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# DIRICHLET REGULARITY AND DEGENERATE DIFFUSION

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $m\colon \Omega \to (0, \infty)$ be measurable and locally bounded. We study a natural realization of the operator $m\triangle$ in $C_0(\Omega) := \{u \in C(\overline{\Omega}) : u_{|\partial\Omega} = 0\}$. If $\Omega$ is Dirichlet regular, then the operator generates a positive contraction semigroup on $C_0(\Omega)$ whenever $\frac{1}{m} \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > \frac{N}{2}$. If $m(x)$ does not go fast enough to 0 as $x \to \partial\Omega$, then Dirichlet regularity is necessary. However, if $|m(x)| \leq c\cdot \mathrm{dist}(x, \partial\Omega)^2$, then we show that $m\triangle_0$ generates a semigroup on $C_0(\Omega)$ without any regularity assumptions on $\Omega$. We show that the condition for degeneration of $m$ near the boundary is optimal.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $m \in L^\infty_{\mathrm{loc}}(\Omega)$ be strictly positive. The aim of this paper is to investigate when a natural realization of the operator $m\triangle$ in $C_0(\Omega) := \{u \in C(\overline{\Omega}) : u\lfloor_{\partial\Omega} = 0\}$ generates a $C_0$-semigroup. If $\Omega$ is Dirichlet regular, then it suffices that $\frac{1}{m} \in L^p_{\mathrm{loc}}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. If $\frac{1}{m} \in L^p(\Omega)$, then Dirichlet regularity is a necessary condition. However, if the *diffusion is weak at a point* $z \in \partial\Omega$ in the sense that $m(x) \leq c \cdot \mathrm{dist}(x, \partial\Omega)^2$ in a neighbourhood of $z$, then Dirichlet regularity is not needed.

In fact, these phenomena are of local nature. Our main result (Theorem 7.1) says the following. Let $m \in L^\infty(\Omega)$ be strictly positive such that $\frac{1}{m} \in L^p_{\mathrm{loc}}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. Assume that for each $z \in \partial\Omega$ one of the following conditions is satisfied:

(a) $z$ is a regular point (in the sense of Wiener) or
(b) the diffusion is weak at $z$.

Then $m\triangle_0$ generates a positive $C_0$-semigroup on $C_0(\Omega)$. Here $m\triangle_0$ is the natural realization of $m\triangle$ in $C_0(\Omega)$ (see Section 4).

Our notion of weak diffusion is optimal. We show that it does not suffice that $m(x) \leq c \cdot \mathrm{dist}(x, \partial\Omega)^\beta$ for some $\beta < 2$ to ensure that $m\triangle_0$ generates a semigroup.

It is much easier to study the operator in the setting of $L^p$ spaces, by which we also start. However, there are good reasons to consider the operator on the space $C_0(\Omega)$. One reason is that we obtain a Feller semigroup in this way with the corresponding relations to stochastic processes (see [14], [16], [17] and [33] for the role of $C_0(\Omega)$ in the theory of Markov processes). Another reason concerns possible applications to non-linear problems and dynamical systems. For semilinear problems

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the space $C_0(\Omega)$ is much better suited than $L^p(\Omega)$-spaces since composition with a locally Lipschitz continuous function is locally Lipschitz continuous on $C_0(\Omega)$ but never on $L^p(\Omega)$; see the treatise of Cazenave-Haraux [10], for example. Studying arbitrary measurable functions $m$ seems to be useful for possible applications to quasilinear equations.

In the present paper nowhere do we suppose that the function $m$ satisfies any regularity assumptions other than measurability. Generation results on $C_0(\Omega)$ for bounded *continuous* functions $m$ have been given previously by Lumer [23] (see also [22]). He uses barriers with respect to the new operator $m\triangle$ (instead of the Laplacian). The methods we use here are very different from those employed in [23].

In the case where $\frac{1}{m} \in L^p(\Omega)$ for some $p > \frac{N}{2}$ we use techniques from [2]. The special case where $m$ is a smooth version of the distance to the boundary had been considered by Davies [12] and Pang [30]. These results were inspiring for us, and we use a smooth version of the distance as comparison when the diffusion is weak at a boundary point.

Our results show in particular that for $m$ larger than a positive constant (even $\frac{1}{m} \in L^p(\Omega)$, $p > \frac{N}{2}$ suffices) the regular points of $m\triangle$ are the same as for $\triangle$. The operator $m\triangle$ is a very special kind of elliptic operator in non-divergence form. For general elliptic operators in non-divergence form this is no longer true in both directions. In fact Miller [26] showed that there may be regular points for the Laplacian which are non-regular for a particular elliptic operator in non-divergence form and vice versa. This is in sharp contrast with the situation for uniformly elliptic operators in *divergence* form; see the results of Littman, Stampacchia, Weinberger [21].

The operator $m\triangle$ obtained further attention in the literature. McIntosh and Nahmod [25] proved $H^\infty$-calculus. Duong and Ouhabaz [15] investigated Gaussian estimates for the semigroup generated by this operator. In both results $m$ is assumed to be larger than a positive constant. We should also point out that non-divergence operators in *one* dimension (also degenerate ones) and their probabilistic interpretation are studied by Mandl [24]. An application to mathematical finance is contained in Cannarsa et al. [9].

## 2. Preliminaries

Here we fix some notation and explain arguments which are frequently used. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. We write $\omega \Subset \Omega$ if $\omega$ is an open subset of $\mathbb{R}^N$ such that $\overline{\omega} \subset \Omega$. The space $C_c(\Omega)$ denotes continuous functions on $\Omega$ with values in $\mathbb{R}$ having compact support. $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ is the space of all test functions and $\mathcal{D}(\Omega)'$ the space of all distributions.

We denote by

$$H^1(\Omega) := \left\{u \in L^2(\Omega) : \ D_j u \in L^2(\Omega), \ j = 1, \dots, d\right\}$$

the first Sobolev space and by $H_0^1(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. We let

$$L^p_{\mathrm{loc}}(\Omega) := \left\{u \colon \Omega \to \mathbb{R} \ \text{ measurable s.t.} \int_\omega |u(x)|^p \, dx < \infty \ \text{ whenever } \omega \Subset \Omega \right\},$$

where $1 \le p < \infty$. Similarly,

$$H^1_{\mathrm{loc}}(\Omega) := \left\{u \in L^2_{\mathrm{loc}}(\Omega) : \ D_j u \in L^2_{\mathrm{loc}}(\Omega) \quad \text{for } j = 1, \dots, d\right\}.$$

We let

$$C_0(\Omega) := \left\{u \in C(\overline{\Omega}) : \ u_{|\partial\Omega} = 0\right\},$$

where $\partial\Omega$ is the boundary of $\Omega$.

Then $H^1(\Omega) \cap C_0(\Omega) \subset H^1_0(\Omega)$, but

$$H^1_0(\Omega) \cap C(\overline{\Omega}) \subset C_0(\Omega) \quad \text{if and only if } \Omega \text{ is regular in capacity}$$

(see [8]). The spaces $H^1_0(\Omega)$ and $H^1(\Omega)$ are sublattices of $L^2(\Omega)$. More precisely,

$$u \in H^1(\Omega) \quad \text{implies } D_j u^+ \in H^1(\Omega) \text{ and } D_j u^+ = \chi_{\{u>0\}} D_j u, \quad j = 1, ..., d,$$

where by $\chi_A$ we denote the characteristic function of a set $A$. If $u \in H^1_0(\Omega)$, then also $u^+ \in H^1_0(\Omega)$.

If $u \in L^1_{\mathrm{loc}}(\Omega)$, then the Laplacian $\triangle u$ is a distribution. By

$$-\triangle u \leq 0 \quad \text{in } \mathcal{D}(\Omega)'$$

we mean that

$$-\langle \triangle u, v\rangle \leq 0 \qquad \text{whenever } 0 \leq v \in \mathcal{D}(\Omega).$$

If $u \in H^1_{\mathrm{loc}}(\Omega)$, this is equivalent to

$$\int_\Omega \nabla u(x) \nabla v(x)\, dx \leq 0 \quad \text{for } 0 \leq v \in \mathcal{D}(\Omega) \tag{2.1}$$

and if $u \in H^1(\Omega)$, both inequalities remain true for all $0 \leq v \in H^1_0(\Omega)$. In fact, the cone $\mathcal{D}(\Omega)_+$ of all positive test functions is dense in $H^1_0(\Omega)_+ := \left\{u \in H^1_0(\Omega) \colon u \geq 0\right\}$.

We frequently use the following **maximum principle:** Let $u \in H^1(\Omega)$ such that

$$-\triangle u \leq 0.$$

If $u^+ \in H^1_0(\Omega)$, then $u \leq 0$.

In fact, taking $v = u$ in (2.1) we obtain $\int_\Omega |\nabla u(x)^+|^2\, dx \leq 0$. By Poincaré's inequality, this implies that $u^+ = 0$.

## 3. The semigroup on $L^2\left(\Omega, \frac{dx}{m(x)}\right)$

Let $m \colon \Omega \to (0, \infty)$ be measurable such that $\frac{1}{m} \in L^1_{\mathrm{loc}}(\Omega)$. We consider the Hilbert space $L^2\left(\Omega, \frac{dx}{m(x)}\right)$ with the scalar product

$$\langle u|v\rangle = \int_\Omega u(x)v(x)\, \frac{dx}{m(x)}.$$

On $L^2\left(\Omega, \frac{dx}{m(x)}\right)$ we define the operator $m\triangle_2$ by

$$\mathcal{D}(m\triangle_2) := \left\{u \in H^1_0(\Omega) \cap L^2\left(\Omega, \frac{dx}{m(x)}\right) \colon \exists f \in L^2\left(\Omega, \frac{dx}{m(x)}\right) \text{ such that } \triangle u = \frac{f}{m}\right\},$$

$$(m\triangle_2)u := f.$$

Note that $\frac{f}{m} \in L^1_{\mathrm{loc}}(\Omega)$ since for $\omega \Subset \Omega$

$$\int_\omega \frac{|f(x)|}{m(x)}\, dx \leq \left(\int_\omega |f(x)|^2 \frac{dx}{m(x)}\right)^{\frac{1}{2}} \left(\int_\omega \frac{dx}{m(x)}\right)^{\frac{1}{2}}.$$

Thus the identity $\triangle u = \frac{f}{m}$ is well defined in $\mathcal{D}(\Omega)'$. The expression $m\triangle_2$ is purely symbolic and has to be understood in the sense of the above definition. In fact, in general $\triangle u$ is merely in $\mathcal{D}(\Omega)'$ and $m\triangle u$ cannot be defined as a distribution.

We will prove the following theorem.

**Theorem 3.1.** *The operator* $m\triangle_2$ *is self-adjoint and generates a positive, contractive* $C_0$*-semigroup* $T_2$ *on* $L^2\big(\Omega, \frac{dx}{m(x)}\big)$. *Moreover, the semigroup is submarkovian.*

Here, an operator $S$ on $L^2\big(\Omega, \frac{dx}{m(x)}\big)$ is called *submarkovian* if $f(x) \leq 1$ a.e. implies $Sf(x) \leq 1$ a.e. This is equivalent to saying that $S$ is positive and

$$\|Sf\|_\infty \leq \|f\|_\infty \text{ for all } f \in L^2\big(\Omega, \tfrac{dx}{m(x)}\big) \cap L^\infty(\Omega).$$

To say that the semigroup $T_2$ is *submarkovian* means that each $T_2(t)$, $t \geq 0$, is submarkovian.

We set $V := H_0^1(\Omega) \cap L^2\big(\Omega, \frac{dx}{m(x)}\big)$. Then $V$ is a Hilbert space for the norm

$$\|u\|_V^2 = \|u\|_{H^1(\Omega)}^2 + \|u\|_{L^2(\Omega, \frac{dx}{m(x)})}^2.$$

We let $\mathcal{D}(\Omega)_+ := \{v \in \mathcal{D}(\Omega) \colon\ v \geq 0\}$ and $V_+ := \{u \in V \colon\ u \geq 0 \text{ a.e.}\}$.

**Proposition 3.2.** $\mathcal{D}(\Omega)$ *is dense in* $V$ *and* $\mathcal{D}(\Omega)_+$ *is dense in* $V_+$.

*Proof.* We prove the second assertion. The first assertion then follows since $V = V_+ - V_+$.

a) Let $u \in V_+$. There exists a sequence $\varphi_n \in \mathcal{D}(\Omega)$ s.t. $\varphi_n \to u$ in $H^1(\Omega)$. Let $u_n := (\varphi_n \wedge u) \vee 0$. Then $0 \leq u_n \leq u$ and $u_n \to u$ in $H^1(\Omega)$. Moreover $u_n \to u$ a.e. (for a subsequence which we denote also by $u_n$). Hence $u_n \to u$ in $L^2\big(\Omega, \frac{dx}{m(x)}\big)$ by the dominated convergence theorem. We have shown that $V_+ \cap L_c^\infty(\Omega)$ is dense in $V_+$, where

$$L_c^\infty(\Omega) := \{u \in L^\infty(\Omega) : \text{supp } u \subset \Omega \text{ is compact}\}.$$

b) Let $u \in V_+ \cap L_c^\infty(\Omega), u_n := \rho_n * u$, where $\rho_n$ is a mollifier. Then $u_n \in \mathcal{D}(\Omega)$, supp $u_n \subset K \Subset \Omega$ (for $n \geq n_0$) and $\|u_n\|_\infty \leq c$ (for $n \geq n_0$), $u_n \to u$ in $H^1(\Omega)$ and $u_n \to u$ a.e. after choosing a subsequence. Hence $u_n \to u$ in $L^2\big(\Omega, \frac{dx}{m(x)}\big)$. □

*Proof of Theorem* 3.1. Let $a \colon V \times V \to \mathbb{R}$ be given by

$$a(u,v) = \int_\Omega \nabla u(x) \nabla v(x)\, dx.$$

Then $a$ is continuous, symmetric and bilinear. Moreover, $a$ is *accretive*, i.e., $a(u,u) \geq 0$ for all $u \in V$ and *elliptic* with respect to $L^2\big(\Omega, \frac{dx}{m(x)}\big)$, i.e.,

$$a(u,u) + \omega \|u\|_{L^2(\Omega, \frac{dx}{m(x)})}^2 \geq \alpha \|u\|_V^2$$

for some $\omega \in \mathbb{R}$ and $\alpha > 0$.

This follows from Poincaré's inequality, which asserts that $\sqrt{\int_\Omega |\nabla u(x)|^2\, dx}$ defines an equivalent norm on $H_0^1(\Omega)$.

Let $A$ be the operator associated with $a$. Then $A$ is self-adjoint and $-A$ generates a contractive semigroup $T_2$ on $L^2\big(\Omega, \frac{dx}{m(x)}\big)$. We show that $-m\triangle_2 = A$. In fact, for $u, f \in L^2\big(\Omega, \frac{dx}{m(x)}\big)$ we have by definition,

$$u \in \mathcal{D}(A) \text{ and } -Au = f \qquad \text{if and only if}$$
$$a(u,v) = -\int_\Omega f(x) v(x) \frac{dx}{m(x)} \qquad \text{for all } v \in V.$$

Taking $v \in \mathcal{D}(\Omega)$, this implies that $\triangle u = \frac{f}{m}$. Hence $u \in \mathcal{D}(m\triangle_2)$ and $(m\triangle_2)u = f$. Conversely, if $u \in \mathcal{D}(m\triangle_2)$ and $(m\triangle_2)u = f$, then $\triangle u = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$. Since $u \in H_0^1(\Omega)$, this implies that

$$\int_\Omega \nabla u(x)\nabla v(x)\,dx = -\langle \triangle u, v\rangle = -\int_\Omega f(x)v(x)\frac{dx}{m(x)}$$

for all $v \in \mathcal{D}(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in $V$ it follows that $u \in \mathcal{D}(A)$ and $Au = f$.

It follows from the Beurling-Deny criterion ([11], Theorem 1.3.3) or ([28], Corollary 2.17) that the semigroup is submarkovian. □

As a consequence we find a consistent family $T_p$, $1 \le p \le \infty$, of semigroups on $L^p\big(\Omega, \frac{dx}{m(x)}\big)$, such that $T_2$ is the given semigroup generated by $m\triangle_2$. Here $T_p$ is a positive, contractive $C_0$-semigroup for $1 \le p < \infty$ and $T_\infty(t) = T_1'(t)$ for all $t \ge 0$. We denote the generator of $T_p$ by $m\triangle_p$. Thus $m\triangle_\infty = (m\triangle_1)'$.

We note that consistency of the semigroups implies consistency of the resolvents. In particular,

$$R(\lambda, m\triangle_\infty)f = R(\lambda, m\triangle_2)f \tag{3.1}$$

for all $\lambda > 0$, $f \in L^\infty(\Omega) \cap L^2\big(\Omega, \frac{dx}{m(x)}\big)$. We also note that

$$R(\lambda, m\triangle_\infty) \ge 0 \quad \text{for all } \lambda > 0.$$

Finally, we will frequently use the following local regularity of the Laplacian.

Let $\frac{N}{2} < p \le \infty$. Then

$$u \in L^1_{\mathrm{loc}}(\Omega),\ \triangle u \in L^p_{\mathrm{loc}}(\Omega) \quad \text{implies } u \in C(\Omega). \tag{3.2}$$

See ([13], II.3 Proposition 6). To avoid confusion in the case $N = 1$ we shall tacitly assume $p \ge 1$ throughout the paper.

If $m \equiv 1$, then the operator $\triangle_p := m\triangle_p$ is just the Dirichlet Laplacian on $L^p(\Omega)$. We need the following properties of this operator.

**Proposition 3.3.** *The operator $\triangle_p$ is invertible. Moreover, for $\frac{N}{2} < p \le \infty$ the following holds:*

(a) $\mathcal{D}(\triangle_p) = \{u \in H_0^1(\Omega) : \triangle u \in L^p(\Omega)\}$ *and* $\triangle_p u = \triangle u$ *in* $\mathcal{D}(\Omega)'$ *for all* $u \in \mathcal{D}(\triangle_p)$.

(b) $\mathcal{D}(\triangle_p) \subset C^{\mathrm{b}}(\Omega) := \{u\colon \Omega \to \mathbb{R} : u$ *is bounded and continuous*$\}$.

*Proof.* The invertibility follows from ([11], Theorem 1.6.3), for example. Note that for $\frac{N}{2} < p \le \infty$

$$\|T_p(t)\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))} \le ct^{-\frac{N}{2p}}e^{-\omega t} \quad (t \ge 0)$$

for some $c > 0$, $\omega > 0$ (see e.g. [28] Lemma 6.5). Thus

$$R(0, \triangle_p) = \int_0^\infty T_p(t)\,dt \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega)).$$

Let $f \in L^p(\Omega)$, $u = R(0,\triangle_p)f$. Then $u \in L^\infty(\Omega)$. Moreover, $-\triangle u = f$ in $\mathcal{D}(\Omega)'$. In fact, let $f_k \to f$ in $L^p(\Omega)$ where $f_k \in L^2(\Omega) \cap L^p(\Omega)$. Then $u_k := R(0,\triangle_p)f_k \to u$ in $L^\infty(\Omega)$. Moreover, since $R(0,\triangle_p)f_k = R(0,\triangle_2)f_k$, one has $u_k \in H_0^1(\Omega)$ and $-\triangle u_k = f_k$ in $\mathcal{D}(\Omega)'$. Since $u_k \to u$ in $L^\infty(\Omega) \hookrightarrow \mathcal{D}(\Omega)'$, it follows that $\triangle u_k \to \triangle u$

in $\mathcal{D}(\Omega)'$. Thus $-\triangle u = f$. It follows from (3.2) that $u \in C(\Omega)$. Finally, by the definition of $\triangle_2$, one has

$$\int_\Omega |\nabla u_k(x)|^2\,dx = \int_\Omega f_k(x)u_k(x)\,dx \le \|f_k\|_{L^p(\Omega)}\|u_k\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{p'}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Thus $(u_k)_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Taking a subsequence, we may assume that $u_k \rightharpoonup w \in H_0^1(\Omega)$. Since $u_k \to u \in L^\infty(\Omega)$, it follows that $u = w \in H_0^1(\Omega)$. Thus (b) and one inclusion in (a) are proved.

Let $u \in H_0^1(\Omega)$ such that $f := \triangle u \in L^p(\Omega)$. It remains to show that $u \in \mathcal{D}(\triangle_p)$ and $\triangle_p u = \triangle u$. Let $w = R(0, \triangle_p)f$. Then $w \in H_0^1(\Omega)$ and $-\triangle w = f$ by what has been proved above. Thus $u + w \in H_0^1(\Omega)$ and $\triangle(u + w) = 0$. By the maximum principle (see the Introduction) this implies $u + w = 0$. □

Now we can add the following local regularity of the Laplacian.

Let $\frac{N}{2} < p \le \infty$. Then

$$u \in L^1_{\mathrm{loc}}(\Omega),\ \triangle u \in L^p_{\mathrm{loc}}(\Omega) \text{ implies } u \in H^1_{\mathrm{loc}}(\Omega). \tag{3.3}$$

In fact, let $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $\triangle u \in L^p_{\mathrm{loc}}(\Omega)$. Let $\omega \Subset \Omega$ be arbitrary and $f = \triangle u_{|\omega} \in L^p(\omega)$. Consider the operator $\triangle_p$ on $L^p(\omega)$. Then $w := \triangle_p^{-1} f \in H_0^1(\omega)$ by Proposition 3.3. Since $\triangle w = f = \triangle u$ in $\mathcal{D}(\Omega)'$, the function $u - w$ is harmonic and hence in $C^\infty(\omega)$. Thus $u \in H^1(\omega)$.

In the following we again consider a function $m\colon \Omega \to (0, \infty)$ satisfying $\frac{1}{m} \in L^1_{\mathrm{loc}}(\Omega)$. We first show how $m\triangle_\infty$ operates on functions.

**Proposition 3.4.** (a) *Let $u \in \mathcal{D}(m\triangle_\infty)$, $f = (m\triangle_\infty)u$. Then*

$$\triangle u = \frac{f}{m} \quad \textit{in } \mathcal{D}(\Omega)'.$$

(b) *If $\frac{1}{m} \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > \frac{N}{2}$, then*

$$\mathcal{D}(m\triangle_\infty) \subset C^{\mathrm{b}}(\Omega) \cap H^1_{\mathrm{loc}}(\Omega).$$

(c) *If $m \in L^\infty_{\mathrm{loc}}(\Omega)$, then $\mathcal{D}(\Omega) \subset \mathcal{D}(m\triangle_\infty)$ and $(m\triangle_\infty)u = m \cdot \triangle u$ for $u \in \mathcal{D}(\Omega)$.*

*Proof.* (a) Let $\lambda > 0$. Define $g := \lambda u - f \in L^\infty(\Omega)$. Then $u = R(\lambda, m\triangle_\infty)g$. If $g \in L^\infty(\Omega) \cap L^2\big(\Omega, \frac{dx}{m(x)}\big)$, then the claim follows from the fact that $R(\lambda, m\triangle_\infty)g = R(\lambda, m\triangle_2)g$. In the general case there exist $g_k \in L^\infty(\Omega) \cap L^2\big(\Omega, \frac{dx}{m(x)}\big)$ such that $g_k \to g$ for $\sigma\big(L^\infty(\Omega), L^1\big(\Omega, \frac{dx}{m(x)}\big)\big)$. Let $u_k = R(\lambda, m\triangle_\infty)g_k$. Then

$$-\triangle u_k = \frac{g_k - \lambda u_k}{m}.$$

Now we use the fact that $R(\lambda, m\triangle_\infty) = R(\lambda, m\triangle_1)'$ is continuous for the *weak*∗-topology $\sigma\big(L^\infty(\Omega), L^1\big(\Omega, \frac{dx}{m(x)}\big)\big)$. Hence $u_k \to u$ for $\sigma\big(L^\infty(\Omega), L^1\big(\Omega, \frac{dx}{m(x)}\big)\big)$. Since $\mathcal{D}(\Omega) \subset L^1\big(\Omega, \frac{dx}{m(x)}\big)$ we conclude that $u_k \to u$ in $\mathcal{D}(\Omega)'$. Hence $\triangle u_k \to \triangle u$ in $\mathcal{D}(\Omega)'$. Since $g_k - \lambda u_k \to g - \lambda u$ for $\sigma\big(L^\infty(\Omega), L^1\big(\Omega, \frac{dx}{m(x)}\big)\big)$, it follows that $\frac{g_k - \lambda u_k}{m} \to \frac{g - \lambda u}{m}$ in $\mathcal{D}(\Omega)'$. Thus

$$-\triangle u = \frac{g - \lambda u}{m} = -\frac{f}{m}.$$

The proof of (a) is complete.

(b) This follows now from (3.2) and (3.3).

(c) Assume that $m \in L^\infty_{\text{loc}}(\Omega)$. Let $u \in \mathcal{D}(\Omega),\ f = m \cdot \triangle u$. Then $u \in H^1_0(\Omega),\ f \in L^2\big(\Omega, \frac{dx}{m(x)}\big)$ and $\triangle u = \frac{f}{m}$. Thus $u \in \mathcal{D}(m\triangle_2)$ and $(m\triangle_2)u = f$. Let $\lambda > 0$ and set $g := \lambda u - f$. Then $g \in L^\infty(\Omega) \cap L^2\big(\Omega, \frac{dx}{m(x)}\big)$ and $R(\lambda, m\triangle_\infty)g = R(\lambda, m\triangle_2)g = u$. Thus $u \in \mathcal{D}(m\triangle_\infty)$ and $\lambda u - (m\triangle_\infty)u = g = \lambda u - f$, i.e., $(m\triangle_\infty)u = f$. □

In Proposition 3.4, the boundary condition is not incorporated. But if $\frac{1}{m} \in L^1(\Omega)$, then $L^\infty(\Omega) \subset L^2\big(\Omega, \frac{dx}{m(x)}\big)$, and the operator $m\triangle_\infty$ is just the part of $m\triangle_2$ in $L^\infty(\Omega)$. Thus, if $\frac{1}{m} \in L^1(\Omega)$, then

$$\mathcal{D}(m\triangle_\infty) = \left\{u \in H^1_0(\Omega) \cap L^\infty(\Omega) :\ \exists f \in L^\infty(\Omega) \text{ s.t. } \triangle u = \frac{f}{m}\right\} \tag{3.4}$$
$$(m\triangle_\infty)u = f.$$

If $\frac{1}{m} \in L^p(\Omega)$ for some $\infty \geq p > \frac{N}{2}$, we can even assert more.

**Proposition 3.5.** *Assume that $\frac{1}{m} \in L^p(\Omega)$, where $\frac{N}{2} < p \leq \infty$. Then $m\triangle_\infty$ is invertible.*

*Proof.* Let $f \in L^\infty(\Omega)$. Then $\frac{f}{m} \in L^p(\Omega)$. Thus by Proposition 3.3 there exists $u \in H^1_0(\Omega)$ such that $\triangle u = \frac{f}{m}$. This shows that $m\triangle_\infty$ is surjective. If $u \in \mathcal{D}(m\triangle_\infty), (m\triangle_\infty)u = 0$, then by (3.4) we have $u \in H^1_0(\Omega)$ and $\triangle u = 0$. This implies that $u = 0$. Thus $(m\triangle_\infty)$ is injective. Since the operator is closed, the proof is finished. □

The positive semigroups $T_p$ generated by $m\triangle_p$ on $L^p\big(\Omega, \frac{dx}{m(x)}\big)$ have many interesting properties. We just mention that they are always irreducible if $\Omega$ is connected (where we assume only $0 < m,\ \frac{1}{m} \in L^1_{\text{loc}}(\Omega)$ as before). This means that

$$(e^{t(m\triangle_p)}f)(x) > 0 \text{ a.e. for all } 0 \leq f \in L^p\Big(\Omega, \frac{dx}{m(x)}\Big),\ f \neq 0, \text{ and for all } t > 0.$$

For $p = 2$ this follows from Ouhabaz' simple criterion that

$$\chi_C \cdot H^1_0(\Omega) \subset H^1_0(\Omega) \quad \text{implies } |C| = 0 \text{ or } |\Omega \backslash C)| = 0$$

for each Borel set $C \subset \Omega$ (see [28], Section 4.2 or [3]). For another proof of irreducibility we refer to [18], and for consequences we refer to [4].

## 4. The operator $m\triangle_0$ on $C_0(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $m\colon \Omega \to (0, \infty)$ be a measurable function such that $m \in L^\infty_{\text{loc}}(\Omega)$ and $\frac{1}{m} \in L^p_{\text{loc}}$, where $p > \frac{N}{2}$. We want to define a maximal realization of $m\triangle$ in $C_0(\Omega)$. If $u \in C_0(\Omega)$, then $\triangle u \in \mathcal{D}(\Omega)'$, but $m\triangle u$ may not be defined as a distribution. Thus the following definition is natural.

**Definition 4.1.** We define the operator $m\triangle_0$ on $C_0(\Omega)$ by

$$\mathcal{D}(m\triangle_0) := \left\{u \in C_0(\Omega) :\ \exists f \in C_0(\Omega) \text{ s.t. } \triangle u = \frac{f}{m}\right\},$$
$$(m\triangle_0)u := f.$$

Since $\frac{f}{m} \in L^1_{\text{loc}} \subset \mathcal{D}(\Omega)'$, this definition makes sense. The notation $(m\triangle_0)$ is purely symbolic. But if $u \in C_0(\Omega) \cap C^2(\Omega)$ such that $m \cdot \triangle u \in C_0(\Omega)$, then $u \in \mathcal{D}(m\triangle_0)$ and $(m\triangle_0)u = m \cdot \triangle u$.

**Proposition 4.2.** *The operator $m\triangle_0$ is closed and dissipative. Moreover, if*

$$R(\lambda_0, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$$

*for some $\lambda_0 > 0$, then $m\triangle_0$ generates a $C_0$-semigroup of positive contractions on $C_0(\Omega)$. In that case*

$$(0,\infty) \subset \rho(m\triangle_0),$$
$$R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega) \quad \textit{for all } \lambda > 0 \qquad \textit{and}$$
$$R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)_{|C_0(\Omega)}.$$

Note that in general, $\mathcal{D}(\Omega) \not\subseteq \mathcal{D}(m\triangle_0)$, since we do not assume that $m$ is continuous. Thus in Proposition 4.2 density of the domain (which is necessary for the generation property) needs a separate argument.

Since $m\triangle_0$ is dissipative, it follows in particular that no proper restriction of $m\triangle_0$ may generate a $C_0$-semigroup on $C_0(\Omega)$.

We first prove dissipativity.

**Lemma 4.3.** *Let $\lambda > 0$, $u = \mathcal{D}(m\triangle_0)$, and $f = \lambda u - (m\triangle_0)u$. Let $c > 0$ be such that*

$$f(x) \le c \quad \textit{for all } x \in \Omega.$$

*Then $\lambda u(x) \le c$ for all $x \in \Omega$.*

*Proof.* By the definition of the operator we have

$$\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \le \frac{c}{m}.$$

Since by (3.3) $u \in H^1_{\mathrm{loc}}(\Omega)$, this implies that for $0 \le v \in \mathcal{D}(\Omega)$

$$\int_\Omega \frac{(\lambda u(x) - c)}{m(x)} v(x)\,dx + \int_\Omega \nabla u(x)\nabla v(x)\,dx \le 0. \tag{4.1}$$

Since $u \in C_0(\Omega)$, $(\lambda u - c)^+$ has compact support. Let $\omega \Subset \Omega$ such that supp $(\lambda u - c)^+ \subset \omega$. Then $(\lambda u - c)^+ \in H^1_0(\omega)$ and $(\lambda u - c) \in H^1(\omega)$. Now (4.1) implies that

$$\int_\omega \frac{(\lambda u(x) - c)}{m(x)} v(x)\,dx + \frac{1}{\lambda}\int_\omega \nabla(\lambda u(x) - c)\nabla v(x)\,dx \le 0$$

for all $0 \le v \in H^1_0(\omega)$. Taking, in particular, $v := (\lambda u - c)^+$, we see that

$$\int_\omega \frac{(\lambda u(x) - c)^{+2}}{m(x)}\,dx + \frac{1}{\lambda}\int_\omega |\nabla(\lambda u(x) - c)^+|^2\,dx \le 0.$$

This implies that $(\lambda u - c)^+ = 0$, i.e., $\lambda u \le c$. □

Applying Lemma 4.3 to $\pm u$, we see that

$$\|\lambda u\|_{L^\infty(\Omega)} \le \|\lambda u - (m\triangle_0)u\|_\infty$$

for all $u \in \mathcal{D}(m\triangle_0)$, i.e., $m\triangle_0$ is dissipative. But in fact, Lemma 4.3 shows that the operator $m\triangle_0$ is *dispersive*. We refer to ([5], [27], Chapter II) for this notion.

*Proof of Proposition* 4.2. The dissipativity has been proved above, and the closedness is easy to see. Now let $R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ for some $\lambda > 0$. We show that $\lambda \in \rho(m\triangle_0)$ and $R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)_{|C_0(\Omega)}$. Let $f \in C_0(\Omega)$ and consider $u = R(\lambda, m\triangle_\infty)f \in C_0(\Omega)$. Then (by Proposition 3.4)

$$\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

It follows that $u \in \mathcal{D}(m\triangle_0)$ and $(\lambda u - (m\triangle_0)u) = f$. We have shown that $\lambda - (m\triangle_0)$ is surjective. Since the injectivity of $(\lambda - m\triangle_0)$ follows from the dissipativity of $m\triangle_0$, the closed graph theorem now implies that $\lambda \in \rho(m\triangle_0)$. The calculation above also shows that $R(\lambda, m\triangle_0)f = u = R(\lambda, m\triangle_\infty)f$.

By the resolvent identity (see [1], Proposition 3.II.2) for $0 \leq f \in C_0(\Omega)$ and $\lambda > \lambda_0$ we have

$$0 \leq R(\lambda, m\triangle_\infty)f \leq R(\lambda_0, m\triangle_\infty)f \in C_0(\Omega).$$

Since by Proposition 3.3 the function $R(\lambda, m\triangle_\infty)f$ is continuous, it follows from the domination property above that $R(\lambda, m\triangle_\infty)f \in C_0(\Omega)$. Thus $C_0(\Omega)$ is invariant for all $\lambda \geq \lambda_0$. Hence $[\lambda_0, \infty) \subset \rho(m\triangle_0)$.

Next we show that $\mathcal{D}(m\triangle_0)$ is dense in $C_0(\Omega)$. Since $m \in L^\infty_{\text{loc}}(\Omega)$, we have $\mathcal{D}(\Omega) \subset \mathcal{D}(m\triangle_\infty)$ by Proposition 3.4. Hence $C_0(\Omega) \subset \overline{\mathcal{D}(m\triangle_\infty)}$. Thus, for $f \in C_0(\Omega)$ one has

$$\lim_{\lambda\to\infty} \lambda R(\lambda, m\triangle_0)f = \lim_{\lambda\to\infty} \lambda R(\lambda, m\triangle_\infty)f = f.$$

Since $\lambda R(\lambda, m\triangle_0)f \in \mathcal{D}(m\triangle_0)$, density of the domain is proved. Now the Lumer-Phillips theorem implies that $m\triangle_0$ generates a contractive $C_0$-semigroup. Since the resolvent of $m\triangle_0$ is positive, this semigroup is positive. It also follows that $(0, \infty) \subset \rho(m\triangle_0)$. □

We will now consider two cases which imply the invariance given in Proposition 4.2, namely that $\Omega$ is Dirichlet regular or that the diffusion coefficient $m(x)$ tends to 0 fast enough as $x$ approaches the boundary. We start by discussing Dirichlet regularity.

## 5. Regular points

Let $\Omega \subset \mathbb{R}^N$ be open, bounded and let $\frac{N}{2} < p \leq \infty$. Let $m\colon \Omega \to (0, \infty)$ be measurable such that $m \in L^\infty_{\text{loc}}(\Omega)$ and $\frac{1}{m} \in L^p_{\text{loc}}(\Omega)$.

**Theorem 5.1.** *If $\Omega$ is Dirichlet regular, then $m\triangle_0$ generates a positive contractive $C_0$-semigroup on $C_0(\Omega)$.*

Thus in the case of a Dirichlet regular set, no condition on $m(x)$ as $x$ approaches the boundary is needed. We merely impose a (very weak) regularity condition on $m$ in the interior of $\Omega$.

It will be useful to prove an individual version of Theorem 5.1 first. For this we have to recall the notion of regular points.

Consider the Dirichlet problem

$$\begin{cases} h \in C(\overline{\Omega}) \cap C^2(\Omega), \\ \triangle h = 0 \text{ in } \Omega, \\ h_{|\partial\Omega} = \varphi, \end{cases} \tag{5.1}$$

where $\varphi \in C(\partial\Omega)$ is given. Recall that $\Omega$ is called *Dirichlet regular* if for each $\varphi \in C(\partial\Omega)$ a (necessarily unique) solution of (5.1) exists. If $\Omega$ has Lipschitz boundary, then $\Omega$ is Dirichlet regular. Much weaker geometric properties of the boundary suffice, though. In dimension $N = 1$ each bounded open subset $\Omega$ of $\mathbb{R}$ is Dirichlet regular. If $N = 2$, then each simply connected bounded open set is Dirichlet regular. This is no longer true in $\mathbb{R}^3$. The Lebesgue cusp gives an example of a simply connected domain with continuous boundary, which is not Dirichlet regular (see [6] for more information).

A function $u \in C(\overline{\Omega})$ is called a *subsolution* if

$$-\triangle u \leq 0 \text{ in } \mathcal{D}(\Omega)' \quad \text{and} \quad \limsup_{x\to z,\, x\in\Omega} u(x) \leq \varphi(z) \quad \text{for all } z \in \partial\Omega.$$

A function $u \in C(\overline{\Omega})$ is called a *supersolution* if

$$-\triangle u \geq 0 \text{ in } \mathcal{D}(\Omega)' \quad \text{and} \quad \liminf_{x\to z,\, x\in\Omega} u(x) \geq \varphi(z) \quad \text{for all } z \in \partial\Omega.$$

**Theorem 5.2** (Perron)**.** *Let $\varphi \in C(\partial\Omega)$. Then for all $x \in \Omega$*

$$h_\varphi(x) := \sup\{u(x) : \quad u \text{ is a subsolution}\}$$

*exists. Moreover,*

$$h_\varphi(x) = \inf\{v(x) : \quad v \text{ is a supersolution}\}.$$

*The function $h_\varphi$ is harmonic and*

$$\inf_{\partial\Omega} \varphi \leq h_\varphi(x) \leq \sup_{\partial\Omega} \varphi$$

*for all $x \in \Omega$. If (5.1) has a solution $h$, then $h_\varphi = h$.*

The function $h_\varphi$ is called the *Perron solution* of (5.1).

A point $z \in \partial\Omega$ is called *regular* if

$$\lim_{x\to z,\, x\in\Omega} h_\varphi(x) = \varphi(z)$$

for all $\varphi \in C(\partial\Omega)$. Thus $\Omega$ is Dirichlet regular if and only if each point $z \in \partial\Omega$ is regular. It is possible to characterize regular points by the existence of a barrier or by a capacity condition (Wiener's theorem). We refer to [20].

Now we can formulate the local version of Theorem 5.1, which we want to prove.

**Theorem 5.3.** *Let $\Omega$ be bounded and open. Let $z \in \partial\Omega$ be a regular point. Let $\lambda > 0$, $f \in C_0(\Omega)$, and $u = R(\lambda, m\triangle_\infty)f$. Then*

$$\lim_{x\to z,\, x\in\Omega} u(x) = 0.$$

Thus, if $\Omega$ is Dirichlet regular, then $C_0(\Omega)$ is invariant under $R(\lambda, m\triangle_\infty)$ and Theorem 5.1 follows from Proposition 4.2.

For the proof of Theorem 5.3 we use the following variational characterization of the Perron solution (see [7]).

**Theorem 5.4.** *Let $\Phi \in C(\overline{\Omega})$ be such that $\triangle\Phi \in H^{-1}(\Omega)$. Let $\varphi = \Phi_{|\partial\Omega}$. Let $u$ be the unique solution of*

$$u \in H_0^1(\Omega),$$
$$-\triangle u = \triangle\Phi.$$

*Then $h_\varphi = \Phi + u$.*

For our purposes the following consequence is important. Recall that by Proposition 3.3, for all $f \in L^p(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that

$$-\triangle u = f \quad \text{in } \mathcal{D}(\Omega)'.$$

In fact, $u = R(0, \triangle_p)f$, where $\triangle_p$ denotes the Dirichlet Laplacian on $L^p(\Omega)$. Moreover, one has $u \in C^{\mathrm{b}}(\Omega)$.

**Corollary 5.5.** *Let $f \in L^p(\Omega)$, $u = R(0, \triangle_p)f$. Then*

$$\lim_{x \to z,\, x \in \Omega} u(x) = 0$$

*for each regular point $z \in \partial\Omega$. Thus, if $\Omega$ is Dirichlet regular, then $u \in C_0(\Omega)$.*

*Proof.* It follows from the Sobolev embedding theorem that $L^p(\Omega) \subset H^{-1}(\Omega)$. Let $f \in L^p(\Omega)$. Let $\Phi = E * f$, where $E$ is the Newtonian potential. Then (by [13], II.3, Proposition 6) $\Phi \in C(\mathbb{R}^N)$, and in $\mathcal{D}(\Omega)'$ we have

$$\triangle\Phi = f \in L^p(\Omega) \subset H^{-1}(\Omega).$$

Let $u = R(0, \triangle_p)f$. Then it follows from Theorem 5.4 that $h_\varphi = \Phi + u$. Thus

$$\lim_{x \to z,\, x \in \Omega} h_\varphi(x) = \varphi(z) \quad \text{ if } z \in \partial\Omega \text{ is regular.}$$

Consequently,[1] $\lim_{x \to z} u(x) = 0$. □

*Remark.* a) In [2] a more special case of Corollary 5.5 is proved with the help of $H^1$-barriers (the proof of Theorem 3.8 in [2]).

b) Special cases of Theorem 5.4 were obtained previously by Hildebrandt [19] and Simader [31].

*Proof of Theorem* 5.3. (a) Let $\lambda > 0$, $0 \le f \in C_{\mathrm{c}}(\Omega)$, and $u = R(\lambda, m\triangle_\infty)f$. Then $u \in H_0^1(\Omega)$ and

$$\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

Moreover, $0 \le u \in C^{\mathrm{b}}(\Omega)$. Observe that $0 \le \frac{f}{m} \in L^p(\Omega)$. Let $w = R(0, \triangle_p)\frac{f}{m}$. Then we know that $0 \le w \in H_0^1(\Omega) \cap C^{\mathrm{b}}(\Omega)$ and, by Corollary 5.5, $\lim_{x \to z} w(x) = 0$ for all regular points $z \in \partial\Omega$. By definition,

$$-\triangle w = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

Thus $-\triangle(u - w) \le 0$ in $\mathcal{D}(\Omega)'$. Since $u - w \in H^1(\Omega)$ and $(u - w)^+ \in H_0^1(\Omega)$, it follows from the maximum principle that $u \le w$. Hence $\lim_{x \to z} u(x) = 0$ for each regular point $z \in \partial\Omega$.

(b) Let $z \in \partial\Omega$ be a regular point. Then by (a)

$$\lim_{x \to z,\, x \in \Omega} (R(\lambda, m\triangle_\infty)f)(x) = 0$$

for each $0 \le f \in C_{\mathrm{c}}(\Omega)$, hence also for each $f \in C_{\mathrm{c}}(\Omega)$. Since $C_{\mathrm{c}}(\Omega)$ is dense in $C_0(\Omega)$, this remains true for all $f \in C_0(\Omega)$. □

Next we show a converse of Theorem 5.1. If the diffusion coefficient $m$ is not weak enough at the boundary, then Dirichlet regularity is necessary for $m\triangle_0$ to generate a $C_0$-semigroup. More precisely, the following holds. Recall that $\frac{N}{2} < p \le \infty$.

[1] We will sometimes use the notation $\lim_{x \to z} f(x) := \lim_{x \to z,\, x \in \Omega} f(x)$ for $f \colon \Omega \to \mathbb{R}$.

**Theorem 5.6.** *Assume that* $\frac{1}{m} \in L^p(\Omega)$*. Then* $m\triangle_0$ *generates a* $C_0$*-semigroup if and only if* $\Omega$ *is Dirichlet regular.*

For the proof we need the following.

**Proposition 5.7.** *Let* $u \in C_0(\Omega)$ *be such that* $-\triangle u = f \in L^p(\Omega)$ *for some* $p > \frac{N}{2}$*. Then* $u \in H_0^1(\Omega)$*, hence* $u = R(0, \triangle_p)f$*.*

This follows from [6], Corollary 1.4, since $L^p(\Omega) \subset H^{-1}(\Omega)$.

*Proof of Theorem* 5.6. Assume that $m\triangle_0$ generates a $C_0$-semigroup. Since $\frac{1}{m} \in L^p(\Omega)$, we know from Proposition 3.5 that $[0, \infty) \subset \rho(m\triangle_\infty)$ and $R(\lambda, m\triangle_\infty) \geq 0$ for all $\lambda \geq 0$.

We now claim $R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ and $R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)_{|C_0(\Omega)}$ for any $\lambda > 0$. Let $f \in C_0(\Omega)$ and $u = R(\lambda, m\triangle_0)f$. Then

$$-\triangle u = \frac{f}{m} - \lambda \frac{u}{m} \in L^p(\Omega).$$

Since $u \in C_0(\Omega)$, it follows from Proposition 5.7 that $u \in H_0^1(\Omega)$. Since $\frac{1}{m} \in L^p(\Omega)$ we have $L^\infty(\Omega) \subset L^2\big(\Omega, \frac{dx}{m(x)}\big)$. Thus by (3.4) we have $u \in \mathcal{D}(m\triangle_\infty)$ and $\lambda u - (m\triangle_\infty)u = f$. Hence $u = R(\lambda, m\triangle_\infty)f$. This proves the claim.

Since $0 \in \rho(m\triangle_\infty)$, the claim implies that

$$\limsup_{\lambda \to 0} \|R(\lambda, m\triangle_0)\|_{\mathcal{L}(C_0(\Omega))} < \infty,$$

hence $0 \in \rho(m\triangle_0)$ and $R(0, m\triangle_0) \geq 0$.

Let $0 \leq f \in C_0(\Omega)$ and $f(x) > 0$ for all $x \in \Omega$ and $u = R(0, m\triangle_0)f$. Then $u \in C_0(\Omega)$ and $-\triangle u = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$. Hence $R(0, \triangle_p)\frac{f}{m} = u \in C_0(\Omega)$ by Proposition 5.7. We deduce that $R(0, \triangle_p)g \in C_0(\Omega)$ for all $g \in L^p(\Omega)$ such that $|g| \leq \frac{f}{m}$ for some $0 \leq f \in C_0(\Omega)$. The space of all such functions $g$ is dense in $L^p(\Omega)$. Thus $R(0, \triangle_p)L^p(\Omega) \subset C_0(\Omega)$. Now it follows from [2], Theorem 2.4, that $\Omega$ is Dirichlet regular. □

## 6. Points of weak diffusion

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $m \colon \Omega \to (0, \infty)$ be a bounded measurable function such that $\frac{1}{m} \in L^p_{\text{loc}}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. Instead of regularity we may assume that $m$ is small in a neighbourhood of a boundary point. We say that $z \in \partial\Omega$ is a *point of weak diffusion* (for the operator $m\triangle$) if there exist $r > 0$ and $c > 0$ such that

$$m(x) \leq c \cdot \text{dist}(x, \partial\Omega)^2 \tag{6.1}$$

for all $x \in \Omega \cap B(z, r)$. If $z \in \partial\Omega$ is a point of weak diffusion, then we show that

$$\lim_{x \to z,\ x \in \Omega} (R(\lambda, m\triangle_\infty)f)(x) = 0 \tag{6.2}$$

for all $f \in C_0(\Omega)$. We will also show that condition (6.1) is optimal in the sense that

$$m(x) \leq c \cdot \text{dist}(x, \partial\Omega)^\alpha$$

for some $0 < \alpha < 2$ does not suffice to enforce (6.2).

We need the notion of a regularized distance function.

**Lemma 6.1.** *There exist a function $\sigma\colon \Omega \to (0,+\infty)$, which is of class $C^\infty(\Omega)$, and a constant $c_\sigma > 0$ such that*

$$\begin{aligned} c_\sigma^{-1} d(x) \leq &\sigma(x) \leq c_\sigma d(x),\\ |\nabla\sigma|^2 &\leq c_\sigma,\\ |\sigma\triangle\sigma| &\leq c_\sigma \end{aligned}$$

*for all $x \in \Omega$, where $d(x) := \inf\left\{\|x-y\|,\, y \in \mathbb{R}^d\backslash\Omega\right\}$.*

See [32], Chapter 6, for a proof based on the Whitney decomposition of $\Omega$.

Since $\sigma \in C_0(\Omega)$, it follows in particular that $\sigma \in H_0^1(\Omega)$. First we consider the case $m(x) := \sigma(x)^2$.

**Proposition 6.2.** *The operator $\sigma^2\triangle_0$ generates a strongly continuous semigroup of positive contractions on $C_0(\Omega)$.*

*Proof.* Let $\lambda \geq c_\sigma+1$, where $c_\sigma$ is a constant from Lemma 6.1. Set $u = R(\lambda, \sigma^2\triangle_\infty)\sigma$. Since $\sigma \in L^2\big(\Omega, \frac{dx}{\sigma(x)^2}\big)$, it follows from (3.1) that $0 \leq u \in H_0^1(\Omega) \cap L^2\big(\Omega, \frac{dx}{\sigma(x)^2}\big)$ and

$$\lambda\frac{u}{\sigma^2} - \triangle u = \frac{\sigma}{\sigma^2} \quad \text{in } \mathcal{D}(\Omega)'.$$

Since $\sigma\triangle\sigma \leq c_\sigma$, it follows that $\sigma \leq \lambda\sigma - c_\sigma\sigma \leq \lambda\sigma - \sigma^2\triangle\sigma$. Thus

$$\lambda\frac{u}{\sigma^2} - \triangle u = \frac{1}{\sigma^2}\sigma \leq \lambda\frac{\sigma}{\sigma^2} - \triangle\sigma \quad \text{in } \mathcal{D}(\Omega)'.$$

Hence

$$\lambda\frac{(u-\sigma)}{\sigma^2} - \triangle(u-\sigma) \leq 0 \quad \text{in } \mathcal{D}(\Omega)'.$$

Since $u - \sigma \in H^1(\Omega)$ and $(u-\sigma)^+ \leq u \in H_0^1(\Omega)$, it follows that $(u-\sigma)^+ \in H_0^1(\Omega)$. Now the maximum principle (see Section 2) implies that $(u-\sigma)^+ \leq 0$, i.e., $u \leq \sigma$.

We have shown that

$$R(\lambda, \sigma^2\triangle_\infty)\sigma \leq \sigma \qquad (\lambda \geq \lambda_0 := 1 + c_\sigma). \tag{6.3}$$

Thus, for $f \in C_0(\Omega)$ such that $|f| \leq c\sigma$, one has

$$|R(\lambda, \sigma^2\triangle_\infty)f| \leq cR(\lambda, \sigma^2\triangle_\infty)\sigma \leq c\sigma.$$

Consequently, $R(\lambda, \sigma^2\triangle_\infty)f \in C_0(\Omega)$ for $\lambda \geq \lambda_0$. Since functions satisfying $|f| \leq c\sigma$ for some $c \geq 0$ are dense in $C_0(\Omega)$, we deduce that $R(\lambda, \sigma^2\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ for $\lambda \geq \lambda_0$. Now the claim follows from Proposition 4.2. □

We comment that the result of Proposition 6.2 may be alternatively deduced from [12], Theorem 5.4. However, our argument given here is quite different from [12].

We need a local extension of the resolvents of $\sigma^2\triangle$. Recall that $\frac{N}{2} < p \leq \infty$.

**Lemma 6.3.** *Let $\omega \Subset \Omega$, $\lambda > 0$. There exists an operator*

$$Q(\lambda, \omega) \in \mathcal{L}(L^p(\omega), C_0(\Omega))$$

*such that*

$$Q(\lambda, \omega)f = R(\lambda, \sigma^2\triangle_0)f \quad \textit{for all } f \in L^p(\omega) \cap C_0(\Omega).$$

*For $f \in L^p(\omega)$ the function $u = Q(\lambda, \omega)f$ is the unique solution of*

$$u \in C_0(\Omega),$$
$$\lambda \frac{u}{\sigma^2} - \triangle u = \frac{f}{\sigma^2} \text{ in } \mathcal{D}(\Omega)'. \tag{6.4}$$

*Moreover, $u \in H_0^1(\Omega)$.*

Here we consider $L^p(\omega)$ as a subspace of $L^p(\Omega)$ extending functions by 0 outside $\omega$. Similarly, we consider $C_c(\omega) \subset C_0(\omega) \subset C_0(\Omega)$.

*Proof.* (a) Let $0 \le f \in C_c(\omega)$. There exists $\delta > 0$ such that $\sigma^2 \ge \delta$ on $\omega$. Let $u = R(\lambda, \sigma^2\triangle_0)f = R(\lambda, \sigma^2\triangle_2)f$. Then $0 \le u \in H_0^1(\Omega)$ and

$$\lambda \frac{u}{\sigma^2} - \triangle u = \frac{f}{\sigma^2} \le \frac{1}{\delta} f.$$

Let $w := \frac{1}{\delta} R(0, \triangle_p)f$, where $\triangle_p$ denotes the Dirichlet Laplacian on $L^p(\Omega)$. Then $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$-\triangle w = \frac{1}{\delta} f \quad \text{ in } \mathcal{D}(\Omega)'.$$

Moreover, $\|w\|_{L^\infty(\Omega)} \le c_1 \|f\|_{L^p(\omega)}$, where $c_1 = \frac{1}{\delta}\|R(0, \triangle_p)\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))}$ (see Proposition 3.3 (b)). We show that $u \le w$. In fact, we have

$$\begin{aligned} -\triangle u &\le \lambda \frac{u}{\sigma^2} - \triangle u \le \frac{1}{\delta} f \quad \text{and} \\ -\triangle w &= \frac{1}{\delta} f, \end{aligned}$$

hence $-\triangle(u - w) \le 0$ in $\mathcal{D}(\Omega)'$. Consequently, by the maximum principle (see Section 2), $u \le w$. Thus

$$\|u\|_{L^\infty(\Omega)} \le \|w\|_{L^\infty(\Omega)} \le c_1 \|f\|_{L^p(\omega)}.$$

We have shown that

$$\|R(\lambda, \sigma^2\triangle_0)f\|_{L^\infty(\Omega)} \le c_1 \|f\|_{L^p(\omega)} \tag{6.5}$$

for $0 \le f \in C_c(\omega)$. Since for arbitrary $f \in C_c(\omega)$,

$$|R(\lambda, \sigma^2\triangle_0)f| \le R(\lambda, \sigma^2\triangle_0)|f|,$$

the estimate (6.5) remains true for all $f \in C_c(\omega)$. By the density of $C_c(\omega)$ in $L^p(\omega)$, the first claim is proved.

(b) In order to prove the second claim, let $f \in L^p(\omega)$, $u = Q(\lambda, \omega)f$. Let $f_k \in C_c(\omega)$ be such that $f_k \to f$ in $L^p(\omega)$. Then $u_k := Q(\lambda, \omega)f_k \to u$ in $C_0(\Omega)$. We have $u_k \in H_0^1(\Omega) \cap C_0(\Omega)$ and

$$\lambda \frac{u_k}{\sigma^2} - \triangle u_k = \frac{f_k}{\sigma^2} \quad \text{ in } \mathcal{D}(\Omega)'. \tag{6.6}$$

Passing to the limit as $k \to \infty$ shows that (6.4) holds.

It remains to show that $u \in H_0^1(\Omega)$. Multiplying (6.6) by $u_k$ and integrating yields

$$\begin{aligned} \lambda \int_\Omega \frac{u_k(x)^2}{\sigma(x)^2}\, dx + \int_\Omega |\nabla u_k(x)|^2\, dx &= \int_\Omega \frac{f_k(x) u_k(x)}{\sigma(x)^2}\, dx \\ &\le \|u_k\|_{L^\infty(\Omega)} \frac{1}{\delta^2} \cdot |\Omega|^{\frac{1}{p'}} \|f_k\|_{L^p(\Omega)}. \end{aligned}$$

This shows that $(u_k)_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Thus, passing to a subsequence we may assume that $u_k \rightharpoonup w \in H_0^1(\Omega)$. Since $u_k \to u$ in $C_0(\Omega)$, it follows that $u = w \in H_0^1(\Omega)$. □

Now we consider a more general function $m$ satisfying the hypothesis formulated in the beginning of this section. We prove regularity of $m\triangle_\infty$ at points of weak diffusion.

**Theorem 6.4.** *Let $z \in \partial\Omega$ be a point of weak diffusion (in the sense of* (6.1)*). Let $f \in C_0(\Omega)$, $\lambda > 0$, and $u = R(\lambda, m\triangle_\infty)f$. Then*

$$\lim_{x\to z,\ x\in\Omega} u(x) = 0.$$

*Proof.* Let $r_1 > 0$ be a large radius such that $\overline{\Omega} + \overline{B}(0,r) \subset B(0,r_1)$. Consider the open set

$$\widetilde{\Omega} := (\Omega \cap B(z,r)) \cup (B(0,r_1)\backslash\overline{B}(z,\frac{r}{2})).$$

Then $\Omega \subset \widetilde{\Omega}$ and $\overline{B}(z,\frac{r}{2}) \cap \partial\Omega \subset \partial\widetilde{\Omega}$. In particular, $z \in \partial\widetilde{\Omega}$. Consider a regularized distance $\widetilde{\sigma}$ with respect to $\widetilde{\Omega}$. Then there exists a constant $c > 0$ such that

$$m(x) \le c\widetilde{\sigma}(x)^2 \quad \text{for all } x \in \Omega. \tag{6.7}$$

In fact, for $x \in B(z,r)\cap\Omega$ this follows from (6.1). But for $x \in \Omega\backslash B(z,\frac{3}{4}r)$, one has $\operatorname{dist}(x,\partial\widetilde{\Omega}) \ge \frac{r}{4}$. Since $m$ is bounded, it follows that

$$m(x) \le c_2(\frac{r}{4})^2 \le c_2 \operatorname{dist}(x,\partial\widetilde{\Omega})^2$$

for all $x \in \Omega\backslash B(z,\frac{3}{4}r)$. This shows that (6.7) is valid for a suitable constant $c > 0$.

Now let $\lambda > 0$. Let $0 \le f \in C_c(\Omega)$ and $u = R(\lambda, m\triangle_\infty)f$. Then $u \in C^b(\Omega) \cap H_0^1(\Omega)$ and

$$\lambda\frac{u}{m} - \triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

Let $\rho := \frac{m}{\widetilde{\sigma}^2}$. Then $0 < \rho \le c$ on $\Omega$ and

$$\frac{1}{c} \le \frac{1}{\rho} = \frac{\widetilde{\sigma}^2}{m} \in L^p_{\mathrm{loc}}(\Omega).$$

Hence

$$\frac{\lambda}{c}\frac{u}{\widetilde{\sigma}^2} \le \frac{\lambda}{\rho}\frac{u}{\widetilde{\sigma}^2} = \frac{\lambda u}{m}.$$

Thus

$$\frac{\lambda}{c}\frac{u}{\widetilde{\sigma}^2} - \triangle u \le \frac{f}{m} = \frac{1}{\widetilde{\sigma}^2}\frac{f}{\rho}.$$

Let $\omega \Subset \Omega$ be such that *supp* $f \subset \omega$. Consider $Q(\lambda,\omega) \in \mathcal{L}(L^p(\omega), C_0(\widetilde{\Omega}))$ of Lemma 6.3 defined with respect to $\widetilde{\sigma}$. Let $w = Q(\frac{\lambda}{c},\omega)\frac{f}{\rho}$. Note that $w$ is well defined, since $\frac{f}{\rho} \in L^p(\omega)$. Then $0 \le w \in C_0(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega})$ and, by (6.4),

$$\frac{\lambda}{c}\frac{w}{\widetilde{\sigma}^2} - \triangle w = \frac{1}{\widetilde{\sigma}^2}\frac{f}{\rho} \quad \text{in } \mathcal{D}(\widetilde{\Omega})'$$

and hence also in $\mathcal{D}(\Omega)'$. Thus

$$\frac{\lambda}{c}\frac{(u-w)}{\widetilde{\sigma}^2} - \triangle(u-w) \le 0 \quad \text{in } \mathcal{D}(\Omega)'.$$

Recall that $u \in H_0^1(\Omega) \cap C^{\mathrm{b}}(\Omega)$. Thus $(u - w) \in H^1(\Omega)$. Hence

$$\frac{\lambda}{c} \int_\Omega \frac{(u(x) - w(x))}{\widetilde{\sigma}(x)^2} v(x)\, dx + \int_\Omega \nabla(u(x) - w(x)) \nabla v(x)\, dx \leq 0 \tag{6.8}$$

for all $0 \leq v \in \mathcal{D}(\Omega)$. Since $(u - w)^+ \in H^1(\Omega)$ and $(u - w)^+ \leq u \in H_0^1(\Omega)$, it follows that $(u - w)^+ \in H_0^1(\Omega)$.

Since $u = R(\lambda, m\triangle_\infty) f = R(\lambda, m\triangle_2) f$, it follows that

$$u \in L^2\Big(\Omega, \frac{dx}{m(x)}\Big) \subset L^2\Big(\Omega, \frac{dx}{\widetilde{\sigma}(x)^2}\Big)$$

because of (6.7). It follows (since also $w \in L^2\big(\Omega, \frac{dx}{\widetilde{\sigma}(x)^2}\big)$) that

$$v_1 := (u - w)^+ \in V := L^2\Big(\Omega, \frac{dx}{\widetilde{\sigma}(x)^2}\Big) \cap H_0^1(\Omega).$$

Since $\mathcal{D}(\Omega)_+$ is dense in $V_+$ by Proposition 3.2, (6.8) remains true for $v := v_1$. This means that

$$\frac{\lambda}{c} \int_\Omega \frac{(u(x) - w(x))^{+2}}{\widetilde{\sigma}(x)^2}\, dx + \int_\Omega |\nabla(u(x) - w(x))^+|^2\, dx \leq 0.$$

This implies that $(u - w)^+ = 0$. Hence $0 \leq u \leq w$.

Since

$$\lim_{x \to z,\, x \in \widetilde{\Omega}} w(x) = 0,$$

it follows that

$$\lim_{x \to z,\, x \in \Omega} u(x) = 0.$$

We have proved the theorem for the case when $0 \leq f \in C_{\mathrm{c}}(\Omega)$. Hence it is also true for arbitrary $f \in C_{\mathrm{c}}(\Omega)$. Since $R(\lambda, m\triangle_\infty) \in \mathcal{L}(L^\infty(\Omega))$, and $C_{\mathrm{c}}(\Omega)$ is dense in $C_0(\Omega)$, it follows that

$$\lim_{x \to z,\, x \in \Omega} (R(\lambda, m\triangle_\infty) f)(x) = 0$$

for all $f \in C_0(\Omega)$. ∎

**Corollary 6.5.** *Assume that each $z \in \partial\Omega$ is a point of weak diffusion (in the sense of* (6.1)*). Then $m\triangle_0$ generates a positive, contractive $C_0$-semigroup on $C_0(\Omega)$.*

## 7. Conclusion

We may now formulate the following general generation theorem. Let $\Omega \subset \mathbb{R}^N$ be bounded, open and $\frac{N}{2} < p \leq \infty$. Let $m \colon \Omega \to (0, \infty)$ be bounded and such that $\frac{1}{m} \in L^p_{\mathrm{loc}}(\Omega)$.

**Theorem 7.1.** *Assume that for each point $z \in \partial\Omega$ one of the following conditions is satisfied:*

(a) *$z$ is a regular point or*
(b) *$z$ is a point of weak diffusion (in the sense of* (6.1)*).*

*Then $m\triangle_0$ generates a positive, contractive $C_0$-semigroup on $C_0(\Omega)$.*

*Proof.* Theorem 5.3 and Theorem 6.4 show that $C_0(\Omega)$ is invariant. Thus the claim follows from Proposition 4.2. ∎

Finally, we show that the condition (6.1) of being a point of weak diffusion is optimal.

Let $N = 2$ and $\Omega = \{x \in \mathbb{R}^2 : \ 0 < |x| < 2\}$. Then $\partial\Omega = \mathbb{T} \cup \{0\}$, where $\mathbb{T} = \{x \in \mathbb{R}^2 : \ |x| = 2\}$. The points in $\mathbb{T}$ are regular, but 0 is not regular.

Consider the function $d$ given by $d(x) = |x|, \ x \in \Omega$. Thus $d(x) = \operatorname{dist}(x, \partial\Omega)$ for $0 < |x| < \frac{1}{2}$. Then $\frac{1}{d} \in L^q(\Omega)$ if and only if $q < 2$. Now let $0 < \beta < 2$. Then $\frac{1}{d^\beta} \in L^p(\Omega)$ for some $p > 1 = \frac{N}{2}$. Since $\Omega$ is not Dirichlet regular, it follows from Theorem 5.6 that $d^\beta \triangle_0$ is not a generator.

On the other hand, if $\beta \geq 2$, then for $m = d^\beta$, the point 0 is of weak diffusion. Since the other boundary points are regular, it follows from Theorem 7.1 that $d^\beta \triangle_0$ generates a $C_0$-semigroup on $C_0(\Omega)$.

An interesting open set in $\mathbb{R}^3$ with continuous boundary and exactly one singular point is the Lebesgue cusp (see e.g. [7] for a detailed investigation).

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