int \nabla(u_1 - \bar u_1) \nabla v = 0$ for all $v \in H^1_0(\Omega_1)$. Consider $v = (u_1 - \bar u_1)^+$. Then $v \in H^1_0(\bar\Omega_1)$ and $v \leq u_1$ since $\bar u_1 \geq 0$. Hence $v \in H^1_0(\Omega_1)$ and

$$\lambda \int v^2 + \int |\nabla v|^2 \, dx = \lambda \int (u_1 - \bar u_1) v \, dx + \int \nabla(u_1 - \bar u_1) \nabla v \, dx = 0.$$

Consequently, $v = (u_1 - \bar u_1)^+ = 0$ and so $u_1 \leq \bar u_1$. Moreover, from the first lines of the proof,

$$\lambda \int (\bar u_1 - u_2) v \, dx + \int \nabla(\bar u_1 - u_2) \nabla v \, dx = 0$$

for all $v \in H^1_0(\bar\Omega_1) \subset H^1_0(\Omega_2)$. To prove the last inequality let now $v = (\bar u_1 - u_2)^+ \in H^1_0(\Omega_2)$. Since $u_2 \geq 0$, we have $0 \leq v \leq \bar u_1$ and consequently, $v \in H^1_0(\bar\Omega_1)$. It follows that

$$\lambda \int v^2 \, dx + \int |\nabla v|^2 \, dx = \lambda \int (\bar u_1 - u_2) v \, dx + \int \nabla(\bar u_1 - u_2) \nabla v \, dx = 0.$$

Thus $(\bar u_1 - u_2)^+ = 0$, that is, $\bar u_1 \leq u_2$. □

Since Ω is bounded, the injections $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and $H_0^1(\bar{\Omega}) \hookrightarrow L^2(\Omega)$ are compact. Thus Δ_Ω and $\Delta_{\bar{\Omega}}$ are self-adjoint and have compact resolvent. We denote by $\lambda(\Omega)$ and $\lambda(\bar{\Omega})$ the first eigenvalues of $-\Delta_\Omega$ and of $-\Delta_{\bar{\Omega}}$, respectively. Then

$$0 < \lambda(\bar{\Omega}) = \inf_{u \in H_0^1(\bar{\Omega})} \frac{\|\nabla u\|_{L_2(\Omega)}^2}{\|u\|_{L_2(\Omega)}^2} \leq \inf_{u \in H_0^1(\Omega)} \frac{\|\nabla u\|_{L_2(\Omega)}^2}{\|u\|_{L_2(\Omega)}^2} = \lambda(\Omega). \tag{2.6}$$

We shall see next that there is equality if and only if (2.5) holds and also relate it to properties of the corresponding first eigenfunctions.

Proposition 2.4. *Let Ω be an open, connected bounded set in $\mathbb{R}^N$. The following statements are equivalent.*

(i) $H_0^1(\Omega) = H_0^1(\bar{\Omega})$;
(ii) $\lambda(\Omega) = \lambda(\bar{\Omega})$;
(iii) *The first eigenfunction u of $\Delta_{\bar{\Omega}}$ lies in $H_0^1(\Omega)$;*
(iv) *There exists $f > 0$ such that $R_{\bar{\Omega}} f \in H_0^1(\Omega)$.*

Proof. (ii) $\Rightarrow$ (i). The semigroups $(e^{t\Delta_\Omega})_{t\geq 0}$ and $(e^{t\Delta_{\bar{\Omega}}})_{t\geq 0}$ are irreducible (see [2] or [24, § 4.2] for two different proofs). Assume that $\lambda_\Omega = \lambda_{\bar{\Omega}}$. If we let $A = \Delta_{\bar{\Omega}} - \lambda_{\bar{\Omega}}$ and $B = \Delta_\Omega - \lambda_\Omega$ in [3, Theorem 1.3], then (i) follows. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (iv). Let u be the first eigenfunction of $\Delta_{\bar{\Omega}}$; that is, $u \in D(\Delta_{\bar{\Omega}})$ and $-\Delta_{\bar{\Omega}} u = \lambda(\bar{\Omega}) u$. Then u is strictly positive and $R_{\bar{\Omega}} u = \lambda_{\bar{\Omega}}^{-1} u$.

(iv) $\Rightarrow$ (i) Let $f > 0$ such that $u = R_{\bar{\Omega}} f \in H_0^1(\Omega)$. Then $u(x) > 0$ almost everywhere by irreducibility. By the resolvent identity

$$R_{\bar{\Omega}}(1) u = R_{\bar{\Omega}}(0) f - R_{\bar{\Omega}}(1) f \leq R_{\bar{\Omega}}(0) f = u,$$

so if $g \in L^2(\Omega)$ and $|g| \leq cu$, then

$$|R_{\bar{\Omega}}(1) g| \leq c R_{\bar{\Omega}}(1) u \leq cu.$$

Hence $R_{\bar{\Omega}}(1) g \in H_0^1(\Omega)$ by the ideal property. The set of all $g \in L^2(\Omega)$ such that $|g| \leq cu$ for some $c > 0$ is dense in $L^2(\Omega)$. It follows that $D(\Delta_{\bar{\Omega}}) = R_{\bar{\Omega}}(1) L^2(\Omega) \subset H_0^1(\Omega)$. ◻

We will need the following elementary regularity properties of the Laplacian (see [19, Chapter II § 3, Proposition 6]).

Proposition 2.5. *Let $u \in \mathcal{D}(\Omega)'$ where $\Omega \subset \mathbb{R}^N$ is open.*

(a) *If $\Delta u \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > N/2$, then $u \in C(\Omega)$.*
(b) *If $\Delta u \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > N$, then $u \in C^1(\Omega)$.*

From this proposition we obtain the following uniform local estimate. We write $\omega \subset\subset \Omega$ if ω is open, bounded and $\bar{\omega} \subset \Omega$.

Proposition 2.6. *Let Ω be open and $\omega \subset\subset \Omega$. Let $p > N$. Then there exists a constant $c > 0$ such that*

$$\|u\|_{C^1(\bar{\omega})} \leq c\big(\|\Delta u\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}\big)$$

for all $u \in L^2(\Omega)$ such that $\Delta u \in L^p(\Omega)$. Here $\|u\|_{C^1(\omega)} = \|u\|_{L^\infty(\omega)} + \|\nabla u\|_{L^\infty(\omega)}$.

Proof. The space $X := \{u \in L^2(\Omega) \colon \Delta u \in L^p(\Omega)\}$ is a Banach space, as well as

$$C^1(\bar{\omega}) = \{u \in C^1(\omega) \colon u, D_j u \text{ have a continuous extension to } \bar{\omega}\}.$$

The mapping

$$T \colon X \to C^1(\bar{\omega}),\ u \mapsto u|_\omega$$

is linear and has a closed graph. Hence T is continuous. □

3. **Stability of the Poisson problem.** In this section we consider a sequence of bounded open sets $\Omega_n \subset \mathbb{R}^N$ which converges to a bounded open set $\Omega \subset \mathbb{R}^N$ in a sense to be made precise.

Definition 3.1. We write $\Omega_n \uparrow \Omega$ if $\Omega_n \subset \Omega$ and if for each compact set $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n \geq n_0$.

Proposition 3.2. *Let $\Omega_n \uparrow \Omega$, $f \in L^2(\Omega)$, $u = R_\Omega f$ and $u_n = R_{\Omega_n} f$, then $u_n \to u$ in $L^2(\Omega)$. If $f \in L^p_{\mathrm{loc}}(\Omega)$ where $p > N$. Then $u_n, u \in C^1(\Omega)$ and*

$$\lim_{n\to\infty} u_n(x) = u(x)$$

uniformly on compact subsets of Ω.

Proof. The first property follows for instance from [18, Proposition 7.1] or [4, proof of Theorem 6.2]. Let $\omega \subset \omega_1 \subset \Omega$ be open sets with $\bar{\omega} \subset \omega_1$ and $\bar{\omega}_1 \subset \Omega$ compact. It follows from Proposition 2.6 that $u_n, u \in C^1(\Omega)$ and $\|u_n\|_{C^1(\bar{\omega}_1)} \leq c\big(\|\Delta u_n\|_{L^p(\Omega)} + \|u_n\|_{L^2(\Omega)}\big)$. Thus $\sup \|u_n\|_{C^1(\bar{\omega}_1)} < \infty$, so the sequence $(u_n)_{n\in\mathbb{N}}$ is equicontinuous on $\bar{\omega}$. Hence there exists a subsequence which converges in $C(\bar{\omega})$. Since $u_n \to u$ in $L^2(\omega)$, it follows that u_n converges to u in $C(\bar{\omega})$. □

Next we consider convergence from the exterior.

Definition 3.3. We write $\Omega_n \downarrow \Omega$ if

(a) $\bar{\Omega} \subset \Omega_n$ for all $n \in \mathbb{N}$ and
(b) $\lambda\big((\Omega_n \cap B) \setminus \bar{\Omega}\big) \to \infty$ as $n \to \infty$ for every ball B.

Remark 3.4. Condition (b) is in particular satisfied if

$$|(\Omega_n \cap B) \setminus \bar{\Omega}| \to 0 \quad \text{as} \quad n \to \infty$$

for each ball B where $|F|$ denote the Lebesgue measure of a Borel set $F \subset \mathbb{R}^N$. This follows from the Faber-Krahn inequality (see [18, Proposition 7.6]).

Proposition 3.5. *Let $\Omega_n \downarrow \Omega$, let $f \in L^2(\mathbb{R}^N)$, $u_n = R_{\Omega_n} f$ and $u = R_{\bar{\Omega}} f$. Then $u_n \to u$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. If $f \in L^p_{\mathrm{loc}}$, $p > N$, then $u_n, u \in C^1(\Omega)$ and convergence is uniform on compact subsets of Ω.*

Proof. The first assertion follows from [18, Theorem 6.1], the second as in Proposition 3.2. □

Thus convergence from above and from below lead to different limits if

$$H^1_0(\Omega) \neq H^1_0(\bar{\Omega}).$$

On the other hand if $H^1(\Omega) = H^1_0(\bar{\Omega})$, then we may simultaneously approximate from the interior and the exterior.

Definition 3.6. We write $\Omega_n \to \Omega$ if the following two conditions are satisfied:

(a) For each compact $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n \geq n_0$.
(b) For each ball $B \subset \mathbb{R}^N$, $\lambda\big((\Omega_n \cap B) \setminus \bar{\Omega}\big) \to \infty$ as $n \to \infty$.

Theorem 3.7. *Let Ω be open and bounded. The following assertions are equivalent.*

(i) $H^1_0(\Omega) = H^1_0(\bar{\Omega})$;
(ii) *For every sequence (Ω_n) with $\Omega_n \to \Omega$ and every $f \in L^2(\mathbb{R}^N)$ we have $R_{\Omega_n} f \to R_\Omega f$ in $L^2(\mathbb{R}^N)$.*

Moreover, in this case, if $f \in L^p_{\mathrm{loc}}$ for some $p > N$, then $R_{\Omega_n} f \in C^1(\Omega_n)$, $R_\Omega f \in C^1(\Omega)$ and the convergence is uniform on each compact subset of Ω.

Proof. (ii) $\Rightarrow$ (i) is clear from the previous discussion.

(i) $\Rightarrow$ (ii) follows from [13, 25] or [18, Theorem 7.5]. The last assertion is proved as Proposition 3.2. □

In view of this characterisation given in Theorem 3.7 we give the following definition.

Definition 3.8. An open set $\Omega \subset \mathbb{R}^N$ is *stable* if $H^1_0(\Omega) = H^1_0(\bar{\Omega})$.

In the following section we will relate the notion of stability to stability of the Dirichlet problem.

4. **Stability of the Dirichlet problem.** Let Ω be a bounded open set with topological boundary $\partial\Omega$. We say that Ω is *Dirichlet regular* if for every $\varphi \in C(\partial\Omega)$ there exists a solution of

$$D(\varphi, \Omega) \qquad\qquad h \in \mathcal{H}(\Omega) \cap C(\bar{\Omega}), \quad h_{|\partial\Omega} = \varphi,$$

where $\mathcal{H}(\Omega) = \{u \in C^2(\Omega) \colon \Delta u = 0\}$ is the space of all harmonic functions. There is at most one solution of $D(\varphi, \Omega)$ by the maximum principle. If for a given $\varphi \in C(\partial\Omega)$ there exists a solution of $D(\varphi, \Omega)$ we say that $D(\varphi, \Omega)$ is *solvable*. We recall the notion of the Perron solution and refer to [19, Chapter II § 4] for the proof of the following proposition. The Perron solution is a generalised solution of $D(\varphi, \Omega)$ which always exists. A function $v \in C(\Omega)$ is called *subharmonic* if $-\Delta v \leq 0$ in $\mathcal{D}(\Omega)'$, that is, $-\int_\Omega v \Delta\omega \, dx \leq 0$ for all $0 \leq \omega \in \mathcal{D}(\Omega)$. The function v is called *superharmonic* if $-v$ is subharmonic. Given $\varphi \in C(\partial\Omega), u \in C(\Omega)$ we write $u \leq \varphi$ on $\partial\Omega$ if

$$\overline{\lim_{x \to z}}\, u(x) \leq \varphi(z)$$

for all $z \in \partial\Omega$ and $u \geq \varphi$ on $\partial\Omega$ if

$$\overline{\lim_{x \to z}}\, u(x) \geq \varphi(z)$$

for all $z \in \partial\Omega$. We let $\mathcal{H}^b(\Omega) := \mathcal{H}(\Omega) \cap L^\infty(\Omega)$, which is a Banach space for the supremum norm.

Theorem 4.1 (Perron solution)**.** *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open bounded set.*

(a) *If $\varphi \in C(\partial\Omega)$, then*

$$h_\varphi(x) = \sup\{u(x) \colon u \text{ subharmonic and } u \leq \varphi \text{ on } \partial\Omega\}$$

exists for all $x \in \Omega$ and defines a harmonic function h_φ on Ω. Moreover $h_\varphi(x) = \inf\{v(x) \colon v \text{ superharmonic}, v \geq \varphi \text{ on } \partial\Omega\}$.

(b) *The mapping $\varphi \in C(\partial\Omega) \mapsto h_\varphi \in \mathcal{H}^b(\Omega)$ is linear, positive and contractive.*

(c) *If $D(\varphi, \omega)$ has a solution h, then $h_\varphi = h$.*

We call the function h_φ the *Perron solution* of $D(\varphi, \Omega)$. Next we want to describe the Perron solution with the help of the solution of the Poisson equation. The following is a generalisation of [6, Lemma 2.2.a)] with a similar proof which we include for completeness. We let $C_0(\Omega) := \{u \in C(\bar{\Omega}) \colon u_{|\partial\Omega} = 0\}$.

Proposition 4.2. *Let Ω be a bounded open set and $u \in C_0(\Omega)$. Suppose that $\Delta u \in L^2(\Omega)$. Then $u \in H^1_0(\Omega)$.*

Proof. Let $v \in H_0^1(\Omega)$ such that $-\Delta v = \Delta u$. Then $h = u + v \in \mathcal{H}(\Omega) \subset C^\infty(\Omega)$. Consequently, $u = h - v \in H^1_{\rm loc}(\Omega)$. Let $\varepsilon > 0$. Since $u \in C_0(\Omega)$, it follows that $(u-\varepsilon)^+ \in C_c(\Omega)$ (the space of all continuous functions on Ω with compact support). Let $\omega \subset\subset \Omega$ such that $\operatorname{supp}(u-\varepsilon)^+ \subset \omega$. Then $(u-\varepsilon)^+ \in H_0^1(\omega)$ by [9, Lemma IX 5] (see Section 2). If we let $f := -\Delta u$, then for $w \in \mathcal{D}(\omega)$,

$$\int \nabla u \nabla w \, dx = -\int u \Delta w \, dx = \int f w \, dx.$$

This identity remains true for $w \in H_0^1(\omega)$. Take $w = (u-\varepsilon)^+$. Then

$$\begin{aligned}\int |\nabla (u-\varepsilon)^+|^2 \, dx = \int \nabla u \nabla (u-\varepsilon)^+ \, dx &= \int f (u-\varepsilon)^+ \, dx \\ &\leq \|f\|_2 \|(u-\varepsilon)^+\|_2 \leq \|f\|_2 |\Omega|^{1/2} \|u\|_\infty.\end{aligned}$$

Thus $\{(u-\varepsilon)^+ : \varepsilon \in (0,1]\}$ is bounded in $H_0^1(\Omega)$. Hence there exists a weak limit point $v \in H_0^1(\Omega)$ as $\varepsilon \downarrow 0$. Since $(u-\varepsilon)^+ \to u^+$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$, it follows that $u^+ = v \in H_0^1(\Omega)$. Applying this to $-u$ instead of u, we obtain that also $u^- \in H_0^1(\Omega)$. □

Corollary 4.3. *Let Ω be Dirichlet regular and $\Phi \in C^2(\bar\Omega), \varphi = \Phi_{|\partial\Omega}$. Let h be the solution of $D(\varphi, \Omega)$. Then $u = h - \Phi \in H_0^1(\Omega)$, that is, $h - \Phi = R_\Omega(\Delta\Phi)$.*

Next we describe the Perron solution by approximation from the interior. Keldyš [22, Theorem 2] refers to the following result as Wiener's Theorem. We give a proof to be complete.

Theorem 4.4 (the Perron solution by approximation from the interior)**.** *Let Ω_n be Dirichlet regular and Ω open and bounded. Suppose that $\Omega_n \uparrow \Omega$ and let $\Phi \in C(\bar\Omega), \varphi = \Phi_{|\partial\Omega}$. Let h_n be the solution of $D(\Phi_{|\partial\Omega_n}, \Omega_n)$. Then*

$$\lim_{n\to\infty} h_n(x) = h_\varphi(x)$$

uniformly on compact subsets of Ω. Here h_φ is the Perron solution of $D(\varphi, \Omega)$.

Proof. (a) If we assume that $\Phi \in C^2(\bar\Omega)$, then $h_n := u_n + \Phi$, where $u_n = R_{\Omega_n}(\Delta\Phi)$ by Corollary 4.3. By Proposition 3.2, $\lim_{n\to\infty} u_n = u$ uniformly on compact subsets of Ω, where $u = R_\Omega(\Delta\Phi)$.

(b) Let now $\Phi \in C(\bar\Omega)$ and $\varepsilon > 0$ be arbitrary. Choose $\tilde\Phi \in C^2(\bar\Omega)$ such that

$$\|\Phi - \tilde\Phi\|_{L^\infty(\bar\Omega)} \leq \varepsilon$$

and let $\tilde h_n$ be the solution of $D(\tilde\Phi_{|\partial\Omega}, \Omega_n)$. Then $\tilde h_n$ converges uniformly on compact subsets of Ω by (a). Let $K \subset \Omega$ be compact. Then there exists $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n \geq n_0$. For $n, m \geq n_0$ we have

$$\begin{aligned}&\|h_n - h_m\|_{L_\infty(K)} \\ &\qquad \leq \|h_n - \tilde h_n\|_{C(K)} + \|\tilde h_n - \tilde h_m\|_{C(K)} + \|\tilde h_m - h_m\|_{C(K)} \\ &\qquad\qquad \leq 2\varepsilon + \|\tilde h_n - \tilde h_m\|_{C^\infty(K)}\end{aligned}$$

by the maximum principle. Thus (h_n) is a Cauchy sequence in $C(K)$ for every compact subset $K \subset \Omega$, showing that h_n converges to a function h uniformly on compact subsets of Ω.

(c) The function h merely depends on φ and not on the choice of the extension $\Phi \in C(\bar\Omega)$ of φ. In fact, let $\tilde\Phi \in C(\bar\Omega)$ such that $\tilde\Phi_{|\partial\Omega} = \varphi$. Let $\tilde h_m$ be the solution of

$D(\tilde{\Phi}_{|\partial\Omega_n}, \Omega_n)$ and fix $\varepsilon > 0$ and $x \in \Omega$ arbitrary. Since $\Phi = \tilde{\Phi}$ on $\partial\Omega$ there exists a compact set $K \subset \Omega$ containing x such that $|\Phi(y) - \tilde{\Phi}(y)| \le \varepsilon$ for all $y \in \Omega \setminus K$. Let $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n \ge n_0$. Then $|h_n(x) - \tilde{h}_n(x)| \le \|\Phi - \tilde{\Phi}\|_{L^\infty(\partial\Omega_n)} \le \varepsilon$. Thus $|h(x) - \tilde{h}(x)| \le \varepsilon$, where $h = \lim_{n\to\infty} h_n$ and $\tilde{h} = \lim_{n\to\infty} \tilde{h}_n$. Since $\varepsilon > 0$ is arbitrary, $h(x) = \tilde{h}(x)$.

(d) Let $h(x) = \lim_{n\to\infty} h_n(x)$ as before. We show that $h = h_\varphi$. Let $\Phi \in C(\bar{\Omega})$ such that $\Phi_{|\partial\Omega} = \varphi$. Let $u \in C(\Omega)$ be a subharmonic function such that $u \le \varphi$ on $\partial\Omega$. Then $\tilde{\Phi} := \Phi \vee u \in C(\bar{\Omega})$ and $\tilde{\Phi}_{|\partial\Omega} = \varphi$. If $\tilde{h}_n$ is the solution of $D(\tilde{\Phi}_{|\partial\Omega_n}, \Omega_n)$, then $u \le \tilde{h}_n$ on $\partial\Omega_n$. Hence $u \le h_n$ on Ω_n by the maximum principle [19, Chapter II § 4.1]. Thus $u(x) \le \lim_{n\to\infty} \tilde{h}_n(x) = h(x)$ for all $x \in \Omega$. We see similarly that $v(x) \ge h(x)$ for all $x \in \Omega$ for each superharmonic function v such that $v \ge \varphi$ on $\partial\Omega$. Now Theorem 4.1 implies that $h = h_\varphi$. □

Recall that there always exist Ω_n of class C^∞ such that $\Omega_n \subset \Omega_{n+1}$ and $\Omega_n \uparrow \Omega$ (see [19, Lemma II.4.2.1]). Thus Theorem 4.4 describes completely the Perron solution. Finally we characterise the Perron solution in terms of solutions of the Poisson problem for boundary data in a dense subspace of $C(\partial\Omega)$.

Theorem 4.5 (Perron solution via Poisson equation)**.** *Let $\varphi \in C(\partial\Omega)$ and assume that there exists $\Phi \in C^2(\bar{\Omega})$ such that $\varphi = \Phi_{|\partial\Omega}$. Let $u \in H_0^1(\Omega)$ such that*

$$-\Delta u = \Delta\Phi \quad \textit{in} \quad \mathcal{D}(\Omega)'.$$

Then $h_\varphi = \Phi + u$.

Proof. Let Ω_n be Dirichlet regular such that $\Omega_n \uparrow \Omega$. Let h_n be the solution of $D(\Phi_{|\partial\Omega_n}, \Omega_n), u_n = h_n - \Phi$. Then $u_n \in H_0^1(\Omega_n)$ and $-\Delta u_n = \Delta\Phi$ by Corollary 4.3. Moreover by Proposition 3.2, u_n converges to u in $L^2(\Omega)$, where u is the solution of

$$-\Delta u = \Delta\Phi \quad \text{in } \mathcal{D}(\Omega)', \quad u \in H_0^1(\Omega).$$

Since $h_n \to h_\varphi$ uniformly on compact subsets of Ω by Theorem 4.4 we conclude that $u = h_\varphi - \Phi$ on Ω as claimed. □

Instead of approximating h_φ from the interior (that is, by considering $\Omega_n \uparrow \Omega$), we might consider $\Omega_n \supset \bar{\Omega}$. This however leads to another harmonic function. We quote the following facts which are proved by Keldyš [22, V]. Let $\Omega_n \supset \bar{\Omega}$ such that $\bigcap_{n\in\mathbb{N}} \Omega_n = \bar{\Omega}$, $\Omega_n \supset \Omega_{n+1}$. Assume that Ω_n is Dirichlet regular. Let $\varphi \in C(\partial\Omega)$ and choose $\Phi \in C(\bar{\Omega}_1)$ such that $\Phi_{|\partial\Omega} = \varphi$. Let h_n be the solution of $D(\Phi_{|\partial\Omega_n}, \Omega_n)$. Then h_n converges to a function $H_\varphi \in \mathcal{H}^b(\Omega)$ uniformly on compact subsets of Ω. This function H_φ does not depend on the choice of the Ω_n and the extension Φ of φ (see the proof of Theorem 4.4). Moreover, the mapping $\varphi \mapsto H_\varphi \colon C(\partial\Omega) \to \mathcal{H}^b(\Omega)$ is linear, contractive and positive. Now consider the case where φ has an extension $\Phi \in C^2(\bar{\Omega}_1)$. Then $h_n = \Phi_{|\Omega_n} + u_n$ where $u_n \in H_0^1(\Omega_n)$ is the solution of $-\Delta u_n = \Delta\Phi$ in $\mathcal{D}(\Omega_n)'$. It follows from Proposition 3.5 that u_n converges uniformly on compact subsets of Ω to the function $u = R_{\bar{\Omega}}(\Delta\Phi)$. We have proved the following result.

Proposition 4.6. *Let Ω be a bounded open set, $\varphi \in C(\partial\Omega)$. Assume that there exists $\Phi \in C^2(\mathbb{R}^N)$ such that $\Phi_{|\partial\Omega} = \varphi$. Then*

$$H_\varphi = \Phi + R_{\bar{\Omega}}\big(\Delta\Phi_{|\Omega}\big)$$

Since the space

$$F := \{\varphi \in C(\partial\Omega) \colon \varphi \text{ has an extension } \Phi \in C^2(\mathbb{R}^N)\}$$

is dense in $C(\partial\Omega)$ by the Stone-Weierstraß Theorem we obtain the following result.

Theorem 4.7. *Let Ω be open and bounded. The following assertions are equivalent.*

(i) $H_0^1(\Omega) = H_0^1(\bar{\Omega})$;
(ii) $H_\varphi = h_\varphi$ *for all* $\varphi \in C(\partial\Omega)$.

Keldyš calls Ω *stable* if property (ii) holds. Thus our terminology coincides with his.

Finally, we consider arbitrary convergence of Ω_n, neither from the outside nor from the inside. It is clear from the preceding theorem that then we have to assume in general that Ω is stable in order to obtain a limit.

Theorem 4.8. *Let Ω be an open, bounded, stable set. Let Ω_n be Dirichlet regular such that $\Omega_n \to \Omega$ (Definition 3.6). Let $\Phi \colon \mathbb{R}^N \to \mathbb{R}$ be uniformly continuous, $\varphi = \Phi_{|\partial\Omega}$. Let h_n be the solution of $D(\Phi_{|\partial\Omega_n}, \Omega_n)$ and let h_φ be the Perron solution of $D(\varphi, \Omega)$. Then*

$$\lim_{n\to\infty} h_n(x) = h_\varphi(x)$$

uniformly on compact subsets of Ω.

Proof. (a) Assume that $\Phi \in C^2(\mathbb{R}^N)$, $f := \Delta\Phi$, $u_n := R_{\Omega_n} f$ and $u := R_\Omega f$. Then $h_n := \Phi + u_n$ is the solution of $D(\Phi_{|\Omega_n}, \Omega_n)$ and $h_\varphi = \Phi + u$ (by Theorem 4.5). Since Ω is stable u_n converges to u uniformly on compact subsets of Ω by Theorem 3.7. Thus also h_n converges to h_φ uniformly on compact subsets of Ω.

(b) Let $\varepsilon > 0$. With the help of a mollifier we find $\tilde{\Phi} \in C^2(\mathbb{R}^N)$ such that $\|\tilde{\Phi} - \Phi\|_{L^\infty(\mathbb{R}^N)} \le \varepsilon$. Let $\tilde{h}_n$ be the solution of $D(\tilde{\Phi}_{|\partial\Omega_n}, \Omega_n)$ and let $h_{\tilde{\varphi}}$ be the Perron solution on Ω with respect to $\tilde{\varphi} = \tilde{\Phi}_{|\partial\Omega}$. Let $K \subset \Omega$ be compact. There exists $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n \ge n_0$. Let $n \ge n_0$, then

$$\begin{aligned}
&\|h_n - h_\varphi\|_{C(K)} \\
&\qquad \le \|h_n - \tilde{h}_n\|_{C(K)} + \|\tilde{h}_n - h_{\tilde{\varphi}}\|_{C(K)} + \|h_{\tilde{\varphi}} - h_\varphi\|_{C(K)} \\
&\qquad\qquad \le 2\varepsilon + \|\tilde{h}_n - h_{\tilde{\varphi}}\|_{C(K)}.
\end{aligned}$$

It follows from (a) that $\limsup_{n\to\infty} \|h_n - h_\varphi\|_{C(K)} \le 2\varepsilon$. □

5. **Uniform convergence.** Let Ω be a bounded open set and let Ω_n be bounded, open and Dirichlet regular and $\Phi \in \mathrm{BUC}(\mathbb{R}^N)$, where $\mathrm{BUC}(\mathbb{R}^N)$ denotes the set of bounded and uniformly continuous functions on $\mathbb{R}^N$. Consider the solution h_n of $D(\Phi_{|\partial\Omega_n}, \Omega_n)$ and the Perron solution h of $D(\Phi_{|\partial\Omega}, \Omega)$. In Section 4 we showed the following. Assume that Ω is stable (that is, $H_0^1(\Omega) = H_0^1(\bar{\Omega})$) and that $\Omega_n \to \Omega$ (in the sense of Definition 3.6). Then by Theorem 4.8 $h_n \to h$ uniformly on compact subsets of Ω. We also saw that stability is a necessary assumption for this convergence to hold. Now assume in addition that Ω is Dirichlet regular. Then $h \in C(\bar{\Omega})$ and $h_{|\partial\Omega} = \Phi_{|\partial\Omega}$. We extend h to a continuous function on $\mathbb{R}^N$ by letting $h(x) = \Phi(x)$ for $x \in \mathbb{R}^N \setminus \bar{\Omega}$. Similarly, we let $h_n(x) = \Phi(x)$ for $x \in \mathbb{R}^N \setminus \bar{\Omega}_n$. Thus $h, h_n \in \mathrm{BUC}(\mathbb{R}^N)$. It is surprising that under these hypotheses h_n converges to h uniformly on $\mathbb{R}^N$. This is the main result of this section (Theorem 5.2). Its proof is based on the following basic lemma, which is a modification and generalisation

of Keldyš Lemma VI in [22]. Our hypotheses are considerably weaker than those given by Keldyš.

Basic Lemma 5.1. *Let Ω_n, Ω bounded, open sets and*

$$h_n, h, \Phi \in \mathrm{BUC}(\mathbb{R}^N)$$

such that $h_n = \Phi$ on $\mathbb{R}^N \setminus \Omega_n$, $h = \Phi$ on $\mathbb{R}^N \setminus \Omega$ and such that h_n is harmonic on Ω_n and h harmonic on Ω. Assume that

(a) *$h_n \to h$ in measure;*
(b) *for each compact set $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n \geq n_0$.*

Then $h_n \to h$ uniformly in $\mathbb{R}^N$ as $n \to \infty$.

Proof. Let $0 < \varepsilon \leq 2$. Choose $\delta > 0$ such that $|x-y| \leq \delta$ implies $|\Phi(x)-\Phi(y)| \leq \varepsilon/2$ and $|h(x) - h(y)| \leq \varepsilon/4$. Such a choice is possible since Φ and h are uniformly continuous. Next choose $c > 0$ such that

$$\frac{1}{\sigma_N \cdot (\delta/2)^{N-1}}\Big(c \cdot \frac{2}{\delta}\Big)\big(2\|\Phi\|_{L^\infty(\mathbb{R}^N)} + 1\big) = \frac{\varepsilon}{2} \tag{5.1}$$

where σ_N is the surface area of the unit sphere in $\mathbb{R}^N$. Define

$$F_n := \{x \in \mathbb{R}^N : |h_n(x) - h(x)| > \varepsilon/4\} \tag{5.2}$$

and note that since $h_n \to h$ in measure, $|F_n| \to 0$ as $n \to \infty$. Hence we can choose $n_1 \in \mathbb{N}$ such that

$$|F_n| < c \tag{5.3}$$

for all $n \geq n_0$. By assumption we can choose $n_2 \in \mathbb{N}$ such that also

$$K := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \geq \delta\} \subset \Omega_n.$$

We set $n_0 := \max\{n_1, n_2\}$. We have to show that

$$|h_n(z) - h(z)| \leq \varepsilon \tag{5.4}$$

for all $n \geq n_0$ and all $z \in \mathbb{R}^N$. If $z \in \Omega_n^c \cap \Omega^c$ and $n \geq n_0$, then $h_n(z) = \Phi(z) = h(z)$ for all $n \in \mathbb{N}$ and thus (5.4) follows. If $z \in \Omega_n^c \cap \Omega$, then $z \notin K$ by definition of K. Hence there exists $y \in \partial\Omega$ such that $|z - y| < \delta$ by choice of δ. As $z \notin \Omega_n$ we have $h_n(z) = \Phi(z)$ and thus

$$|h(z) - h_n(z)| \leq |h(z) - h_n(y)| + |\Phi(y) - \Phi(z)| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon.$$

as claimed. We finally need to show (5.4) for $z \in \Omega_n$. We fix $z \in \Omega_n$ and $n \geq n_0$ and define

$$f(x) := |h_n(x) - h(z)|$$

Then $f(z) \leq \varepsilon$ is equivalent to (5.4). For $0 < \varrho \leq \delta$ we consider $B_\varrho := \{x \in \mathbb{R}^N : |x - z| < \delta\}$ and $S_\varrho := \partial B_\varrho$. Let $\omega_\varrho \in \mathcal{H}(B_\varrho) \cap C(\bar{B}_\varrho)$ such that

$$\omega_\varrho = f \vee \varepsilon/2 \quad \text{on} \quad S_\varrho.$$

We now show that

$$f \leq \omega_\varrho \quad \text{on} \quad \Omega_n \cap B_\varrho. \tag{5.5}$$

Since $h_n(x) - h(z)$ is harmonic on $\Omega_n \cap B_\varrho$ and thus f subharmonic on $\Omega_n \cap B_\varrho$ it is sufficient to show that (5.5) holds on $\partial(\Omega_n \cap B_\varrho)$. If $x \in \partial B_\varrho$, then $\omega_\varrho(x) =$

$f(x) \vee \varepsilon/2 \geq f(x)$. If $x \in \partial\Omega_n \cap B_\varrho \cap \Omega$, then $x \notin K$. Thus there exists $y \in \partial\Omega$ such that $|x-y| < \delta$. As $h_n(x) = \Phi(x)$ we get

$$f(x) = |\Phi(x) - h(z)| \leq |\Phi(x) - \Phi(y)| + |h(y) - h(z)| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon$$

by choice of δ. Finally, if $x \in \partial\Omega_n \cap B_\varrho$, but $x \notin \Omega$. Then $f(x) = |\Phi(x) - \Phi(x)| = 0$, so (5.5) is satisfied.

To prove (5.4) for $z \in \Omega_n$ we therefore need to find $0 < \varrho \leq \delta$ such that $\omega_\varrho(z) \leq \varepsilon$. For this we will use the spherical mean-value property of the harmonic function ω_ϱ. Recall that $|F_n| < c$ by (5.3) and so there exists $\varrho \in [\delta/2, \delta]$ such that

$$\sigma(F_\varrho) < c \cdot \frac{2}{\delta} \tag{5.6}$$

where $F_\varrho = F_n \cap S_\varrho$ and σ is the surface measure of S_ϱ. Indeed, assume that

$$\sigma(F_\varrho) \geq c \cdot \frac{2}{\delta} \quad \text{for all} \quad \varrho \in [\delta/2, \delta].$$

Then $|F_n| \geq \int_{\delta/2}^{\delta} \sigma(F_\varrho)\, d\varrho \geq (c \cdot \frac{2}{\delta})\frac{\delta}{2} = c$, contradicting (5.3). Hence (5.6) follows. By the mean-value theorem we have

$$\begin{aligned}\omega_\varrho(z) &= \frac{1}{\sigma(S_\varrho)} \int_{S_\varrho} w_\varrho(y)\, d\sigma(y) \\ &= \frac{1}{\sigma(S_\varrho)} \int_{S_\varrho \setminus F_\varrho} w_\varrho(y)\, d\sigma(y) + \frac{1}{\sigma(S_\varrho)} \int_{F_\varrho} w_\varrho(y)\, d\sigma(y).\end{aligned}$$

The second integral is estimated by (5.6) and by using that $w_\varrho \leq f \vee \varepsilon/2 \leq 2\|\Phi\|_{L^\infty(\mathbb{R}^N)} + 1$. Hence by (5.1)

$$\begin{aligned}\frac{1}{\sigma(S_\varrho)} \int_{F_\varrho} w_\varrho(y)\, d\sigma(y) &\leq \frac{1}{\sigma_N \cdot \varrho^{N-1}} \Big(c \cdot \frac{2}{\delta}\Big)\big(2\|\Phi\|_{L^\infty(\mathbb{R}^N)} + 1\big) \\ &\leq \frac{1}{\sigma_N (\delta/2)^{N-1}} \Big(c \cdot \frac{1}{\delta}\Big)\big(2\|\Phi\|_{L^\infty(\mathbb{R}^N)} + 1\big) = \frac{\varepsilon}{2}.\end{aligned}$$

In order to estimate the first integral let $y \in S_\varrho \setminus F_\varrho$. Then $|h_n(y) - h(y)| \leq \varepsilon/4$ since $y \notin F_n$. Hence

$$f(y) = |h_n(y) - h(z)| \leq |h_n(y) - h(y)| + |h(y) - h(z)| \leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

Hence $\omega_\varrho(y) = f(y) \vee \varepsilon/2 = \varepsilon/2$ on $S_\varrho \setminus F_\varrho$. Consequently,

$$\frac{1}{\sigma(S_\varrho)} \int_{S_\varrho \setminus F_\varrho} w_\varrho(y)\, d\sigma(y) \leq \varepsilon/2,$$

completing the proof of the basic lemma. □

Now the L^2-theory of Section 3 gives us simple criteria for h_n to converge to h in measure. When doing so we will identify $C_0(\Omega)$ with a subspace of

$$C_0(\mathbb{R}^N) := \{u \in C(\mathbb{R}^N) \colon \lim_{|x| \to \infty} u(x) = 0\}$$

extending functions by 0, that is,

$$C_0(\Omega) = \{u \in C(\mathbb{R}^N) : u(x) = 0 \text{ if } x \notin \Omega\}.$$

Theorem 5.2. *Let Ω_n, Ω be bounded open sets such that $\Omega_n \to \Omega$. Assume that Ω is stable. Let $\Phi \in \mathrm{BUC}(\mathbb{R}^N)$. Assume that the Dirichlet problems $D(\Phi_{|\partial\Omega_n}, \Omega_n)$ and $D(\Phi_{|\partial\Omega}, \Omega)$ are solvable. Let h_n, h be the solutions extended to $\mathbb{R}^N$ by Φ. Then $h_n \to h$ uniformly on $\mathbb{R}^N$.*

Proof. As a first step we assume that $\Phi \in C^2(\mathbb{R}^N)$. Then $h_n = \Phi + u_n$ where $u_n \in C_0(\Omega_n), -\Delta u_n = \Delta\Phi$ in $\mathcal{D}(\Omega_n)', h = \Phi + u$ where $u \in C_0(\Omega)$ and $-\Delta u = \Delta\Phi$ in $\mathcal{D}(\Omega)'$. It follows from Proposition 4.2 that $u_n \in H_0^1(\Omega_n)$ and $u \in H_0^1(\Omega)$. Now Theorem 3.7 implies that $u_n \to u$ in $L^2(\mathbb{R}^N)$ and hence also in measure. Thus $h_n \to h$ in measure and the claim follows from the Basic Lemma 5.1. If $\Phi \in \mathrm{BUC}(\mathbb{R}^N)$, then the assertion is reduced to the above exactly as in the proof of Theorem 4.8. □

Finally we want to establish a result on the convergence in the operator norm (Theorem 5.6). We need some preparation.

Proposition 5.3. *Let Ω be a Dirichlet regular, bounded open set. Let $N/2 < p < \infty$. Then for each $f \in L^p(\Omega)$ there is a unique solution of the Poisson problem*

$$u \in C_0(\Omega), \quad -\Delta u = f \quad \text{in} \quad \mathcal{D}(\Omega)'. \tag{5.7}$$

Proof. Uniqueness is clear from the maximum principle. The solution can be obtained as follows. Denote by $E_N \in C^\infty(\mathbb{R}^N \setminus \{0\})$ the Newtonian potential. Then $E_N \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^N)$. Thus $\Phi := E_N * f \in C(\mathbb{R}^N)$. Let h be the solution of $D(\Phi_{|\partial\Omega}, \Omega)$. Then $u := h - \Phi \in C_0(\Omega)$ and $-\Delta u = \Delta\Phi = f$. □

We denote by $R^p_\Omega \in \mathcal{L}(L^p(\mathbb{R}^N), C_0(\mathbb{R}^N))$ the operator which to each $f \in L^p(\Omega)$ associates the solution u of (5.7).

Proposition 5.4. *R^p_Ω is a compact operator.*

Proof. Let $T(t) = e^{-t\Delta_\Omega}$. Since T is dominated by the Gaussian semigroup (see Proposition 2.5), we have

$$\|T(t)\|_{\mathcal{L}(L^p, L^\infty)} \le Ct^{-N/2p}e^{-\omega t}$$

for all $t \ge 0$ for some $\omega > 0$ (see [5, § 7.3]). We have $R^p_\Omega = \int_0^\infty T(t)\,dt$. Let $f_n \rightharpoonup f$ in $L^p(\Omega)$ weakly. Since $T(t) = T(t/2)T(t/2) \in \mathcal{L}(L^p, C_0(\Omega)) \circ \mathcal{L}(L^p(\Omega))$ is compact (since $T(t/2)\colon L^p(\Omega) \to L^p(\Omega)$ is compact) it follows that $T(t)f_n \to T(t)f$. By Lebesgue's Theorem we deduce that $R^p_\Omega f_n \to R^p_\Omega f$. Thus R^p_Ω is completely continuous and so compact since $L^p(\Omega)$ is reflexive. □

We will use the following simple interpolation inequality.

Lemma 5.5. *Let $1 \le p_1 < p_2 \le \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Let X be a Banach space and $T \in \mathcal{L}(L^{p_1}(\Omega), X)$. Then*

$$\|T\|_{\mathcal{L}(L^p(\Omega),X)} \le \|T\|^\theta_{\mathcal{L}(L^{p_1},X)} \cdot \|T\|^{1-\theta}_{\mathcal{L}(L^{p_2},X)}.$$

Proof. Consider $T' \in \mathcal{L}(X', L^{p_1}(\Omega)')$. Let $x' \in X', \|x'\| \le 1$. Then

$$\begin{aligned}\|T'x'\|_{L^{p'}} &\le \|T'x'\|^\theta_{L^{p_1'}}\|T'x'\|^{1-\theta}_{L^{p_2'}} \\ &\le \|T'\|^\theta_{\mathcal{L}(X',L^{p_1'})}\|T'\|^{1-\theta}_{\mathcal{L}(X',L^{p_2'})} = \|T'\|^\theta_{\mathcal{L}(L^{p_1},X)}\|T'\|^{1-\theta}_{\mathcal{L}(L^{p_2},X)}.\end{aligned}$$

Since $\|T\|_{\mathcal{L}(L^p,X)} = \|T'\|_{\mathcal{L}(X,L^{p'})} = \sup_{\|x'\|\le 1}\|T'x'\|_{L^{p'}}$ the claim follows. □

Theorem 5.6. *Let Ω_n, Ω be open, Dirichlet regular sets, all contained in a large ball B. Assume that Ω is stable. Let $p > N/2$. If $\Omega_n \to \Omega$, then $R^p_{\Omega_n} \to R^p_\Omega$ in $\mathcal{L}(L^p(\mathbb{R}^N), C_0(\mathbb{R}^N))$.*

Proof. (a) First assume that $p > N$. Let $f_n \rightharpoonup f$ in $L^p(\mathbb{R}^N)$. We show that

$$u_n = R^p_{\Omega_n} f_n \to u = R^p_\Omega f \quad \text{in } C_0(\mathbb{R}^N).$$

This implies the claim (see [18, Proposition B1]). We may assume that f_n and f vanish outside the ball B. Let $\Phi_n = E_N * f_n$, $\Phi = E_N * f$. Since $f_n \rightharpoonup f$ in $L^p(\mathbb{R}^N)$, and since $E_N \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^N)$, it follows that

$$\Phi_n(x) = \int_B E_N(x-y) f_n(y)\, dy \to \Phi(x) = \int_B E_N(x-y) f(y)\, dy$$

as $n \to \infty$ for all $x \in \mathbb{R}^N$. Since $p > N$, we have $D_j E_N \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^N)$. This shows that $\Phi_n \in C^1(\mathbb{R}^n)$ and $D_j \Phi_n = D_j E_N * \Phi$ is bounded in $C(\bar{B})$. Now Arzela's Theorem implies that $\Phi_n \to \Phi$ uniformly on $\bar{B}$. We modify Φ_n and Φ outside $\bar{B}$ such that $\Phi_n \to \Phi$ uniformly on $\mathbb{R}^N$. Let $h_n, h, \tilde{h}_n \in C(\mathbb{R}^N)$ such that h_n and $\tilde{h}_n$ are harmonic on Ω_n and $h_n = \Phi_n$ on $\mathbb{R}^N \setminus \Omega_n$ and $\tilde{h}_n = \Phi$ outside Ω_n, h harmonic on Ω and equal to Φ outside Ω. Then $h_n = \Phi_n + u_n$, $h = \Phi + u$. Since $\Phi_n \to \Phi$ uniformly, it follows that $\tilde{h}_n - h_n \to 0$ uniformly on $\mathbb{R}^N$. It follows from Theorem 5.2 that $\tilde{h}_n \to h$ uniformly on $\mathbb{R}^N$. Hence $h_n \to h$ uniformly on $\mathbb{R}^N$.

(b) Now let $N \geq p > N/2$. Choose $q_1 > N$ and $1 < q_2 < p$, $0 < \theta < 1$ such that

$$\frac{1}{p} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Then by the preceding lemma

$$\|R^p_{\Omega_n} - R^p_\Omega\|_{\mathcal{L}(L^p, C_0)} \leq \|R^{q_1}_{\Omega_n} - R^{q_1}_\Omega\|^\theta_{\mathcal{L}(L^{q_1}, C_0)} \|R^{q_2}_{\Omega_n} - R^{q_2}_\Omega\|^{1-\theta}_{\mathcal{L}(L^{q_2}, C_0)}$$

which converges to 0 as $n \to \infty$ by the first part (a) of the proof. ☐

6. **Conclusion: The diverse notions of stability.** Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set. Here we collect the diverse notions of stability and discuss their relationships. In particular, we compare the results with those established by Hedberg [21] with the help of abstract potential theory. At first we consider the case where Ω is stable but possibly not Dirichlet regular. Lebesgue's cusp is a concrete example which will be discussed in detail in Section 7. Note that in (c) and (d) below we assume that the Dirichlet problems are solvable.

Theorem 6.1. *For the purpose of this theorem we write $\Omega_n \searrow \Omega$ if $\Omega_n \supset \Omega_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} \Omega_n = \bar{\Omega}$. Consider the following statements.*

(a) *Ω is stable, that is, $H^1_0(\Omega) = H^1_0(\bar{\Omega})$;*
(b) *If $\Omega_n \searrow \Omega$, $\Phi \in C(\bar{\Omega}_1)$, $h_n \in C(\bar{\Omega}_n) \cap \mathcal{H}(\Omega_n)$ and $h_{n|\partial\Omega_n} = \Phi_{|\partial\Omega_n}$, then h_n converges to h_φ uniformly on compact subsets, where $\varphi = \Phi_{|\partial\Omega}$.*
(c) *If $\Omega_n \searrow \Omega$, $\Phi \in C(\bar{\Omega}_1)$, $h_n \in C(\bar{\Omega}_n) \cap \mathcal{H}(\Omega_n)$, $h_{n|\partial\Omega_n} = \Phi_{|\partial\Omega_n}$, and $h \in C(\bar{\Omega}) \cap \mathcal{H}(\Omega)$, $h_{|\partial\Omega} = \Phi_{|\partial\Omega}$, then $h_n \to h$ uniformly on $\bar{\Omega}$.*
(d) *If $h \in C(\bar{\Omega}) \cap \mathcal{H}(\Omega)$, then for each $\varepsilon > 0$ there exist an open set $U \supset \bar{\Omega}$ and $k \in \mathcal{H}(U)$ such that*

$$\|h - k\|_{C(\bar{\Omega})} \leq \varepsilon.$$

(e) *If* $\Omega_n \searrow \Omega$, $f \in L^p(\Omega_1)$ *for some* $p > N/2$, $u_n \in C_0(\Omega_n)$ *such that* $-\Delta u_n = f$ *in* $\mathcal{D}(\Omega_n)'$ *and* $u \in C_0(\Omega)$ *such that* $-\Delta u = f$ *in* $\mathcal{D}(\Omega)'$, *then* $u_n \to u$ *uniformly on* $\bar{\Omega}$.

Then (*a*) $\Leftrightarrow$ (*b*) $\Rightarrow$ (*c*) $\Leftrightarrow$ (*d*) $\Rightarrow$ (*e*).

Proof. (a) $\Rightarrow$ (b) follows from Theorem 4.7 and (a) $\Leftrightarrow$ (c) from Theorem 5.2. We prove (c) $\Rightarrow$ (d). Let $h \in C(\bar{\Omega}) \bigcap_{n\in\mathbb{N}} \mathcal{H}(\Omega_n)$. Choose Ω_n Dirichlet regular such that $\Omega_n \searrow \Omega$. Then h_n from (c) converges to h uniformly on $\bar{\Omega}$. Given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\|h_n - h\|_{C(\bar{\Omega})} \le \varepsilon$. Choose $U = \Omega_n$ and $k = h_n$.

Next we prove (d) $\Rightarrow$ (c). Consider the setting of (c). Let $\varepsilon > 0$. By assumption (d) there exists $k \in \mathcal{H}(U)$ such that $\|k - h\|_{C(\bar{\Omega})} \le \varepsilon$ where U is open such that $\bar{\Omega} \subset U$. Choose $n_0 \in \mathbb{N}$ such that $\bar{\Omega}_{n_0} \subset U$. By the uniform continuity of k and Φ on $\bar{\Omega}$ there exists $\delta > 0$ such that

$$|k(x) - k(z)| \le \varepsilon, |\Phi(x) - \Phi(z)| \le \varepsilon \quad \text{whenever} \quad |x - z| \le \delta, \quad x, z \in \bar{\Omega}_{n_0}.$$

Let $n_1 \ge n_0$ such that $\operatorname{dist}(\partial\Omega, \partial\Omega_{n_1}) < \delta$. Let $n \ge n_1$. We claim that

$$\|h - h_n\|_{C(\bar{\Omega})} \le 4\epsilon.$$

For that, it suffices to show that $\|k - h_n\|_{C(\bar{\Omega}_n)} \le 3\epsilon$. (In fact, then $\|h - h_n\|_{C(\bar{\Omega})} \le \|h - k\|_{C(\bar{\Omega})} + \|k - h_n\|_{C(\bar{\Omega}_n)} \le 4\epsilon$). Let $x \in \partial\Omega_n$. By the maximum principle it suffices to show that $|k(x) - h_n(x)| \le 3\epsilon$. There exists $z \in \partial\Omega$ such that $|z - x| \le \delta$. Hence $|k(x) - h_n(x)| = |k(x) - \Phi(x)| \le |k(x) - k(z)| + |k(z) - h(z)| + |\Phi(z) - \Phi(x)| \le 3\epsilon$, where we use that $h(z) = \Phi(z)$.

(*c*) $\Rightarrow$ (*e*) Let $\Phi = E * f \in C(\mathbb{R}^N), h_n = \Phi + u_n, h = \Phi + u$. Then $h_n \in C(\bar{\Omega}_n) \cap \mathcal{H}(\Omega_n), h \in C(\bar{\Omega}) \cap \mathcal{H}(\Omega)$. It follows from (c) that $h_n \to h$ uniformly on $\bar{\Omega}$. Hence $u_n \to u$ uniformly on $\bar{\Omega}$. ∎

If Ω is topologically regular, that is, $\mathring{\bar{\Omega}} = \Omega$, then Hedberg proves the equivalence of (a), (b) and (d) by using abstract potential theory (see [21, Theorem 11.9], where (i) is our (a), (iv) our (d)), and by a capacity condition which is equivalent to Keldyšh's notion of stability (c). This may be interpreted in the following way. Assume that Ω is not stable. Then (c) is violated. Thus we find $\varphi \in C(\partial\Omega)$ such that $h_\varphi \in C(\bar{\Omega})$ (that is, h_φ is a classical solution of the Dirichlet problem) but $h_\varphi \ne H_\varphi$. This shows that the harmonic function H_φ obtained by approximating from the exterior does not coincide with the classical solution. So the good generalised solution is the Perron solution which is obtained by approximating from the interior.

We do not know whether (e) implies (a). For that we should know that the φ above has an extension $\Phi \in C(\mathbb{R}^N)$ such that $\Delta\Phi \in L^p(\Omega)$ on Ω for some $p > \frac{N}{2}$. But φ is obtained in a very indirect way in the work of Hedberg. Things are different if Ω is Dirichlet regular. Then we obtain the following entire characterisation with a complete proof.

Theorem 6.2. *Let* Ω *be a bounded open set, which is Dirichlet regular. Then the statements* (a)–(e) *of Theorem 6.1 are equivalent.*

Proof. We have to show that (e) $\Rightarrow$ (a). Let $f \in L^\infty(\Omega)$, $f > 0$, $\bar{u} = R_{\bar{\Omega}} f$. By Proposition 2.4 it suffices to show that $u \in H_0^1(\Omega)$. Let Ω_n be Dirichlet regular such that $\Omega_n \searrow \Omega$ and $u_n := R^p_{\Omega_n} f$, where $p > N/2$. Then $u_n \in C_0(\Omega_n)$ and $-\Delta u_n = f$ in $\mathcal{D}'(\Omega_n)$. Moreover $u_n \to \bar{u}$ in $L^2(\mathbb{R}^N)$ by Proposition 3.5. It follows from the assumption (e) that $u_n \to u = R_\Omega f$ uniformly on $\bar{\Omega}$. Thus $u = \bar{u}$. ∎

7. **Lebesgue's cusp: An example for a non-regular domain.** In this section we provide a detailed discussion on a domain with one singular point. This means there exists a function $\varphi \in C(\partial\Omega)$ such that the Dirichlet problem $D(\Omega, \varphi)$ does not have a classical solution. The fact which surprised us is that φ can be the restriction of a very smooth function on $\mathbb{R}^N$ which is constant in the neighbourhood of the singular point. Even more surprisingly it can be locally near each point of $\partial\Omega$ be the restriction of a real analytic function. This shows that the solvability of the Dirichlet problem does not depend on local properties of φ near the singular point, but on global properties, which was another surprise to us.

The explicit construction is based on remarks in Keldyš [22, page 6], who attributes it to an unidentified paper of Lebesgue published in 1912.

The domain is constructed with the help of the level surfaces of the function

$$u(x, y, z) = r + z \log(r - z)$$

with $r := \sqrt{x^2 + y^2 + z^2}$. This function is analytic on its natural domain which is $\mathbb{R}^3$ minus the positive z-axis. An explicit calculation shows that u is harmonic on its domain. Indeed, an elementary computation shows that

$$u_x = \frac{x}{r - z}, \quad u_y = \frac{y}{r - z}, \quad u_z = \log(r - z) \tag{7.1}$$

and

$$u_{xx} = \frac{r^2 - rz - x^2}{(r - z)^2 r}, \quad u_{yy} = \frac{r^2 - rz - y^2}{(r - z)^2 r}, \quad u_{zz} = -\frac{1}{r}. \tag{7.2}$$

Adding the latter up we see that $\Delta u = 0$. The function u is radially symmetric with respect to the z-axis, so in order to determine the level surfaces it will be enough to look at the function of two variables $v(x, z) := u(x, 0, z)$. By (7.2) the function $z \mapsto v(x, z)$ is strictly concave attaining a maximum at $z = (x^2 - 1)/2$. Since $v(x, 0) = |x| > 0$ if $x \neq 0$ we have that given $c \leq 0$, there exists a unique $z = z(x) > 0$ such that $v(z, x) = c$. We now look at the properties of that function $z(x)$ for some fixed $c \leq 0$.

Since that $v(z, x) = r - z \log(r - z) = c \leq 0$ with $z, x > 0$, clearly $\log(r - z) < 0$. Thus by the the implicit function theorem

$$z'(x) = -\frac{v_x}{v_z} = \frac{x}{(r - z)\log(r - z)} > 0,$$

showing that $z(x)$ is strictly increasing in the first quadrant. It therefore has a limit as $x \to 0+$ which can only be zero as otherwise $z \log(r - z)$ becomes unbounded. We next determine $z'(0)$. Since $v(x, z)$ is decreasing in z for $0 < |x| < 1$ we have for $0 < z < |x|^\alpha$, $\alpha \in (0, 1)$,

$$x > v(x, z) > v(x, |x|^\alpha) > 2|x| + |x| \log\Big(\frac{|x|^{2-\alpha}}{\sqrt{1 + |x|^{2-2\alpha}} + 1}\Big) \to 0$$

as $x \to 0$. Hence, if $(x, z) \to (0, 0)$ and $z < |x|^\alpha$ for some $\alpha \in (0, 1)$ we have $v(x, z) \to 0$. This means that for the function $z(x)$ defined above,

$$\lim_{x \to 0+} z'(x) = \infty.$$

The contour map of $v(x, z)$ is shown in Figure 1. Hence the surface $u(x, y, z)$ forms a cusp at $(0, 0, 0)$ which is in fact stronger than any polynomial (The general theory of singular points also implies that). We now take a function $\psi \in \mathcal{D}(\mathbb{R})$ with $\psi(x) = \psi(-x)$, $\psi = 1$ on $(0, 1/2)$ and $\psi = 0$ for $x > 3/4$. The curve given by $\psi(x)z(x) + (1 - \psi(x))\sqrt{1 - x^2}$ in the upper half plane and $-\sqrt{1 - x^2}$ in the lower

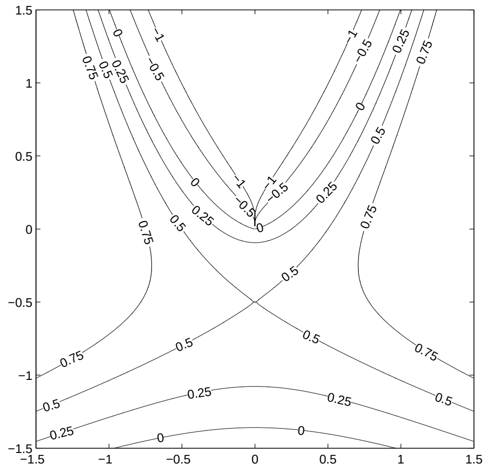

FIGURE 1. Contour map of $v(x,z)$

half plane is a closed C^∞-curve except for $(0,0)$. Now let Ω be the open set enclosed by the surface obtained by revolving that curve about the z-axis. Figure 2 shows $\partial\Omega$ cut open to reveal the inwards pointing cusp at $(0,0,0)$. From the above discussion

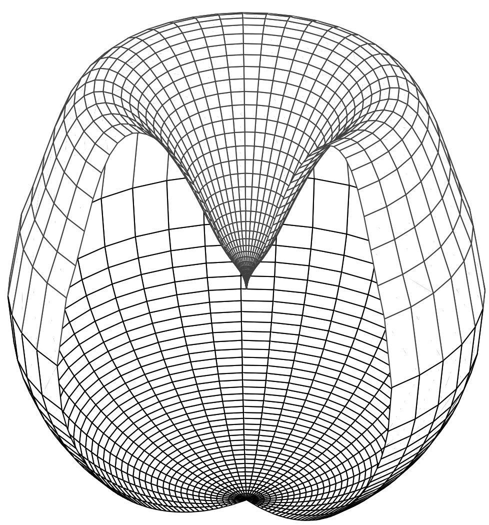

FIGURE 2. Domain with a singular point (cut open)

we know the following.

Lemma 7.1. *The function $u = r + z\log(r-z)$ is harmonic on Ω and continuous on $\bar{\Omega}\setminus\{0\}$. Moreover, $u = 1$ on $B(0,1/2)\cap\partial\Omega$, but*

$$-1 = \varliminf_{(x,y,z)\to(0,0,0)} u(x,y,z) < \varlimsup_{(x,y,z)\to(0,0,0)} u(x,y,z) = 0,$$

so u is not continuous at $(0,0,0)$.

To conclude that u coincides with the Perron solution of

$$\Delta w = 0, \quad w = u|_{\partial\Omega}$$

we use the following theorem proved in [22, Theorem IX].

Theorem 7.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $\varphi \in C(\partial\Omega)$. Moreover, suppose that h is a bounded harmonic function on Ω such that $\lim_{x\to z} h(x) = \varphi(z)$ for every regular point z of $\partial\Omega$. Then $h = h_\varphi$ is the Perron solution.*

Since u is continuous on $\bar{\Omega}\setminus\{0\}$ and harmonic in Ω we know that it coincides with the Perron solution. We now use this to make some observations on the solvability of the Dirichlet problem.

Remark 7.3. Let now Ψ be a C^∞-function such that $\Psi = 1$ if $r < 1/2$ or $\sqrt{x^2+y^2} < 1/2$ and $z > 0$, and $\Psi = 0$ if $r > 3/4$ or $\sqrt{x^2+y^2} > 3/4$ and $z > 0$. If we set

$$\Phi := -\Psi + (1-\Psi)u,$$

then $\Phi \in C^\infty(\mathbb{R}^3)$ and $\Phi|_{\partial\Omega} = u|_{\partial\Omega}$. Hence the Dirichlet problem $\Delta u = 0$ in Ω, $u = \varphi$ on $\partial\Omega$ does not need to be solvable even if φ is the restriction of a C^∞-function which is constant in the neighbourhood of a singular point. This means that the local behaviour of the boundary function φ near a singular point has no influence whatsoever on the solvability of the Dirichlet problem! Also the smoothness of Φ has no influence.

We further observe that $\varphi = u|_{\partial\Omega}$ can be seen locally as the restriction of a harmonic function in the neighbourhood of each point. Indeed, $\varphi = u|_{\partial\Omega}$ is the restriction of the harmonic function u in the neighbourhood of every point of $\partial\Omega$ except for $(0,0,0)$. In a neighbourhood of $(0,0,0)$, φ is the restriction of the constant function with value minus one.

The conclusion is that solvability depends on *global*, not local properties of φ, and also smoothness does not imply solvability at all. In particular, taking -1 as the boundary function, the solution is -1 which is continuous on $\bar{\Omega}$, but if we take the above example which is also -1 in a neighbourhood of the singular point, then the Dirichlet problem is not solvable.

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