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Gaussian estimates for elliptic operators with unbounded drift

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Abstract

We consider a strictly elliptic operator

$$\mathcal{A}u = \sum_{ij} D_i(a_{ij} D_j u) - b \cdot \nabla u + \operatorname{div}(c \cdot u) - Vu,$$

where $0 \leqslant V \in L^\infty_{\mathrm{loc}}$, $a_{ij} \in C^1_b(\mathbb{R}^N)$, $b, c \in C^1(\mathbb{R}^N, \mathbb{R}^N)$. If $\operatorname{div} b \leqslant \beta V$, $\operatorname{div} c \leqslant \beta V$, $0 < \beta < 1$, then a natural realization of $\mathcal{A}$ generates a positive C_0-semigroup T in $L^2(\mathbb{R}^N)$. The semigroup satisfies pseudo-Gaussian estimates if

$$|b| \leqslant k_1 V^\alpha + k_2, \qquad |c| \leqslant k_1 V^\alpha + k_2,$$

where $\frac{1}{2} \leqslant \alpha < 1$. If $\alpha = \frac{1}{2}$, then Gaussian estimates are valid. The constant $\alpha = \frac{1}{2}$ is optimal with respect to this property.

Keywords: Gaussian estimates; Pseudo-Gaussian estimates; Strictly elliptic operator

0. Introduction

We consider a strictly elliptic operator of the form

$$\mathcal{A}u = \sum_{i,j=1}^{N} D_i(a_{ij} D_j u) - b \cdot \nabla u + \operatorname{div}(cu) - Vu$$

on $L^2(\mathbb{R}^N)$ where $a_{ij} \in C^1_b(\mathbb{R}^N)$, $b, c \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ are real coefficients. If b, c, V are bounded, then this is a classical elliptic operator and semigroup properties have been studied extensively. In particular, it is known that the canonical realization of $\mathcal{A}$ in $L^2(\mathbb{R}^N)$ generates a positive C_0-semigroup satisfying Gaussian estimates (see e.g. [4,7,16] and the survey [3]). Here we are interested in the case where the drift terms b and c are

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unbounded. Then one still obtains a semigroup satisfying various regularity properties if the potential V compensates the unbounded drift. We consider the assumption

$$\operatorname{div} b \leqslant \beta V, \qquad \operatorname{div} c \leqslant \beta V \tag{H_1}$$

where $0 < \beta < 1$. Then we show that there is a natural unique realization A of the differential operator $\mathcal{A}$ which generates a minimal positive semigroup T on $L^2(\mathbb{R}^N)$. This semigroup as well as its adjoint are submarkovian. We say that T satisfies *pseudo-Gaussian* estimates of order $m \geqslant 2$ if $T(t)$ has a kernel k_t satisfying

$$0 \leqslant k_t(x, y) \leqslant c_1 e^{\omega t} t^{-N/2} \exp\bigl\{-c_2\bigl(|x-y|^m/t\bigr)^{1/m-1}\bigr\}$$

for all $x, y \in \mathbb{R}^N$, $t > 0$ and some constants $c_1, c_2 > 0$, $\omega \in \mathbb{R}$. In the case where $m = 2$ we say that T satisfies *Gaussian estimates*. In order to obtain such pseudo-Gaussian estimates we impose an additional growth condition on the drift terms b and c, namely,

$$|b| \leqslant k_1 V^\alpha + k_2, \qquad |c| \leqslant k_1 V^\alpha + k_2 \tag{H_2}$$

where $\frac{1}{2} \leqslant \alpha < 1$, $k_1, k_2 \geqslant 0$. If $\alpha = \frac{1}{2}$, then it was proved in [2] that T has Gaussian estimates. The purpose of this paper is to show on one hand that $\alpha = \frac{1}{2}$ is optimal for this property (Section 3). On the other hand, if $\frac{1}{2} < \alpha < 1$, then we show that T still satisfies pseudo-Gaussian estimates even though T need not be holomorphic in that case. Pseudo-Gaussian estimates of order $m > 2$ are still of interest. For instance, they imply that the realizations A_p of A in $L^p(\mathbb{R}^N)$ have all the same spectrum, $1 \leqslant p \leqslant \infty$, at least if $m < \frac{2N}{N-2}$. For elliptic operators with moderately growing drift terms but no compensating V such pseudo-Gaussian estimates had been obtained before by Karrmann [9]. Here we do not study regularity properties of the operator A. For this we refer to [2,14,15]. We also mention the works by Liskevich, Sobol and Vogt [12,13,18] where a different approximation is used and spectral properties are studied.

1. Elliptic operators with unbounded drift

In this section we define the realization of an elliptic operator with unbounded drift in $L^2(\mathbb{R}^N)$. The construction is similar to the one in [2] but we ask for less regularity. Moreover, we establish an additional coerciveness property which is used later to prove quasi-Gaussian estimates. We assume throughout this section that $a_{ij} \in L^\infty(\mathbb{R})$ and

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geqslant \nu|\xi|^2 \tag{1.1}$$

for all $x \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$, where $\nu > 0$ is a fixed constant. Let $b = (b_1, \ldots, b_N)$, $c = (c_1, \ldots, c_N) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, and let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$. We assume in this section that

$$\operatorname{div} b \leqslant V, \qquad \operatorname{div} c \leqslant V. \tag{H_0}$$

Later in Section 2 we will replace (H_0) by a stronger assumption (H_1) and require more regularity on the diffusion coefficients a_{ij} and positivity of the potential. Define the elliptic operator

$$\mathcal{A} : H^1_{\mathrm{loc}}\bigl(\mathbb{R}^N\bigr) \to \mathcal{D}\bigl(\mathbb{R}^N\bigr)',$$
$$\mathcal{A}u = \sum_{i,j=1}^N D_i(a_{ij}D_j u) - b \cdot \nabla u + \operatorname{div}(cu) - Vu,$$

i.e., for $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $v \in \mathcal{D}(\mathbb{R}^N)$ we have

$$-\langle \mathcal{A}u, v\rangle = \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij} D_j u D_i v\, dx + \int_{\mathbb{R}^N} \left\{ \sum_{j=1}^N (b_j D_j u v + c_j u D_j v) + Vuv \right\} dx.$$

We define the maximal operator $A_{\max}$ in $L^2(\mathbb{R}^N)$ by

$$D(A_{\max}) := \bigl\{u \in L^2\bigl(\mathbb{R}^N\bigr) \cap H^1_{\mathrm{loc}}\bigl(\mathbb{R}^N\bigr),\ \mathcal{A}u \in L^2\bigl(\mathbb{R}^N\bigr)\bigr\},$$
$$A_{\max}u = \mathcal{A}u.$$

Now we describe the minimal realization of $\mathcal{A}$ in $L^2(\mathbb{R}^N)$ as follows.

Theorem 1.1. *There exists a unique operator A on $L^2(\mathbb{R}^N)$ such that*

(a) $A \subset A_{\max}$;
(b) *A generates a positive C_0-semigroup T on $L^2(\mathbb{R}^N)$;*
(c) *if $B \subset A_{\max}$ generates a positive C_0-semigroup S, then $T(t) \leqslant S(t)$ for all $t \geqslant 0$.*

We call A the minimal realization *of $\mathcal{A}$ in $L^2(\mathbb{R}^N)$.*

When giving the proof we also establish important properties of A and of T.

Proposition 1.2 *(Coerciveness). One has $D(A) \subset H^1(\mathbb{R}^N)$ and*

$$-(Au|u) \geqslant \nu \|u\|_{H^1}^2 \tag{1.2}$$

for all $u \in D(A)$.

Proposition 1.3 *(Ultracontractivity). The semigroup T and its adjoint are submarkovian. Moreover T is* ultracontractive*, namely*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leqslant c_\nu t^{-N/2} \quad (t > 0), \tag{1.3}$$

where $c_\nu > 0$ depends only on the space dimension and the ellipticity constant ν.

Recall that a C_0-semigroup S on $L^2(\mathbb{R}^N)$ is called *submarkovian* if S is positive and

$$\|S(t)f\|_\infty \leqslant \|f\|_\infty \quad (t > 0),$$

for all $f \in L^\infty \cap L^2$. If B is an operator on $L^2(\mathbb{R}^N)$ we let

$$\|B\|_{\mathcal{L}(L^p,L^q)} := \sup_{\substack{\|f\|_p \leqslant 1 \\ f \in L^2}} \|Bf\|_q.$$

Since T and T^* are submarkovian, it follows from the Riesz–Thorin Theorem that

$$\|T(t)\|_{\mathcal{L}(L^p)} \leqslant 1 \quad (t \geqslant 0),$$

for all $1 \leqslant p \leqslant \infty$.

The remainder of this section is devoted to the proofs of Theorem 1.1 and Propositions 1.2, 1.3. As in [2] we approximate the operator A by realizations of $\mathcal{A}$ on balls whose radii go to ∞. However, here we do not study regularity properties of A and we restrict ourselves to the Hilbert space case $L^2(\mathbb{R}^N)$ (whereas $L^p(\mathbb{R}^N)$ was considered in [2]). Our assumptions on V and a_{ij} are more general than in [2]. Denote by $B_r = \{x \in \mathbb{R}^N\colon |x| < r\}$ the ball of radius $r > 0$. The bilinear form

$$a_r(u,v) := \int_{B_r} \sum_{i,j=1}^N a_{ij} D_j u D_i v\,dx + \int_{B_r} \left\{ \sum_{j=1}^N (b_j D_j u v + c_j u D_j v) + Vuv \right\} dx$$

is continuous on $H_0^1(B_r)$. We show that

$$a_r(u,u) \geqslant \nu \int_{B_r} |\nabla u|^2\,dx \tag{1.4}$$

for all $u \in H_0^1(B_r)$. In fact, let $u \in H_0^1(B_r)$. Then

$$a_r(u,u) \geqslant \nu \int_{B_r} |\nabla u|^2\,dx + \int_{B_r} \left\{ \sum_{j=1}^N (b_j + c_j) \frac{1}{2} D_j u^2 + V u^2 \right\} dx$$
$$= \nu \int_{B_r} |\nabla u|^2\,dx + \int_{B_r} \left(-\operatorname{div} \frac{b+c}{2} + V \right) u^2\,dx \geqslant \nu \int_{B_r} |\nabla u|^2\,dx.$$

In view of Poincaré's inequality, (1.4) implies that a_r is coercive. Denote by $-A_r$ the associated operator on $L^2(B_r)$. Then A_r generates a C_0-semigroup T_r on $L^2(B_r)$. Since $u \in H_0^1(B_r)$ implies that $u^+, u^- \in H_0^1(B_r)$ and $a(u^+, u^-) = 0$ the semigroup T_r is positive by the first Beurling–Deny criterion on forms [16, Theorem 2.6]. Since a_r is coercive, T_r is contractive [16, Chapter 1]. Next we show that for $0 < r_1 < r_2$

$$T_{r_1}(t) \leqslant T_{r_2}(t), \tag{1.5}$$

or, equivalently,

$$R(\lambda, A_{r_1}) \leqslant R(\lambda, A_{r_2}) \quad (\lambda > 0). \tag{1.6}$$

Here we identify $L^2(B_r)$ with a subspace of $L^2(\mathbb{R}^N)$ and extend an operator B on $L^2(B_r)$ to $L^2(\mathbb{R}^N)$ by defining it as 0 on $L^2(B_r)^\perp = \{u \in L^2(\mathbb{R}^N)\colon u|_{B_r} = 0\}$. Similarly, we may identify $H_0^1(B_{r_1})$ with a subspace of $H_0^1(B_{r_2})$, see [5, Proposition IX.18].

Proof of (1.6). Let $0 \leqslant f \in L^2(\mathbb{R}^N)$, $\lambda > 0$, $u_1 = R(\lambda, A_{r_1})f$, $u_2 = R(\lambda, A_{r_2})f$. We want to show that $u_1 \leqslant u_2$. One has by definition of A_{r_1}, A_{r_2},

$$\lambda \int_{B_{r_1}} u_k v + \int_{B_{r_1}} \sum_{i,j=1}^N a_{ij} D_i u_k D_j v + \int_{B_{r_1}} \sum_{i=1}^N b_i D_i u_k v + \int_{B_{r_1}} \sum_{i=1}^N c_i D_i v u_k + \int_{B_{r_1}} V u_k v = \int_{B_{r_1}} f v$$

for all $v \in H_0^1(B_{r_1})$, $k = 1, 2$. Since $u_2 \geqslant 0$ one has $(u_1 - u_2)^+ \leqslant u_1$, hence $(u_1 - u_2)^+ \in H_0^1(B_{r_1})$. Taking $v = (u_1 - u_2)^+$ and subtracting the two identities we obtain

$$\lambda \int_{B_{r_1}} (u_1 - u_2)(u_1 - u_2)^+ + \int_{B_{r_1}} \sum_{i,j=1}^N a_{ij} D_i(u_1 - u_2) \cdot D_j(u_1 - u_2)^+ + \int_{B_{r_1}} \sum_{i=1}^N b_i D_i(u_1 - u_2)(u_1 - u_2)^+$$
$$+ \int_{B_{r_1}} \sum_{i=1}^N c_i D_i(u_1 - u_2)^+(u_1 - u_2) + \int_{B_{r_1}} V(u_1 - u_2)(u_1 - u_2)^+ = 0.$$

Since $D_i(u_1 - u_2)(u_1 - u_2)^+ = D_i(u_1 - u_2)^+(u_1 - u_2)^+$ this gives

$$\lambda \int_{B_{r_1}} (u_1 - u_2)^{+2} + \int_{B_{r_1}} \nu \left|\nabla(u_1 - u_2)^+\right|^2 dx + \int_{B_{r_1}} \left\{ \sum_{j=1}^N \frac{(b_i + c_i)}{2} D_i(u_1 - u_2)^{+2} + V(u_1 - u_2)^{+2} \right\} \leqslant 0.$$

The third term equals

$$\int_{B_{r_1}} \left(-\operatorname{div} \frac{b+c}{2} + V \right) (u_1 - u_2)^{+2}\,dx$$

which is $\geqslant 0$ by the hypothesis (H_0). Thus $(u_1 - u_2)^+ \leqslant 0$, hence $u_1 \leqslant u_2$ on B_{r_1}. □

Next we show that

$$\lim_{r \uparrow \infty} T_r(t) f =: T(t) f \tag{1.7}$$

exists in $L^2(\mathbb{R}^N)$ for all $f \in L^2(\mathbb{R}^N)$ and defines a positive contraction C_0-semigroup whose generator we denote by A.

Proof of (1.7). (a) Let $0 \leqslant f \in L^2(\mathbb{R}^N)$. Since $T_{r_1}(t)f \leqslant T_{r_2}(t)f$ for $0 < r_1 \leqslant r_2$ and $\|T_r(t)f\|_2 \leqslant \|f\|_2$, the limit in (1.7) exists in $L^2(\mathbb{R}^N)$. It follows that $T(t)$ is a positive contraction and $T(t+s) = T(t)T(s)$ for $s, t \geqslant 0$. In order to show that T is strongly continuous, let $0 \leqslant f \in \mathcal{D}(\mathbb{R}^N)$. Let $t_n \downarrow 0$, $f_n = T(t_n)f$. We have to show that $f_n \to f$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. Let $r > 0$ such that $\operatorname{supp} f \subset B_r$. Observe that $0 \leqslant g_n := T_r(t_n)f \leqslant f_n$. Since T_r is strongly continuous, $\lim_{n\to\infty} g_n = f$. Moreover, $\|f_n\|_2 \leqslant \|f\|_2$. Hence $\limsup_{n\to\infty} \|g_n - f_n\|_2^2 = \limsup_{n\to\infty}\{\|g_n\|_2^2 + \|f_n\|_2^2 - 2(g_n|f_n)_2\} \leqslant \limsup_{n\to\infty}\{2\|f\|_2^2 - 2(g_n|g_n)_2\} = 0$. □

We mention that, by dominated convergence as in [1, Section 3.6], property (1.7) implies that

$$R(\lambda, A)f = \lim_{r\uparrow\infty} R(\lambda, A_r)f \tag{1.8}$$

for all $\lambda > 0$, $f \in L^2(\mathbb{R}^N)$. Next we show that

$$D(A) \subset H^1(\mathbb{R}^N) \quad \text{and} \quad \nu \int_{\mathbb{R}^N} |\nabla u|^2\, dx \leqslant (-Au|u) \tag{1.9}$$

for all $u \in D(A)$. Moreover,

$$A \subset A_{\max}. \tag{1.10}$$

(a) We prove (1.9). Let $f \in L^2(\mathbb{R}^N)$, $u_n = R(1, A_{r_n})f$, $u = R(1, A)f$ where $r_n \uparrow \infty$. Then $u_n \to u$ in $L^2(\mathbb{R}^N)$ by (1.8). Since $u_n - A_{r_n}u_n = f$ and $u - Au = f$ in $L^2(B_{r_n})$, it follows that

$$A_{r_n}u_n \to Au \quad \text{in} \quad L^2(\mathbb{R}^N).$$

By (1.4) we have

$$\nu \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \leqslant -(A_{r_n}u_n|u_n).$$

Since $-(A_{r_n}u_n|u_n) \to (-Au \mid u)$ as $n \to \infty$, it follows that

$$\nu \limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \leqslant (-Au|u). \tag{1.11}$$

Thus $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. Considering a subsequence, we may assume that $u_n \to u$ weakly in $H^1(\mathbb{R}^N)$. Let $h = (h_1, \dots, h_N) \in L^2(\mathbb{R}^N)^N$ such that $\|h\|_2 \leqslant 1$. Then by (1.11),

$$\int_{\mathbb{R}^N} \nabla u \cdot h\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla u_n \cdot h\, dx \leqslant \overline{\lim_{n\to\infty}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^{1/2} \leqslant \left[-(Au|u)/\nu\right]^{1/2}.$$

Hence

$$\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx \right)^{1/2} = \sup_{\substack{h\in L^2(\mathbb{R}^N)^N \\ \|h\|_2\leqslant 1}} \int_{\mathbb{R}^N} \nabla u \cdot h\, dx \leqslant \left[-(Au|u)/\nu\right]^{1/2}.$$

Thus (1.9) is proved.

(b) In order to prove (1.10) we keep the notations of (a) and have to show that $u \in D(A_{\max})$ and $Au = A_{\max}u$. Let $v \in \mathcal{D}(\mathbb{R}^N)$. Then

$$(-A_{r_n}u_n|v) = \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij} D_j u_n D_i v\, dx + \int_{\mathbb{R}^N} \left\{ \sum_{j=1}^N (b_j D_j u_n v + c_j u_n D_j v) + V u_n v \right\} dx.$$

Since $u_n \to u$ weakly in $H^1(\mathbb{R}^N)$ and $A_{r_n}u_n \to Au$ in $L^2(\mathbb{R}^N)$, it follows that $(-Au|v) = (\mathcal{A}u|v)$.

Next we show the minimality property in Theorem 1.1. Assume that S is a positive semigroup whose generator B satisfies $B \subset A_{\max}$. Then

$$0 \leqslant T(t) \leqslant S(t) \quad (t \geqslant 0). \tag{1.12}$$

Proof of (1.12). We have to show that

$$R(\lambda, A) \leqslant R(\lambda, B) \tag{1.13}$$

for $\lambda > 0$ sufficiently large. Let $r > 0$; because of (1.8) it suffices to show that

$$R(\lambda, A_r) \leqslant R(\lambda, B). \tag{1.14}$$

Let $f \in L^2(\mathbb{R}^N)$, $f \geqslant 0$, $u_1 = R(\lambda, A_r)f$, $u_2 = R(\lambda, B)f$. Then $0 \leqslant u_1 \in H_0^1(B_r)$, $0 \leqslant u_2 \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$. We have to show that $u_1 \leqslant u_2$. Since $B \subset A_{\max}$ we have $\lambda u_2 - \mathcal{A}u_2 = f$ in $\mathcal{D}(B_r)'$, and also $\lambda u_1 - \mathcal{A}u_1 = f$ in $\mathcal{D}(B_r)'$ by the definition of A_r. Hence

$$\lambda \int_{B_r} (u_1 - u_2)v\,dx + \int_{B_r} \sum_{i,j=1}^N a_{ij} D_j(u_1 - u_2) D_i v\,dx + \int_{B_r} \sum_{j=1}^N \bigl(b_j D_j(u_1 - u_2)v + c_j(u_1 - u_2)D_j v\bigr)\,dx$$
$$+ \int_{B_r} V(u_1 - u_2)v\,dx = 0$$

for all $v \in \mathcal{D}(B_r)$. This identity remains true for $v \in H_0^1(B_r)$ by passing to the limit. Since $u_2 \geqslant 0$ one has $(u_1 - u_2)^+ \leqslant u_1$, hence $(u_1 - u_2)^+ \in H_0^1(B_r)$. Choosing $v = (u_1 - u_2)^+$ in the identity above we obtain

$$\lambda \int_{B_r} (u_1 - u_2)^{+2} + \int_{B_r} \sum_{i,j=1}^N a_{ij} D_j(u_1 - u_2)^+ D_j(u_1 - u_2)^+\,dx$$
$$+ \int_{B_r} \sum_{j=1}^N \bigl(b_j D_j(u_1 - u_2)^+(u_1 - u_2)^+ + c_j D_j(u_1 - u_2)^+(u_1 - u_2)^+\bigr)\,dx + \int_{B_r} V(u_1 - u_2)^{+2}\,dx$$
$$= 0.$$

Consequently

$$\lambda \int_{B_r} (u_1 - u_2)^{+2}\,dx + \nu \int_{B_r} \bigl|\nabla(u_1 - u_2)^+\bigr|^2\,dx + \int_{B_r} \left(-\operatorname{div}\left(\frac{b+c}{2}\right) + V\right)(u_1 - u_1)^{+2}\,dx \leqslant 0.$$

Since $-\operatorname{div}(\frac{b+c}{2}) + V \geqslant 0$ this implies that $(u_1 - u_2)^+ = 0$; i.e., $u_1 \leqslant u_2$. $\square$

The proofs of Theorem 1.1 and Proposition 1.2 are complete.

We now show that T is submarkovian. Because of (1.7), it suffices to show that T_r is submarkovian. By the second criterion of Beurling–Deny–Ouhabaz on forms (see [16]) this is equivalent to

$$a_r\bigl(u \wedge 1, (u-1)^+\bigr) \geqslant 0 \tag{1.15}$$

for all $u \in H_0^1(B_r)$.

Proof of (1.15). Since $D_j(u \wedge 1) = D_j u 1_{\{u<1\}}$, $D_j((u-1)^+) = D_j u 1_{\{u>1\}}$ and $D_j u = 0$ a.e. on $\{u = 1\}$, one has

$$a_r\bigl(u \wedge 1, (u-1)^+\bigr) = \int_{\mathbb{R}^N} \left\{ \sum_{j=1}^N c_j(u \wedge 1) D_j(u-1)^+ + V(u \wedge 1)(u-1)^+ \right\} dx$$

$$= \int_{\mathbb{R}^N} \left\{ \sum_{j=1}^{N} c_j D_j (u-1)^+ + V(u-1)^+ \right\} dx$$

$$= \int_{\mathbb{R}^N} (-\operatorname{div} c + V)(u-1)^+ \, dx \geqslant 0$$

in view of the hypothesis (H_1). □

Next we show that the adjoint semigroup $T^* = (T(t)^*)_{t \geqslant 0}$ is generated by the minimal realization of the adjoint differential operator $\mathcal{A}^*$ which is defined by replacing a_{ij} by a_{ji} and by interchanging b and c, i.e.

$$\mathcal{A}^* u = \sum_{i,j=1}^{N} D_i(a_{ji} D_j u) + c\nabla u - \operatorname{div}(bu) - Vu \quad \big(u \in H^1_{\text{loc}}\big). \tag{1.16}$$

Lemma 1.4. *The minimal realization in $L^2(\mathbb{R}^N)$ of $\mathcal{A}^*$ is the adjoint A^* of A.*

Proof. The adjoint $-A_r^*$ of $-A_r$ is associated with the form a_r^* defined on $H_0^1(B_r) \times H_0^1(B_r)$ by

$$a_r^*(u, v) = a_r(v, u).$$

The semigroup generated by A_r^* is the adjoint T_r^* of T_r. Let B be the minimal realization of $\mathcal{A}^*$ in $L^2(\mathbb{R}^N)$ and S the semigroup generated by B. Then

$$S(t)f = \lim_{r \uparrow \infty} T_r(t)^* f = T(t)^* f$$

for all $f \in L^2(\mathbb{R}^N)$. □

As a consequence, we deduce that also T^* is submarkovian. Finally, we have to show ultracontractivity. We use the following criterion (cf. [6,19], [3, Section 7], [17]).

Proposition 1.5. *For each $\delta > 0$ there exists a constant $c_\delta > 0$ such that the following holds. Let S be a C_0-semigroup on $L^2(\mathbb{R}^N)$ such that S and S^* are submarkovian. Assume that the generator B of S satisfies*

(a) $D(B) \subset H^1(\mathbb{R}^N)$;
(b) $(-Bu|u) \geqslant \delta \|u\|_{H^1}^2$ $(u \in D(B))$;
(c) $(-B^*u|u) \geqslant \delta \|u\|_{H^1}^2$ $(u \in D(B^*))$.

Then

$$\big\| S(t) \big\|_{\mathcal{L}(L^1, L^\infty)} \leqslant c_\delta t^{-N/2} \quad (t > 0). \tag{1.17}$$

The proof of Proposition 1.5 is based on Nash's inequality

$$\|u\|_2^{2+4/N} \leqslant c_N \|u\|_{H^1}^2 \|u\|_1^{4/N} \tag{1.18}$$

for all $u \in H^1(\mathbb{R}^N)$ and some constant $c_N > 0$, and one may choose $c_\delta = (\frac{c_N \cdot N}{\delta})^{N/2}$.

Proof of Proposition 1.5. (i) $D(B) \cap L^1$ is dense in $L^1 \cap L^2$. In fact, the semigroup S extrapolates to a C_0-semigroup on L^1 (see [8], [3, Section 7.2]). Hence for $f \in L^1 \cap L^2$, $\lambda R(\lambda, B) f \to f$ in L^1 and in L^2 as $\lambda \to \infty$. But $\lambda R(\lambda, B) f \in D(B)$.

(ii) Now we modify the proof of [4, Proposition 3.8] to show that

$$\big\| S(t) f \big\|_2 \leqslant \left(\frac{N c_N}{4\delta} \right)^{N/4} t^{-N/4} \|f\|_1 \tag{1.19}$$

for all $f \in D(B) \cap L^1$. Let $f \in D(B) \cap L^1$. Then, by (1.18)

$$\frac{d}{dt}\left\|S(t)f\right\|_2^2 = \left(BS(t)f|S(t)f\right) + \left(S(t)f|B^*S(t)f\right) \leqslant -2\delta\left\|S(t)f\right\|_{H^1}^2 \leqslant -\frac{2\delta}{c_N}\frac{\|S(t)f\|_2^{2+4/N}}{\|S(t)f\|_1^{4/N}}.$$

Hence

$$\frac{d}{dt}\left(\left\|S(t)f\right\|_2^2\right)^{-2/N} = -\frac{2}{N}\left\|S(t)f\right\|_2^{2(-2/N-1)}\frac{d}{dt}\left\|S(t)f\right\|_2^2 \geqslant \frac{4\delta}{Nc_N}\frac{1}{\|S(t)f\|_1^{4/N}} \geqslant \frac{4\delta}{Nc_N}\frac{1}{\|f\|_1^{4/N}}.$$

Integrating, we obtain

$$\left(\left\|S(t)f\right\|_2^2\right)^{-2/N} \geqslant t\frac{4\delta}{Nc_N}\frac{1}{\|f\|_1^{4/N}}$$

which implies (1.19).

It follows from (i) that (1.19) remains true for $f \in L^1 \cap L^2$.

(iii) Applying (b) to S^* instead of S shows that

$$\left\|S^*(t)f\right\|_2 \leqslant \left(\frac{Nc_N}{4\delta}\right)^{N/4} t^{-N/4}\|f\|_1 \tag{1.20}$$

($f \in L^1 \cap L^2$). Hence

$$\left\|S(t)f\right\|_\infty \leqslant \left(\frac{Nc_N}{4\delta}\right)^{N/4} t^{-N/4}\|f\|_2 \tag{1.21}$$

($f \in L^2 \cap L^\infty$). Concluding, for $f \in L^1 \cap L^2$,

$$\begin{aligned}\left\|S(t)f\right\|_\infty &= \left\|S\left(\frac{t}{2}\right)S\left(\frac{t}{2}\right)f\right\|_\infty \leqslant \left(\frac{Nc_N}{4\delta}\right)^{N/4}\left(\frac{t}{2}\right)^{-N/4}\left\|S\left(\frac{t}{2}\right)f\right\|_2 \leqslant \left[\left(\frac{Nc_N}{4\delta}\right)^{N/4}\left(\frac{t}{2}\right)^{-N/4}\right]^2\|f\|_1 \\ &= c_\delta t^{-N/2}\|f\|_1. \qquad \square\end{aligned}$$

Proposition 1.5 implies the ultracontractivity property (1.3) with $c_\nu = (\frac{c_N \cdot N}{\nu})^{N/2}$ since by (1.9) and Lemma 1.4 the hypotheses (a), (b), (c) in Proposition 1.5 are satisfied for the operator $B = A$. Thus the proofs of Theorem 1.1 and Propositions 1.2, 1.3 are complete.

2. Pseudo-Gaussian estimates

Let T be a positive C_0-semigroup on $L^2(\mathbb{R}^N)$. We say that T satisfies *pseudo-Gaussian estimates* of type $m \geqslant 2$ if there exist real constants $c_1 > 0$, $c_2 > 0$, $\omega \in \mathbb{R}$ and a measurable kernel $k_t \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying

$$0 \leqslant k_t(x,y) \leqslant c_1 e^{\omega t} t^{-N/2} \exp\left(-\frac{c_2|x-y|^m}{t}\right)^{1/m-1} \tag{2.1}$$

x, y-a.e. for all $t > 0$ such that

$$\left(T(t)f\right)(x) = \int_{\mathbb{R}^N} k_t(x,y)f(y)\,dy \tag{2.2}$$

x-a.e. for all $t > 0$, $f \in L^2(\mathbb{R}^N)$. If $m = 2$, then we say that T satisfies *Gaussian estimates*.

In fact, the Gaussian semigroup satisfies such an estimate for $m = 2$. It is the best case as the following monotonicity property shows.

Proposition 2.1. *Let $b_1, b_2 > 0$ and let $m_2 > m_1 \geqslant 2$ be real constant. Then there exists $\omega \geqslant 0$ such that*

$$\exp\left(-b_1\left(\frac{|z|^{m_1}}{t}\right)^{1/(m_1-1)}\right) \leqslant \exp\left(-b_2\left(\frac{|z|^{m_2}}{t}\right)^{1/(m_2-1)}\right)e^{\omega t} \tag{2.3}$$

for all $z \in \mathbb{R}^N$, $t > 0$.

Proof. We have to find a constant ω such that

$$-b_1\left(\frac{|z|^{m_1}}{t}\right)^{1/(m_1-1)} \leqslant -b_2\frac{|z|^{m_2}}{t^{-1/(m_2-1)}} + \omega t.$$

Let

$$f_t(x) = b_2 x^{m_2/(m_2-1)} t^{-1/(m_2-1)} - b_1 x^{m_1/(m_1-1)} t^{-1/(m_1-1)} \quad (x \geqslant 0),$$

where $t > 0$. Since $\frac{m_2}{m_2-1} < \frac{m_1}{m_1-1}$, $f_t(\infty) = -\infty$. Moreover, $f_t(0) \leqslant 0$. Let $x \geqslant 0$ such that $f_t'(x) = 0$. Then

$$b_2\frac{m_2}{m_2-1}x^{\frac{1}{m_2-1}}t^{-\frac{1}{m_2-1}} = b_1\frac{m_1}{m_1-1}x^{\frac{1}{m_1-1}}t^{-\frac{1}{m_1-1}}.$$

Hence $\alpha_2(\frac{x}{t})^{\frac{1}{m_2-1}} = \alpha_1(\frac{x}{t})^{\frac{1}{m_1-1}}$. Thus $\frac{\alpha_2}{\alpha_1} = (\frac{x}{t})^{\frac{1}{m_2-1}-\frac{1}{m_1-1}}$. This implies that $x = \beta t$ for some $\beta > 0$ independent of $t > 0$. Thus $\max_{y>0} f_t(y) = f_t(\beta t) = \tilde{b}_2 t - \tilde{b}_1 t$ where $\tilde{b}_2, \tilde{b}_1 \in \mathbb{R}$ are constants. Choose $\omega \geqslant \tilde{b}_2 - \tilde{b}_1$. □

Pseudo-Gaussian estimates can be established with the help of a version of Davies' trick which goes as follows. Let

$$\mathcal{W} := \{\psi \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N):\ \|D_j\psi\|_\infty \leqslant 1,\ \|D_iD_j\psi\|_\infty \leqslant 1,\ i,j = 1,\dots,N\}.$$

Let S be a positive C_0-semigroup on $L^2(\mathbb{R}^N)$. For $\varrho \in \mathbb{R}$, $\psi \in \mathcal{W}$ we denote by S^ϱ the C_0-semigroup given by

$$S^\varrho(t)f = e^{-\varrho\psi}S(t)(e^{\varrho\psi}f). \tag{2.4}$$

We keep in mind that $S^\varrho(t)$ also depends on ψ, but the estimates should not. In fact, we have the following.

Proposition 2.2. *Let $m \geqslant 2$ be a real constant. Assume that there exist $c > 0$, $\omega \in \mathbb{R}$, such that*

$$\|S^\varrho(t)\|_{\mathcal{L}(L^1,L^\infty)} \leqslant ct^{-N/2}e^{\omega(1+\varrho^m)t} \tag{2.5}$$

for all $\varrho \in \mathbb{R}$, $\psi \in \mathcal{W}$, $t > 0$. Then S satisfies pseudo-Gaussian estimates of order m.

We recall the Dunford–Pettis criterion which says that an operator B on $L^2(\mathbb{R}^N)$ is given by a measurable kernel $k \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ if and only if $\|B\|_{\mathcal{L}(L^1,L^\infty)} < \infty$. In that case,

$$\|k\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} = \|B\|_{\mathcal{L}(L^1,L^\infty)}.$$

Proof of Proposition 2.2. This is a modification of [4, Proposition 3.3]. It follows from the Dunford–Pettis criterion applied to the operator $S(t)$ that $S(t)$ is given by a measurable kernel k. Consequently, $S^\varrho(t)$ is given by the kernel

$$k^\varrho(t,x,y) = k(t,x,y)e^{\varrho(\psi(y)-\psi(x))}.$$

Since by the Dunford–Pettis criterion again one has

$$k^\varrho(t,x,y) \leqslant ct^{-N/2}e^{\omega(1+\varrho^m)t},$$

it follows that

$$k(t,x,y) \leqslant ct^{-N/2}e^{\omega t}e^{\omega\varrho^m t \pm \varrho(\psi(y)-\psi(x))}$$

for all $\varrho \in \mathbb{R}$. Now, $d(x,y) = \sup\{\psi(x) - \psi(y):\ \psi \in W\}$ defines a metric on $\mathbb{R}^N$ which is equivalent to the given metric, see [17, pp. 200–202]. Hence $d(x,y) \leqslant \beta|x-y|$ for all $x, y \in \mathbb{R}^N$ and some $\beta > 0$. Thus

$$k(t,x,y) \leqslant ct^{-N/2}e^{\omega t}e^{\omega\varrho^m t - \varrho\beta|y-x|}$$

a.e. Choosing

$$\varrho = \left(\frac{\beta|x-y|}{t\omega m}\right)^{\frac{1}{m-1}}$$

we obtain

$$k(t,x,y) \leqslant ct^{-N/2}e^{\omega t}\exp\left\{-\frac{c_2|y-x|^m}{t}\right\}^{\frac{1}{m-1}}$$

where $c_2 = \beta^{\frac{m}{m-1}}(m^{-\frac{1}{m-1}} - m^{-\frac{m}{m-1}})$. □

Now we have to consider a stronger hypothesis than (H_0), namely

$$\operatorname{div} b \leqslant \beta V, \qquad \operatorname{div} c \leqslant \beta V \tag{H_1}$$

for some constant $0 < \beta < 1$. We also need a condition on the growth of the drift terms b and c with respect to V (assumed nonnegative), namely

$$V \geqslant 0, \quad |b| \leqslant k_1 V^\alpha + k_2, \quad |c| \leqslant k_1 V^\alpha + k_2, \tag{H_2}$$

where $\frac{1}{2} \leqslant \alpha < 1$, $k_1, k_2 \geqslant 0$, as well as some more regularity on the diffusion coefficients:

$$a_{ij} \in C_b^1(\mathbb{R}^N). \tag{H_3}$$

The following result extends [2, Theorem 5.2] from the case $\alpha = \frac{1}{2}$ (i.e., $m = 2$) to $\frac{1}{2} \leqslant \alpha < 1$. Note however, that in contrast to the situation when $\alpha = \frac{1}{2}$, if $\alpha > \frac{1}{2}$ then the operator $-A$ is not associated with a form and the semigroup T may not be holomorphic (see [2, Section 6] and Section 3 below).

Theorem 2.3. *Let A be the minimal realization of the elliptic operator whose coefficients satisfy* (1.1)*,* (H_1)*,* (H_2) *and* (H_3)*. Let T be the semigroup generated by A. Then T satisfies a pseudo-Gaussian estimate of order $m = \frac{1}{1-\alpha}$.*

Proof. Let $\varrho \in \mathbb{R}$, $\psi \in \mathcal{W}$. It is obvious that

$$T^\varrho(t)f = \lim_{r\uparrow\infty} T_r^\varrho(t)f.$$

Thus the generator A^ϱ of T^ϱ is the minimal realization of the elliptic operator $\mathcal{A}^\varrho$ with coefficients

$$a_{ij}^\varrho = a_{ij},$$

$$b_i^\varrho = b_i - \varrho\sum_{j=1}^N a_{ij}\psi_j,$$

$$c_i^\varrho = c_i + \varrho\sum_{i,j=1}^N a_{ki}\psi_k,$$

$$V^\varrho = V - \varrho^2\sum_{i,j=1}^N a_{ij}\psi_i\psi_j + \varrho\sum_{i=1}^N b_i\psi_i - \varrho\sum_{i=1}^N c_i\psi_i,$$

where $\psi_i = D_i\psi$, cf. [4, Lemma 3.6]. We will find $\omega \in \mathbb{R}$ such that for

$$W^\varrho = V^\varrho + (1+\varrho^m)\omega$$

one has

$$\operatorname{div} b^\varrho \leqslant W^\varrho, \qquad \operatorname{div} c^\varrho \leqslant W^\varrho, \tag{2.6}$$

where ω is independent of $\varrho \in \mathbb{R}$ and $\psi \in \mathcal{W}$. Then Proposition 1.3 applied to $A^\varrho - (1+\varrho^m)\omega$ implies that

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leqslant c_\nu t^{-N/2}e^{\omega(1+\varrho^m)t} \quad (t > 0). \tag{2.7}$$

Then Proposition 2.2 proves the claim. In order to prove (2.6) we proceed in several steps. We first show that

$$\varrho V^\alpha \leqslant \varepsilon^{1/\alpha}\alpha V + (1-\alpha)\varepsilon^{-m}\varrho^m \tag{2.8}$$

for all $\varepsilon > 0$. In fact, let $q = \frac{1}{\alpha}$, $\frac{1}{p} = 1 - \frac{1}{q}$ and recall that $m = \frac{1}{1-\alpha} = p$. Then by Hölder's inequality

$$\varrho V^\alpha = \frac{1}{\varepsilon}\varrho V^\alpha \varepsilon \leqslant \frac{1}{p}\frac{1}{\varepsilon^p}\varrho^p + \frac{1}{q}V^{\alpha q}\varepsilon^q = (1-\alpha)\varepsilon^{-m}\varrho^m + \alpha V \varepsilon^{1/\alpha}.$$

Next we show that there exists $\omega_1 \in \mathbb{R}$ such that

$$\beta V \leqslant V^\varrho + \omega_1\big(1 + \varrho^m\big) \tag{2.9}$$

for all $\varrho \in \mathbb{R}$, $\psi \in \mathcal{W}$, where $\beta \in (0, 1)$ is the constant in (H_1). In fact, by (H_2) and (2.8),

$$\begin{aligned} V^\varrho &\geqslant V - k_3\varrho^2 - k_3\varrho V^\alpha - k_4\varrho \\ &\geqslant V - k_3\varrho^2 - k_3\varepsilon^{1/\alpha}\alpha V - k_3(1-\alpha)\varepsilon^{-m}\varrho^m - k_4\varrho \\ &\geqslant \beta V - \omega_1\big(1+\varrho^m\big) \end{aligned}$$

for suitable constants $k_3, k_4\omega_1$ where $\varepsilon > 0$ is chosen such that $\beta = 1 - k_3\varepsilon^{1/\alpha}\alpha$. Now we show (2.6). One has by (2.9),

$$\begin{aligned} \operatorname{div} b^\varrho &= \operatorname{div} b - \varrho \sum_{i,j=1}^{N} D_i(a_{ij}\psi_j) \\ &\leqslant \beta V + k_4\varrho \\ &\leqslant V^\varrho + \omega_1\big(1+\varrho^m\big) + k_5\varrho \\ &\leqslant V^\varrho + \omega\big(1+\varrho^m\big) \end{aligned}$$

for all $\varrho \in \mathbb{R}$, $\psi \in \mathcal{W}$ where k_5, ω are suitable constants. The estimate for $\operatorname{div} c^\varrho$ is the same. □

Remark 2.4. It is obvious from the definition that a semigroup S satisfies (pseudo-) Gaussian estimates if and only if $(e^{\omega t}S(t))_{t\geqslant 0}$ does so for some $\omega \in \mathbb{R}$. Thus in Theorem 2.3 we may replace condition (H_1) by the weaker condition

$$\operatorname{div} b \leqslant \beta V + \beta', \qquad \operatorname{div} c \leqslant \beta V + \beta' \tag{H_1'}$$

where $0 < \beta < 1$, $\beta' \in \mathbb{R}$ and the result remains valid.

As application we obtain a result on p-independence of the spectrum. Assume that assumptions (1.1) and (H_1) are satisfied. Let A be the minimal realization of the elliptic operator $\mathcal{A}$. Then A generates a C_0-semigroup T on $L^2(\mathbb{R}^N)$ and T as well as T^* are submarkovian. As a consequence there exists a consistent family $T_p = (T_p(t))_{t\geqslant 0}$ of semigroups on $L^p(\mathbb{R}^N)$ such that $T_2 = T$. Here T_p is a C_0-semigroup if $1 \leqslant p < \infty$ and T_∞ is a dual C_0-semigroup. We denote by A_p the generator of T_p, $1 \leqslant p \leqslant \infty$.

Corollary 2.5. *Assume that* (1.1)*,* (H_1)*,* (H_2) *and* (H_3) *are satisfied. Assume that* $\alpha < \frac{N+2}{2N}$*. Then* $\sigma(A_p) = \sigma(A)$ *for all* $p \in [1, \infty]$*. Here* $\frac{1}{2} \leqslant \alpha < 1$ *is the constant occurring in hypothesis* (H_2)*.*

Proof. This follows from a result of Karrmann [9, Corollary 6.2] which in turn is a consequence of a result of Kunstmann [10, Theorem 1.1]. □

The restriction

$$\alpha < \frac{N+2}{2N}$$

is due to the fact that Karrmann proves spectral p-independence in the case of quasi-Gaussian estimates of order m if $m < \frac{2N}{N-2}$. We do not know whether these conditions are optimal.

3. An example

In order to show that Theorem 2.3 is optimal we consider the one-dimensional example

$$\mathcal{A}u = u'' - x^3 u' + |x|^\gamma u,$$

where $\gamma > 2$. Then condition (H_1') is satisfied (see Remark 2.4). Let A be the minimal realization of $\mathcal{A}$ in $L^2(\mathbb{R})$ and let T be the semigroup generated by A. If $\gamma \geqslant 6$, then it follows from Theorem 2.3 that T satisfies Gaussian estimates. If $6 > \gamma > 3$, then Theorem 2.3 says that T satisfies pseudo-Gaussian estimates of order $m = \frac{\gamma}{\gamma - 3}$. We show that T does not satisfy Gaussian estimates in that case.

Proposition 3.1. *Let $3 < \gamma < 6$. Then T does not satisfy Gaussian estimates.*

Proof. Assume that $T(t)$ is given by a kernel k_t satisfying

$$0 \leqslant k_t(x, y) \leqslant c_1 e^{\omega t} \frac{1}{\sqrt{t}} e^{-c_2|x-y|^2/t}. \tag{3.1}$$

Consider the operator $I_n \in \mathcal{L}(L^2)$ given by

$$(I_n u)(x) = u\left(\frac{x-n}{\lambda_n}\right),$$

where $\lambda_n = n^{3-\beta}$, $\gamma < \beta < 6$. Then

$$\|I_n u\|_2 = \sqrt{\lambda_n}\|u\|_2 \quad \bigl(u \in L^2(\mathbb{R})\bigr)$$

and $(I_n^{-1}u)(x) = u(\lambda_n x + n)$. Define the semigroup T_n on $L^2(\mathbb{R})$ by

$$T_n(t) = I_n^{-1} T(r_n t) I_n,$$

where $r_n = n^{-\beta}$. It follows from the Trotter–Kato Theorem that

$$\lim_{n\to\infty} T_n(t) f = S(t) f \tag{3.2}$$

for all $f \in L^2(\mathbb{R})$ where S is the shift semigroup given by $(S(t)u)(x) = u(x - t)$ (see [2, Proposition 6.4]) One has for $f \in L^2(\mathbb{R})$

$$\begin{aligned} T_n(t) f(x) &= \bigl(T(r_n t)(I_n f)\bigr)(n + \lambda_n x) \\ &= \int_{\mathbb{R}} k_{r_n t}(n + \lambda_n x, y) f\left(\frac{y-n}{\lambda_n}\right) dy \\ &= \int_{\mathbb{R}} \lambda_n k_{r_n t}(n + \lambda_n x, n + \lambda_n y) f(y)\, dy \\ &= \int_{\mathbb{R}} k_t^n(x, y) f(y)\, dy \end{aligned}$$

where $k_t^n(x, y) = \lambda_n k_{r_n t}(n + \lambda_n x, n + \lambda_n y)$. By (3.1) we obtain

$$\begin{aligned} k_t^n(x, y) &\leqslant n^{3-\beta} c_1 e^{\omega t r_n} \frac{1}{\sqrt{r_n t}} e^{-c_2 \lambda_n^2 |x-y|^2/n^{-\beta} t} \\ &= n^{3-\beta/2} c_1 e^{\omega t r_n} \frac{1}{\sqrt{t}} e^{-c_2 n^{6-\beta}|x-y|^2/t}. \end{aligned}$$

Denoting by $G = (G(t))_{t \geqslant 0}$ the Gaussian semigroup, this implies that for $0 \leqslant f \in L^2(\mathbb{R}^N)$,

$$\bigl(T_n(t) f\bigr)(x) \leqslant c e^{\omega t r_n} \bigl(G\bigl(t/4c_2 n^{6-\beta}\bigr) f\bigr)(x).$$

Thus

$$S(t)f = \lim_{n\to\infty} T_n(t)f \leqslant \lim_{n\to\infty} ce^{\omega t r_n} G\big(t/4c_2 n^{6-\beta}\big) f = c_1 f.$$

This is a contradiction. □

Remark 3.2. It was shown in [2, Proposition 6.4] that for $2 \leqslant \gamma < 6$, the semigroup T is not holomorphic. It seems not to be known whether Gaussian estimates for positive semigroups imply holomorphy. They do not without positivity assumption as Voigt's example

$$Au = u'' + ix$$

on $L^2(\mathbb{R})$ shows (see Liskevich and Manavi [11] for more details).

References

[1] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Problems, Monogr. Math., Birkhäuser, Basel, 2001.

[2] W. Arendt, G. Metafune, D. Pallara, Schrödinger operators with unbounded drift, J. Operator Theory 55 (2006) 185–211.

[3] W. Arendt, Semigroups and evolution equations: Functional calculus, regularity and kernel estimates, in: C.M. Dafermos, E. Feireisl (Eds.), Handbook of Differential Equations: Evolutionary Equations, vol. I, Elsevier, Amsterdam, 2004, pp. 1–85.

[4] W. Arendt, A.F.M. ter Elst, Gaussian estimates for second order elliptic operators with boundary conditions, J. Operator Theory 38 (1997) 87–130.

[5] H. Brézis, Analyse Fonctionelle, Masson, Paris, 1983.

[6] T. Coulhon, Dimension à l'infini d'un semi-groupe analytique, Bull. Sci. Math. 114 (1990) 485–500.

[7] D. Daners, Heat kernel estimates for operators with boundary conditions, Math. Nachr. 217 (2000) 13–41.

[8] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge Univ. Press, 1989.

[9] S. Karrmann, Gaussian estimates for second-order operators with unbounded coefficients, J. Math. Anal. Appl. 258 (2001) 320–348.

[10] P.C. Kunstmann, Kernel estimates and L^p-spectral independence of differential and integral operators, in: Proceedings of the 7th OT Conference, Theta, 2000.

[11] V. Liskevich, A. Manavi, Dominated semigroups with singular complex potentials, J. Funct. Anal. 151 (1997) 281–305.

[12] V. Liskevich, Z. Sobol, Estimates of integral kernels for semigroups associated with second-order elliptic operators with singular coefficients, Potential Anal. 18 (2003) 359–390.

[13] V. Liskevich, Z. Sobol, H. Vogt, On the L^p-theory of C_0-semigroups associated with second-order elliptic operators II, J. Funct. Anal. 193 (2002) 55–76.

[14] G. Metafune, E. Priola, Some classes of nonanalytic Markov semigroups, J. Math. Anal. Appl. 294 (2004) 596–613.

[15] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt, L^p-regularity for elliptic operators with unbounded coefficients, Adv. Difference Equ. 10 (2005) 1131–1164.

[16] E. Ouhabaz, Analysis of Heat Equations on Domains, Princeton Univ. Press, Oxford, 2005.

[17] D.W. Robinson, Elliptic Operators on Lie Groups, Oxford Univ. Press, 1991.

[18] Z. Sobol, H. Vogt, L_p-theory of C_0-semigroups associated with second-order elliptic operators, J. Funct. Anal. 193 (2002) 24–54.

[19] N. Varopoulos, L. Saloff-Coste, T. Coulhon, Geometry and Analysis on Groups, Cambridge Univ. Press, 1993.