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Fourier Series in Banach spaces and Maximal Regularity

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Abstract. We consider Fourier series of functions in $L^p(0, 2\pi; X)$ where X is a Banach space. In particular, we show that the Fourier series of each function in $L^p(0, 2\pi; X)$ converges unconditionally if and only if $p = 2$ and X is a Hilbert space. For operator-valued multipliers we present the Marcinkiewicz theorem and give applications to differential equations. In particular, we characterize maximal regularity (in a slightly different version than the usual one) by R-sectoriality. Applications to non-autonomous problems are indicated.

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0. Introduction

The study of vector-valued Fourier transforms is on one hand motivated by the structure theory of Banach spaces where the validity of certain classical properties reflects geometric properties of the Banach spaces; on the other hand, it has fundamental applications in PDE. Of particular importance is the subject of operator-valued Fourier multipliers which have immediate applications to properties of maximal regularity for evolution equations. The aim of this article is to introduce into this subject and to show how it can be applied.

The approach we use here is based on Fourier series. Given a Banach space X, the Fourier series of each $f \in L^p(0, 2\pi; X)$ converges in $L^p(0, 2\pi; X)$ if and only if X is a UMD-space and $1 < p < \infty$. We show here that the Fourier series converges unconditionally in $L^p(0, 2\pi; X)$ if and only if $p = 2$ and X is a Hilbert space (Theorem 1.5). Even though this result is known to specialists of unconditional

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structures in Banach spaces, it seems not be contained in the literature. The result can be reformulated by saying that each bounded sequence of scalar operators $(\lambda_k I)_{k\in\mathbb{Z}}$ is a multiplier for $L^p(0,2\pi;X)$ if and only if $p=2$ and X is a Hilbert space.

The phenomenon that operator-valued versions of certain classical multiplier theorems are only valid in Hilbert spaces was first observed by Pisier (unpublished) as a consequence of Kwapien's deep characterization of Hilbert spaces. In recent years the subject saw a spectacular revival. A break-through was the operator-valued Michlin multiplier theorem proved by Weis [W01] in 2001. Here we will concentrate on periodic multipliers, i.e., operator-valued versions of the Marcinkiewicz multiplier Theorem.

If $1<p<\infty$ and X is a Hilbert space, we present the Marcinkiewicz multiplier Theorem for operator-valued sequences $(M_k)_{k\in\mathbb{Z}} \in \mathcal{L}(X)$ (Theorem 1.7 and Corollary 1.8). This result has a version which also holds in UMD-spaces but the notion of R-boundedness is needed. This is the content of Section 2.

Then we show how the operator-valued Marcinkiewicz multiplier Theorem (Theorem 2.3) can be applied to vector-valued differential equations. Most natural is the periodic case (Section 3), but our emphasis is on the classical maximal regularity problem:

$$P_0(\tau,p) \quad \begin{cases} \dot u(t) = Au(t) + f(t), & t\in(0,\tau) \text{ a.e.} \\ u(0) = 0 \ . \end{cases}$$

Here we consider an operator $A \in \mathcal{L}(D,X)$, where D is a Banach space which is continuously embedded in X. If for $1\le p<\infty$, $P_0(\tau,p)$ is well posed (i.e., for all $f\in L^p(0,\tau;X)$ there is a unique solution $u\in W^{1,p}(0,\tau;X)\cap L^p(0,\tau;D)$ of $P_0(\tau,p)$), then we show that the operator A is closed and R-sectorial without any assumptions on the spaces. This defers somehow from the usual setting since a priori, we do not consider A as an unbounded operator on X. To do so is motivated by the recent interesting applications of maximal regularity to the non-autonomous problem which we explain briefly in Section 5.

In Section 4, we also show that conversely, problem $P_0(\tau,p)$ is well posed whenever $1<p<\infty$, X is a UMD-space and A is R-sectorial.

In the present paper, we explain and complement results of [AB02] and the approach to maximal regularity via periodic multipliers chosen there. There is an alternative way based on the Michlin multiplier theorem, and we refer to [W01] and [KW04] for further information.

1. Vector-valued Fourier series and operator-valued Fourier multipliers

Let X be a Banach space and let $L^p_{2\pi}(X) := \big\{f:\mathbb{R}\to X \text{ measurable}, f(t+2\pi) = f(t) \text{ a.e. and } \int_0^{2\pi} \|f(t)\|^p dt < \infty\big\}$ the space of all X-valued 2π-periodic locally p-integrable functions on $\mathbb{R}$, $1\le p<\infty$. Then $L^p_{2\pi}(X)$ is a Banach space for the norm

$$\|f\|_p := \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(t)\|^p dt \right)^{1/p}.$$

If $f \in L^p_{2\pi}(X)$, then we denote by

$$\hat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt$$

the kth Fourier coefficient of f, where $k \in \mathbb{Z}$.

The Fourier series of f

$$\sum_{k=-\infty}^{+\infty} e_k \otimes \hat{f}(k)$$

converges in the sense of Cesàro to f in $L^p_{2\pi}(X)$; i.e., $\|\sigma_n - f\|_p \to 0$ $(n \to \infty)$, where $\sigma_n = \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes \hat{f}(k)$. Here we let $e_k(t) = e^{ikt}$ $(t \in \mathbb{R})$ and for $x \in X$ we define $e_k \otimes x$ by $(e_k \otimes x)(t) = e^{ikt} x$ $(t \in \mathbb{R})$, where $k \in \mathbb{Z}$. In particular, the space of all X-valued **trigonometric polynomials**

$$\mathcal{T}(X) := \Big\{ \sum_{k=-n}^{n} e_k \otimes x_k : \quad n \in \mathbb{N},\ x_{-n}, \ldots, x_n \in X \Big\}$$

is dense in $L^p_{2\pi}(X)$ for all $1 \le p < \infty$. Moreover, the Uniqueness Theorem holds: If $f \in L^p_{2\pi}(X)$ is such that $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$, then $f = 0$ a.e. This allows us to define operator-valued multipliers in the following way. If Y is another Banach space, we denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X to Y. When $X = Y$, we will simply denote it by $\mathcal{L}(X)$.

Definition 1.1. Let X, Y be Banach spaces and let $1 \le p < \infty$. A sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ is an L^p**-multiplier** if for each $f \in L^p_{2\pi}(X)$, there exists $g \in L^p_{2\pi}(Y)$ such that $\hat{g}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Now the Uniqueness Theorem guarantees that g is unique. It follows that the mapping $f \mapsto g$ is linear. Thus, by the Closed Graph Theorem there exists a unique linear operator $M \in \mathcal{L}(L^p_{2\pi}(X), L^p_{2\pi}(Y))$ such that

$$(Mf)^\wedge(k) = M_k \hat{f}(k) \quad (k \in \mathbb{Z}) \tag{1.1}$$

for all $f \in L^p_{2\pi}(X)$. The operators obtained in this way are exactly the translation invariant operators. To make this precise we consider the C_0-group $\mathcal{U}$ on $L^p_{2\pi}(X)$ given by

$$(\mathcal{U}(\tau)f)(t) = f(t+\tau) \quad (t \in \mathbb{R}).$$

The same group is also considered on $L^p_{2\pi}(Y)$ without changing the name. Then an operator $T \in \mathcal{L}\big(L^p_{2\pi}(X), L^p_{2\pi}(Y)\big)$ is called **translation invariant**, if

$$\mathcal{U}(t)T = T\mathcal{U}(t)$$

for all $t \in \mathbb{R}$.

Proposition 1.2. *Let $T \in \mathcal{L}\big(L^p_{2\pi}(X), L^p_{2\pi}(Y)\big)$, where X, Y are Banach spaces and $1 \le p < \infty$. The following assertions are equivalent*

(i) *T is translation invariant;*

(ii) *there exist $M_k \in \mathcal{L}(X, Y)$, such that*

$$(Tf)^\wedge(k) = M_k \hat{f}(k), \quad (k \in \mathbb{Z})$$

for all $f \in L^p_{2\pi}(X)$.

The following lemma is needed for the proof of Proposition 1.2.

Lemma 1.3. *Let $g \in L^p_{2\pi}(Y)$, $k \in \mathbb{Z}$ be such that*

$$g(t+\tau) = e^{ik\tau} g(t) \quad t\text{-a.e.}$$

for all $\tau \in \mathbb{R}$. Then there exists a unique $y \in Y$ such that $g = e_k \otimes y$.

Proof. Let $\mathcal{B}$ be the generator of $\mathcal{U}$. Then $D(\mathcal{B}) = W^{1,p}_{2\pi}(Y) := \big\{u \in C(\mathbb{R}; Y) : u(t+2\pi) = u(t)$ for all $t \in \mathbb{R}$, and $u' \in L^p_{2\pi}(Y)\big\}$ and $\mathcal{B}u = u'$ for all $u \in D(\mathcal{B})$. Here u' is understood in the sense of distributions. Now the assumption on g says that $\mathcal{U}(\tau)g = e^{ik\tau} g$ for all $\tau \in \mathbb{R}$. Thus $g \in D(\mathcal{B})$ and $g' = ikg$. It follows that g' is continuous, hence $g \in C^1(\mathbb{R}; Y)$. Consequently, $g(t) = e^{ikt} g(0)$ for all $t \in \mathbb{R}$. □

Proof of Proposition 1.2. (i)⇒(ii). Assume that T is translation invariant. Let $k \in \mathbb{Z}$. For $x \in X$ consider $g := T(e_k \otimes x)$. Then $\mathcal{U}(\tau)g = T\mathcal{U}(\tau)(e_k \otimes x) = e^{ik\tau} T(e_k \otimes x) = e^{ik\tau} g$ for all $\tau \in \mathbb{R}$. By Lemma 1.3 there exists a unique $y \in Y$ such that

$$T(e_k \otimes x) = e_k \otimes y. \tag{1.2}$$

We let $M_k x := y$. Then $M_k : X \to Y$ is linear and continuous. Moreover, by (1.2) one has

$$(Tf)^\wedge(k) = M_k \hat{f}(k) \tag{1.3}$$

for all $f \in \mathcal{T}(X)$. Since $\mathcal{T}(X)$ is dense in $L^p_{2\pi}(X)$ and T is continuous from $L^p_{2\pi}(X)$ to $L^p_{2\pi}(Y)$, the identity (1.3) remains true for all $f \in L^p_{2\pi}(X)$.

(ii)⇒(i). Let $\tau \in \mathbb{R}$ and $f \in L^p_{2\pi}(X)$. Then for $k \in \mathbb{Z}$ one has

$$\begin{aligned} \Big(\mathcal{U}(\tau)Tf\Big)^\wedge(k) &= e^{ik\tau}(Tf)^\wedge(k) = e^{ik\tau} M_k \hat{f}(k) \\ &= M_k\Big(e^{ik\tau}\hat{f}(k)\Big) = M_k\Big(\mathcal{U}(\tau)f\Big)^\wedge(k) \\ &= \Big(T\mathcal{U}(\tau)f\Big)^\wedge(k). \end{aligned}$$

The Uniqueness Theorem implies that $\mathcal{U}(\tau)Tf = T\mathcal{U}(\tau)f$. □

Next we want to describe criteria which insure that a given sequence $(M_k)_{k\in\mathbb{Z}}$ in $\mathcal{L}(X, Y)$ is an L^p-multiplier. We say that a Banach space X is a **Hilbert space**, if there exists a scalar product $\langle \cdot\,, \cdot \rangle$ on X such that $\langle x, x\rangle^{1/2}$ defines an equivalent norm on X. If $p \ne 2$, then even in the scalar case there are bounded sequences which are not L^p-multipliers. If both X, Y are Hilbert spaces, then each bounded sequence $(M_k)_{k\in\mathbb{Z}}$ is an L^2-multiplier. The following result shows that the converse remains true.

Theorem 1.4. *Let X, Y be Banach spaces. Assume that each bounded sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ is an L^2-multiplier. Then both spaces X, Y are Hilbert spaces.*

Proof. It follows from the assumption that there exists $C > 0$ such that for every finite sequence $(x_k)_{k\in\mathbb{Z}}$ in X and $M_k \in \mathcal{L}(X,Y)$ satisfying $\|M_k\| \le 1$, we have

$$\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes M_k x_k\Big\|_{L^2(0,2\pi;Y)} \le C\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes x_k\Big\|_{L^2(0,2\pi;X)}. \tag{1.4}$$

Let $(x_k)_{k\in\mathbb{Z}}$ be a finite sequence in X. There exist $f_k \in X'$ such that $f_k(x_k) = \|x_k\|$ and $\|f_k\| = 1$. Let $u \in X,\ \|u\| = 1$ be fixed. Consider the linear operators $M_k \in \mathcal{L}(X,Y)$ given by $M_k(x) := f_k(x)u$ $(x \in X)$. Then $\|M_k\|_{\mathcal{L}(X,Y)} = 1$. Moreover, $M_k x_k = f_k(x_k)u = \|x_k\|u$. It follows from (1.4) that

$$\Big(\sum_{k\in\mathbb{Z}} \|x_k\|^2\Big)^{1/2} \le C\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes x_k\Big\|_{L^2(0,2\pi;X)}.$$

Thus X is of Fourier type 2. Hence X is a Hilbert space [Pie07, page 317].

Now we are going to show that Y is also a Hilbert space. For this we let $(y_k)_{k\in\mathbb{Z}}$ be a finite sequence in Y and let $u \in X$ and $f \in X'$ be such that $\|u\| = \|f\| = f(u) = 1$. Consider the linear operators $N_k \in \mathcal{L}(X,Y)$ given by $N_k(x) := f(x)y_k/\|y_k\|$ $(x \in X)$ when $y_k \ne 0$, and $N_k = 0$ when $y_k = 0$. Then $\|N_k\|_{\mathcal{L}(X,Y)} \le 1$. It follows from (1.4) that

$$\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes N_k x_k\Big\|_{L^2(0,2\pi;Y)} \le C\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes x_k\Big\|_{L^2(0,2\pi;X)}$$

for $x_k \in X$. Taking $x_k = \|y_k\|u$. Then $N_k x_k = y_k$. It follows that

$$\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes y_k\Big\|_{L^2(0,2\pi;Y)} \le C\Big(\sum_{k\in\mathbb{Z}} \|y_k\|^2\Big)^{1/2}.$$

We have shown that Y is of Fourier cotype 2. Thus Y is of Fourier type 2 and hence Y is a Hilbert space [Pie07, p. 316]. ☐

When $X = Y$, the preceding result can be improved. Instead of considering all bounded linear operator sequences, we only need to consider $(\lambda_k I)_{k\in\mathbb{Z}}$ where $(\lambda_k)_{k\in\mathbb{Z}}$ is a bounded scalar sequence. For the proof we need to introduce Rademacher functions. We denote by r_k the kth **Rademacher function** on $[0,1]$ given by $r_k(t) = \operatorname{sgn}\big(\sin(2^k\pi t)\big),\ k = 1,2,3,\ldots$. Here we recall two fundamental properties of Rademacher functions which will be used in the proof of the next result. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a bijection and let $(x_k)_{k\ge1}$ be a finite sequence in X. Then it follows easily from the definition hat

$$\Big\|\sum_{k} r_k x_k\Big\|_{L^2(0,1;X)} = \Big\|\sum_{k} r_{\pi(k)} x_k\Big\|_{L^2(0,1;X)}.$$

Kahane's contraction principle states that for every finite sequence $(x_k)_{k\geq 1}$ in X and $\lambda_k \in \mathbb{C}$ with $|\lambda_k| \leq 1$,

$$\Big\| \sum_k \lambda_k r_k x_k \Big\|_{L^2(0,1;X)} \leq 2 \Big\| \sum_k r_k x_k \Big\|_{L^2(0,1;X)}.$$

(see, e.g., [LT79]). Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two measure spaces. We consider subsets $\mathcal{J}_1 \subset L^2(\Omega_1)$, $\mathcal{J}_2 \subset L^2(\Omega_2)$. Then by

$$\Big\| \sum_k f_k x_k \Big\|_{L^2(\Omega_1;X)} \simeq \Big\| \sum_k g_k x_k \Big\|_{L^2(\Omega_2;X)}$$

we mean that there exists a constant $C > 0$ depending only on X such that for every finite number of $x_k \in X$, and every sequence $(f_k) \subset \mathcal{J}_1$ and $(g_k) \subset \mathcal{J}_2$ (satisfying possibly some further restriction to be made precise), one has

$$\frac{1}{C} \Big\| \sum_k f_k x_k \Big\|_{L^2(\Omega_1;X)} \leq \Big\| \sum_k g_k x_k \Big\|_{L^2(\Omega_2;X)} \leq C \Big\| \sum_k f_k x_k \Big\|_{L^2(\Omega_1;X)}.$$

Theorem 1.5. *Let X be a Banach space. Assume that each bounded sequence $(\lambda_k I)_{k\in\mathbb{Z}}$ defines an L^2-multiplier. Then X is a Hilbert space.*

Proof. We claim that for every finite number of trigonometric polynomials $f_k \in L^2(0, 2\pi)$ and every finite number of $x_k \in X$, we have

$$\Big\| \sum_k r_k f_k x_k \Big\|_{L^2([0,1]\times[0,2\pi];X)} \simeq \Big\| \sum_{n,k} r_{m_{n,k}} \hat{f}_k(n) x_k \Big\|_{L^2([0,1];X)}, \tag{1.5}$$

where $m_{n_1,k_1} \neq m_{n_2,k_2}$ when $(n_1, k_1) \neq (n_2, k_2)$. Indeed, it follows from the Kahane's contraction principle that for every finite number of $x_k \in X$,

$$\Big\| \sum_k r_k e_k \otimes x_k \Big\|_{L^2([0,1]\times[0,2\pi];X)} \simeq \Big\| \sum_k r_k x_k \Big\|_{L^2([0,1];X)}. \tag{1.6}$$

On the other hand, since every bounded scalar sequence defines an L^2-multiplier by assumption, it follows that

$$\Big\| \sum_k r_k e_k \otimes x_k \Big\|_{L^2([0,1]\times[0,2\pi];X)} \simeq \Big\| \sum_k e_k \otimes x_k \Big\|_{L^2([0,2\pi];X)}. \tag{1.7}$$

If $f \in L^2(0, 2\pi)$, we let $\Delta(f) := \{n \in \mathbb{Z} : \hat{f}(n) \neq 0\}$ be the Fourier spectrum of f. Then there exist $N_1, N_2, \ldots \in \mathbb{N}$ sufficiently large, so that $\{0\} < \Delta(e_{N_k} f_k) < \Delta(e_{N_l} f_l)$ whenever $k < l$. Here by $M_1 < M_2$ we mean that $k < l$ for every $k \in M_1$ and $l \in M_2$, where M_1 and M_2 are subsets of $\mathbb{N}$.

It follows from Kahane's contraction principle, (1.6) and (1.7) that

$$\begin{aligned}
&\Big\|\sum_k r_k f_k x_k\Big\|_{L^2([0,1]\times[0,2\pi];X)}\\
&\quad\simeq \Big\|\sum_k r_k e_{N_k}\Big(\sum_n e_n\otimes \hat f_k(n)\Big)x_k\Big\|_{L^2([0,1]\times[0,2\pi];X)}\\
&\quad= \Big\|\sum_{k,n} r_k e_{N_k+n}\otimes \hat f_k(n)x_k\Big\|_{L^2([0,1]\times[0,2\pi];X)}\\
&\quad\simeq \Big\|\sum_{k,n} r_{m_{n,k}} r_k e_{N_k+n}\otimes \hat f_k(n)x_k\Big\|_{L^2([0,1]\times[0,1]\times[0,2\pi];X)}\\
&\quad\simeq \Big\|\sum_{k,n} r_{m_{n,k}} e_{N_k+n}\otimes \hat f_k(n)x_k\Big\|_{L^2([0,1]\times[0,2\pi];X)}\\
&\quad\simeq \Big\|\sum_{k,n} r_{m_{n,k}} \hat f_k(n)x_k\Big\|_{L^2([0,1];X)}.
\end{aligned}$$

We have shown that our claim (1.5) is true.

Now under the assumption of the theorem, X is a UMD-space as the Hilbert transform is bounded on $L^2(0,2\pi;X)$. It follows that X has a non trivial cotype $q<\infty$. This implies that for every finite number of $x_k\in X$, one has

$$\Big\|\sum_k r_k x_k\Big\|_{L^2([0,1];X)}\simeq \Big\|\sum_k g_k x_k\Big\|_{L^2(\Omega;X)} \tag{1.8}$$

where the g_k's are independent complex standard Gaussian variables on some probability space (Ω,Σ,p) [LeTa91, p. 253]. It follows from this and (1.5) that if $f_k\in L^2(0,2\pi)$ are trigonometric polynomials, then

$$\Big\|\sum_k g_k f_k x_k\Big\|_{L^2([0,2\pi]\times\Omega;X)}\simeq \Big\|\sum_{n,k} g_{n,k}\hat f_k(n)x_k\Big\|_{L^2(\Omega';X)} \tag{1.9}$$

where the $g_{n,k}$'s are independent complex standard Gaussian variables on some probability space (Ω',Σ',p'). We remark that (1.9) remains true for arbitrary $f_k\in L^2(0,2\pi)$ by an approximation argument. Now assume that $f_k\in L^2(0,2\pi)$ satisfy $\|f_k\|_{L^2(0,2\pi)}=1$ and let $h_k:=\sum_n g_{n,k}\hat f_k(n)$. Then the h_k's are also independent complex standard Gaussian variables on (Ω',Σ',p') as $\big(\sum_n|\hat f_k(n)|^2\big)^{1/2}=\|f_k\|_2=1$. Thus we have by (1.8)

$$\Big\|\sum_{n,k} g_{n,k}\hat f_k(n)x_k\Big\|_{L^2(\Omega';X)}=\Big\|\sum_k h_k x_k\Big\|_{L^2(\Omega';X)}\simeq \Big\|\sum_k r_k x_k\Big\|_{L^2([0,1];X)}.$$

On the other hand, if we let $f_k:=\sqrt{2N\pi}\,\chi_{[\frac{k-1}{N},\frac{k}{N}]}$ for $1\le k\le N$. Then $\|f_k\|_2=1$ and

$$\Big\|\sum_k g_k f_k x_k\Big\|_{L^2([0,2\pi]\times\Omega;X)}=\Big(\sum_k\|x_k\|^2\Big)^{1/2}.$$

It follows from (1.9) that

$$\Big(\sum_k \|x_k\|^2\Big)^{1/2} \simeq \Big\| \sum_k r_k x_k \Big\|_{L^2([0,1];X)}.$$

It follows that X is of cotype 2 and type 2. Thus X is a Hilbert space by a result of S. Kwapien [Kw72]. □

Recall that a series $\sum_{k=-\infty}^{+\infty} x_k$ in a Banach space X is called **unconditionally convergent** if

$$\lim_{n\to\infty} \sum_{k=-n}^{n} x_{\pi(k)}$$

exists for each bijection $\pi : \mathbb{Z} \to \mathbb{Z}$. This is equivalent to the fact that

$$\lim_{n\to\infty} \sum_{k=-n}^{n} \lambda_k x_k$$

exists for each $(\lambda_k)_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$. Thus if the Fourier series

$$\sum_{k=-\infty}^{+\infty} e_k \otimes \hat{f}(k)$$

converges unconditionally in $L^p_{2\pi}(X)$, then each bounded sequence $(\lambda_k I)_{k\in\mathbb{Z}}$ with I the identity operator on X, is an L^p-multiplier. This implies that $p = 2$ (as a consequence of the scalar result mentioned above) and X is a Hilbert space by Theorem 1.5.

Next we discuss convergence of the Fourier series. For $f \in L^p_{2\pi}(X)$ and $n \in \mathbb{N}$, we let $S_n(f) := \sum_{|k|\le n} e_k \otimes \hat{f}(k)$ be the partial sum of the Fourier series; hence $S_n \in \mathcal{L}(L^p_{2\pi}(X))$ for $n \in \mathbb{N}$.

Theorem 1.6. *Let X be a Banach space. The following conditions are equivalent.*

(i) *X is a UMD-space;*

(ii) $\sup_{n\in\mathbb{N}} \|S_n\| < \infty$ *for some $1 < p < \infty$ (equivalently for all $1 < p < \infty$);*

(iii) *for each $1 < p < \infty$ and each $f \in L^p_{2\pi}(X)$, the Fourier series $\sum_{k=-\infty}^{+\infty} e_k \otimes \hat{f}(k)$ converges to f in $L^p_{2\pi}(X)$;*

(iv) *there exists $1 < p < \infty$ and for each $f \in L^p_{2\pi}(X)$, the Fourier series $\sum_{k=-\infty}^{+\infty} e_k \otimes \hat{f}(k)$ converges to f in $L^p_{2\pi}(X)$;*

(v) *the sequence $(M_k)_{k\in\mathbb{Z}}$ given by $M_k = I$ for $k \ge 0$, $M_k = -I$ for $k < 0$, is an L^p-multiplier for some $1 < p < \infty$ (equivalently for all $1 < p < \infty$).*

We refer to the literature (e.g., [Bur01]) for definition of UMD-spaces. Here we just mention that each L^p-space is a UMD-space whenever $1 < p < \infty$. Moreover, each UMD-space is reflexive and closed subspaces and quotient spaces of UMD-spaces are UMD-spaces.

The operator associated with the multiplier $M_k = \mathrm{sgn}(k)I$ on $L^p_{2\pi}(X)$ is called the **Hilbert transform**. Thus the Hilbert transform is bounded on $L^p_{2\pi}(X)$ if and only if $1 < p < \infty$ and X is a UMD-space.

For the operator-valued multiplier theorems we want to present here we need the notion of **Rademacher type** (or briefly **type**) and **Rademacher cotype** (or briefly **cotype**) of a Banach space. We refer to [LT79] [Pie07, p. 308] for the definitions and further properties of these notions. We just recall that every L^q-space with $1 \le q \le 2$ is of type q and cotype 2. Every L^q-space with $2 \le q < \infty$ is of cotype q and type 2. A Banach space X is of type 2 and of cotype 2 if and only if X is a Hilbert space [Kw72].

Next we formulate the variational form of the Marcinkiewicz multiplier theorem.

Theorem 1.7. *Let X, Y be UMD-spaces. Assume that X is of cotype 2 and Y is of type 2. Let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ be a bounded sequence such that*

$$\sup_{n\in\mathbb{N}} \Big(\sum_{k=2^{n-1}}^{2^n-1} \Big\| M_{k+1} - M_k \Big\| + \sum_{k=-2^n}^{-2^{n-1}-1} \Big\| M_{k+1} - M_k \Big\| \Big) < \infty. \tag{1.10}$$

Then $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.

The hypothesis on X, Y are satisfied in particular when both spaces X and Y are Hilbert spaces. In that case Theorem 1.7 was proved by J. Schwartz [Sch61] in 1961. The scalar case is due to J. Marcinkiewicz and appeared in 1939 in Studia Mathematica.

The more general case we present here can be obtained by an inspection of the proof of [AB02, Theorem 1.3] stopping on page 318 line 9 and [AB02, Proposition 1.13]. We mention a special case which turns out to the most suitable for generalizations

Corollary 1.8 ([AB02, Theorem 1.3]). *Let X, Y be UMD-spaces, X is of cotype 2 and Y is of type 2. Let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ be a bounded sequence such that*

$$\tau := \sup_{k\in\mathbb{Z}} |k| \|M_{k+1} - M_k\| < \infty. \tag{1.11}$$

Then $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier whenever $1 < p < \infty$.

Proof. For $n \in \mathbb{N}$ one has

$$\sum_{k=2^{n-1}}^{2^n-1} \Big\| M_{k+1} - M_k \Big\| \le \tau \sum_{k=2^{n-1}}^{2^n-1} \frac{1}{k} \le \tau \frac{2^n - 2^{n-1}}{2^{n-1}} = \tau,$$

and similarly

$$\sum_{k=-2^n}^{-2^{n-1}-1} \Big\| M_{k+1} - M_k \Big\| \le \tau.$$

Now the result follows from Theorem 1.7. □

Concerning operator-valued versions of the variational version of Marcinkiewicz Theorem (Corollary 1.8) on UMD-spaces instead of Hilbert spaces we refer to Z. Strkalj and L. Weis [SW07] (see also [CPSW00] for results on operator-valued Fourier multipliers with respect to more general Schauder decompositions).

2. The Marcinkiewicz multiplier theorem in the general case

In order to describe the general periodic multiplier theorem we need the notion of R-boundedness. We recall that r_k is the kth Rademacher function on $[0,1]$.

Definition 2.1. Let X, Y be Banach spaces. A set $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called **R-bounded** if for all (equivalently for one) $q \in [1,\infty)$ there exists a constant $C > 0$ such that

$$\Big\| \sum_{k=1}^{n} r_k T_k x_k \Big\|_{L^q(0,1;Y)} \leq C \Big\| \sum_{k=1}^{n} r_k x_k \Big\|_{L^q(0,1;X)}$$

for all $n \in \mathbb{N},\ x_1, \ldots, x_n \in X,\ T_1, \ldots, T_n \in \mathcal{T}$.

This notion of unconditional boundedness of a family of operators implies boundedness but is stronger in general. More precisely the following holds.

Proposition 2.2 ([AB02, Proposition 1.13]). *Let X, Y be Banach spaces. The following assertions are equivalent.*

(i) *Each bounded $\mathcal{T} \subset \mathcal{L}(X,Y)$ is R-bounded;*
(ii) *X is of cotype 2 and Y is of type 2.*

One immediate consequence of Proposition 2.2 is that each bounded subset of $\mathcal{L}(X)$ is R-bounded if and only if X is a Hilbert space. We note however that for an arbitrary Banach space a family $\mathcal{T} \subset \mathcal{L}(X)$ of scalar operators is R-bounded if and only if it is bounded. Here we call $T \in \mathcal{L}(X)$ **scalar**, if it is of the form λI for some $\lambda \in \mathbb{C}$. Now we can formulate the operator-valued Marcinkiewicz Theorem.

Theorem 2.3 ([AB02, Theorem 1.3]). *Let X, Y be UMD-spaces, $1 < p < \infty$. Assume that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ is such that both*

$$\Big\{ M_k : \ k \in \mathbb{Z} \Big\}, \quad \Big\{ k(M_{k+1} - M_k) : \ k \in \mathbb{Z} \Big\}$$

are R-bounded in $\mathcal{L}(X,Y)$. Then $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier.

It is known that when $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ is an L^p-multiplier, then the set $\{M_k : \ k \in \mathbb{Z}\}$ must be R-bounded [AB02, Proposition 1.11]. In general, it is not possible to replace the R-boundedness in Theorem 2.3 by norm boundedness. More precisely, the following is true.

Proposition 2.4 ([AB02, Proposition 1.17]). *Let X, Y be Banach space and $1 \leq p < \infty$. Assume that every sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ satisfying $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$ and $\sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty$, is an L^p-multiplier. Then X is of cotype 2 and Y is of type 2. In particular, when $X = Y$, then X is a Hilbert space.*

3. The periodic non-homogeneous problems

We are now going to show how the multiplier theorem can be applied. Let X be a UMD-space and let D be a Banach space which is continuously embedded into X; we write $D \hookrightarrow X$ for short. Let $A \in \mathcal{L}(D, X)$, $1 < p < \infty$. We consider the following problem. Given $f \in L^p(0, 2\pi; X)$ we want to find a solution u of the problem

$$P_{\text{per}} \quad \begin{cases} u \in W^{1,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D) \\ \dot{u}(t) = Au(t) + f(t) \ \text{ a.e.} \\ u(0) = u(2\pi). \end{cases}$$

Here $W^{1,p}(0, 2\pi; X)$ consists of those continuous functions $u : [0, 2\pi] \to X$ for which there exists $u' \in L^p(0, 2\pi; X)$ such that

$$u(t) = u(0) + \int_0^t u'(s)ds \quad (t \in [0, 2\pi]).$$

Equivalently, $W^{1,p}(0, 2\pi; X)$ consists of those functions $u \in L^p(0, 2\pi; X)$ for which $u' \in L^p(0, 2\pi; X)$, where u' is defined in the sense of distributions. We say that problem P_{per} is **well posed** if for each $f \in L^p(0, 2\pi; X)$, there exists a unique solution u of P_{per}. The following result characterizes well-posedness of the problem P_{per}.

Theorem 3.1. *Let $1 < p < \infty$. The following assertions are equivalent.*

(i) *For each $f \in L^p(0, 2\pi; X)$ there exists a unique solution of P_{per};*
(ii) *for each $k \in \mathbb{Z}$ the operator $ik - A \in \mathcal{L}(D, X)$ is invertible and the family $\{(ik - A)^{-1} : k \in \mathbb{Z}\}$ is R-bounded in $\mathcal{L}(X, D)$.*

Theorem 3.1 is a consequence of the multiplier theorem. It is similar to [AB02, Theorem 2.3], where a stronger hypothesis on A is imposed, namely that A is closed as an unbounded operator on X. This means by definition that the graph of A

$$G(A) := \{(x, Ax) : \ x \in D\}$$

is closed in $X \times X$. Condition (ii) does imply closedness of A. In fact, taking $k = 0$, (ii) implies that A is invertible. Now let $(x, y) \in \overline{G(A)}$. Then there exist $x_n \in D$ such that $x_n \to x$ and $Ax_n \to y$ in X. It follows that $x_n = A^{-1}(Ax_n) \to A^{-1}y$ in D. Hence $x = A^{-1}y \in D$ and $Ax = y$. We now give a proof of Theorem 3.1.

Proof of Theorem 3.1. (ii)⇒(i). Assume condition (ii). Then A is invertible. It follows that the graph norm

$$\|x\|_A := \|Ax\| \qquad (x \in D)$$

defines an equivalent norm on D. In fact, since A is an isometric isomorphism from $(D, \|\cdot\|_A)$ to X, it follows that $(D, \|\cdot\|_A)$ is complete. Since $\|x\|_A \leq \|A\| \|x\|_D$, it follows from the Open Mapping Theorem that both norms on D are equivalent.

Thus A is an isomorphism from D to X. It follows from (ii) that the family $\{A(ik-A)^{-1}:\ k\in\mathbb{Z}\}$ is R-bounded in $\mathcal{L}(X)$. Since

$$ik(ik-A)^{-1} - A(ik-A)^{-1} = I_X, \tag{3.1}$$

we conclude that the family

$$\Big\{k(ik-A)^{-1}:\ k\in\mathbb{Z}\Big\}$$

is R-bounded in $\mathcal{L}(X)$. Now [AB02, Theorem 2.3] implies that P_{per} is well posed.

(i)⇒(ii). Assume that P_{per} is well posed.

a) We claim that A is bijective. Let $y\in X$ be given. Then for $f(t)\equiv -y$, there exists $u\in W^{1,p}(0,2\pi;X)\cap L^p(0,2\pi;D)$ such that $\dot u(t) = Au(t)-y$ a.e. and $u(0)=u(2\pi)$. Then $x := \frac{1}{2\pi}\int\limits_0^{2\pi} u(t)dt\in D$ and

$$\begin{aligned} Ax &= \frac{1}{2\pi}\int\limits_0^{2\pi} Au(t)dt = \frac{1}{2\pi}\int\limits_0^{2\pi}\dot u(t)dt - \frac{1}{2\pi}\int\limits_0^{2\pi} f(t)dt \\ &= \frac{1}{2\pi}\Big(u(2\pi)-u(0)\Big) + y = y. \end{aligned}$$

We have shown that A is surjective. Injectivity can be seen as follows. Let $x\in D$ be such that $Ax=0$. Then $u(t):=x$ satisfies $0=\dot u(t) = Au(t)+0$. Hence u is a solution of P_{per} for $f\equiv 0$. Since also $v\equiv 0$ is a solution, it follows that $x\equiv u\equiv 0$.

b) It follows from a) that A is invertible and hence that A is closed. Now [AB02, Theorem 2.3] implies that $ik-A$ is bijective and the set $\{k(ik-A)^{-1}:\ k\in\mathbb{Z}\}$ is R-bounded in $\mathcal{L}(X)$, and so $\{(ik-A)^{-1}:\ k\in\mathbb{Z}\}$ is R-bounded in $\mathcal{L}(X,D)$ since $A^{-1}: X\to D$ is an isomorphism. □

Two consequences of Theorem 3.1 are remarkable. First of all it follows that well-posedness of P_{per} is independent of $p\in(1,\infty)$. Secondly, D is dense in X. This follows from condition (ii) of Theorem 3.1, [ABHN01, Proposition 3.3.8] and the fact that X is reflexive.

4. Maximal regularity

Instead of Dirichlet boundary conditions we consider now the initial value problem. Again X is a UMD-space and D is a Banach space such that $D\hookrightarrow X$. Let $A\in\mathcal{L}(D,X)$ be an operator, $1<p<\infty$ and $\tau>0$. Given $f\in L^p(0,\tau;X)$ we want to find a solution of the problem

$$P_0(\tau,p)\qquad \left\{\begin{array}{l} u\in W^{1,p}(0,\tau;X)\cap L^p(0,\tau;D)\\ \dot u(t) = Au(t)+f(t)\quad \text{a.e.}\\ u(0)=0\ . \end{array}\right.$$

We say that problem $P_0(\tau,p)$ is **well posed** if for every $f\in L^p(0,\tau;X)$, there exists a unique solution u of $P_0(\tau,p)$. This can be characterized as follows.

Theorem 4.1. *Let $1 < p < \infty$ and let $\tau > 0$ be fixed. The following assertions concerning the operator A are equivalent.*

(i) *Problem $P_0(\tau, p)$ is well posed;*
(ii) *there exists $\omega \in \mathbb{R}$ such that $(\lambda - A)$ is invertible whenever $\operatorname{Re}\lambda > \omega$ and the set $\{(\lambda - A)^{-1} : \operatorname{Re}\lambda > \omega\}$ is R-bounded in $\mathcal{L}(X, D)$.*

If these equivalent conditions are satisfied, then D is dense in X and A (with domain D) generates a holomorphic C_0-semigroup on X.

Implication (ii)⇒(i) is due to L. Weis [W01] who uses an operator-valued multiplier theorem on $L^p(\mathbb{R}, X)$. Here we will deduce this implication from the periodic multiplier theorem, Theorem 3.1. The proof of the implication (i)⇒(ii) is given in two steps. At first we show that A generates a holomorphic C_0-semigroup. This result is due to G. Dore [Do93, Theorem 2.2], who states it under the additional hypothesis that A is closed and D is dense in X. However, the proof sketched in [Do93] can be carried over to the more general situation. We give all the details to convince the reader. Once it is known that A generates a holomorphic C_0-semigroup one may again use Theorem 3.1 (or use [CP00]).

Proof of Theorem 4.1. (i)⇒(ii). By the Closed Graph Theorem there exists a constant $c_1 \geq 0$ such that

$$\|u\|_{W^{1,p}(0,\tau;X)} \leq c_1 \|f\|_{L^p(0,\tau;X)} \tag{4.1}$$

for every $f \in L^p(0, \tau; X)$, where u denotes the unique solution of $P_0(\tau, p)$.
a) First we show that there exists $\omega_1 \in \mathbb{R}$ such that $\lambda - A$ is injective whenever $\operatorname{Re}\lambda \geq \omega_1$. In fact, let $\lambda \in \mathbb{C}, x \in D$, such that $\lambda x - Ax = 0$. Then $u(t) = \frac{1}{\lambda}(e^{\lambda t} - 1)x$ is the unique solution of $P_0(\tau, p)$ for $f \equiv x$. Hence $\|u\|_{W^{1,p}(0,\tau;X)} \leq c_1 \|f\|_{L^p(0,\tau;X)} = c_1 \tau^{1/p} \|x\|$. Hence by (4.1)

$$\begin{aligned} \|x\| \left(\frac{1}{p\operatorname{Re}\lambda} \left(e^{\operatorname{Re}\lambda p\tau} - 1 \right) \right)^{\frac{1}{p}} &= \|\dot{u}\|_{L^p(0,\tau;X)} \\ &\leq c_1 \|f\|_{L^p(0,\tau;X)} = c_1 \tau^{1/p} \|x\|. \end{aligned}$$

Thus if we choose $\omega_1 > 0$ so large that

$$\frac{1}{p\operatorname{Re}\lambda} \left(e^{\operatorname{Re}\lambda p\tau} - 1 \right) > c_1^p \tau,$$

whenever $\operatorname{Re}\lambda \geq \omega_1$, then $x = 0$.
b) Since $W^{1,p}(0, \tau; X) \hookrightarrow C([0, \tau]; X)$, there exists $c_2 \geq 0$ such that

$$\|u(\tau)\|_X \leq c_2 \|u\|_{W^{1,p}(0,\tau;X)} \tag{4.2}$$

for all $u \in W^{1,p}(0, \tau; X)$. Let

$$f_\lambda(t) := \begin{cases} e^{\lambda t} & \text{if } t \leq \frac{1}{\operatorname{Re}\lambda} \\ 0 & \text{if } \frac{1}{\operatorname{Re}\lambda} < t \leq \tau \end{cases}$$

where $\operatorname{Re}\lambda > \max\left\{\frac{1}{\tau}, \omega_1\right\} =: \omega_2$. Then for $x \in X$ one has

$$\|f_\lambda \otimes x\|_{L^p(0,\tau;X)} = \frac{c_3\|x\|}{(\operatorname{Re}\lambda)^{\frac{1}{p}}},$$

where $c_3 = \left(\frac{1}{p}(e^p - 1)\right)^{\frac{1}{p}}$. Let u be the solution of $P_0(\tau, p)$ for the inhomogeneity $f_\lambda \otimes x$. Define $R(\lambda)x := \operatorname{Re}\lambda \int_0^\tau e^{-\lambda t} u(t)\,dt \in D$. Then

$$\begin{aligned} (\lambda - A)R(\lambda)x &= -\operatorname{Re}\lambda \int_0^\tau (e^{-\lambda t})' u(t)\,dt - \operatorname{Re}\lambda \int_0^\tau e^{-\lambda t} Au(t)\,dt \\ &= -\operatorname{Re}\lambda \int_0^\tau (e^{-\lambda t})' u(t)\,dt - \operatorname{Re}\lambda \int_0^\tau e^{-\lambda t} u'(t)\,dt \\ &\quad +\operatorname{Re}\lambda \int_0^{\frac{1}{\operatorname{Re}\lambda}} e^{-\lambda t} f_\lambda(t)\,dt\,x \\ &= -\operatorname{Re}\lambda e^{-\lambda\tau} u(\tau) + x. \end{aligned}$$

Define $Sx := \operatorname{Re}\lambda e^{-\lambda\tau} u(\tau)$. Then $S : X \to X$ is linear and by (4.1) and (4.2)

$$\begin{aligned} \|Sx\| &\leq c_2 \operatorname{Re}\lambda e^{-\operatorname{Re}\lambda\tau} \|u\|_{W^{1,p}(0,\tau;X)} \\ &\leq c_1 c_2 \operatorname{Re}\lambda e^{-\operatorname{Re}\lambda\tau} \|f_\lambda\|_{L^p(0,\tau;X)} \|x\| \\ &= c_1 c_2 c_3 (\operatorname{Re}\lambda)^{\frac{1}{p'}} e^{-\operatorname{Re}\lambda\tau} \|x\|, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Thus there exists $\omega \geq \omega_2$ such that $\|S\|_{\mathcal{L}(X)} \leq \frac{1}{2}$ whenever $\operatorname{Re}\lambda \geq \omega$. Let $\operatorname{Re}\lambda \geq \omega$. Since

$$(\lambda - A)R(\lambda)x = x - Sx \tag{4.3}$$

for all $x \in X$, and since $I - S$ is surjective, it follows that $(\lambda - A)$ is surjective. Hence by a) the operator $(\lambda - A) : D \to X$ is bijective. By using a similar argument used before the proof of Theorem 3.1, this implies already that the unbounded operator $\lambda - A$ on X with domain D is closed. Thus the unbounded operator A on X with domain D is also closed. Moreover, by (4.3)

$$(\lambda - A)^{-1}x = R(\lambda)x + (\lambda - A)^{-1}Sx\ .$$

Hence

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq \|R(\lambda)\|_{\mathcal{L}(X)} + \frac{1}{2}\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)}.$$

Consequently

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq 2\|R(\lambda)\|_{\mathcal{L}(X)}\ . \tag{4.4}$$

Let $x \in X$, then for u as above,

$$\begin{aligned} R(\lambda)x &= \frac{\operatorname{Re}\lambda}{\lambda} \int_0^\tau -(e^{-\lambda t})' u(t)\,dt \\ &= \frac{\operatorname{Re}\lambda}{\lambda} \Big\{ \int_0^\tau e^{-\lambda t} u'(t)\,dt - e^{-\lambda\tau} u(\tau) \Big\} . \end{aligned}$$

Hence by (4.1) and (4.2)

$$\begin{aligned} \|\lambda R(\lambda)x\| &\le \operatorname{Re}\lambda \Big\{ \Big(\int_0^\tau e^{-\operatorname{Re}\lambda p' t}\,dt \Big)^{\frac{1}{p'}} \|u'\|_{L^p(0,\tau;X)} + c_2 e^{-\operatorname{Re}\lambda\tau} \|u\|_{W^{1,p}(0,\tau;X)} \Big\} \\ &\le \operatorname{Re}\lambda \Big\{ \Big(\frac{1}{p'\operatorname{Re}\lambda} \Big)^{1/p'} + c_2 e^{-\operatorname{Re}\lambda\tau} \Big\} c_1 \|x\| \|f_\lambda\|_{L_p(0,\tau;X)} \\ &\le c_1 c_3 \Big(\Big(\frac{1}{p'} \Big)^{1/p'} + c_2 (\operatorname{Re}\lambda)^{1/p'} e^{-\operatorname{Re}\lambda\tau} \Big) \|x\| \\ &\le c_4 \|x\| \end{aligned}$$

for some constant $c_4 > 0$ whenever $\operatorname{Re}\lambda \ge \omega$. Thus it follows from (4.4) that

$$\sup_{\operatorname{Re}\lambda > \omega} \|\lambda(\lambda - A)^{-1}\|_{\mathcal{L}(X)} < \infty .$$

By [ABHN01, Proposition 3.3.8] this implies that D is dense in X (since X is reflexive) and that A generates a holomorphic C_0-semigroup (by [ABHN01, Corollary 3.7.17]). Now assertion (ii) can be deduced from Theorem 3.1 as in [AB02, Corollary 5.2 and the following lines]. Alternatively one can use [CP00].

(ii)⇒(i). Let $1 < p < \infty$ be fixed and let $\tau > 0$. We have to show that problem $P_0(\tau, p)$ is well posed. The assumption (ii) implies that A generates a holomorphic C_0-semigroup T (keep in mind that X is reflexive, so the domain D of A is dense in X by [ABHN01, Proposition 3.3.8]). This shows in particular that $P_0(\tau, p)$ has at most one solution. In fact, if $u \in W^{1,p}(0,\tau;X) \cap L^p(0,\tau;D)$ such that $\dot u(t) = Au(t)$ and $u(0) = 0$, consider $v(t) = \int_0^t u(s)\,ds$. Then $v \in C^1([0,\tau];X) \cap C([0,\tau];D)$ is a classical solution of $\dot v(t) = Av(t)$ and $v(0) = 0$. Hence $v \equiv 0$ by [ABHN01, Theorem 3.1.12] and so $u \equiv 0$. Thus it remains to show existence which we do now.

1^{st} **case:** $\omega < 0, \tau = 2\pi$. If the constant ω of condition (ii) is negative, then we can apply Theorem 3.1. Thus, given $f \in L^p(0,2\pi;X)$ there exists $w \in W^{1,p}(0,2\pi;X) \cap L^p(0,2\pi;D)$ satisfying $\dot w(t) = Aw(t) + f(t)$ a.e. Let $v(t) := T(t)w(0)$. Then by [Lun95, 1.2.2 and 2.2.1], one has $v \in W^{1,p}(0,2\pi;X) \cap L^p(0,2\pi;D)$ and $\dot v(t) = Av(t)$ a.e., $v(0) = w(0)$. Thus $u := w - v$ is a solution of $P_0(2\pi, p)$.

2^{nd} **case:** $\omega < 0, \tau > 0$ arbitrary. Let $f \in L^p(0,\tau;X)$. Define $g(t) = rf(rt)$ where $r := \frac{\tau}{2\pi}$. The operator rA satisfies condition (ii) as well. Then $g \in L^p(0,2\pi;X)$.

By the first case there exists $v \in W^{1,p}(0,2\pi;X) \cap L^p(0,2\pi;D)$ such that $\dot{v}(t) = rAv(t) + g(t)$ a.e. and $v(0) = 0$. Let $u(t) := v(t/r)$. Then $u \in W^{1,p}(0,\tau;X) \cap L^p(0,\tau;D), u(0) = 0$ and $\dot{u}(t) = \frac{1}{r}\dot{v}\left(\frac{t}{r}\right) = Av\left(\frac{t}{r}\right) + \frac{1}{r}g\left(\frac{t}{r}\right) = Au(t) + f(t)$ a.e. Thus u is a solution of $P_0(\tau,p)$.

3$^{\text{rd}}$ **case:** The constant $\omega \in \mathbb{R}$ and $\tau > 0$ are arbitrary. Let $\omega_1 > \omega$. Then the operator $A - \omega_1$ satisfies the assumptions of the 2^{nd} case. Let $f \in L^p(0,\tau;X)$. Then there exists $v \in W^{1,p}(0,\tau;X) \cap L^p(0,\tau;D)$ satisfying $\dot{v}(t) = Av(t) - w_1 v(t) + g(t), v(0) = 0$ where $g(t) = e^{-\omega_1 t} f(t)$. Then $u(t) = e^{\omega_1 t} v(t)$ is a solution of $P_0(\tau,p)$. ☐

An immediate consequence of Theorem 4.1 is the following.

Corollary 4.2. *If $P_0(\tau,p)$ is well posed for some $\tau > 0, 1 < p < \infty$, then it is well posed for all $\tau > 0, 1 < p < \infty$.*

Remark 4.3. It can be seen from the proof of Theorem 4.1 that the implication (i)⇒(ii) is always true, without any assumption on the Banach space X. We know that (i) does not imply that D is dense in X, in general. If X is reflexive, though, then (ii) implies density of D.

5. The non-autonomous equations

Let X be a UMD-space and D a Banach space such that $D \hookrightarrow X$. Given an operator $A \in \mathcal{L}(D,X)$ we say that A satisfies **maximal regularity** if condition (i) of Theorem 4.1 is satisfied. We know that this is independent of the choice of $1 < p < \infty$ and $\tau > 0$ (Corollary 4.2). Given $1 < p < \infty, \ \tau > 0$ we define the maximal regularity space

$$MR_p(0,\tau) := W^{1,p}(0,\tau;X) \cap L^p(0,\tau;D)$$

which is a Banach space for the sum norm

$$\|u\|_{MR} := \|u\|_{W^{1,p}(0,\tau;X)} + \|u\|_{L^p(0,\tau;D)}.$$

By $Tr_p := \{u(0) : \ u \in MR_p(0,\tau)\}$ we define the trace space. Then the following result on well-posedness can be obtained by a simple perturbation argument (see [Am04] or [ACFP07]).

Theorem 5.1. *Let $A : [0,\tau] \to \mathcal{L}(D,X)$ be continuous and assume that $A(t)$ satisfies maximal regularity for each $t \in [0,\tau]$. Then for each $f \in L^p(0,\tau;X)$ and each initial value $u_0 \in Tr_p$, there exists a unique $u \in MR_p(0,\tau)$ satisfying*

$$\begin{cases} \dot{u}(t) = A(t)u(t) + f(t) & a.e. \\ u(0) = u_0. \end{cases}$$

The non-autonomous problem with periodic boundary conditions is not as simple. Let $A : [0,2\pi] \to \mathcal{L}(D,X)$ be continuous such that $A(0) = A(2\pi)$. Furthermore we want to assume that the injection $D \hookrightarrow X$ is compact. Then there are two type of results. In the first one we assume the same condition on $A(t)$ as in Theorem 5.1 for Dirichlet boundary conditions. We define for $1 < p < \infty$ the periodic maximal regularity space

$$MR_{\mathrm{per},p} := \Big\{ u \in W^{1,p}(0,2\pi;X) \cap L^p(0,2\pi;D) : \ u(0) = u(2\pi) \Big\}$$

which is a Banach space for the natural norm

$$\|u\|_{MR} := \|u\|_{W^{1,p}(0,2\pi;X)} + \|u\|_{L^p(0,2\pi;D)}.$$

Theorem 5.2 ([AR09, Corollary 9.3]). *Assume that*

(a) $A(t) + \epsilon$ *is dissipative for all* $t \in [0,2\pi]$ *and some* $\epsilon > 0$ *and that*
(b) $A(t)$ *satisfies maximal regularity (as defined above) for all* $t \in [0,2\pi]$.

Then for $1 < p < \infty$ *and each* $f \in L^p(0,2\pi;X)$, *there exists a unique solution of*

$$P_{\mathrm{per}} \quad \begin{cases} u \in MR_{\mathrm{per},p} \\ \dot{u}(t) = A(t)u(t) + f(t) \quad a.e. \end{cases}$$

If P_{per} is well posed for A then it is so for $-A$ as well. But conditions (a) and (b) are not invariant by this inversion. This shows that conditions (a) and (b) are too strong in general. In the autonomous case, condition (ii) of Theorem 3.1 is equivalent to well-posedness of P_{per}. So also in the non-autonomous case, it is natural to assume that each $A(t)$ satisfies this condition. However, this condition alone is too weak to deduce that P_{per} is well posed in this case, and in the finite-dimensional case Floquet Theory is available. Still some significant results on the solutions of problem P_{per} are valid. In order to formulate them, we consider the operator

$$D_A : \ MR_{\mathrm{per},p} \to L^p(0,2\pi;X)$$

given by $D_A u := \dot{u} - A(\cdot)u(\cdot)$. This is a bounded linear operator between the two Banach spaces $MR_{\mathrm{per},p}$ and $L^p(0,2\pi;X)$. Note that $MR_{\mathrm{per},p}$ is continuously (and compactly) embedded in $L^p(0,2\pi;X)$. So we may consider D_A as an unbounded operator on the Banach space $L^p(0,2\pi;X)$. If each $A(t)$ satisfies condition (ii) of Theorem 3.1, then this operator is indeed closed (as unbounded operator on $L^p(0,2\pi;X)$). This is not obvious but a particular result of maximal regularity. Some more can be said. The operator D_A is a **Fredholm operator**, that is, D_A has finite-dimensional kernel and closed image $R(D_A)$ with finite codimension in $L^p(0,2\pi;X)$ for each $1 < p < \infty$. To say that P_{per} is well posed means that D_A is invertible. This is stronger than Fredholm, but knowing the Fredholm property helps a lot to prove invertibility (under further assumptions as in Theorem 5.2 for example). We collect the information given here in the following concluding theorem.

Theorem 5.3 ([AR09, Theorem 4.1 and Corollary 3.7]). *Assume that* $A(t)$ *satisfies condition* (ii) *of Theorem* 3.1 *for each* $t \in [0,2\pi]$. *Let* $1 < p < \infty$. *Then the following holds.*

(a) D_A *is closed seen as unbounded operator on* $L^p(0,2\pi;X)$;
(b) D_A *is a Fredholm operator;*
(c) *if* $\sigma(D_A) \neq \mathbb{C}$, *then* D_A *has compact resolvent and in particular Fredholm's alternative holds:* P_{per} *is well posed if and only if* D_A *is injective.*

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