DIRICHLET REGULARITY AND DEGENERATE DIFFUSION

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $m: \Omega \to (0, \infty)$ be measurable and locally bounded. We study a natural realization of the operator $m\Delta$ in $C_0(\Omega) := \{ u \in C(\overline{\Omega}) : u|_{\partial \Omega} = 0 \}$. If $\Omega$ is Dirichlet regular, then the operator generates a positive contraction semigroup on $C_0(\Omega)$ whenever $\frac{1}{m} \in L^p_{\text{loc}}(\Omega)$ for some $p > \frac{N}{2}$. If $m(x)$ does not go fast enough to 0 as $x \to \partial \Omega$, then Dirichlet regularity is necessary. However, if $|m(x)| \leq c \cdot \text{dist}(x, \partial \Omega)^2$, then we show that $m\Delta_0$ generates a semigroup on $C_0(\Omega)$ without any regularity assumptions on $\Omega$. We show that the condition for degeneration of $m$ near the boundary is optimal.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $m \in L^\infty_{\text{loc}}(\Omega)$ be strictly positive. The aim of this paper is to investigate when a natural realization of the operator $m\Delta$ in $C_0(\Omega) := \{ u \in C(\overline{\Omega}) : u|_{\partial \Omega} = 0 \}$ generates a $C_0$-semigroup. If $\Omega$ is Dirichlet regular, then it suffices that $\frac{1}{m} \in L^p_{\text{loc}}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. If $\frac{1}{m} \in L^p(\Omega)$, then Dirichlet regularity is a necessary condition. However, if the diffusion is weak at a point $z \in \partial \Omega$ in the sense that $m(x) \leq c \cdot \text{dist}(x, \partial \Omega)^2$ in a neighbourhood of $z$, then Dirichlet regularity is not needed.

In fact, these phenomena are of local nature. Our main result (Theorem 7.1) says the following. Let $m \in L^\infty(\Omega)$ be strictly positive such that $\frac{1}{m} \in L^p_{\text{loc}}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. Assume that for each $z \in \partial \Omega$ one of the following conditions is satisfied:

(a) $z$ is a regular point (in the sense of Wiener) or 
(b) the diffusion is weak at $z$.

Then $m\Delta_0$ generates a positive $C_0$-semigroup on $C_0(\Omega)$. Here $m\Delta_0$ is the natural realization of $m\Delta$ in $C_0(\Omega)$ (see Section 4).

Our notion of weak diffusion is optimal. We show that it does not suffice that $m(x) \leq c \cdot \text{dist}(x, \partial \Omega)^\beta$ for some $\beta < 2$ to ensure that $m\Delta_0$ generates a semigroup.

It is much easier to study the operator in the setting of $L^p$ spaces, by which we also start. However, there are good reasons to consider the operator on the space $C_0(\Omega)$. One reason is that we obtain a Feller semigroup in this way with the corresponding relations to stochastic processes (see [14], [16], [17] and [33] for the role of $C_0(\Omega)$ in the theory of Markov processes). Another reason concerns possible applications to non-linear problems and dynamical systems. For semilinear problems...
the space $C_0(\Omega)$ is much better suited than $L^p(\Omega)$-spaces since composition with a locally Lipschitz continuous function is locally Lipschitz continuous on $C_0(\Omega)$ but never on $L^p(\Omega)$; see the treatise of Cazenave-Haraux [10], for example. Studying arbitrary measurable functions $m$ seems to be useful for possible applications to quasilinear equations.

In the present paper nowhere do we suppose that the function $m$ satisfies any regularity assumptions other than measurability. Generation results on $C_0(\Omega)$ for bounded continuous functions $m$ have been given previously by Lumer [23] (see also [22]). He uses barriers with respect to the new operator $m\Delta$ (instead of the Laplacian). The methods we use here are very different from those employed in [23].

In the case where $\frac{1}{m} \in L^p(\Omega)$ for some $p > \frac{N}{2}$ we use techniques from [2]. The special case where $m$ is a smooth version of the distance to the boundary had been considered by Davies [12] and Pang [30]. These results were inspiring for us, and we use a smooth version of the distance as comparison when the diffusion is weak at a boundary point.

Our results show in particular that for $m$ larger than a positive constant (even $\frac{1}{m} \in L^p(\Omega)$, $p > \frac{N}{2}$ suffices) the regular points of $m\Delta$ are the same as for $\Delta$. The operator $m\Delta$ is a very special kind of elliptic operator in non-divergence form. For general elliptic operators in non-divergence form this is no longer true in both directions. In fact Miller [26] showed that there may be regular points for the Laplacian which are non-regular for a particular elliptic operator in non-divergence form and vice versa. This is in sharp contrast with the situation for uniformly elliptic operators in divergence form; see the results of Littman, Stampacchia, Weinberger [21].

The operator $m\Delta$ obtained further attention in the literature. McIntosh and Nahmod [25] proved $H^\infty$-calculus. Duong and Ouhabaz [15] investigated Gaussian estimates for the semigroup generated by this operator. In both results $m$ is assumed to be larger than a positive constant. We should also point out that non-divergence operators in one dimension (also degenerate ones) and their probabilistic interpretation are studied by Mandl [24]. An application to mathematical finance is contained in Cannarsa et al. [9].

2. Preliminaries

Here we fix some notation and explain arguments which are frequently used. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. We write $\omega \Subset \Omega$ if $\omega$ is an open subset of $\mathbb{R}^N$ such that $\overline{\omega} \subset \Omega$. The space $C_c(\Omega)$ denotes continuous functions on $\Omega$ with values in $\mathbb{R}$ having compact support. $D(\Omega) = C_c^\infty(\Omega)$ is the space of all test functions and $D(\Omega)'$ the space of all distributions.

We denote by

$$H^1(\Omega) := \{ u \in L^2(\Omega) : D_j u \in L^2(\Omega), \ j = 1, \ldots, d \}$$

the first Sobolev space and by $H^1_0(\Omega)$ the closure of $D(\Omega)$ in $H^1(\Omega)$. We let

$$L^p_{\text{loc}}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable s.t. } \int_\omega |u(x)|^p \, dx < \infty \text{ whenever } \omega \Subset \Omega \right\},$$

where $1 \leq p < \infty$. Similarly,

$$H^1_{\text{loc}}(\Omega) := \left\{ u \in L^2_{\text{loc}}(\Omega) : D_j u \in L^2_{\text{loc}}(\Omega) \text{ for } j = 1, \ldots, d \right\}.$$
We let $C_0(\Omega) := \{ u \in C(\overline{\Omega}) : u_{|\partial\Omega} = 0 \}$, where $\partial\Omega$ is the boundary of $\Omega$.

Then $H^1(\Omega) \cap C(\Omega) \subset H^1_0(\Omega)$, but $H^1_0(\Omega) \cap C(\Omega) \subset C_0(\Omega)$ if and only if $\Omega$ is regular in capacity (see [3]). The spaces $H^1_0(\Omega)$ and $H^1(\Omega)$ are sublattices of $L^2(\Omega)$. More precisely,

$u \in H^1(\Omega)$ implies $D_j u^+ \in H^1(\Omega)$ and $D_j u^+ = \chi_{\{u > 0\}} D_j u, \quad j = 1, \ldots, d,$

where by $\chi_A$ we denote the characteristic function of a set $A$. If $u \in H^1_0(\Omega)$, then also $u^+ \in H^1_0(\Omega)$.

If $u \in L^1_{\text{loc}}(\Omega)$, then the Laplacian $\triangle u$ is a distribution. By $-\triangle u \leq 0$ in $\mathcal{D}(\Omega)'$ we mean that

$- \langle \triangle u, v \rangle \leq 0 \quad \text{whenever} \quad 0 \leq v \in \mathcal{D}(\Omega).$

If $u \in H^1_{\text{loc}}(\Omega)$, this is equivalent to

$\int_\Omega \nabla u(x) \nabla v(x) \, dx \leq 0 \quad \text{for} \quad 0 \leq v \in \mathcal{D}(\Omega)$

and if $u \in H^1(\Omega)$, both inequalities remain true for all $0 \leq v \in H^1_0(\Omega)$. In fact, the cone $\mathcal{D}(\Omega)_+$ of all positive test functions is dense in $H^1_0(\Omega)_+ := \{ u \in H^1_0(\Omega) : u \geq 0 \}$.

We frequently use the following maximum principle: Let $u \in H^1(\Omega)$ such that $-\triangle u \leq 0$.

If $u^+ \in H^1_0(\Omega)$, then $u \leq 0$.

In fact, taking $v = u$ in (2.1) we obtain $\int_\Omega \nabla u(x)^+ |\nabla u(x)| \, dx \leq 0$. By Poincaré’s inequality, this implies that $u^+ = 0$.

3. The semigroup on $L^2(\Omega, \frac{dx}{m(x)})$

Let $m : \Omega \rightarrow (0, \infty)$ be measurable such that $\frac{1}{m} \in L^1_{\text{loc}}(\Omega)$. We consider the Hilbert space $L^2(\Omega, \frac{dx}{m(x)})$ with the scalar product

$\langle u | v \rangle = \int_\Omega u(x) v(x) \frac{dx}{m(x)}.$

On $L^2(\Omega, \frac{dx}{m(x)})$ we define the operator $m \triangle_2$ by

$\mathcal{D}(m \triangle_2) := \left\{ u \in H^1_0(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)}) : \exists f \in L^2(\Omega, \frac{dx}{m(x)}) \quad \text{such that} \quad \triangle u = \frac{f}{m} \right\},$

$(m \triangle_2) u := f.$

Note that $\frac{f}{m} \in L^1_{\text{loc}}(\Omega)$ since for $\omega \subset \Omega$

$\int_\omega \frac{|f(x)|}{m(x)} \, dx \leq \left( \int_\omega |f(x)|^2 \frac{dx}{m(x)} \right)^{\frac{1}{2}} \left( \int_\omega \frac{dx}{m(x)} \right)^{\frac{1}{2}}.$

Thus the identity $\triangle u = \frac{f}{m}$ is well defined in $\mathcal{D}(\Omega)'$. The expression $m \triangle_2$ is purely symbolic and has to be understood in the sense of the above definition. In fact, in general $\triangle u$ is merely in $\mathcal{D}(\Omega)'$ and $m \triangle u$ cannot be defined as a distribution.

We will prove the following theorem.
The operator $m \triangle_2$ is self-adjoint and generates a positive, contractive $C_0$-semigroup $T_2$ on $L^2(\Omega, \frac{dx}{m(x)})$. Moreover, the semigroup is submarkovian.

Here, an operator $S$ on $L^2(\Omega, \frac{dx}{m(x)})$ is called submarkovian if $f(x) \leq 1$ a.e. implies $Sf(x) \leq 1$ a.e. This is equivalent to saying that $S$ is positive and

$$\|Sf\|_{L^\infty} \leq \|f\|_{L^\infty}$$

for all $f \in L^2(\Omega, \frac{dx}{m(x)}) \cap L^\infty(\Omega)$.

To say that the semigroup $T_2$ is submarkovian means that each $T_2(t), t \geq 0$, is submarkovian.

We set $V := H_0^1(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$. Then $V$ is a Hilbert space for the norm

$$\|u\|^2_V = \|u\|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\Omega, \frac{dx}{m(x)})}.$$

We let $D(\Omega)_+ := \{v \in D(\Omega) : v \geq 0\}$ and $V_+ := \{u \in V : u \geq 0 \text{ a.e.}\}$.

**Proposition 3.2.** $D(\Omega)$ is dense in $V$ and $D(\Omega)_+$ is dense in $V_+$.

**Proof.** We prove the second assertion. The first assertion then follows since $V = V_+ - V_+$.

a) Let $u \in V_+$. There exists a sequence $\varphi_n \in D(\Omega)$ s.t. $\varphi_n \to u$ in $H^1(\Omega)$. Let $u_n := (\varphi_n \wedge u) \vee 0$. Then $0 \leq u_n \leq u$ and $u_n \to u$ in $H^1(\Omega)$. Moreover $u_n \to u$ a.e. (for a subsequence which we denote also by $u_n$). Hence $u_n \to u$ in $L^2(\Omega, \frac{dx}{m(x)})$ by the dominated convergence theorem. We have shown that $V_+ \cap L^\infty(\Omega)$ is dense in $V_+$, where

$$L^\infty(\Omega) := \{u \in L^\infty(\Omega) : \text{supp} u \subset \Omega \text{ is compact}\}.$$

b) Let $u \in V_+ \cap L^\infty(\Omega), u_n := \rho_n * u$, where $\rho_n$ is a mollifier. Then $u_n \in D(\Omega)_+$, $\text{supp} u_n \subset K \Subset (\Omega$ (for $n \geq n_0$) and $\|u_n\|_{L^\infty} \leq c$ (for $n \geq n_0$), $u_n \to u$ in $H^1(\Omega)$ and $u_n \to u$ a.e. after choosing a subsequence. Hence $u_n \to u$ in $L^2(\Omega, \frac{dx}{m(x)})$.

**Proof of Theorem 3.1.** Let $a : V \times V \to \mathbb{R}$ be given by

$$a(u, v) = \int_\Omega \nabla u(x) \nabla v(x) \, dx.$$

Then $a$ is continuous, symmetric and bilinear. Moreover, $a$ is accretive, i.e., $a(u, u) \geq 0$ for all $u \in V$ and elliptic with respect to $L^2(\Omega, \frac{dx}{m(x)})$, i.e.,

$$a(u, u) + \omega \|u\|^2_{L^2(\Omega, \frac{dx}{m(x)})} \geq \alpha \|u\|^2_V$$

for some $\omega \in \mathbb{R}$ and $\alpha > 0$.

This follows from Poincaré’s inequality, which asserts that $\sqrt{\int_\Omega |\nabla u(x)|^2 \, dx}$ defines an equivalent norm on $H^1_0(\Omega)$.

Let $A$ be the operator associated with $a$. Then $A$ is self-adjoint and $-A$ generates a contractive semigroup $T_2$ on $L^2(\Omega, \frac{dx}{m(x)})$. We show that $-m \triangle_2 = A$. In fact, for $u, f \in L^2(\Omega, \frac{dx}{m(x)})$ we have by definition,

$$u \in D(A) \text{ and } -Au = f \quad \text{if and only if}$$

$$a(u, v) = -\int_\Omega f(x) v(x) \frac{dx}{m(x)} \quad \text{for all } v \in V.$$
Taking \( v \in \mathcal{D}(\Omega) \), this implies that \( \Delta u = \frac{f}{m} \). Hence \( u \in \mathcal{D}(m\Delta_2) \) and \( m\Delta_2 u = f \). Conversely, if \( u \in \mathcal{D}(m\Delta_2) \) and \( m\Delta_2 u = f \), then \( \Delta u = \frac{f}{m} \) in \( \mathcal{D}(\Omega)' \). Since \( u \in H^1_0(\Omega) \), this implies that

\[
\int_{\Omega} \nabla u(x) \nabla v(x) \, dx = -\langle \Delta u, v \rangle = -\int_{\Omega} f(x)v(x) \frac{dx}{m(x)}
\]

for all \( v \in \mathcal{D}(\Omega) \). Since \( \mathcal{D}(\Omega) \) is dense in \( V \) it follows that \( u \in \mathcal{D}(A) \) and \( Au = f \).

Proof. For \( p \) as in \( \mathcal{A} \), see \( \mathcal{A} \), II.3 Proposition 6). To avoid confusion in the case \( u \in V \) the following holds:

\[ R(\lambda, m\Delta_\infty) f = R(\lambda, m\Delta_2) f \]

for all \( \lambda > 0 \), \( f \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)}) \). We also note that

\[ R(\lambda, m\Delta_\infty) \geq 0 \quad \text{for all } \lambda > 0. \]

Finally, we will frequently use the following local regularity of the Laplacian.

Let \( \frac{N}{2} < p \leq \infty \). Then

\[ u \in L^1_{\text{loc}}(\Omega), \quad \Delta u \in L^p_{\text{loc}}(\Omega) \quad \text{implies } u \in C(\Omega). \]

See (\[ \mathcal{A} \]), II.3 Proposition 6). To avoid confusion in the case \( N = 1 \) we shall tacitly assume \( p \geq 1 \) throughout the paper.

If \( m \equiv 1 \), then the operator \( \Delta_p := m\Delta_p \) is just the Dirichlet Laplacian on \( L^p(\Omega) \). We need the following properties of this operator.

**Proposition 3.3.** The operator \( \Delta_p \) is invertible. Moreover, for \( \frac{N}{2} < p \leq \infty \) the following holds:

(a) \( \mathcal{D}(\Delta_p) = \{ u \in H^1_0(\Omega) : \Delta u \in L^p(\Omega) \} \) and \( \Delta_p u = \Delta u \) in \( \mathcal{D}(\Omega)' \) for all \( u \in \mathcal{D}(\Delta_p) \).

(b) \( \mathcal{D}(\Delta_p) \subset C^0(\Omega) := \{ u : \Omega \to \mathbb{R} : u \text{ is bounded and continuous} \} \).

**Proof.** The invertibility follows from (\[ \mathcal{A} \]), Theorem 1.6.3), for example. Note that for \( \frac{N}{2} < p \leq \infty \)

\[
\| T_p(t) \|_{L^p(\Omega), L^\infty(\Omega)} \leq c t^{-\frac{N}{p}} e^{-\omega t} \quad (t \geq 0)
\]

for some \( c > 0 \), \( \omega > 0 \) (see e.g. \[ \mathcal{A} \] Lemma 6.5). Thus

\[
R(0, \Delta_p) = \int_0^\infty T_p(t) \, dt \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega)).
\]

Let \( f \in L^p(\Omega) \), \( u = R(0, \Delta_p) f \). Then \( u \in L^\infty(\Omega) \). Moreover, \( -\Delta u = f \) in \( \mathcal{D}(\Omega)' \).

In fact, let \( f_k \to f \) in \( L^p(\Omega) \) where \( f_k \in L^2(\Omega) \cap L^p(\Omega) \). Then \( u_k := R(0, \Delta_p) f_k \to u \) in \( L^\infty(\Omega) \). Moreover, since \( R(0, \Delta_p) f_k = R(0, \Delta_2) f_k \), one has \( u_k \in H^1_0(\Omega) \) and \( -\Delta u_k = f_k \) in \( \mathcal{D}(\Omega)' \). Since \( u_k \to u \) in \( L^\infty(\Omega) \hookrightarrow \mathcal{D}(\Omega)' \), it follows that \( \Delta u_k \to \Delta u \) as \( k \to \infty \).
in \( \mathcal{D}(\Omega)' \). Thus \(-\Delta u = f\). It follows from (3.2) that \( u \in C(\Omega) \). Finally, by the definition of \( \Delta_2 \), one has
\[
\int_{\Omega} |\nabla u_k(x)|^2 \, dx = \int_{\Omega} f_k(x) u_k(x) \, dx \leq \| f_k \|_{L^p(\Omega)} \| u_k \|_{L^1(\Omega)} |\Omega|^\frac{1}{p},
\]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Thus \((u_k)_{k \in \mathbb{N}}\) is bounded in \( H^1_0(\Omega) \). Taking a subsequence, we may assume that \( u_k \to w \in H^1_0(\Omega) \). Since \( u_k \to u \in L^\infty(\Omega) \), it follows that \( u = w \in H^1_0(\Omega) \). Thus (b) and one inclusion in (a) are proved.

Let \( u \in H^1_0(\Omega) \) such that \( f := -\Delta u \in L^p(\Omega) \). It remains to show that \( u \in \mathcal{D}(\Delta_p) \) and \( \Delta_p u = u \). Let \( w = R(0, \Delta_p) f \). Then \( w \in H^1_0(\Omega) \) and \(-\Delta w = f\) by what has been proved above. Thus \( u + w \in H^1_0(\Omega) \) and \( \Delta(u + w) = 0 \). By the maximum principle (see the Introduction) this implies \( u + w = 0 \).

Now we can add the following local regularity of the Laplacian.

Let \( \frac{m}{2} < p \leq \infty \). Then
\[
(3.3) \quad u \in L^1_{\text{loc}}(\Omega), \ \Delta u \in L^p_{\text{loc}}(\Omega) \implies u \in H^1_{\text{loc}}(\Omega).
\]
In fact, let \( u \in L^1_{\text{loc}}(\Omega) \) such that \( \Delta u \in L^p_{\text{loc}}(\Omega) \). Let \( \omega \in \Omega \) be arbitrary and \( f = \Delta u\vert_\omega \in L^p(\omega) \). Consider the operator \( \Delta_p \) on \( L^p(\omega) \). Then \( w := \Delta_p^{-1} f \in H^1_0(\omega) \) by Proposition 3.3. Since \( \Delta w = f = \Delta u \) in \( \mathcal{D}(\Omega)' \), the function \( u - w \) is harmonic and hence in \( C^\infty(\omega) \). Thus \( u \in H^1(\omega) \).

In the following we again consider a function \( m : \Omega \to (0, \infty) \) satisfying \( \frac{1}{m} \in L^1_{\text{loc}}(\Omega) \). We first show how \( m\Delta_\infty \) operates on functions.

**Proposition 3.4.**

(a) Let \( u \in \mathcal{D}(m\Delta_\infty) \), \( f = (m\Delta_\infty) u \). Then
\[
\Delta u = \frac{f}{m} \quad \text{in} \ \mathcal{D}(\Omega)',
\]
(b) If \( \frac{1}{m} \in L^p_{\text{loc}}(\Omega) \) for some \( p > \frac{N}{2} \), then
\[
\mathcal{D}(m\Delta_\infty) \subset C^0(\Omega) \cap H^1_{\text{loc}}(\Omega).
\]
(c) If \( m \in L^\infty_{\text{loc}}(\Omega) \), then \( \mathcal{D}(\Omega) \subset \mathcal{D}(m\Delta_\infty) \) and \( (m\Delta_\infty) u = m \cdot \Delta u \) for \( u \in \mathcal{D}(\Omega) \).

**Proof.** (a) Let \( \lambda > 0 \). Define \( g := \lambda u - f \in L^\infty(\Omega) \). Then \( u = R(\lambda, m\Delta_\infty) g \). If \( g \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)}) \), then the claim follows from the fact that \( R(\lambda, m\Delta_\infty) g = R(\lambda, m\Delta_2) g \). In the general case there exist \( g_k \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)}) \) such that \( g_k \to g \) for \( \sigma(\mathcal{L}(\infty), L^1(\Omega, \frac{dx}{m(x)}) ) \). Let \( u_k = R(\lambda, m\Delta_\infty) g_k \). Then
\[
-\Delta u_k = \frac{g_k - \lambda u_k}{m}.
\]
Now we use the fact that \( R(\lambda, m\Delta_\infty) = R(\lambda, m\Delta_1)' \) is continuous for the weak-*-topology \( \sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}) ) \). Hence \( u_k \to u \) for \( \sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}) ) \). Since \( \mathcal{D}(\Omega) \subset L^1(\Omega, \frac{dx}{m(x)}) \) we conclude that \( u_k \to u \) in \( \mathcal{D}(\Omega)' \). Hence \( \Delta u_k \to \Delta u \) in \( \mathcal{D}(\Omega)' \). Since \( g_k - \lambda u_k \to g - \lambda u \) for \( \sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}) ) \), it follows that \( \frac{g_k - \lambda u_k}{m} \to \frac{g - \lambda u}{m} \) in \( \mathcal{D}(\Omega)' \). Thus
\[
-\Delta u = \frac{g - \lambda u}{m} = -\frac{f}{m}.
\]
The proof of (a) is complete.
(b) This follows now from (3.2) and (3.3).
(c) Assume that \( m \in L^\infty_{\text{loc}}(\Omega) \). Let \( u \in \mathcal{D}(\Omega) \), \( f = m \cdot \Delta u \). Then \( u \in H^1_0(\Omega) \), \( f \in L^2(\Omega, \frac{dx}{m(x)}) \) and \( \Delta u = \frac{f}{m} \). Thus \( u \in \mathcal{D}(m\Delta_2) \) and \( (m\Delta_2)u = f \). Let \( \lambda > 0 \) and set \( g := \lambda u - f \). Then \( g \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)}) \) and \( R(\lambda, m\Delta_\infty)g = R(\lambda, m\Delta_2)g = u \). Thus \( u \in \mathcal{D}(m\Delta_\infty) \) and \( \lambda u - (m\Delta_\infty)u = g = \lambda u - f \), i.e., \( (m\Delta_\infty)u = f \). \( \square \)

In Proposition 3.4, the boundary condition is not incorporated. But if \( \frac{1}{m} \in L^1(\Omega) \), then \( L^\infty(\Omega) \subset L^2(\Omega, \frac{dx}{m(x)}) \), and the operator \( m\Delta_\infty \) is just the part of \( m\Delta_2 \) in \( L^\infty(\Omega) \). Thus, if \( \frac{1}{m} \in L^1(\Omega) \), then

\[
\mathcal{D}(m\Delta_\infty) = \left\{ u \in H^1_0(\Omega) \cap L^\infty(\Omega) : \exists f \in L^\infty(\Omega) \text{ s.t. } \Delta u = \frac{f}{m} \right\}
\]

(3.4)

If \( \frac{1}{m} \in L^p(\Omega) \) for some \( \infty \geq p > \frac{N}{2} \), we can even assert more.

**Proposition 3.5.** Assume that \( \frac{1}{m} \in L^p(\Omega) \), where \( \frac{N}{2} < p \leq \infty \). Then \( m\Delta_\infty \) is invertible.

**Proof.** Let \( f \in L^\infty(\Omega) \). Then \( \frac{f}{m} \in L^p(\Omega) \). Thus by Proposition 3.3 there exists \( u \in H^1_0(\Omega) \) such that \( \Delta u = \frac{f}{m} \). This shows that \( m\Delta_\infty \) is surjective. If \( u \in \mathcal{D}(m\Delta_\infty), (m\Delta_\infty)u = 0 \), then by (3.4) we have \( u \in H^1_0(\Omega) \) and \( \Delta u = 0 \). This implies that \( u = 0 \). Thus \( (m\Delta_\infty) \) is injective. Since the operator is closed, the proof is finished. \( \square \)

The positive semigroups \( T_t \) generated by \( m\Delta_\rho \) on \( L^p(\Omega, \frac{dx}{m(x)}) \) have many interesting properties. We just mention that they are always irreducible if \( \Omega \) is connected (where we assume only \( 0 < m, \frac{1}{m} \in L^1_{\text{loc}}(\Omega) \) as before). This means that

\[
(e^{t(m\Delta_\rho)}f)(x) > 0 \text{ a.e. for all } 0 \leq f \in L^p(\Omega, \frac{dx}{m(x)}), f \neq 0, \text{ and for all } t > 0.
\]

For \( p = 2 \) this follows from Ouhabaz’ simple criterion that

\[
\chi_C \cdot H^1_0(\Omega) \subset H^1_0(\Omega) \impliedby |C| = 0 \text{ or } |\Omega \setminus C| = 0
\]

for each Borel set \( C \subset \Omega \) (see [28], Section 4.2 or [3]). For another proof of irreducibility we refer to [13], and for consequences we refer to [4].

### 4. The operator \( m\Delta_0 \) on \( C_0(\Omega) \)

Let \( \Omega \subset \mathbb{R}^N \) be open and bounded. Let \( m : \Omega \to (0, \infty) \) be a measurable function such that \( m \in L^\infty(\Omega) \) and \( \frac{1}{m} \in L^p_{\text{loc}}, \) where \( p > \frac{N}{2} \). We want to define a maximal realization of \( m\Delta \) in \( C_0(\Omega) \). If \( u \in C_0(\Omega) \), then \( \Delta u \in \mathcal{D}(\Omega)' \), but \( m\Delta u \) may not be defined as a distribution. Thus the following definition is natural.

**Definition 4.1.** We define the operator \( m\Delta_0 \) on \( C_0(\Omega) \) by

\[
\mathcal{D}(m\Delta_0) := \left\{ u \in C_0(\Omega) : \exists f \in C_0(\Omega) \text{ s.t. } \Delta u = \frac{f}{m} \right\},
\]

\[(m\Delta_0)u := f.\]

Since \( \frac{1}{m} \in L^1_{\text{loc}} \subset \mathcal{D}(\Omega)' \), this definition makes sense. The notation \( (m\Delta_0) \) is purely symbolic. But if \( u \in C_0(\Omega) \cap C^2(\Omega) \) such that \( m \cdot \Delta u \in C_0(\Omega) \), then \( u \in \mathcal{D}(m\Delta_0) \) and \( (m\Delta_0)u = m \cdot \Delta u \).
Proposition 4.2. The operator $m\triangle_0$ is closed and dissipative. Moreover, if

$$R(\lambda_0, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$$

for some $\lambda_0 > 0$, then $m\triangle_0$ generates a $C_0$-semigroup of positive contractions on $C_0(\Omega)$. In that case

$$(0, \infty) \subset \rho(m\triangle_0),$$
$$R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega) \quad \text{for all } \lambda > 0 \quad \text{and}$$
$$R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)|_{C_0(\Omega)}.$$

Note that in general, $D(\Omega) \not\subseteq D(m\triangle_0)$, since we do not assume that $m$ is continuous. Thus in Proposition 4.2 density of the domain (which is necessary for the generation property) needs a separate argument.

Since $m\triangle_0$ is dissipative, it follows in particular that no proper restriction of $m\triangle_0$ may generate a $C_0$-semigroup on $C_0(\Omega)$.

We first prove dissipativity.

Lemma 4.3. Let $\lambda > 0$, $u = D(m\triangle_0)$, and $f = \lambda u - (m\triangle_0)u$. Let $c > 0$ be such that

$$f(x) \leq c \quad \text{for all } x \in \Omega.$$ 

Then $\lambda u(x) \leq c$ for all $x \in \Omega$.

Proof. By the definition of the operator we have

$$\frac{\lambda u}{m} - \triangle u = \frac{f}{m} \leq \frac{c}{m}.$$ 

Since by $u \in H^1_{\text{loc}}(\Omega)$, this implies that for $0 \leq v \in D(\Omega)$

$$\int_\Omega \frac{(\lambda u(x) - c)}{m(x)} v(x) dx + \int_\Omega \nabla u(x) \nabla v(x) dx \leq 0. \quad (4.1)$$

Since $u \in C_0(\Omega)$, $(\lambda u - c)^+$ has compact support. Let $\omega \in \Omega$ such that $\text{supp} (\lambda u - c)^+ \subset \omega$. Then $(\lambda u - c)^+ \in H^1_0(\omega)$ and $(\lambda u - c) \in H^1(\omega)$. Now (4.1) implies that

$$\int_\omega \frac{(\lambda u(x) - c)}{m(x)} v(x) dx + \frac{1}{\lambda} \int_\omega \nabla (\lambda u(x) - c) \nabla v(x) dx \leq 0$$

for all $0 \leq v \in H^1_0(\omega)$. Taking, in particular, $v := (\lambda u - c)^+$, we see that

$$\int_\omega \frac{(\lambda u(x) - c)^+}{m(x)} dx + \frac{1}{\lambda} \int_\omega |\nabla (\lambda u(x) - c)^+|^2 dx \leq 0.$$ 

This implies that $(\lambda u - c)^+ = 0$, i.e., $\lambda u \leq c$.

Applying Lemma 4.3 to $\pm u$, we see that

$$\|\lambda u\|_{L^\infty(\Omega)} \leq \|\lambda u - (m\triangle_0)u\|_\infty$$

for all $u \in D(m\triangle_0)$, i.e., $m\triangle_0$ is dissipative. But in fact, Lemma 4.3 shows that the operator $m\triangle_0$ is dispersive. We refer to ([5], [27], Chapter II) for this notion.

Proof of Proposition 4.2. The dissipativity has been proved above, and the closedness is easy to see. Now let $R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ for some $\lambda > 0$. We show
that \( \lambda \in \rho(m\triangle_0) \) and \( R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)|_{C_0(\Omega)} \). Let \( f \in C_0(\Omega) \) and consider \( u = R(\lambda, m\triangle_\infty)f \in C_0(\Omega) \). Then (by Proposition 3.3)

\[
\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \quad \text{in } D(\Omega)'.
\]

It follows that \( u \in D(m\triangle_0) \) and \( (\lambda u - (m\triangle_0)u) = f \). We have shown that \( \lambda - (m\triangle_0) \) is surjective. Since the injectivity of \( (\lambda - m\triangle_0) \) follows from the dissipativity of \( m\triangle_0 \), the closed graph theorem now implies that \( \lambda \in \rho(m\triangle_0) \). The calculation above also shows that \( R(\lambda, m\triangle_0)f = u = R(\lambda, m\triangle_\infty)f \).

By the resolvent identity (see [1], Proposition 3.II.2) for \( 0 < \lambda < \lambda_0 \), the closed graph theorem now implies that \( R(\lambda, m\triangle_0) \) is surjective. Since the injectivity of \( (\lambda, m\triangle_0) \) is continuous, it follows from the domination property above that \( R(\lambda, m\triangle_\infty)f \in C_0(\Omega) \). Thus \( C_0(\Omega) \) is invariant for all \( \lambda \geq \lambda_0 \). Hence \( [\lambda_0, \infty) \subset \rho(m\triangle_0) \).

Next we show that \( D(m\triangle_0) \) is dense in \( C_0(\Omega) \). Since \( m \in L^\infty_{\text{loc}}(\Omega) \), we have \( D(\Omega) \subset D(m\triangle_\infty) \) by Proposition 3.4. Hence \( C_0(\Omega) \subset \overline{D(m\triangle_\infty)} \). Thus, for \( f \in C_0(\Omega) \) one has

\[
\lim_{\lambda \to \infty} \lambda R(\lambda, m\triangle_0)f = \lim_{\lambda \to \infty} \lambda R(\lambda, m\triangle_\infty)f = f.
\]

Since \( \lambda R(\lambda, m\triangle_0)f \in D(m\triangle_0) \), density of the domain is proved. Now the Lumer-Phillips theorem implies that \( m\triangle_0 \) generates a contractive \( C_0 \)-semigroup. Since the resolvent of \( m\triangle_0 \) is positive, this semigroup is positive. It also follows that \( (0, \infty) \subset \rho(m\triangle_0) \). \( \square \)

We will now consider two cases which imply the invariance given in Proposition 4.2, namely that \( \Omega \) is Dirichlet regular or that the diffusion coefficient \( m(x) \) tends to 0 fast enough as \( x \) approaches the boundary. We start by discussing Dirichlet regularity.

5. \textbf{Regular points}

Let \( \Omega \subset \mathbb{R}^N \) be open, bounded and let \( \frac{N}{2} < p \leq \infty \). Let \( m: \Omega \to (0, \infty) \) be measurable such that \( m \in L^\infty_{\text{loc}}(\Omega) \) and \( \frac{1}{m} \in L^p_{\text{loc}}(\Omega) \).

\textbf{Theorem 5.1.} If \( \Omega \) is Dirichlet regular, then \( m\triangle_0 \) generates a positive contractive \( C_0 \)-semigroup on \( C_0(\Omega) \).

Thus in the case of a Dirichlet regular set, no condition on \( m(x) \) as \( x \) approaches the boundary is needed. We merely impose a (very weak) regularity condition on \( m \) in the interior of \( \Omega \).

It will be useful to prove an individual version of Theorem 5.1 first. For this we have to recall the notion of regular points.

Consider the Dirichlet problem

\[
\begin{cases}
  h \in C(\Omega) \cap C^2(\Omega), \\
  \triangle h = 0 \text{ in } \Omega, \\
  h|_{\partial\Omega} = \varphi,
\end{cases}
\]

where \( \varphi \in C(\partial\Omega) \) is given. Recall that \( \Omega \) is called \textit{Dirichlet regular} if for each \( \varphi \in C(\partial\Omega) \) a (necessarily unique) solution of (5.1) exists. If \( \Omega \) has Lipschitz boundary,
then \( \Omega \) is Dirichlet regular. Much weaker geometric properties of the boundary suffice, though. In dimension \( N = 1 \) each bounded open subset \( \Omega \) of \( \mathbb{R} \) is Dirichlet regular. If \( N = 2 \), then each simply connected bounded open set is Dirichlet regular. This is no longer true in \( \mathbb{R}^3 \). The Lebesgue cusp gives an example of a simply connected domain with continuous boundary, which is not Dirichlet regular (see [6] for more information).

A function \( u \in C(\bar{\Omega}) \) is called a \textit{subsolution} if
\[
-\Delta u \leq 0 \text{ in } D(\Omega) \quad \text{and} \quad \limsup_{x \to z, x \in \Omega} u(x) \leq \varphi(z) \quad \text{for all } z \in \partial \Omega.
\]

A function \( u \in C(\bar{\Omega}) \) is called a \textit{supersolution} if
\[
-\Delta u \geq 0 \text{ in } D(\Omega) \quad \text{and} \quad \liminf_{x \to z, x \in \Omega} u(x) \geq \varphi(z) \quad \text{for all } z \in \partial \Omega.
\]

\textbf{Theorem 5.2} (Perron). Let \( \varphi \in C(\partial \Omega) \). Then for all \( x \in \Omega \)
\[
h_\varphi(x) := \sup \{ u(x) : u \text{ is a subsolution} \}
\]
exists. Moreover,
\[
h_\varphi(x) = \inf \{ v(x) : v \text{ is a supersolution} \}.
\]
The function \( h_\varphi \) is harmonic and
\[
\inf_{\partial \Omega} \varphi \leq h_\varphi(x) \leq \sup_{\partial \Omega} \varphi
\]
for all \( x \in \Omega \). If (5.1) has a solution \( h \), then \( h_\varphi = h \).

The function \( h_\varphi \) is called the \textit{Perron solution} of (5.1).

A point \( z \in \partial \Omega \) is called \textit{regular} if
\[
\lim_{x \to z, x \in \Omega} h_\varphi(x) = \varphi(z)
\]
for all \( \varphi \in C(\partial \Omega) \). Thus \( \Omega \) is Dirichlet regular if and only if each point \( z \in \partial \Omega \) is regular. It is possible to characterize regular points by the existence of a barrier or by a capacity condition (Wiener’s theorem). We refer to [20].

Now we can formulate the local version of Theorem 5.1, which we want to prove.

\textbf{Theorem 5.3}. Let \( \Omega \) be bounded and open. Let \( z \in \partial \Omega \) be a regular point. Let \( \lambda > 0, \ f \in C_0(\Omega) \), and \( u = R(\lambda, m\Delta_\infty) f \). Then
\[
\lim_{x \to z, x \in \Omega} u(x) = 0.
\]

Thus, if \( \Omega \) is Dirichlet regular, then \( C_0(\Omega) \) is invariant under \( R(\lambda, m\Delta_\infty) \) and Theorem 5.1 follows from Proposition 4.2.

For the proof of Theorem 5.3 we use the following variational characterization of the Perron solution (see [7]).

\textbf{Theorem 5.4}. Let \( \Phi \in C(\bar{\Omega}) \) be such that \( \Delta \Phi \in H^{-1}(\Omega) \). Let \( \varphi = \Phi|_{\partial \Omega} \). Let \( u \) be the unique solution of
\[
u \in H^1_0(\Omega), \quad -\Delta u = \Delta \Phi.
\]

Then \( h_\varphi = \Phi + u \).
For our purposes the following consequence is important. Recall that by Proposition 3.3 for all \( f \in L^p(\Omega) \) there exists a unique \( u \in H_0^1(\Omega) \) such that
\[-\triangle u = f \quad \text{in } \mathcal{D}(\Omega)').\]
In fact, \( u = R(0, \triangle_p) f \), where \( \triangle_p \) denotes the Dirichlet Laplacian on \( L^p(\Omega) \). Moreover, one has \( u \in C^0(\Omega) \).

**Corollary 5.5.** Let \( f \in L^p(\Omega) \), \( u = R(0, \triangle_p) f \). Then
\[\lim_{x \to z, x \in \Omega} u(x) = 0\]
for each regular point \( z \in \partial \Omega \). Thus, if \( \Omega \) is Dirichlet regular, then \( u \in C_0(\Omega) \).

**Proof.** It follows from the Sobolev embedding theorem that \( L^p(\Omega) \subset H^{-1}(\Omega) \). Let \( f \in L^p(\Omega) \). Let \( \Phi = E \ast f \), where \( E \) is the Newtonian potential. Then (by [13], II.3, Proposition 6) \( \Phi \in C(\mathbb{R}^N) \), and in \( \mathcal{D}(\Omega)' \) we have
\[\triangle \Phi = f \in L^p(\Omega) \subset H^{-1}(\Omega) .\]
Let \( u = R(0, \triangle_p) f \). Then it follows from Theorem 5.4 that \( h_\varphi = \Phi + u \). Thus
\[\lim_{x \to z, x \in \Omega} h_\varphi(x) = \varphi(z) \quad \text{if } z \in \partial \Omega \text{ is regular.}\]
Consequently,\(^1\) \( \lim_{x \to z} u(x) = 0 \).

**Remark.** a) In [2] a more special case of Corollary 5.5 is proved with the help of \( H^1 \)-barriers (the proof of Theorem 3.8 in [2]).

b) Special cases of Theorem 5.4 were obtained previously by Hildebrandt [10] and Simader [31].

**Proof of Theorem 5.3.** (a) Let \( \lambda > 0 \), \( 0 \leq f \in C_c(\Omega) \), and \( u = R(\lambda, m \triangle \infty) f \). Then \( u \in H_0^1(\Omega) \) and
\[\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)' .\]
Moreover, \( 0 \leq u \in C^0(\Omega) \). Observe that \( 0 \leq \frac{f}{m} \in L^p(\Omega) \). Let \( w = R(0, \triangle_p) \frac{f}{m} \). Then we know that \( 0 \leq w \in H_0^1(\Omega) \cap C^0(\Omega) \) and, by Corollary 5.5 \( \lim_{x \to z} w(x) = 0 \) for all regular points \( z \in \partial \Omega \). By definition,
\[-\triangle w = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)' .\]
Thus \(-\triangle (u - w) \leq 0 \) in \( \mathcal{D}(\Omega)' \). Since \( u - w \in H^1(\Omega) \) and \( (u - w)^+ \in H_0^1(\Omega) \), it follows from the maximum principle that \( u \leq w \). Hence \( \lim_{x \to z} u(x) = 0 \) for each regular point \( z \in \partial \Omega \).

(b) Let \( z \in \partial \Omega \) be a regular point. Then by (a)
\[\lim_{x \to z, x \in \Omega} (R(\lambda, m \triangle \infty) f)(x) = 0\]
for each \( 0 \leq f \in C_c(\Omega) \), hence also for each \( f \in C_c(\Omega) \). Since \( C_c(\Omega) \) is dense in \( C_0(\Omega) \), this remains true for all \( f \in C_0(\Omega) \).

Next we show a converse of Theorem 5.1. If the diffusion coefficient \( m \) is not weak enough at the boundary, then Dirichlet regularity is necessary for \( m \triangle \infty \) to generate a \( C_0 \)-semigroup. More precisely, the following holds. Recall that \( \frac{2}{p} < p \leq \infty \).

\(^1\)We will sometimes use the notation \( \lim_{x \to z} f(x) := \lim_{x \to z, x \in \Omega} f(x) \) for \( f : \Omega \to \mathbb{R} \).
Theorem 5.6. Assume that $\frac{1}{m} \in L^p(\Omega)$. Then $m\Delta_0$ generates a $C_0$-semigroup if and only if $\Omega$ is Dirichlet regular.

For the proof we need the following.

Proposition 5.7. Let $u \in C_0(\Omega)$ be such that $-\Delta u = f \in L^p(\Omega)$ for some $p > \frac{N}{2}$. Then $u \in H^1_0(\Omega)$, hence $u = R(0, \Delta_p)f$.

This follows from [6], Corollary 1.4, since $L^p(\Omega) \subset H^{-1}(\Omega)$.

Proof of Theorem 5.6. Assume that $m\Delta_0$ generates a $C_0$-semigroup. Since $\frac{1}{m} \in L^p(\Omega)$, we know from Proposition 5.7 that $[0, \infty) \subset \rho(m\Delta_0)$ and $R(\lambda, m\Delta_0) \geq 0$ for all $\lambda \geq 0$.

We now claim $R(\lambda, m\Delta_\infty)C_0(\Omega) \subset C_0(\Omega)$ and $R(\lambda, m\Delta_0) = R(\lambda, m\Delta_\infty)\mid_{C_0(\Omega)}$ for any $\lambda > 0$. Let $f \in C_0(\Omega)$ and $u = R(\lambda, m\Delta_0)f$. Then

$$-\Delta u = \frac{f}{m} - \frac{u}{m} \in L^p(\Omega).$$

Since $u \in C_0(\Omega)$, it follows from Proposition 5.7 that $u \in H^1_0(\Omega)$. Since $\frac{1}{m} \in L^p(\Omega)$ we have $L^\infty(\Omega) \subset L^2(\Omega, \frac{dx}{m(x)})$. Thus by [6], (4) we have $u \in D(m\Delta_\infty)$ and $\lambda u - (m\Delta_\infty)u = f$. Hence $u = R(\lambda, m\Delta_\infty)f$. This proves the claim.

Since $0 \in \rho(m\Delta_\infty)$, the claim implies that

$$\lim_{\lambda \to 0} \|R(\lambda, m\Delta_0)\|_{L(C_0(\Omega))} < \infty,$$

hence $0 \in \rho(m\Delta_0)$ and $R(0, m\Delta_0) \geq 0$.

Let $0 \leq f \in C_0(\Omega)$ and $f(x) > 0$ for all $x \in \Omega$ and $u = R(0, m\Delta_0)f$. Then $u \in C_0(\Omega)$ and $-\Delta u = \frac{f}{m}$ in $D(\Omega)'$. Hence $R(0, \Delta_p)\frac{f}{m} = u \in C_0(\Omega)$ by Proposition 5.7. We deduce that $R(0, \Delta_p)g \in C_0(\Omega)$ for all $g \in L^p(\Omega)$ such that $|g| \leq \frac{f}{m}$ for some $0 \leq f \in C_0(\Omega)$. The space of all such functions $g$ is dense in $L^p(\Omega)$. Thus $R(0, \Delta_p)L^p(\Omega) \subset C_0(\Omega)$. Now it follows from [2], Theorem 2.4, that $\Omega$ is Dirichlet regular.

6. Points of weak diffusion

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $m: \Omega \to (0, \infty)$ be a bounded measurable function such that $\frac{1}{m} \in L^{p}_{\text{loc}}(\Omega)$ for some $\frac{N}{2} < p \leq \infty$. Instead of regularity we may assume that $m$ is small in a neighbourhood of a boundary point. We say that $z \in \partial\Omega$ is a point of weak diffusion (for the operator $m\Delta$) if there exist $r > 0$ and $c > 0$ such that

$$m(x) \leq c \cdot \text{dist}(x, \partial\Omega)^2$$

for all $x \in \Omega \cap B(z, r)$. If $z \in \partial\Omega$ is a point of weak diffusion, then we show that

$$\lim_{x \to z, x \in \Omega} (R(\lambda, m\Delta_\infty)f)(x) = 0$$

for all $f \in C_0(\Omega)$. We will also show that condition (6.1) is optimal in the sense that

$$m(x) \leq c \cdot \text{dist}(x, \partial\Omega)^{\alpha}$$

for some $0 < \alpha < 2$ does not suffice to enforce (6.2).

We need the notion of a regularized distance function.
Lemma 6.1. There exist a function $\sigma: \Omega \to (0, +\infty)$, which is of class $C^\infty(\Omega)$, and a constant $c_\sigma > 0$ such that
\[
c_\sigma^{-1}d(x) \leq \sigma(x) \leq c_\sigma d(x),
\]
\[
|\nabla \sigma|^2 \leq c_\sigma,
\]
\[
|\sigma \Delta \sigma| \leq c_\sigma
\]
for all $x \in \Omega$, where $d(x) := \inf \{\|x - y\|, y \in \mathbb{R}^d \setminus \Omega\}$.

See [32], Chapter 6, for a proof based on the Whitney decomposition of $\Omega$.

Since $\sigma \in C_0(\Omega)$, it follows in particular that $\sigma \in H^1_0(\Omega)$. First we consider the case $m(x) := \sigma(x)^2$.

Proposition 6.2. The operator $\sigma^2 \Delta_0$ generates a strongly continuous semigroup of positive contractions on $C_0(\Omega)$.

Proof. Let $\lambda \geq c_\sigma + 1$, where $c_\sigma$ is a constant from Lemma 6.1. Set
\[
u = R(\lambda, \sigma^2 \Delta_\infty)\sigma.
\]
Since $\sigma \in L^2(\Omega, \frac{dx}{\sigma^2})$, it follows from [31] that $0 \leq u \in H^1_0(\Omega) \cap L^2(\Omega, \frac{dx}{\sigma^2})$ and
\[
\lambda \frac{\nu}{\sigma^2} - \Delta u = \frac{\sigma}{\sigma^2} \quad \text{in } D(\Omega)'.
\]
Since $\sigma \Delta \sigma \leq c_\sigma$, it follows that $\sigma \leq \lambda \sigma - c_\sigma \sigma \leq \lambda \sigma - \sigma^2 \Delta \sigma$. Thus
\[
\lambda \frac{\nu}{\sigma^2} - \Delta u = \frac{\sigma}{\sigma^2} \leq \lambda \frac{\sigma}{\sigma^2} - \Delta \sigma \quad \text{in } D(\Omega)'.
\]
Hence
\[
\lambda \frac{(u - \sigma)}{\sigma^2} - \Delta (u - \sigma) \leq 0 \quad \text{in } D(\Omega)'.
\]
Since $u - \sigma \in H^1(\Omega)$ and $(u - \sigma)^+ \leq u \in H^1_0(\Omega)$, it follows that $(u - \sigma)^+ \in H^1_0(\Omega)$.

Now the maximum principle (see Section 2) implies that $(u - \sigma)^+ \leq 0$, i.e., $u \leq \sigma$.

We have shown that
\[
R(\lambda, \sigma^2 \Delta_\infty)\sigma \leq \sigma \quad (\lambda \geq \lambda_0 := 1 + c_\sigma).
\]
Thus, for $f \in C_0(\Omega)$ such that $|f| \leq c\sigma$, one has
\[
|R(\lambda, \sigma^2 \Delta_\infty)f| \leq c |R(\lambda, \sigma^2 \Delta_\infty)\sigma| \leq c\sigma.
\]
Consequently, $R(\lambda, \sigma^2 \Delta_\infty)f \in C_0(\Omega)$ for $\lambda \geq \lambda_0$. Since functions satisfying $|f| \leq c\sigma$ for some $c \geq 0$ are dense in $C_0(\Omega)$, we deduce that $R(\lambda, \sigma^2 \Delta_\infty)C_0(\Omega) \subset C_0(\Omega)$ for $\lambda \geq \lambda_0$. Now the claim follows from Proposition 4.2.

We comment that the result of Proposition 6.2 may be alternatively deduced from [12], Theorem 5.4. However, our argument given here is quite different from [12].

We need a local extension of the resolvents of $\sigma^2 \Delta$. Recall that $\frac{\lambda}{2} < p \leq \infty$.

Lemma 6.3. Let $\omega \Subset \Omega$, $\lambda > 0$. There exists an operator
\[
Q(\lambda, \omega) \in L(L^p(\omega), C_0(\Omega))
\]
such that
\[
Q(\lambda, \omega)f = R(\lambda, \sigma^2 \Delta_0)f \quad \text{for all } f \in L^p(\omega) \cap C_0(\Omega).
\]
For \( f \in L^p(\omega) \) the function \( u = Q(\lambda, \omega)f \) is the unique solution of
\[
\lambda \frac{u}{\sigma^2} - \triangle u = \frac{f}{\sigma^2} \text{ in } D(\Omega)'.
\]
Moreover, \( u \in H^1_0(\Omega) \).

Here we consider \( L^p(\omega) \) as a subspace of \( L^p(\Omega) \) extending functions by 0 outside \( \omega \). Similarly, we consider \( C_c(\omega) \subset C_0(\omega) \subset C_0(\Omega) \).

Proof. (a) Let \( 0 \leq f \in C_c(\omega) \). There exists \( \delta > 0 \) such that \( \sigma^2 \geq \delta \) on \( \omega \). Let \( u = R(\lambda, \sigma^2 \triangle_0)f = R(\lambda, \sigma^2 \triangle_2)f \). Then \( 0 \leq u \in H^1_0(\Omega) \) and
\[
\lambda \frac{u}{\sigma^2} - \triangle u = \frac{f}{\sigma^2} \leq \frac{1}{\delta} f.
\]
Let \( w := \frac{1}{\delta} R(0, \triangle_p)f \), where \( \triangle_p \) denotes the Dirichlet Laplacian on \( L^p(\Omega) \). Then \( w \in H^1_0(\Omega) \cap L^\infty(\Omega) \) and
\[
-\triangle w = \frac{1}{\delta} f \text{ in } D(\Omega)'.
\]
Moreover, \( \|w\|_{L^\infty(\Omega)} \leq c_1 \|f\|_{L^p(\omega)} \), where \( c_1 = \frac{1}{\delta} \|R(0, \triangle_p)\|_{L(L^p(\Omega), L^\infty(\Omega))} \) (see Proposition 3.3 (b)). We show that \( u \leq w \). In fact, we have
\[
- \triangle u \leq \lambda \frac{u}{\sigma^2} - \triangle u \leq \frac{1}{\delta} f \quad \text{and}
\]
\[
- \triangle w = \frac{1}{\delta} f,
\]
hence \( -\triangle(u - w) \leq 0 \) in \( D(\Omega)' \). Consequently, by the maximum principle (see Section 2), \( u \leq w \). Thus
\[
\|u\|_{L^\infty(\Omega)} \leq \|w\|_{L^\infty(\Omega)} \leq c_1 \|f\|_{L^p(\omega)}.
\]
We have shown that
\[
\|R(\lambda, \sigma^2 \triangle_0)f\|_{L^\infty(\Omega)} \leq c_1 \|f\|_{L^p(\omega)} \quad \text{for } 0 \leq f \in C_c(\omega),
\]
the estimate \((6.5)\) remains true for all \( f \in C_c(\omega) \). By the density of \( C_c(\omega) \) in \( L^p(\omega) \), the first claim is proved.

(b) In order to prove the second claim, let \( f \in L^p(\omega) \), \( u = Q(\lambda, \omega)f \). Let \( f_k \in C_c(\omega) \) be such that \( f_k \to f \) in \( L^p(\omega) \). Then \( u_k := Q(\lambda, \omega)f_k \to u \) in \( C_0(\Omega) \). We have \( u_k \in H^1_0(\Omega) \cap C_0(\Omega) \) and
\[
\lambda \frac{u_k}{\sigma^2} - \triangle u_k = \frac{f_k}{\sigma^2} \text{ in } D(\Omega)'.
\]
Passing to the limit as \( k \to \infty \) shows that \((6.4)\) holds.

It remains to show that \( u \in H^1_0(\Omega) \). Multiplying \((6.6)\) by \( u_k \) and integrating yields
\[
\lambda \int_\Omega \frac{u_k(x)^2}{\sigma(x)^2} \, dx + \int_\Omega |\nabla u_k(x)|^2 \, dx = \int_\Omega \frac{f_k(x)u_k(x)}{\sigma(x)^2} \, dx \leq \|u_k\|_{L^\infty(\Omega)} \frac{1}{\delta^2} \cdot |\Omega|^{1/p} \|f_k\|_{L^p(\Omega)}.
\]
This shows that $(u_k)_{k \in \mathbb{N}}$ is bounded in $H^1_0(\Omega)$. Thus, passing to a subsequence we may assume that $u_k \to w \in H^1_0(\Omega)$. Since $u_k \to u$ in $C_0(\Omega)$, it follows that $u = w \in H^1_0(\Omega)$. \hfill \Box

Now we consider a more general function $m$ satisfying the hypothesis formulated in the beginning of this section. We prove regularity of $m\Delta_\infty$ at points of weak diffusion.

**Theorem 6.4.** Let $z \in \partial \Omega$ be a point of weak diffusion (in the sense of (0.1)). Let $f \in C_0(\Omega)$, $\lambda > 0$, and $u = R(\lambda, m\Delta_\infty)f$. Then

$$\lim_{x \to z, \ z \in \Omega} u(x) = 0.$$ 

**Proof.** Let $r_1 > 0$ be a large radius such that $\Omega + B(0, r) \subset B(0, r_1)$. Consider the open set

$$\tilde{\Omega} := (\Omega \cap B(z, r)) \cup (B(0, r_1) \setminus B(z, r_2)).$$

Then $\Omega \subset \tilde{\Omega}$ and $B(z, r_2) \cap \partial \Omega \subset \partial \tilde{\Omega}$. In particular, $z \in \partial \tilde{\Omega}$. Consider a regularized distance $\tilde{\sigma}$ with respect to $\tilde{\Omega}$. Then there exists a constant $c > 0$ such that

$$m(x) \leq c\tilde{\sigma}(x)^2 \quad \text{for all } x \in \Omega.$$ 

In fact, for $x \in B(z, r) \cap \Omega$ this follows from (0.1). But for $x \in \Omega \setminus B(z, \frac{3}{4}r)$, one has $\text{dist}(x, \partial \tilde{\Omega}) \geq \frac{r}{4}$. Since $m$ is bounded, it follows that

$$m(x) \leq c_2 \left(\frac{r}{4}\right)^2 \leq c_2 \text{dist}(x, \partial \tilde{\Omega})^2$$

for all $x \in \Omega \setminus B(z, \frac{3}{4}r)$. This shows that (6.7) is valid for a suitable constant $c > 0$.

Now let $\lambda > 0$. Let $0 \leq f \in C_c(\Omega)$ and $u = R(\lambda, m\Delta_\infty)f$. Then $u \in C^0(\Omega) \cap H^1_0(\Omega)$ and

$$\frac{\lambda}{m} \frac{u}{\sigma^2} - \Delta u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$ 

Let $\rho := \frac{m}{\sigma^2}$. Then $0 < \rho \leq c$ on $\Omega$ and

$$\frac{1}{c} \leq \frac{1}{\rho} = \frac{\tilde{\sigma}^2}{m} \in L^p_{\text{loc}}(\Omega).$$

Hence

$$\frac{\lambda}{c} \frac{u}{\sigma^2} \leq \frac{\lambda}{\rho} \tilde{\sigma}^2 = \frac{\lambda u}{m}.$$ 

Thus

$$\frac{\lambda}{c} \frac{u}{\sigma^2} - \Delta u \leq \frac{f}{m} = \frac{1}{\sigma^2} \frac{f}{\rho}.$$ 

Let $\omega \subset \Omega$ be such that $\text{supp } f \subset \omega$. Consider $Q(\lambda, \omega) \in L^p(\rho, C_0(\tilde{\Omega}))$ of Lemma 6.3 defined with respect to $\tilde{\sigma}$. Let $w = Q(\frac{\lambda}{c}, \omega)\frac{f}{\rho}$. Note that $w$ is well defined, since $\frac{f}{\rho} \in L^p(\omega)$. Then $0 \leq w \in C_0(\tilde{\Omega}) \cap H^1_0(\tilde{\Omega})$ and, by (6.4),

$$\frac{\lambda}{c} \frac{w}{\sigma^2} - \Delta w = \frac{1}{\sigma^2} \frac{f}{\rho} \quad \text{in } \mathcal{D}(\tilde{\Omega})',$$

and hence also in $\mathcal{D}(\Omega)'$. Thus

$$\frac{\lambda}{c} \frac{(u - w)}{\sigma^2} - \Delta (u - w) \leq 0 \quad \text{in } \mathcal{D}(\Omega)'.$$
Recall that \( u \in H^1_0(\Omega) \cap C^b(\Omega) \). Thus \( (u - w) \in H^1(\Omega) \). Hence

\[
\frac{\lambda}{c} \int_\Omega \frac{(u(x) - w(x))^2}{\sigma(x)^2} v(x) \, dx + \int_\Omega \nabla(u(x) - w(x)) \nabla v(x) \, dx \leq 0
\]

for all \( 0 \leq v \in \mathcal{D}(\Omega) \). Since \( (u - w)^+ \in H^1(\Omega) \) and \( (u - w)^+ \leq u \in H^1_0(\Omega) \), it follows that \( (u - w)^+ \in H^1_0(\Omega) \).

Since \( u = R(\lambda, m \Delta_{\infty}) f = R(\lambda, m \Delta_2) f \), it follows that

\[
u \in L^2 \left( \Omega, \frac{dx}{m(x)} \right) \subset L^2 \left( \Omega, \frac{dx}{\sigma(x)^2} \right)
\]

because of (6.7). It follows (since also \( w \in L^2(\Omega, \frac{dx}{\sigma(x)^2}) \)) that

\[
\nu : = (u - w)^+ \in V := L^2 \left( \Omega, \frac{dx}{\sigma(x)^2} \right) \cap H^1_0(\Omega).
\]

Since \( \mathcal{D}(\Omega)_+ \) is dense in \( V_+ \) by Proposition 5.2, (6.8) remains true for \( v := \nu_1 \). This means that

\[
\frac{\lambda}{c} \int_\Omega \frac{(u(x) - w(x))^2}{\sigma(x)^2} \, dx + \int_\Omega |\nabla(u(x) - w(x))^+|^2 \, dx \leq 0
\]

This implies that \( (u - w)^+ = 0 \). Hence \( 0 \leq u \leq w \).

Since

\[
\lim_{x \to z, x \in \Omega} w(x) = 0,
\]

it follows that

\[
\lim_{x \to z, x \in \Omega} u(x) = 0.
\]

We have proved the theorem for the case when \( 0 \leq f \in C_c(\Omega) \). Hence it is also true for arbitrary \( f \in C_c(\Omega) \). Since \( R(\lambda, m \Delta_{\infty}) \in \mathcal{L}(L^\infty(\Omega)) \), and \( C_c(\Omega) \) is dense in \( C_0(\Omega) \), it follows that

\[
\lim_{x \to z, x \in \Omega} (R(\lambda, m \Delta_{\infty}) f)(x) = 0
\]

for all \( f \in C_0(\Omega) \). \( \square \)

**Corollary 6.5.** Assume that each \( z \in \partial \Omega \) is a point of weak diffusion (in the sense of (6.1)). Then \( m \Delta_0 \) generates a positive, contractive \( C_0 \)-semigroup on \( C_0(\Omega) \).

**7. Conclusion**

We may now formulate the following general generation theorem. Let \( \Omega \subset \mathbb{R}^N \) be bounded, open and \( \frac{N}{2} < p \leq \infty \). Let \( m : \Omega \to (0, \infty) \) be bounded and such that \( \frac{\lambda}{m} \in L^p_{\text{loc}}(\Omega) \).

**Theorem 7.1.** Assume that for each point \( z \in \partial \Omega \) one of the following conditions is satisfied:

(a) \( z \) is a regular point or
(b) \( z \) is a point of weak diffusion (in the sense of (6.1)).

Then \( m \Delta_0 \) generates a positive, contractive \( C_0 \)-semigroup on \( C_0(\Omega) \).

**Proof.** Theorem 5.3 and Theorem 6.4 show that \( C_0(\Omega) \) is invariant. Thus the claim follows from Proposition 4.2. \( \square \)
Finally, we show that the condition (6.1) of being a point of weak diffusion is optimal.

Let $N = 2$ and $\Omega = \{x \in \mathbb{R}^2 : 0 < |x| < 2\}$. Then $\partial \Omega = T \cup \{0\}$, where $T = \{x \in \mathbb{R}^2 : |x| = 2\}$. The points in $T$ are regular, but 0 is not regular.

Consider the function $d$ given by $d(x) = |x|$, $x \in \Omega$. Thus $d(x) = \text{dist}(x, \partial \Omega)$ for $0 < |x| < \frac{1}{2}$. Then $\frac{1}{2} \in L^q(\Omega)$ if and only if $q < 2$. Now let $0 < \beta < 2$. Then $\frac{1}{\beta} \in L^p(\Omega)$ for some $p > 1 = \frac{2}{\beta}$. Since $\Omega$ is not Dirichlet regular, it follows from Theorem 5.6 that $d^3 \triangle_0$ is not a generator.

On the other hand, if $\beta \geq 2$, then for $m = d^3$, the point 0 is of weak diffusion. Since the other boundary points are regular, it follows from Theorem 7.1 that $d^3 \triangle_0$ generates a $C_0$-semigroup on $C_0(\Omega)$.

An interesting open set in $\mathbb{R}^3$ with continuous boundary and exactly one singular point is the Lebesgue cusp (see e.g. [1] for a detailed investigation).

REFERENCES


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