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Equivalent complete norms and positivity

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Dedicated to Professor Heinz König on the occasion of his 80th birthday

Abstract. In the first part of the article we characterize automatic continuity of positive operators. As a corollary we consider complete norms for which a given cone E_+ in an infinite dimensional Banach space E is closed and we obtain the following result: every two such norms are equivalent if and only if $E_+ \cap (-E_+) = \{0\}$ and $E_+ - E_+$ has finite codimension.
Without preservation of an order structure, on an infinite dimensional Banach space one can always construct infinitely many mutually non-equivalent complete norms. We use different techniques to prove this. The most striking is a set theoretic approach which allows us to construct infinitely many complete norms such that the resulting Banach spaces are mutually non-isomorphic.

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1. Introduction. Each two norms on a finite dimensional real vector space are equivalent. Conversely, if $(E, \|\cdot\|_1)$ is an infinite dimensional Banach space, then there exists a discontinuous linear functional $\varphi\colon E \to \mathbb{R}$. Take $u \in E$ such that $\varphi(u) = 1$. Then $Sx := x - 2\varphi(x)u$ defines a linear, discontinuous map $S\colon E \to E$, and $S^2 = I$. Thus $\|x\|_2 := \|Sx\|_1$ defines a complete norm on E which is not equivalent to the given norm $\|\cdot\|_1$.

In this article we present several approaches to construct many mutually non-equivalent, complete norms on a Banach space. An extension of the above argument is one technique which yields an infinite number of complete, mutually non-equivalent norms, see Section 4.1. If we want that the resulting Banach spaces are even mutually non-isomorphic, the task is more difficult. We use set theory and assume the Generalized Continuum Hypothesis in Section 4.2 to characterize those cardinals κ for which there exists a Banach space E such that $\mathrm{card}(E) = \kappa$. This answers a question of Laugwitz from 1955. As a corollary, given an arbitrary infinite dimensional Banach space E, we find mutually non-isomorphic, complete norms $\|\cdot\|_p$ for $1 \leq p < \infty$ on E.

König and Wittstock [14] showed in 1992 by what amount one has to modify the norm of a Banach space in order to make a discontinuous functional continuous. Here we prove a somehow dual result, showing how a given set of continuous functionals can be made non-continuous by changing the norm, see Section 3. This is an extension of results due to Donoghue and Masani [8] and Alpay and Mills [2].

We start the article by giving positive results in Section 2. Given an infinite dimensional Banach space $(E, \|\cdot\|)$ with a closed cone E_+, we show that the following two assertions are equivalent.

(i) Each complete norm for which E_+ is closed is equivalent to the given one;
(ii) E_+ is proper and $E_+ - E_+$ is of finite codimension.

Here E_+ is called *proper* if $E_+ \cap (-E_+) = \{0\}$. On the way to this result we characterize under which conditions positive linear maps between ordered Banach spaces are automatically continuous.

2. Positivity, Automatic Continuity and Equivalence of Norms. In this section we investigate which additional properties of a norm determine it uniquely (up to equivalence) among all complete norms, i.e., within which classes of norms there do not exist non-equivalent complete norms. This question is closely related to automatic continuity of operators enjoying additional properties. A beautiful theory exists on automatic continuity of algebra homomorphisms. We will not discuss this here but refer to [20] and [7], for example.

Instead, we will focus on automatic continuity of positive operators and describe the precise necessary and sufficient conditions on ordered Banach spaces E and F such that every positive linear operator $T\colon E \to F$ is continuous. In this section we restrict ourselves to real vector spaces. Recall that a *cone* C is a non-empty subset of a vector space E such that $C + C \subset C$ and $\alpha C \subset C$ for all $\alpha \geq 0$. We say that C is *proper* if $C \cap (-C) = \{0\}$. We call C *generating* if $C + (-C) = E$. A pair (E, E_+), where E is a Banach space and $E_+ \subset E$ is a closed cone, is called an *ordered Banach space*. We call the elements of E_+ *positive* and write $x \leq y$ to express that $y - x \in E_+$. For ordered Banach spaces (E, E_+) and (F, F_+) an operator $T\colon E \to F$ is called *positive* if $T(E_+) \subset F_+$. On the field of scalars $\mathbb{R}$ we always consider the usual positive cone $\mathbb{R}_+ := [0, \infty)$.

It is known that if E_+ is closed and generating and F_+ is closed and proper, then every positive linear operator $T\colon E \to F$ is continuous, see Arendt [3, Appendix] or Aliprantis and Tourky [4, Theorem 2.32]. We will slightly improve this result to necessary and sufficient conditions for automatic continuity. For the sake of completeness, we repeat the proofs of the auxiliary results that we need, despite the fact that they are well-known.

2.1. Positive Functionals. First we only consider the automatic continuity of positive linear functionals.

Lemma 2.1. *Let E be a Banach space and let F be a subspace of E that has finite codimension in E. Assume that there exists a complete norm $\|\cdot\|_F$ on F and a constant $c > 0$ such that $\|x\|_E \leq c\|x\|_F$ for all $x \in F$. Then F is closed in E (with respect to $\|\cdot\|_E$).*

Proof. Let G be a subspace of E such that $F + G = E$ and $F \cap G = \{0\}$. By assumption, G is finite dimensional and hence closed in E. Thus the vector space $V := F \times G$ is a Banach space for the two norms $\|(y, z)\|_1 := \|y + z\|_E$ and $\|(y, z)\|_2 := \|y\|_F + \|z\|_E$. Obviously we have $\|(y, z)\|_1 \leq \max\{1, c\}\|(y, z)\|_2$ for all $(y, z) \in V$. Hence by the open mapping theorem the two norms are equivalent. Since F is closed in $(V, \|\cdot\|_2)$ by construction, it is closed also in $(V, \|\cdot\|_1) = (E, \|\cdot\|_E)$. □

Lemma 2.2. *Let (E, E_+) be an ordered Banach space, $F := E_+ - E_+$. Then*

$$\|x\|_F := \inf\{\|y\|_E + \|z\|_E : y, z \in E_+,\ x = y - z\}$$

defines a complete norm on F. Moreover, $F \hookrightarrow E$, and E_+ is closed in F.

Proof. It is easily checked that $\|\cdot\|_F$ is a norm on F and $\|x\|_E \leq \|x\|_F$ for all $x \in F$. Hence it suffices to show that $(F, \|\cdot\|_F)$ is complete. This is equivalent to the fact that every absolutely convergent series is convergent. So let (x_n) be a sequence in F such that $\sum_{n=1}^{\infty} \|x_n\|_F < \infty$. By definition there exist sequences (y_n) and (z_n) in E_+ such that $x_n = y_n - z_n$ and $\|y_n\|_E + \|z_n\|_E \leq \|x_n\|_F + 2^{-n}$. Thus $\sum_{n=1}^{\infty} \|y_n\|_E < \infty$ and $\sum_{n=1}^{\infty} \|z_n\|_E < \infty$. Since E is complete, the limits $y := \sum_{n=1}^{\infty} y_n$ and $z := \sum_{n=1}^{\infty} z_n$ exist in E, and since E_+ is closed, $y, z \in E_+$. Let $x := y - z \in F$. Then

$$\Big\|x - \sum_{n=1}^{N-1} x_n\Big\|_F = \Big\|\sum_{n=N}^{\infty} y_n - \sum_{n=N}^{\infty} z_n\Big\|_F \leq \sum_{n=N}^{\infty} \|y_n\|_E + \sum_{n=N}^{\infty} \|z_n\|_E \to 0$$

as $N \to \infty$. Hence $\sum_{n=1}^{\infty} x_n$ converges in $(F, \|\cdot\|_F)$. □

Lemma 2.3. *Let (E, E_+) be an ordered Banach space with generating cone. Then every positive linear functional on E is continuous.*

Proof. Let φ be a positive linear functional. We claim that there exists $c > 0$ such that $\varphi(x) \leq c\|x\|$ for all $x \in E_+$. In fact, otherwise there exists a sequence (x_n) in E_+ such that $\|x_n\| \leq 2^{-n}$ and $\varphi(x_n) \geq n$. Since E is complete and E_+ is closed, $x := \sum_{n=1}^{\infty} x_n$ exists and satisfies $x \geq x_m$ for every $m \in \mathbb{N}$. This implies $\varphi(x) \geq \varphi(x_m) \geq m$ for every $m \in \mathbb{N}$, which is absurd.

Now let $\|x\|_+ := \inf\{\|y\|+\|z\| : y, z \in E_+,\ x = y-z\}$. According to Lemma 2.2, $(E, \|\cdot\|_+)$ is a Banach space. Since $\|x\| \le \|x\|_+$ for every $x \in E$, the open mapping theorem implies that $\|\cdot\|$ and $\|\cdot\|_+$ are equivalent. Let $\alpha > 0$ be such that $\|x\|_+ \le \alpha\|x\|$ for all $x \in E$.

Let $x \in E$ be arbitrary. There exist $y, z \in E_+$ such that $x = y - z$. Thus

$$|\varphi(x)| \le |\varphi(y)| + |\varphi(z)| \le c(\|y\| + \|z\|)$$

by what we have already proved. Taking the infimum over all all choices of y and z, we obtain $|\varphi(x)| \le c\|x\|_+ \le c\alpha\|x\|$. We have proved that φ is continuous. □

Theorem 2.4. *Let (E, E_+) be an ordered Banach space. The following assertions are equivalent.*

(i) $\operatorname{codim}_E(E_+ - E_+) < \infty$*;*
(ii) each positive linear functional on E is continuous.

In this case, $E_+ - E_+$ is a closed subspace of E.

Proof. "(i) ⇒ (ii)" According to Lemma 2.2, $F := E_+ - E_+$ equipped with the norm $\|\cdot\|_F$ defined in the lemma is a Banach space. Since $\|x\|_E \le \|x\|_F$ for every $x \in F$, Lemma 2.1 shows that F is closed in E. As F has finite codimension, this implies that F is complemented, i.e., there exists a bounded linear projection P from E onto F. Moreover, $F_+ := E_+$ is a closed, generating cone of F.

Now let φ be any positive linear functional on E. Then $\varphi|_F$ is continuous according to Lemma 2.3, and $\varphi|_{\operatorname{Ker} P}$ is continuous because $\operatorname{Ker} P$ is finite dimensional. Thus $\varphi = \varphi \circ I = \varphi|_F \circ P + \varphi|_{\operatorname{Ker} P} \circ (I - P)$ is continuous.

"(ii) ⇒ (i)" Set $F := E_+ - E_+$ and assume $\operatorname{codim}_E(F) = \infty$. Fix a subspace G of E such that $E = F + G$ and $F \cap G = \{0\}$, and let P be the (possibly unbounded) linear projection of E along F onto G. Since G is infinite dimensional by assumption, there exists an unbounded linear functional ψ on G. Define $\varphi := \psi \circ P$. Then $\varphi|_F = 0$, whence φ is positive. But φ is unbounded on the subspace G, so it cannot be continuous. □

Remark 2.5. One cannot drop the condition that E_+ be closed. In fact, let E_+ equal $\operatorname{Ker}\varphi$ for some discontinuous linear functional φ on E. Then $E_+ - E_+ = E_+$ has codimension 1 and φ is a positive, yet discontinuous linear functional on E.

The following example shows that, if condition (i) is not satisfied, $E_+ - E_+$ may fail to be closed even if E_+ is closed.

Example 2.6. For $x \in \ell^2$ define

$$\|x\|_s := \sum_{n=1}^{\infty} n^2 |x_{2n} + x_{2n+1}|^2,$$

which may be infinite. Then $E := \{x \in \ell^2 : \|x\|_s < \infty\}$ is a Hilbert space for the norm $\|x\|_E := \|x\|_{\ell^2} + \|x\|_s$, the canonical unit vectors (e_n) are total in E, and

$E_+ := \{x \in E : x_n \geq 0 \text{ for } n \in \mathbb{N}\}$ is a closed proper cone in E. Since all e_n are in $F := E_+ - E_+$, F is a dense subspace of E. If $x \in E_+$, then $\sum_{n=1}^{\infty} n^2 x_n^2 < \infty$, as can be seen from the definition. Hence $\sum_{n=1}^{\infty} n^2 |x_n|^2 < \infty$ for all $x \in F$. Thus $u := (u_n)$ with $u_n := (-1)^n/n$ is in E, but not in F. This shows that $F \neq \overline{F} = E$, i.e., F is not closed even though E_+ is a closed cone.

2.2. Positive Operators. Now we are ready to apply the results of the previous section to obtain automatic continuity of positive linear operators.

Lemma 2.7. *Let (E, E_+) be an ordered Banach space. Denote by E'_+ the cone of all positive, continuous linear functionals on E. Then E_+ is proper if and only if $E'_+ - E'_+$ is $\sigma(E', E)$-dense in E'.*

Proof. For $U \subset E$ we denote by $U^\circ := \{x' \in E' : \langle x', u \rangle \leq 1 \text{ for all } u \in U\}$ the *polar set* of U. Since E_+ and $-E_+$ are closed, convex sets, they are weakly closed. Thus $(E_+ \cap -E_+)^\circ$ is the $\sigma(E', E)$-closure of the convex hull of $(E_+)^\circ \cup (-E_+)^\circ$, see [19, IV.1.5, Corollary 2]. Since

$$(E_+)^\circ = \{x' \in E' : \langle x', x \rangle \leq 0 \text{ for all } x \in E_+\},$$

we see that $(E_+)^\circ = -E'_+$ and $(-E_+)^\circ = E'_+$. Because E'_+ and $-E'_+$ are convex cones, the convex hull of $E'_+ \cup -E'_+$ equals $E'_+ + (-E'_+)$. Thus we have shown the identity

$$(E_+ \cap -E_+)^\circ = \operatorname{clo}_{\sigma(E', E)}(E'_+ - E'_+).$$

Hence, if E_+ is proper, $E'_+ - E'_+$ is $\sigma(E', E)$-dense, since $\{0\}^\circ = E'$. If, on the other hand, $E'_+ - E'_+$ is $\sigma(E', E)$-dense in E', then $(E_+ \cap -E_+)^\circ = E'$, whence $E_+ \cap -E_+ \subset \{0\}$ by [19, IV.1.5, Theorem], which shows that E_+ is proper. ☐

Theorem 2.8. *Let (E, E_+) and (F, F_+) be ordered Banach spaces. The following assertions are equivalent.*

- *(i) At least one of the following conditions is fulfilled.*
  - *(1) E is finite dimensional;*
  - *(2) $F_+ = \{0\}$;*
  - *(3) $\operatorname{codim}_E(E_+ - E_+) < \infty$ and F_+ is proper.*
- *(ii) Every positive linear operator from E to F is continuous.*

Proof. "(i) ⇒ (ii)" If one of the first two conditions is fulfilled, the automatic continuity is trivial. So assume (3) and let $T\colon E \to F$ be a positive linear operator. It suffices to show that T is closed. For this, let (x_n) be a sequence in E such that $x_n \to x$ and $Tx_n \to y$. Let $y' \in F'_+$. Then $y' \circ T$ is a positive linear functional on E, hence continuous by Theorem 2.4, and thus $\langle y', Tx_n \rangle \to \langle y', Tx \rangle$. Since $Tx_n \to y$, this shows that $\langle y', y \rangle = \langle y', Tx \rangle$ for all $y' \in F'_+$ and thus for all y' in the $\sigma(F', F)$-closure of $F'_+ - F'_+$, which is F' according to Lemma 2.7. Since F' separates the points of F, we obtain $Tx = y$. Thus T is closed.

"(ii) ⇒ (i)" We may assume that neither E is finite dimensional nor $F_+ = \{0\}$; otherwise there is nothing to show. Fix $f \in F_+$, $f \neq 0$, and choose $f' \in F'$ such that $\langle f', f\rangle = 1$. Let φ be any positive linear functional on E. Then $Tx := \varphi(x)f$ is a positive linear operator from E to F, hence continuous, which implies that $\varphi = f' \circ T$ is continuous. We have shown that any positive linear functional on E is continuous. This implies $\operatorname{codim}_E(E_+ - E_+) < \infty$ according to Theorem 2.4.

Now assume that F_+ is not proper. This means that $G := F_+ \cap -F_+$ is a non-trivial subspace of F. Fix $g \in G$, $g \neq 0$. Since E is infinite dimensional there exists a discontinuous linear functional φ on E. Then $Tx := \varphi(x)g$ defines a discontinuous positive linear operator, contradicting our assumption. Hence F_+ is proper. □

2.3. Examples. We list some spaces which satisfy the assumptions of Theorem 2.8.

(a) In Banach lattices the positive cone is closed, proper and generating.

(b) Let A be a C*-algebra and $E := \{x \in A : x = x^*\}$ be the (closed) subspace of all selfadjoint elements. Then $E_+ := \{x^*x : x \in A\}$ is a closed, proper, generating cone of E.

(c) Let $\Omega \subset \mathbb{R}^d$ be a non-empty open set, $1 \leq p \leq \infty$. The first Sobolev space

$$E := W^{1,p}(\Omega) := \left\{u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega) \text{ for } i = 1, \dots, d\right\}$$

is a Banach space and $E_+ := \{u \in E : u \geq 0 \text{ almost everywhere}\}$ is a closed, proper, generating cone. In fact, E is a sublattice of $L^p(\Omega)$. Note that E_+ is not normal, i.e., there exist unbounded order intervals.

(d) Let $\Omega \subset \mathbb{R}^d$ be a non-empty, open, bounded set, $1 \leq p \leq \infty$ and $k \geq 2$. The higher order Sobolev space

$$E := W^{k,p}(\Omega) := \left\{u \in W^{1,p}(\Omega) : \frac{\partial u}{\partial x_i} \in W^{k-1,p}(\Omega) \text{ for } i = 1, \dots, d\right\}$$

is a Banach space and $E_+ := \{u \in E : u \geq 0 \text{ almost everywhere}\}$ is a closed, proper cone. Note that $W^{k,p}(\Omega)$ is not a lattice. In fact, pick $x_0 \in \mathbb{R}^d$ and $r > 0$ such that $B(x_0, 2r) \subset \Omega$, and define $u(x) := |x - x_0|^2 - r^2$ and $v(x) := \max\{u(x), 0\}$. Then $u \in C^\infty(\overline{\Omega}) \subset E$. Assume that $u^+ \in W^{k,p}(\Omega)$ exists in the sense of vector lattices. There exists a sequence of functions u_n in E such that u_n converges to v pointwise from above. Thus $u^+ \leq v$. But $u^+ \geq v$ by definition. Hence $u^+ = v$. On the other hand, every representative of ∇v is discontinuous on the sphere $\partial B(x_0, r)$, which is a set of positive 1-capacity [10, §5.6.3]. This shows that $v \notin W^{2,1}(\Omega) \supset W^{k,p}(\Omega)$, see [10, §4.8]. This contradicts $v = u^+ \in W^{k,p}(\Omega)$.

But even though the space is not a lattice, the cone is generating for $kp \geq d$ if Ω has sufficiently regular boundary, for example Lipschitz regular boundary. In fact, if $kp > d$ or $p = 1$, then $W^{k,p}(\Omega) \subset L^\infty(\Omega)$ [5.4, Theorem 5.4]. Hence we can write $u = (u + c\mathbb{1}_\Omega) - c\mathbb{1}_\Omega \in E_+ - E_+$ for $u \in W^{k,p}(\Omega)$, where c is an arbitrary constant larger than $\|u\|_\infty$.

The case $kp = d$ and $p \neq 1$ is a little bit more challenging. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a monotonic, non-negative C^∞-function such that $\varphi(x) = x$ for $x \geq 1$ and $\varphi(x) = 0$ for $x \leq 0$. Such a function can be constructed as the antiderivative of a mollification of the function $c\mathbb{1}_{(\frac{1}{4},\frac{3}{4})} + \mathbb{1}_{(\frac{3}{4},\infty)}$ for an adequate choice of $c > 0$. Let u be in E and α be a multiindex. The Sobolev embeddings [1, Theorem 5.4] show that $D^\alpha u \in L^{d/|\alpha|}(\Omega)$ for $|\alpha| \leq k$. Using the chain rule

$$D^\alpha(\varphi \circ u) = \sum_{i=1}^{|\alpha|} (\varphi^{(i)} \circ u) \sum_{\alpha_1+\cdots+\alpha_i=\alpha} \prod_{j=1}^{i} D^{\alpha_j} u,$$

which holds in the sense of distributions, we see that $D^\alpha(\varphi \circ u)$ lies in $L^{d/|\alpha|} \subset L^p$ for $|\alpha| \leq k$, since $\varphi^{(i)}$ is bounded. Thus the functions $v := \varphi \circ u + 1$ and $w := \varphi \circ u - u + 1$ are elements of E_+, hence $u = v - w \in E_+ - E_+$.

(e) Let $\Omega \subset \mathbb{R}^d$ be a non-empty, open, bounded set, and $k \geq 0$. Then $E := C^k(\overline{\Omega})$ is a Banach space and $E_+ := \{u \in E : u \geq 0 \text{ on } \overline{\Omega}\}$ is a closed, proper, generating cone. In fact, since functions in E are bounded, a similar argument as for $W^{k,p}(\Omega)$ applies here.

(f) Let (Ω, d) be a metric space and $\alpha \in (0, 1]$. For $f\colon \Omega \to \mathbb{R}$, set

$$[f]_\alpha := \inf\bigl\{c \geq 0 : |f(x) - f(y)| \leq c\, d(x,y)^\alpha \text{ for all } x, y \in \Omega\bigr\}$$

and $E := C^\alpha(\Omega) := \{f : \Omega \to \mathbb{R} : [f]_\alpha < \infty\}$. Fix an arbitrary $x_0 \in \Omega$. Then $\|f\|_\alpha := [f]_\alpha + |f(x_0)|$ defines a complete norm on E. The space E is a lattice for the pointwise ordering. In fact, for $f \in C^\alpha(\Omega)$,

$$\Bigl||f(x)| - |f(y)|\Bigr| \leq |f(x) - f(y)| \leq [f]_\alpha\, d(x,y)^\alpha$$

for all $x, y \in \Omega$, hence $|f| \in E$. Thus

$$E_+ := \{f \in E : f(x) \geq 0 \text{ for all } x \in \Omega\}$$

is a closed, proper, generating cone. But for example for a non-empty open set $\Omega \subset \mathbb{R}^d$ and $d(x,y) := |x - y|$, E_+ is not normal.

(g) Let $E = W^{1,p}(0,1)$ $(1 \leq p \leq \infty)$ or $E = C^\alpha[0,1]$ $(0 < \alpha \leq 1)$. Let E_0 be the subspace of functions in E that vanish at 0 and 1, and let E_+ denote the elements of E_0 that are pointwise positive. Then E_+ is a closed, proper cone of finite codimension. More precisely, $E_+ - E_+ = E_0$ has codimension 2 in E.

It is a consequence of Theorem 2.8 that in all the above examples there is a unique complete norm on E (up to equivalence) such that E_+ is closed. We make this precise in the following theorem.

Theorem 2.9. *Let (E, E_+) be an ordered, infinite dimensional Banach space. The following assertions are equivalent.*

(i) E_+ is proper and $E_+ - E_+$ has finite codimension;
(ii) each complete norm for which E_+ is closed is equivalent to the given one.

Proof. "(i) $\Rightarrow$ (ii)" Let $|||\cdot|||$ be a second complete norm for which E_+ is closed. Then the assumptions of Theorem 2.8 are fulfilled for the identity mapping I from $(E, \|\cdot\|)$ to $(E, |||\cdot|||)$. According to the open mapping theorem, this implies that $\|\cdot\|$ and $|||\cdot|||$ are equivalent.

"(ii) $\Rightarrow$ (i)" If φ is a linear, discontinuous functional on E, $u \in E$, $\varphi(u) = 1$, then $Sx := x - 2\varphi(x)u$ defines a discontinuous linear mapping $S\colon E \to E$ such that $S^2 = I$. Thus $|||x||| := \|Sx\|$ defines a complete norm on E which is not equivalent to $\|\cdot\|$.

a) Assume that $F := E_+ - E_+$ has infinite codimension. Then there exists an infinite dimensional subspace G of E such that $F \cap G = \{0\}$ and $F + G = E$. Fix any discontinuous linear functional ψ on G, let $u \in G$ be such that $\psi(u) = 1$, and extend ψ to a linear functional φ on E such that $\varphi|_F = 0$. Define S as above. Then $Sx = x$ for $x \in E_+ \subset F$. Hence S is positive and $\|x\| = |||x|||$ for $x \in F$. This shows that E_+ is closed for the norm $|||\cdot|||$. But $|||\cdot|||$ is not equivalent to $\|\cdot\|$. This contradicts our assumption.
b) Assume that $F := E_+ \cap (-E_+) \neq \{0\}$. Pick $u \in F$, $u \neq 0$, let φ be a discontinuous linear functional on E, and define S as above. Then S is positive. Since $S = S^{-1}$, we have even $SE_+ = E_+$. Since S is an isometry between $(E, |||\cdot|||)$ and $(E, \|\cdot\|)$, it follows that E_+ is closed for $|||\cdot|||$. But $|||\cdot|||$ is not equivalent to $\|\cdot\|$. This is a contradiction. □

3. Discontinuous Linear Functionals. In this short section we generalize a result due to Donoghue and Masani [8]. For the proof, we generalize a technique used by Alpay and Mills [2]. We consider a problem which is in some sense dual to the considerations in [14]. There, the authors change the topology in order to make given functionals continuous. Here, we ask conversely whether it is possible to make given functionals discontinuous.

Theorem 3.1. *Let $(E, \|\cdot\|)$ be an infinite dimensional normed space with Hamel basis $\mathcal{B}$ and let A be a subset of $(E, \|\cdot\|)'$ which is of strictly smaller cardinality than $\mathcal{B}$. Then there exists a norm $|||\cdot|||$ on E such that $(E, \|\cdot\|)$ and $(E, |||\cdot|||)$ are isometrically isomorphic and none of the non-zero functionals in A is continuous, i.e., $A \cap (E, |||\cdot|||)' \subset \{0\}$.*

Proof. Without loss of generality let $0 \notin A = \{\varphi_\alpha : \alpha \in I\}$, where $\varphi_\alpha \neq \varphi_\beta$ for $\alpha \neq \beta$. For every $\alpha \in I$ there exists a vector $v_\alpha \in E$ such that $\varphi_\alpha(v_\alpha) \neq 0$. Let $V := \operatorname{span}\{v_\alpha : \alpha \in I\}$. Then $\dim V < \dim E$ by assumption (in the sense of cardinal numbers). Pick an arbitrary subspace U such that $V + U = E$ and $V \cap U = \{0\}$. Then

$$\dim U = \dim E \geq \max\{\aleph_0, \operatorname{card}(I)\} = \operatorname{card}(\mathbb{N} \times I).$$

This shows that there exists an injective mapping from $\mathbb{N} \times I$ into a Hamel basis of U, i.e., we can find a linearly independent family $(u_{n,\alpha})_{n\in\mathbb{N},\alpha\in I}$ in U. By rescaling we can arrange that $\|u_{n,\alpha}\| = \frac{1}{n}$ for all n in $\mathbb{N}$ and all $\alpha \in I$. It is not difficult to see that the vectors $v_{n,\alpha} := v_\alpha + u_{n,\alpha}$ are linearly independent. Extend $(v_{n,\alpha})_{n\in\mathbb{N},\alpha\in I}$ to a Hamel basis of E, so that $(v_\beta)_{\beta\in J}$ is a Hamel basis of E, where $\mathbb{N}\times I \subset J$. Define the linear operator T on E by $Tv_{n,\alpha} := nv_{n,\alpha}$ and $Tv_\beta := v_\beta$ for $\beta \in J \setminus (\mathbb{N} \times I)$. It is obvious that T is a vector space automorphism. Defining $|||Tx||| := \|x\|$, T is an isometric isomorphism from $(E, \|\cdot\|)$ to $(E, |||\cdot|||)$.

We claim that none of the φ_α is continuous with respect to $|||\cdot|||$. Fix $\alpha \in I$. By construction, $v_{n,\alpha} \to v_\alpha$ in $\|\cdot\|$ as $n \to \infty$. By the choice of v_α this implies that there exists $\mu > 0$ such that eventually $|\varphi_\alpha(v_{n,\alpha})| \geq \mu$. In particular, $\varphi_\alpha(v_{n,\alpha}) \not\to 0$. Moreover, $v_{n,\alpha} \to v_\alpha$ implies that $(v_{n,\alpha})_{n\in\mathbb{N}}$ is bounded in $\|\cdot\|$, say $\|v_{n,\alpha}\| \leq M$. Thus $|||v_{n,\alpha}||| = \|T^{-1}v_{n,\alpha}\| \leq M/n$ for all $n \in \mathbb{N}$, which shows $v_{n,\alpha} \to 0$ in $|||\cdot|||$ as $n \to \infty$. This proves that φ_α is not continuous on $(E, |||\cdot|||)$. □

4. Construction of Non-equivalent norms.

4.1. Analytic Approach. Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space. We are going to construct a large number of mutually non-equivalent norms on E such that all the corresponding spaces are isometrically isomorphic to the original one.

Let $(u_\alpha)_{\alpha\in I}$ be a Hamel basis of E. Then every vector $x \in E$ has a unique representation $x = \sum_{\alpha\in I} \lambda_\alpha(x)u_\alpha$ where only finitely many $\lambda_\alpha(x)$ do not vanish.

Proposition 4.1. *At most finitely many coordinate functionals $\lambda_\alpha \colon E \to \mathbb{K}$ are continuous. Moreover, there exists a Hamel basis $(\tilde{u}_\alpha)$ such that none of the corresponding coordinate functionals $\tilde{\lambda}_\alpha$ is continuous.*

Proof. Assume that we can find an infinite subset J of I such that λ_α is continuous for all $\alpha \in J$. We can choose J to be countable, say $J = \{\alpha_1, \alpha_2, \dots\}$. Let

$$x_n := \sum_{k=1}^{n} \frac{u_{\alpha_k}}{\|u_{\alpha_k}\| \cdot k^2}.$$

Since the corresponding series converges absolutely, its partial sums (x_n) converge to some $x \in E$. We can write $x = \sum_{\alpha\in I} \lambda_\alpha(x)u_\alpha$, where $\lambda_\alpha(x) = 0$ for all but finitely many $\alpha \in I$. On the other hand, since λ_{α_k} is continuous, $\lambda_{\alpha_k}(x) = 1/(\|u_{\alpha_k}\|k^2)$ for all $k \in \mathbb{N}$. This is a contradiction. Thus we have proved the first claim.

For the second claim, let u_β be a basis vector with continuous coordinate functional λ_β. By the above argument, there exists a basis vector u_γ such that λ_γ is not continuous. Define $\tilde{u}_\alpha := u_\alpha$ for $\alpha \neq \gamma$ and $\tilde{u}_\gamma := u_\beta + u_\gamma$. It is easy to check that $(\tilde{u}_\alpha)_{\alpha\in I}$ is linearly independent. From

$$x = \sum_{\alpha\in I} \lambda_\alpha(x)u_\alpha = \sum_{\alpha\neq\gamma} \lambda_\alpha(x)\tilde{u}_\alpha + \lambda_\gamma(x)\tilde{u}_\gamma - \lambda_\gamma(x)\tilde{u}_\beta$$

we see that $(\tilde{u}_\alpha)_{\alpha\in I}$ is a Hamel basis and $\tilde{\lambda}_\alpha = \lambda_\alpha$ for $\alpha \neq \beta$, whereas $\tilde{\lambda}_\beta = \lambda_\beta - \lambda_\gamma$. In particular, $\tilde{\lambda}_\beta$ is not continuous. Thus for the Hamel basis $(\tilde{u}_\alpha)_{\alpha\in I}$, the number of continuous coordinate functionals is reduced by one compared to the original Hamel basis $(u_\alpha)_{\alpha\in I}$. We have already seen that only finitely many coordinate functionals of $(u_\alpha)_{\alpha\in I}$ are continuous. So we obtain a Hamel basis without continuous coordinate functionals by successively applying the above argument. □

According to the preceding proposition, we can find a Hamel basis $(u_\alpha)_{\alpha\in I}$ of E such that no coordinate functional λ_α is continuous. For $A \subset I$ we consider the operator S_A defined by

$$S_A x := \sum_{\alpha\in I\setminus A} \lambda_\alpha(x)u_\alpha - \sum_{\alpha\in A} \lambda_\alpha(x)u_\alpha$$

Then $S_A^2 = I$ for all $A \subset I$. Thus S_A is a vector space automorphism of E and $S_A^{-1} = S_A$. We introduce the norms $\|x\|_A := \|S_A x\|$. Note that $\|\cdot\|_\emptyset = \|\cdot\|$. By definition, S_A is an isometry from $(E, \|\cdot\|_A)$ onto $(E, \|\cdot\|)$, so all the spaces $(E, \|\cdot\|_A)$, $A \subset I$, are isometrically isomorphic.

Let $\mathcal{J}$ be the collection of all finite subsets of I. We claim that S_A is discontinuous for $A \in \mathcal{J}$, $A \neq \emptyset$. In fact, let $A \in \mathcal{J}$ and $\alpha \in A$. By Hahn-Banach's theorem there exists $\varphi \in E'$ such that $\varphi(u_\alpha) = 1$ and $\varphi(u_\beta) = 0$ for all $\beta \in A$, $\beta \neq \alpha$. Note that

$$\varphi(S_A x) = \varphi\left(x - 2\sum_{\beta\in A} \lambda_\beta(x)u_\beta\right) = \varphi(x) - 2\lambda_\alpha(x).$$

So if S_A was continuous, then also $\varphi \circ S_A$ and λ_α, contradicting the choice of the Hamel basis.

Now we show that no pair of the norms $\|\cdot\|_A$, $A \subset \mathcal{J}$ is equivalent. To this end, let $A, B \in \mathcal{J}$, $A \neq B$, and assume that $\|\cdot\|_A \sim \|\cdot\|_B$. Then in particular $\|S_A x\| \leq c\|S_B x\|$ for some $c > 0$, and hence $\|S_A S_B x\| \leq c\|x\|$, for all $x \in E$. Thus $S_A S_B = S_{A\Delta B}$ is a bounded operator. Since $A\Delta B \in \mathcal{J}$, this contradicts our last result.

The following theorem summarizes our observations.

Theorem 4.2. *Let E be an infinite dimensional Banach space. There exist mutually non-equivalent norms on E all making the space into an isometric copy of the original one. More precisely, the cardinality of such a set of norms can be chosen to be as large as the dimension of E.*

However, this result is not optimal. One can construct an even larger set of norms which still have the mentioned properties, namely as large as the power set of a Hamel basis. Moreover, every set of mutually non-equivalent complete norms has at most the cardinality of the power set of a Hamel basis. This result is due to Laugwitz [15].

4.2. Set-Theoretic Approach. We now take a different route towards the construction of non-equivalent norms in Banach spaces. Recall that the previous approach was based on the construction of discontinuous vector space automorphisms. Using this idea, however, one can only obtain isometric copies of the original space.

In this section we construct our Banach spaces directly and transfer the norms onto a common underlying vector space. We have to make sure, however, that there are sufficiently many Banach spaces for which the underlying vector spaces are isomorphic. In order to achieve this we characterize those cardinalities which may occur as the cardinality of a Banach space. This answers a question of Laugwitz [15].

First we recall a few definitions and facts from set theory. Proofs can be found in [12, Chapter 5]. Unless otherwise stated, we assume only (ZFC), i.e., the axioms of Zermelo–Fraenkel set theory, including the axiom of choice.

We denote the *cardinality* of a set A by $\mathrm{card}(A)$, and we write $\mathfrak{c} := \mathrm{card}(\mathbb{R})$ for the cardinality of the continuum. For an infinite limit ordinal α we define the *cofinality* $\mathrm{cf}(\alpha)$ *of* α to be the least limit ordinal β such that there is a monotonic function $f\colon \beta \to \alpha$ satisfying $\sup_{\gamma\in\beta} f(\gamma) = \alpha$. An infinite cardinal κ is called *regular* if $\mathrm{cf}(\kappa) = \kappa$; otherwise it is called *singular*. It can be shown that $\mathrm{cf}(\alpha)$ is a regular infinite cardinal.

For this section we need an additional set-theoretic assumption, which we call Hypothesis (H).

Hypothesis (Hypothesis (H)). Let κ be a cardinal number, $\kappa > \mathfrak{c}$.

a) If $\mathrm{cf}(\kappa) > \aleph_0$, then $\kappa^{\aleph_0} = \kappa$.
b) If $\mathrm{cf}(\kappa) = \aleph_0$, then $\mu^{\aleph_0} < \kappa$ for every cardinal $\mu < \kappa$.

We remark that the first part of the hypothesis is always true for $\kappa = \mathfrak{c}$, i.e., $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$. This follows from $\mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\cdot\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$.

The authors do not know whether Hypothesis (H) can be deduced from (ZFC). Nevertheless, Hypothesis (H) holds if we add the *Singular Cardinal Hypothesis* (SCH), an axiom which is strictly weaker than the *Generalized Continuum Hypothesis* (GCH). We mention that it is customary to accept at least the *Continuum Hypothesis* (CH) in the context of automatic continuity results. More precisely, the construction of discontinuous operators often depends on (CH), for example the celebrated result due to Esterle [11] and Dales [6] saying that for every infinite compact space Ω there exists a discontinuous unital algebra homomorphism from $\mathrm{C}(\Omega)$ into some Banach algebra. In fact, the existence of such a homomorphism is independent of (ZFC), cf. [9]. Thus one requires an additional axiom like for example (CH) to prove its existence. Since we treat spaces of arbitrary cardinality, it is only natural that an axiom regarding large cardinals, for example (GCH) or (SCH), comes in.

Proposition 4.3. *In (ZFC) + (SCH), Hypothesis (H) is fulfilled.*

Proof. Let $\kappa > \mathfrak{c}$. If $\mathrm{cf}(\kappa) > \aleph_0$, then $\kappa^{\aleph_0} = \kappa$ according to [12, Theorem 5.22].

Now let $\mathrm{cf}(\kappa) = \aleph_0$ and $\mu < \kappa$. Then κ is singular and hence a limit cardinal [12, Corollary 5.3]. If $\mu \leq \mathfrak{c}$, then $\mu^{\aleph_0} \leq \mathfrak{c}^{\aleph_0} = \mathfrak{c} < \kappa$. If, on the other hand, $\mu > \mathfrak{c}$, then either $\mu^{\aleph_0} = \mu$ or $\mu^{\aleph_0} = \mu^+$, where μ^+ denotes the successor cardinal of μ [12, Theorem 5.22]. Then also $\mu^{\aleph_0} < \kappa$ since κ is a limit cardinal. □

Finally, we provide some information about the cardinality of a Banach space. Let $\mathcal{B}$ be a Hamel basis of a vector space E. Then $\mathrm{card}(E) = \max\{\mathrm{card}(\mathcal{B}), \mathfrak{c}\}$, see Löwig [16]. If E is an infinite dimensional Banach space, then $\mathrm{card}(\mathcal{B}) \geq \mathfrak{c}$. Of course, the latter result is a direct consequence of Baire's theorem if we accept (CH), but it is true already in (ZFC) by a result of Mackey [17, Theorem I-1]. The following fact is an immediate consequence.

Theorem 4.4. *Let E be an infinite dimensional Banach space and $\mathcal{B}$ be a Hamel basis of E. Then* $\mathrm{card}(E) = \mathrm{card}(\mathcal{B})$.

In particular, it follows that the Hamel basis of an infinite dimensional, separable Banach space E is of cardinality $\mathfrak{c}$. In fact, let X be a dense, countable subset of E. Then there is a surjection from the set of sequences in X to the elements of E, hence $\mathrm{card}(E) \leq \mathrm{card}(X)^{\aleph_0} = \aleph_0^{\aleph_0} = \mathfrak{c}$. More is true. In fact, all separable Banach spaces are mutually homeomorphic by a result due to Kadec, cf. [5, III.§8].

Even the Hamel bases of ℓ^∞ are of cardinality $\mathfrak{c}$, since ℓ^∞ is a subspace of the space of all scalar sequences, which has cardinality $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$. Since ℓ^∞ and $L^\infty(0,1)$ are isomorphic as Banach spaces according to a result of Pełczyński [18], also each Hamel basis of $L^\infty(0,1)$ is of cardinality $\mathfrak{c}$. Altogether, almost all Banach spaces occurring in applications can be considered to share the same underlying vector space.

Proposition 4.5. *Assume Hypothesis (H). Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space, $\kappa := \mathrm{card}(E)$. Then $\kappa \geq \mathfrak{c}$ and $\mathrm{cf}(\kappa) > \aleph_0$.*

Proof. We already know that $\kappa \geq \mathfrak{c}$. If $\kappa = \mathfrak{c}$, then $\mathrm{cf}(\kappa) > \aleph_0$ by [12, Corollary 5.12]. Now let $\kappa > \mathfrak{c}$ and assume that $\mathrm{cf}(\kappa) = \aleph_0$. Then there exists an increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ of ordinals in κ such that $\sup_n \alpha_n = \kappa$, i.e., $\bigcup_n \alpha_n = \kappa$. Of course, $\mu_i := \mathrm{card}(\alpha_i) < \kappa$ for all $i \in \mathbb{N}$.

Let $\mathcal{B}$ be a Hamel basis of E. Then $\mathrm{card}(\mathcal{B}) = \kappa$ by Theorem 4.4, so that we can write $\mathcal{B} = (v_\beta)_{\beta\in\kappa}$. Let $U_i := \mathrm{span}\{v_\beta : \beta \in \alpha_i\}$ and $V_i := \overline{U}_i$. By construction, $\bigcup_{i=1}^\infty U_i = E$. Since every element of V_i is the limit of a sequence in U_i,

$$\mathrm{card}(V_i) \leq \mathrm{card}(U_i)^{\aleph_0} = \max\{\mu_i, \mathfrak{c}\}^{\aleph_0} = \max\{\mu_i^{\aleph_0}, \mathfrak{c}^{\aleph_0}\} < \kappa = \mathrm{card}(E),$$

where we have used Hypothesis (H). Thus $V_i \neq E$ for every $i \in \mathbb{N}$. This shows that E is the union of countably many nowhere dense subspaces. This contradicts Baire's theorem and thus proves that the assumption $\mathrm{cf}(\kappa) = \aleph_0$ is false. □

Proposition 4.6. *Assume Hypothesis (H). Let* $\kappa \geq \mathfrak{c}$ *be a cardinal such that* $\mathrm{cf}(\kappa) > \aleph_0$*, let* I *be a set of cardinality* κ*, and let* $p \in [1, \infty)$*. Define* $\ell^p(I)$ *to be the Banach space of all* $x = (x_\alpha)_{\alpha \in I}$ *such that* $\|x\|_p^p := \sum_\alpha |x_\alpha|^p$ *is finite. Then* $\mathrm{card}(\ell^p(I)) = \kappa$*.*

Proof. Since elements of $\ell^p(I)$ are functions from I to $\mathbb{K}$ with at most countable support, we obtain that $\kappa \leq \mathrm{card}(\ell^p(I)) \leq \kappa^{\aleph_0} \cdot \mathfrak{c}^{\aleph_0} = \kappa$. □

Combining these two results, we obtain a characterization of the possible cardinalities for a Banach space, thus answering Laugwitz's question [15, p. 131].

Theorem 4.7. *Assume Hypothesis (H). There exists an infinite dimensional Banach space* E *with* $\mathrm{card}(E) = \kappa$ *if and only if* $\kappa \geq \mathfrak{c}$ *and* $\mathrm{cf}(\kappa) > \aleph_0$*.*

This shows that, assuming Hypothesis (H), there is no Banach space of cardinality $\aleph_\omega$. The cardinals for which there is no Banach space even form a proper class. In fact, if μ is an arbitrary cardinal, then the cardinal $\kappa := \sup\{\mu, 2^\mu, 2^{2^\mu}, \dots\}$ has cofinality $\mathrm{cf}(\kappa) = \aleph_0$, compare [12, p. 58].

Another consequence of our considerations is that every Banach space can be equipped with completely different norms, i.e., norms such that the resulting spaces are mutually non-isomorphic. In particular, the norms are mutually non-equivalent, so at most one of them is equivalent to the original one.

Theorem 4.8. *Assume Hypothesis (H). Let* $(E, \|\cdot\|)$ *be an infinite dimensional Banach space. Then there exist norms* $\|\cdot\|_p$*,* $p \in [1, \infty)$*, on the vector space* E *such that* $(E, \|\cdot\|_p)$ *is a Banach space and the spaces* $(E, \|\cdot\|_p)$ *and* $(E, \|\cdot\|_q)$ *are non-isomorphic for* $p \neq q$*.*

Proof. Let $\mathcal{B}$ be a Hamel basis of E and let $\kappa := \mathrm{card}(\mathcal{B}) = \mathrm{card}(E)$. Then $\kappa = \mathrm{card}(\ell^p(\mathcal{B}))$ for every $p \in [1, \infty)$ by Propositions 4.5 and 4.6. Since each Hamel basis of $\ell^p(\mathcal{B})$ has cardinality κ, too, there exists a vector space isomorphism T_p from E onto $\ell^p(\mathcal{B})$ for every p. Defining $\|x\|_p := \|T_p x\|_p$ on E, the space $(E, \|\cdot\|_p)$ is isometrically isomorphic to $\ell^p(\mathcal{B})$. It is a consequence of Pitt's Theorem [13, §42.3.(10)] that, if $p < q$, then every bounded operator T from $(E, \|\cdot\|_q)$ to $(E, \|\cdot\|_p)$ is compact. In particular, the spaces are not isomorphic. □

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